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STABILITY OF TIME-PERIODIC PARALLEL FLOW OF COMPRESSIBLE VISCOELASTIC SYSTEM IN AN INFINITE LAYER

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Abstract. The stability of a time-periodic parallel flow of a compressible viscoelastic system in an infinite layer is considered under non-slip boundary conditions on the top and bottom and space-periodic condition in unbounded directions. We show that the time-periodic parallel flow is asymptotically stable and its perturbation decays exponentially as time goes to infinity if the initial data are sufficiently close to the time-periodic flow, provided that the Reynolds number and Mach number are sufficiently small, and the strength of the elasticity is sufficiently strong.

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1 Introduction

In this paper, we investigate the stability of time-periodic flow of the compressible viscoelastic system

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \rho(\partial_t v + v\cdot\nabla v) - \nu\Delta v - (\nu+\nu')\nabla \mathrm{div} v + \nabla p(\rho) = \beta^2 \mathrm{div}(\rho F^\top F) + \rho g, \\ \partial_t F + v\cdot\nabla F = (\nabla v)F \end{cases} \tag{1.1}$$

in an infinite layer Ω:

$$\Omega = \{x = (x', x_3);\ x' = (x_1, x_2) \in \mathbb{R}^2,\ 0 < x_3 < 1\}.$$

Here the superscript ${}^\top\cdot$ stands for the transposition; $\rho = \rho(x,t)$, $v = v(x,t) = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$, and $F = F(x,t) = (F^{ij}(x,t))_{1\le i,j\le 3}$ are the unknown density, velocity field, and deformation tensor, respectively, at the time $t \ge 0$ and $x \in \Omega$; $p = p(\rho)$ is the given pressure; ν and ν' are the viscosity coefficients satisfying

$$\nu > 0,\ 2\nu + 3\nu' \ge 0.$$

We also assume that $\frac{\nu'}{\nu} \le \nu_0$ for some positive constant $\nu_0 > 0$. $\beta > 0$ is the strength of the elasticity. In paticular, if we set $\beta = 0$, the system (1.1) reduces to the usual compressible Navier-Stokes equations. We assume that $p'(1) > 0$, and we set $\gamma \equiv \sqrt{p'(1)}$. g is a given external force which has the form

$$g = g^1(x_3,t)e_1,\ e_1 = {}^\top(1,0,0),\ g^1(0,t) = g^1(1,t) = 0, \tag{1.2}$$

with g^1 being T-periodic function of time t, where $T > 0$.

The system is considered under the non-slip boundary conditions

$$v|_{x_3=0,1} = 0, \tag{1.3}$$

and the initial condition

$$(\rho, v, F)|_{t=0} = (\rho_0, v_0, F_0). \tag{1.4}$$

We also assume that (ρ_0, F_0) satisfies the following identities

$$\mathrm{div}(\rho_0 {}^\top F_0) = 0,\ \rho_0 \mathrm{det} F_0 = 1. \tag{1.5}$$

According to [15, Proposition 1], the conditions (1.5) are invariant for $t \ge 0$:

$$\mathrm{div}(\rho^\top F) = 0, \rho \mathrm{det} F = 1.$$

Under a suitable condition on g, the problem (1.1)–(1.3) has a T-periodic solution $\overline{u} = (\bar\rho, \bar v, \bar F)$ satisfying the following properties:

$$\bar\rho = 1,\ \bar v = \bar v^1(x_3,t)e_1,\ \bar F = \bar F(x_3,t) = \left(\nabla(x - \bar\psi^1(x_3,t)e_1)\right)^{-1}.$$

Here $\bar{\psi}^1$ is a function satisfying

$$\bar{\psi}^1(x_3, t+T) = \bar{\psi}^1(x_3, t), \ \partial_t \bar{\psi}^1 = \bar{v}^1.$$

The aim of this paper is to study the stability of the time-periodic parallel flow $\bar{u}$.

The system (1.1) is given as one of the basic models describing motion of compressible viscoelastic fluids. We refer to [5, 12, 16] for the physical details and [3, 6, 7, 8, 14, 15] for the recent progresses related to the mathematical analysis.

Let us mention the related work about the stability of parallel flows. The parallel flow is given as the simple example of the shear flows in the hydrodynamic theory. For the usual compressible case, Kagei [10] studied the stability of stationary parallel flow, and Brezina and Kagei [2] and Brezina [1] investigated the stability of time-periodic parallel flow. As for the mathematical study of the stability of parallel flows of viscoelastic fluid, the incompressible case is studied by [4]. It was shown in [4] that the parallel flow is exponentially stable even if the space-periodic condition is not assumed. The compressible viscoelastic case with large β is considered in [9]. It was shown that the velocity part of the non-stationary parallel flow in [9] decays exponentially as $t \to \infty$ and its stability holds for $\nu \gg 1, \ \gamma \gg 1, \ \beta \gg 1$. Moreover, the result in [9] differs to that in [10] when $\nu \gg 1, \ \gamma \gg 1$. In fact, when $\beta = 0$, the solution behaves like the stationary parallel flow with non-zero velocity as $t \to \infty$. On the other hand, if $\beta \gg 1$, the solution tends to some motionless state with a nontrivial deformation as $t \to \infty$.

In this paper we extend the analysis of [9] to the case of time-periodic parallel flows. We shall show that the time-periodic parallel flow is exponentially stable under sufficiently small perturbations, if ν, γ and β are assumed to be sufficiently large compared to g^1. We briefly explain the main result of this paper in a more precise way.

We denote a periodic cell by D:

$$D = \{x = (x', x_3); x' = (x_1, x_2) \in \Pi_{j=1}^2 \mathbb{T}_{\frac{2\pi}{\alpha_j}}, \ 0 < x_3 < 1\}.$$

Here $\mathbb{T}_{\frac{2\pi}{\alpha_j}} = \mathbb{R}/\left(\frac{2\pi}{\alpha_j}\right)\mathbb{Z}$, $\alpha_j > 0$, $j = 1, 2$.

The main result of this paper states that if $\nu \gg 1, \ \gamma \gg 1, \ \beta \gg 1$, then (1.1)–(1.5) has a unique global solution (ρ, v, F) such that $(\rho, v, F) \in C([0,\infty), H^2(D))$ and $\|(\rho(t), v(t), F(t)) - (1, \bar{v}(t), \bar{F}(t))\|_{H^2} \to 0$ exponentially as $t \to \infty$ under a small initial perturbation $(\rho_0 - 1, v_0 - \bar{v}(0), F_0 - \bar{F}(0)) \in H^2(D)$.

The proof of the main result of this paper is given by an argument similar to that of Qian [14] which is based on the Matsumura-Nishida energy method [13]. To establish a priori estimates, we consider the following problem for the perturbation $u(t) = (\phi(t), w(t), G(t)) = \big(\rho(t) - 1, v(t) - \bar{v}(t), F(t) - \bar{F}(t)\big)$:

$$\begin{cases} \partial_t \phi + \bar{v}^1 \partial_{x_1} \phi + \mathrm{div} w = f_1, \\ \partial_t w + \bar{v}^1 \partial_{x_1} w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \mathrm{div} G \\ +(w^3 \partial_{x_3} \bar{v}^1) e_1 + \nu(\phi \partial_{x_3}^2 \bar{v}^1) e_1 - \beta^2 \mathrm{div}(G^\top \bar{E}) - \beta^2 (G^{33} \partial_{x_3}^2 \bar{\psi}^1) e_1 = f_2, \\ \partial_t G + \bar{v}^1 \partial_{x_1} G - \nabla w - (\nabla w) \bar{E} + w^3 \partial_{x_3} \bar{E} - (\nabla \bar{v}) G = f_3, \\ \nabla \phi = -\mathrm{div}^\top G + {}^\top \bar{E} \mathrm{div}^\top G + f_4, \\ w|_{x_3 = 0,1} = 0, \quad u|_{t=0} = u_0 = (\phi_0, w_0, G_0). \end{cases} \tag{1.6}$$

Here $\tilde{\nu} = \nu + \nu'$ and $\bar{E} = \bar{F} - I$; I is the 3×3 identity matrix; $f_i,\ i = 1,2,3,4$ are nonlinear terms. To derive the L^2 energy estimate of u, we make use of the displacement vector $\psi(x,t) = x - X(x,t)$ as in [3, 14, 16]. Here, $X(x,t)$ is the inverse of the material coordinate $x(X,t)$ which is constructed by the flow map:

$$\begin{cases} \dfrac{dx}{dt}(X,t) = v(x(X,t),t),\ t > 0 \\ x(X,0) = X \in \Omega. \end{cases}$$

Under suitable conditions on F and X, F is written by using ψ as follows

$$F = \bar{F} - \bar{F}\nabla\left(\psi - \bar{\psi}^1 e_1\right)\bar{F} + h\left(\nabla(\psi - \bar{\psi}^1 e_1)\right).$$

Here h satisfies $h(\nabla(\psi - \bar{\psi}^1(t)e_1)) = O(|\nabla(\psi - \bar{\psi}^1(t)e_1)|^2)$. Applying a variant of the Matsumura-Nishida energy method [9, 14] to (1.6) and estimating the interaction between the time-periodic parallel flow and the perturbation, we obtain the estimate:

$$\|u(t)\|^2_{H^2\times H^2\times H^3} + \int_0^t e^{-c_1(t-s)}\|u(s)\|^2_{H^2\times H^3\times H^3}\,\mathrm{d}s \le Ce^{-c_1 t}\|u_0\|^2_{H^2\times H^2\times H^3},$$

provided that $\nu \gg 1$, $\gamma \gg 1$, $\beta \gg 1$, and the initial perturbation is sufficiently small. In [9], it is assumed that the external force g converges to some stationary force exponentially as $t \to \infty$, from which it is expected that the time dependence of g can be regarded as a simple perturbation of the stationary force. Indeed, by taking into account the exponential convergence of g, the non-stationary parallel flow is decomposed into the stationary solution with zero velocity field and the exponentially decaying part; and the latter part is treated as a simple perturbation of the former part because of its exponential decay in time; in particular, the velocity field of the parallel flow converges to zero, together with its time derivative. In contrast to the situation of [9], in this paper, the parallel flow $\bar{u}$ under consideration is time-periodic, and hence, the time fluctuation of the parallel flow $\bar{u}$ remains for all t. This requires a more detail analysis of the interaction between the time-periodic parallel flow and the perturbation.

This paper is organized as follows. In Section 2 we introduce some notation and function spaces. In Section 3, we state the existence of the time-periodic flow and then give the main result of this paper. In Sections 4 and 5, we give a proof of the stability of the time-periodic parallel flow based on the Matsumura-Nishida energy method [13]. In Appendix, we prove the estimates for the time-periodic parallel flows.

2 Preliminaries

In this section, we prepare notations, function spaces and lemmata which will be used throughout the paper. In this paper, we consider functions that are a_j-periodic in x_j $(j = 1,2)$ and therefore, we define D as

$$D := \prod_{j=1}^{2} \mathbb{T}_{a_j} \times (0,1),$$

where $\mathbb{T}_{a_j} := \mathbb{R}/(a_j)\mathbb{Z},\ a_j > 0\ (j = 1, 2)$.
$L^p(D)$ $(1 \le p \le \infty)$ denotes the usual Lebesgue space on D, and its norm is denoted by $\|\cdot\|_{L^2}$. Similarly $H^m(D)$ denotes the m-th order Sobolev space, and its norm is denoted by $\|\cdot\|_{H^m}$.

$H^1_0(D)$ denotes the completion of $C_0^\infty(D)$ in $H^1(D)$, where

$$C_0^\infty(D) := \{f \in C^\infty |\ \mathrm{supp}(f) : \text{compact in } D\}.$$

We abbreviate $\{w = {}^\top(w^1, w^2, w^3)|\ w^j \in L^p(D)\}$, and $\{w = {}^\top(w^1, w^2, w^3)|\ w^j \in H^m(D)\}$ as $L^p(D)$ and $H^m(D)$, respectively, and their norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^m}$, respectively.

The inner product of $L^2(D)$ is denoted by

$$(f, g) := \int_D f(x)g(x)dx,\ \ f, g \in L^2(D).$$

The partial derivatives of a function u in x_i $(i = 1, 2, 3)$ and t are denoted by $\partial_{x_i} u$ and $\partial_t u$, respectively.

We denote the usual divergence, gradient and Laplacian with respect to x by div, ∇, and Δ, respectively. For any functions ϕ, $w = {}^\top(w^1, w^2, w^2)$, $G = (G^{ij})_{1\le i,j\le 3}$, we set

$$\begin{aligned}
\partial\phi &:= \{\partial_{x_1}\phi, \partial_{x_2}\phi\},\\
\partial w &:= \{\partial_{x_j} w^i | i = 1, 2, 3, j = 1, 2\},\\
\partial G &:= \{\partial_{x_k} G^{ij} | i, j = 1, 2, 3, k = 1, 2\},\\
\Delta' &:= \partial^2_{x_1} + \partial^2_{x_2},\\
\nabla' \cdot w' &:= \partial_{x_1} w^1 + \partial_{x_2} w^2.
\end{aligned}$$

We recall the Sobolev inequalities.

Lemma 2.1. *The following inequalities hold:*
$\|u\|_{L^\infty(0,1)} \le C\|u\|_{H^1(0,1)}$ for $u \in H^1(0, 1)$,
$\|u\|_{L^\infty(D)} \le C\|u\|_{H^2(D)}$ for $u \in H^2(D)$,
$\|u\|_{L^4(D)} \le C\|u\|_{H^1(D)}$ for $u \in H^1(D)$.

To estimate the linear terms whose coefficients include the time periodic parallel flow, we use the following lemma.

Lemma 2.2. *Let $f \in H^1(D)$ satisfy the Poincaré type inequality $\|f\|_{L^2(D)} \le C\|\nabla f\|_{L^2(D)}$, and $g \in L^2(0, 1)$. The following inequality holds:*

$$\|gf\|_{L^2(D)} \le C\|g\|_{L^2(0,1)}\|\nabla f\|_{L^2(D)}.$$

This can be proved in a similar manner to [4, Lemma 7.6]. So we omit the proof.

Lemma 2.3. [10, Lemma 8.3.] *Let m be the nonnegative integer and $1 \le \sigma < s$. Suppose that $F(x, t, y)$ is a smooth function on $\Omega \times (0, \infty) \times I$, where I is a compact interval in $\mathbb{R}$. For $|\alpha| + 2j = k$ there hold*

$$\|\partial_x^\alpha \partial_t^j [F(x, t, f_1)]f_2\|_{L^2}$$

$$\leq \begin{cases} C_0(t,f_1(t))[[f_2]]_{k-1} + C_1(t,f_1(t))\{1+|||Df_1|||_{m-1}^{|\alpha|+j-1}\}|||Df_1|||_{m-1}[[f_2]]_k, \\ C_0(t,f_1(t))[[f_2]]_{k-1} + C_1(t,f_1(t))\{1+|||Df_1|||_{m-1}^{|\alpha|+j-1}\}|||Df_1|||_m[[f_2]]_{k-1}, \end{cases}$$

where

$$[[f_2(t)]]_k := (\sum_{k=0}^{[\frac{k}{2}]} \|\partial_t^j f_2(t)\|^2_{H^{k-2j}})^{\frac{1}{2}},$$

$$|||Df_1|||_m := \begin{cases} \|\nabla f_1(t)\|_{L^2}, & m=0, \\ ([[\nabla f_1(t)]]_m^2 + [[\partial_t f_1(t)]]_{m-1}^2)^{\frac{1}{2}}, & m \geq 1, \end{cases}$$

$$C_0(t,f_1(t)) := \sum_{(\beta,l)\leq(\alpha,j),(\beta,l)\neq(0,0)} \sup_{x\in\Omega} |\partial_x^\beta \partial_t^l F(x,t,f_1(x,t))|,$$

$$C_1(t,f_1(t)) := \sum_{(\beta,l)\leq(\alpha,j),1\leq p\leq|\alpha|+j} \sup_{x\in\Omega} |\partial_x^\beta \partial_t^l \partial_y^p F(x,t,f_1(x,t))|.$$

3 Main Result

In this section, we first introduce a time-periodic parallel flow, and then give the main result on the stability of the time-periodic parallel flow.

We consider the time-periodic parallel flow defined as $(\bar\rho, \bar v, \bar F)$, where

$$\bar\rho = 1,\ \bar v = \bar v^1(x_3,t)e_1,\ \bar F = \bar F(x_3,t) = \left(\nabla(x - \bar\psi^1(x_3,t)e_1)\right)^{-1},$$

$$\bar v^1(x_3,t+T) = \bar v^1(x_3,t),\ \bar\psi^1(x_3,t+T) = \bar\psi^1(x_3,t),\ \partial_t\bar\psi^1 = \bar v^1.$$

Here $T>0$ is some constant.

We also assume the compatibility conditions for g^1:

$$\partial_{x_3}^{2j} g^1(0,t) = \partial_{x_3}^{2j} g^1(1,t) = 0,\ t\geq 0,\ j=0,1,2. \tag{3.1}$$

Then the following assertions hold true.

Proposition 3.1. *Let* $\kappa = \min\left\{\nu, \frac{\beta^2}{\nu}\right\}$. *Assume that* $g^1 \in H^1(0,T;H^4(0,1))$ *satisfies the compatibility conditions* (3.1). *Then the system* (1.1) *has a time-periodic flow* $(\bar\rho,\bar v,\bar F) = (1, \bar v^1(x_3,t)e_1, \left(\nabla(x-\bar\psi^1(x_3,t)e_1)\right)^{-1})$ *satisfying the following estimates*:

$$\sup_{t\in[0,T]} \|\bar\psi^1(t)\|^2_{H^6(0,1)} \leq \frac{C}{\beta^4}\left(1+\frac{1}{\kappa^2}\right)\|g^1\|^2_{H^1(0,T;H^4(0,1))},$$

$$\sup_{t\in[0,T]} \|\bar v^1(t)\|^2_{H^4(0,1)} \leq \frac{C}{\nu^2}\left(1+\frac{1}{\kappa^2}\right)\|g^1\|^2_{H^1(0,T;H^4(0,1))},$$

$$\sup_{t\in[0,T]} \|\partial_t\bar v^1(t)\|^2_{H^2(0,1)} \leq C\left(1+\frac{1}{\kappa^2}\right)\|g^1\|^2_{H^1(0,T;H^4(0,1))}.$$

The proof of Proposition 3.1 can be shown as in [9]. The detail will be given in the Appendix.

We next consider the stability of the time-periodic parallel flow $(1, \bar{v}, \bar{F})$.

We set $U(t) = (\phi(t), w(t), G(t)) = (\rho(t) - 1, v(t) - \bar{v}(t), F(t) - \bar{F}(t))$. Then the perturbation $U(t)$ should satisfy the following initial-boundary problem

$$\begin{cases} \partial_t \phi + \bar{v}^1 \partial_{x_1} \phi + \mathrm{div} w = f_1, & (3.2) \\ \partial_t w + \bar{v}^1 \partial_{x_1} w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \mathrm{div} G & \\ \quad + (w^3 \partial_{x_3} \bar{v}^1) e_1 + \nu (\phi \partial_{x_3}^2 \bar{v}^1) e_1 - \beta^2 \mathrm{div}(G^\top \bar{E}) - \beta^2 (G^{33} \partial_{x_3}^2 \bar{\psi}^1) e_1 = f_2, & (3.3) \\ \partial_t G + \bar{v}^1 \partial_{x_1} G - \nabla w - \nabla w \bar{E} + w^3 \partial_{x_3} \bar{E} - \nabla \bar{v} G = f_3, & (3.4) \\ \nabla \phi = -\mathrm{div}^\top G + {}^\top \bar{E} \mathrm{div}^\top G + f_4, & (3.5) \\ w|_{x3=0,1} = 0, U|_{t=0} = U_0 = (\phi_0, w_0, G_0). & \end{cases}$$

Here $\tilde{\nu} = \nu + \nu'$, $\bar{E} = \bar{F} - I = \nabla(\bar{\psi}^1 e_1)$ and $f_i, i = 1, 2, 3, 4$ denote the nonlinear terms;

$$\begin{aligned} f_1 &= -\mathrm{div}(\phi w), \\ f_2 &= -w \cdot \nabla w + \frac{\nu \phi}{1+\phi}(-\Delta w + \partial_{x_3}^2 \bar{v}^1 e_1) - \frac{\tilde{\nu}\phi}{1+\phi} \nabla \mathrm{div} w - \frac{\gamma^2 \phi}{1+\phi} \nabla \phi - \frac{\gamma^2}{1+\phi} \nabla Q(\phi) \\ &\quad + \frac{\beta^2 \phi}{1+\phi} \mathrm{div}((G^\top \bar{E}) + (\partial_{x_3}^2 \bar{\psi}^1) G^{33} e_1) + \frac{\beta^2}{1+\phi} \mathrm{div}(G^\top G + \phi(\bar{F}^\top G + G^\top \bar{F} + G^\top G)), \\ f_3 &= -w \cdot \nabla G + \nabla w G, \\ f_4 &= -\mathrm{div}(\phi^\top G), \end{aligned}$$

where

$$Q(\phi) = \phi^2 \int_0^1 P''(1 + s\phi) \mathrm{d}s, \ \ \nabla Q = O(\phi) \nabla \phi \text{ for } |\phi| \ll 1.$$

Now we mention the main result of this paper about the stability of the time-periodic parallel flow $(1, \bar{v}, \bar{F})$.

Theorem 3.2. *Under the assumption of Proposition* 3.1*, there are positive numbers* ν_1*,* γ_1*, and* $\beta_1 > 0$ *such that if* $\nu \geq \nu_1$*,* $\frac{\gamma^2}{\nu + \tilde{\nu}} \geq \gamma_1^2$*,* $\frac{\beta^2}{\gamma^2} \geq \beta_1^2$*, the following assertion holds. There is a positive number* ϵ_1 *such that if* $u_0 = (\phi_0, w_0, G_0)$ *satisfies* $\|u_0\|_{H^2 \times H^2 \times H^2} \leq \epsilon_1, w_0 \in H_0^1(\Omega)$*,* $\int_\Omega \phi_0 dx = 0$*, then there exists a unique solution* $(\phi(t), w(t), G(t)) \in C([0, \infty); H^2(D))$ *of the problem* (3.2)-(3.5)*, and the perturbation* $U(t) = (\phi(t), w(t), G(t))$ *satisfies*

$$\|u(t)\|_{H^2 \times H^2 \times H^2}^2 + \int_0^t e^{-c_1(t-s)} \|u(s)\|_{H^2 \times H^3 \times H^2}^2 \mathrm{d}s \leq C e^{-c_1 t} \|u_0\|_{H^2 \times H^2 \times H^2}^2.$$

for $t \geq 0$.

Theorem 3.2 is shown by combining local in time existence theorem and a priori estimate.

Proposition 3.3. *Let* $t_0 \geq 0$ *and* $(\phi, w, G)|_{t=t_0} = (\phi_0, w_0, G_0)$*. If we assume* $|\phi_0| \leq \frac{1}{2}, \int_\Omega \phi_0 dx = 0, w_0 \in H_0^1(\Omega), \mathrm{div}((1 + \phi_0)^\top (G_0 + \overline{F}(t_0))) = 0, (1 + \phi_0) \mathrm{div}(G_0 + \overline{F}_0(t_0)) = 1,$

then there are some numbers t_1, C independent of t_0 such that the solution $(\phi, w, G) \in C([t_0, t_1]; H^2 \times H^2 \times H^2)$ of (3.2)–(3.4) satisfying $(\phi, w, G)|_{t=t_0} = (\phi_0, w_0, G_0)$ exists uniquely and the following assertions hold:

$$\begin{aligned}
&\partial_t\phi, \partial_t G \in C([t_0, t_1]; L^2(\Omega)),\\
&w \in L^2([t_0, t_1]; H^3(\Omega)),\ \partial_t w \in C([t_0, t_1]; L^2(\Omega)) \cap L^2([t_0, t_1]; H^1(\Omega)),\\
&\|(\phi(t), w(t), G(t))\|_{H^2\times H^2\times H^2} \le C\|(\phi_0, w_0, G_0)\|_{H^2\times H^2\times H^2}.
\end{aligned}$$

Proposition 3.3 is proved by using Proposition 3.1 in a similar manner to [11, 14, 17]. We omit it.

We next state the a priori estimate.

Proposition 3.4. *There are positive numbers ν_1, γ_1, β_1 such that if $\nu \ge \nu_1$, $\frac{\gamma^2}{\nu+\tilde\nu} \ge \gamma_1^2$, $\frac{\beta^2}{\gamma^2} \ge \beta_1^2$, the following assertion holds:*

There exists a positive number δ_0 such that if $\|u_0\|_{H^2\times H^2\times H^2} \le \delta_0$, then the following inequality holds:

$$\|u(t)\|^2_{H^2\times H^2\times H^2} + \int_0^t e^{-C_1(t-s)}\|u(t)\|^2_{H^2\times H^3\times H^2} ds \le Ce^{-C_1 t}\|u_0\|^2_{H^2\times H^2\times H^2}$$

for $t \ge 0$.

4 Basic estimates

In this section, we establish the basic estimates to show the Proposition 3.4. We first introduce notations:

$$E_0 := \sum_{j=0}^{2} E_0^j, D_0 := \sum_{j=0}^{2} D_0^j, N_0 := \sum_{j=0}^{2} N_0^j.$$

Here

$$\begin{aligned}
E_0^j &:= \gamma^2\|\partial^j\phi\|^2_{L^2} + \|\partial^j w\|^2_{L^2} + \beta^2\|\partial^j G\|^2_{L^2},\\
D_0^j &:= \nu\|\nabla\partial^j w\|^2_{L^2} + \tilde\nu\|\mathrm{div}\partial^j w\|^2_{L^2},\\
N_0^j &:= \gamma^2|(\partial^j f_1, \partial^j\phi)| + |(\partial^j f_2, \partial^j w)| + \beta^2|(\partial^j f_3, \partial^j G)|.
\end{aligned}$$

Proposition 4.1. *There exists $\nu_1 > 0$ such that if $\nu \ge \nu_1$, the following estimate holds:*

$$\frac{d}{dt}E_0 + D_0 \le \frac{c}{\nu}\Big(\sum_{j=1}^{2}\|\partial^j\phi\|^2_{L^2} + \|\partial_{x_3}\phi\|^2_{L^2}\Big) + c\Big(\frac{1}{\nu} + \frac{\beta^2}{\nu}\Big)\sum_{j=0}^{2}\|\partial^j G\|^2_{L^2} + CN_0.$$

Proof. We take the inner product of (3.2) with ϕ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2_{L^2} + (\overline{v}^1\partial_{x_1}\phi, \phi) + (\mathrm{div}w, \phi) = (f_1, \phi).$$

By integration by parts, we have $(\overline{v}^1\partial_{x_1}\phi,\phi)=0$, and therefore

$$\frac{1}{2}\frac{d}{dt}\|\phi\|_{L^2}^2+(\mathrm{div}w,\phi)=(f_1,\phi). \tag{4.1}$$

We take the inner product of (3.3) with w to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2+(\overline{v}^1\partial_{x_1}w,w)+(w^3\partial_{x_3}\overline{v}^1,w^1)+(-\nu\Delta w-\tilde{\nu}\nabla\mathrm{div}w,w)\\&\quad+\nu(\phi\partial_{x_3}^2\overline{v}^1,w^1)+\gamma^2(\nabla\phi,w)-\beta^2(\mathrm{div}(G^\top\overline{F}),w)-\beta^2(G^{33}\partial_{x_3}^2\overline{\psi},w^1)=(f_2,w).\end{aligned}$$

By integration by parts, we have

$$(\overline{v}^1\partial_{x_1}w,w)=0,$$

$$(-\nu\Delta w-\tilde{\nu}\nabla\mathrm{div}w,w)=\nu\|\nabla w\|_{L^2}^2+\tilde{\nu}\|\mathrm{div}w\|_{L^2}^2,$$

and

$$(\nabla\phi,w)=-(\mathrm{div}w,\phi).$$

We thus obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2+(w^3\partial_{x_3}\overline{v}^1,w^1)+\nu\|\nabla w\|_{L^2}^2+\tilde{\nu}\|\mathrm{div}w\|_{L^2}^2+\nu(\phi\partial_{x_3}^2\overline{v}^1,w^1)\\&\quad-\gamma^2(\mathrm{div}w,\phi)-\beta^2(\mathrm{div}(G^\top\overline{F}),w)-\beta^2(G^{33}\partial_{x_3}^2\overline{\psi},w^1)=(f_2,w).\end{aligned} \tag{4.2}$$

We take the inner product of (3.4) with G to obtain

$$\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2+(\overline{v}^1\partial_{x_1}G,G)+(w^3\partial_{x_3}^2\overline{\psi}^1,G^{13})-(\nabla w\overline{F},G)-(\nabla\overline{v}G,G)=(f_3,G).$$

By integration by parts, we have

$$(\overline{v}^1\partial_{x_1}G,G)=0,\ \ (\nabla w\overline{F},G)=-(\mathrm{div}(G^\top\overline{F}),w).$$

We thus obtain

$$\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2+(w^3\partial_{x_3}^2\overline{\psi}^1,G^{13})+(\mathrm{div}(G^\top\overline{F}),w)-(\nabla\overline{v}G,G)=(f_3,G). \tag{4.3}$$

It follows from $\gamma^2\times(4.1)+(4.2)+\beta^2\times(4.3)$ that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\gamma^2\|\phi\|_{L^2}^2+\|w\|_{L^2}^2+\beta^2\|G\|_{L^2}^2)+\nu\|\nabla w\|_{L^2}^2+\tilde{\nu}\|\mathrm{div}w\|_{L^2}^2\\&=-(w^3\partial_{x_3}\overline{v}^1,w^1)-\nu(\phi\partial_{x_3}^2\overline{v}^1,w^1)\\&\quad+\beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1,w^1)-\beta^2(w^3\partial_{x_3}^2\overline{\psi}^1,G^{13})+\beta^2(\nabla\overline{v}G,G)\\&\quad+\gamma^2(f_1,\phi)+(f_2,w)+\beta^2(f_3,G).\end{aligned} \tag{4.4}$$

Since $\|\bar{\psi}^1\|_{W^{5,\infty}(0,1)}=O\left(\frac{1}{\beta^2}\right)$ and $\|\overline{v}^1\|_{W^{3,\infty}(0,1)}=O\left(\frac{1}{\nu}\right)$ by combining Proposition 3.1 and Lemma 2.1, we obtain

$$\frac{1}{2}\frac{d}{dt}(\gamma^2\|\phi\|_{L^2}^2+\|w\|_{L^2}^2+\beta^2\|G\|_{L^2}^2)+\nu\|\nabla w\|_{L^2}^2+\tilde{\nu}\|\mathrm{div}w\|_{L^2}^2$$

$$
\begin{aligned}
&\leq |(w^3\partial_{x_3}\overline{v}^1, w^1)| + \nu|(\phi\partial_{x_3}^2\overline{v}^1, w^1)| \\
&\quad + \beta^2|(G^{33}\partial_{x_3}^2\overline{\psi}^1, w^1)| + \beta^2|(w^3\partial_{x_3}^2\overline{\psi}^1, G^{13})| + \beta^2|(\nabla\overline{v}G, G)| \\
&\quad + \gamma^2|(f_1, \phi)| + |(f_2, w)| + \beta^2|(f_3, G)| \\
&\leq \left(\frac{C}{\nu} + \frac{3\nu}{8}\right)\|\nabla w\|_{L^2}^2 + \frac{C}{\nu}\|\phi\|_{L^2}^2 + C\left(\frac{1}{\nu} + \frac{\beta^2}{\nu}\right)\|G\|_{L^2}^2 \\
&\quad + \gamma^2|(f_1, \phi)| + |(f_2, w)| + \beta^2|(f_3, G)|.
\end{aligned}
$$

We take $\nu > 0$ so that $\frac{C}{\nu^2} \leq \frac{1}{8}$. It then follows from (4.4) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\gamma^2\|\phi\|_{L^2}^2 + \|w\|_{L^2}^2 + \beta^2\|G\|_{L^2}^2) + \nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 \\
&\leq \frac{C}{\nu}\|\phi\|_{L^2}^2 + C(\frac{1}{\nu} + \frac{\beta^2}{\nu})\|G\|_{L^2}^2 \\
&\quad + \gamma^2|(f_1, \phi)| + |(f_2, w)| + \beta^2|(f_3, G)|.
\end{aligned}
$$

Hence we obtain

$$
\frac{d}{dt}E_0^0 + D_0^0 \leq \frac{C}{\nu}\|\nabla\phi\|_{L^2}^2 + C\left(\frac{1}{\nu} + \frac{\beta^2}{\nu}\right)\|G\|_{L^2}^2 + CN_0^0.
$$

Similarly we can show that

$$
\frac{d}{dt}E_0^j + D_0^j \leq \frac{C}{\nu}\|\partial^j\phi\|_{L^2}^2 + C\left(\frac{1}{\nu} + \frac{\beta^2}{\nu}\right)\|\partial^j G\|_{L^2}^2 + CN_0^j, \ j = 1, 2.
$$

This completes the proof. □

We next estimate $\sum_{j=0}^{2}\|\partial^j G\|_{L^2}^2$. Let $x = x(X, t)$ be the material coordinate defined by the solution of the flow map:

$$
\begin{cases}
\frac{dx}{dt}(X, t) = v(x(X, t), t) \\
x(X, 0) = X.
\end{cases}
$$

We set $\psi = x - X$. Then we see that ψ satisfies

$$
\begin{cases}
\partial_t\psi - v = -v\cdot\nabla\psi \\
\psi|_{x_3=0,1} = 0.
\end{cases}
$$

F is rewritten as $F = I + \nabla\psi + (\nabla\psi)^2(I - \nabla\psi)^{-1}$. We set $\zeta = \psi - \overline{\psi}$. We then see that ζ satisfies $\zeta|_{\{x_3=0,1\}} = 0$ and

$$
\partial_t\zeta + \overline{v}^1\partial_{x_1}\zeta - w + w^3\partial_{x_3}\overline{\psi}^1 e_1 = f_5, \tag{4.5}
$$

where $f_5 = -v\cdot\nabla\zeta$. By using ζ, G is rewritten as

$$
G = F - \overline{F} = \nabla\zeta + \overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E} + h_1, \tag{4.6}
$$

where $h_1 = -(\overline{F}\nabla\zeta)^2(I - \overline{F}\nabla\zeta)^{-1}\overline{F}$.

Lemma 4.2. *There are positive numbers β_0 and δ_0 such that if $\beta^2 \geq \beta_0^2, \|\nabla\zeta\|_{H^2} \leq \delta_0$, the following inequality holds:*

$$C^{-1}\sum_{j=0}^{2}\|\nabla\partial^j\zeta\|_{L^2}^2 \leq \sum_{j=0}^{2}\|\partial^j G\|_{L^2}^2 \leq C\sum_{j=0}^{2}\|\nabla\partial^j\zeta\|_{L^2}^2.$$

Proof. Let δ be the positive number. We assume that $\|\nabla\zeta\|_{H^2} \leq \delta_0 < 1$ and $\beta \geq 1$. It follows from (4.6) that

$$\begin{aligned}
\|G\|_{L^2} &\geq \|\nabla\zeta\|_{L^2} - C\Big(\frac{1}{\beta^2}+\frac{1}{\beta^4}\Big)\|\nabla\zeta\|_{L^2} - C\|\nabla\zeta\|_{L^\infty}\|\nabla\zeta\|_{L^2}\\
&\geq \|\nabla\zeta\|_{L^2} - C\Big(\frac{1}{\beta^2}+\frac{1}{\beta^4}+\|\nabla\zeta\|_{H^2}\Big)\|\nabla\zeta\|_{L^2}.\\
\|G\|_{L^2} &\leq \|\nabla\zeta\|_{L^2} + C\Big(\frac{1}{\beta^2}+\frac{1}{\beta^4}+\|\nabla\zeta\|_{H^2}\Big)\|\nabla\zeta\|_{L^2}.
\end{aligned}$$

We take β and δ_0 so that $1 - C\Big(\frac{1}{\beta^2}+\frac{1}{\beta^4}+\delta_0\Big) \geq \frac{1}{\sqrt{2}}$. We thus have $\|G\|_{L^2}^2 \geq \frac{1}{2}\|\nabla\zeta\|_{L^2}^2$. Similarly we obtain $\|G\|_{L^2}^2 \leq 2\|\nabla\zeta\|_{L^2}^2$. Similary, we estimate $\partial^j G$ to have

$$\sum_{j=0}^{2}\|\partial^j G\|_{L^2}^2 \leq C\sum_{j=0}^{2}\|\nabla\partial^j\zeta\|_{L^2}^2.$$

□

We set

$$E_1 := \sum_{j=0}^{2}E_1^j, D_1 := \sum_{j=0}^{2}D_1^j, N_1 := \sum_{j=0}^{2}N_1^j,$$

where

$$\begin{aligned}
&E_1^j := (\partial^j w, \partial^j\zeta),\ D_1^j := \beta^2\|\nabla\partial^j\zeta\|_{L^2}^2,\\
&N_1^j := (\partial^j f_2, \partial^j\zeta) - \gamma^2(\partial^j f_4, \partial^j\zeta) + (\partial^j f_5, \partial^j w) - \gamma^2(\partial^j h_1, \nabla\partial^j\zeta).
\end{aligned}$$

We note that $E_0 + E_1$ and D_1 are equivalent to E_0 and $\sum_{j=0}^2\|\partial^j G\|_{L^2}^2$, respectively under the assumption in Lemma 4.2.

Proposition 4.3. *There are positive numbers $\beta_1, \gamma_1, \eta_1, \nu_2, \delta_1 > 0$ such that if $\beta^2 \geq \beta_1^2, \gamma^2 \geq \gamma_1^2, \frac{\beta^2}{\gamma^2} \geq \eta^2, \frac{\beta^2}{\nu+\tilde{\nu}} \geq \nu_2, \|\nabla\zeta\|_{H^2} \leq \delta_1$, the following estimate holds true:*

$$\frac{d}{dt}E_1 + \frac{1}{4}D_1 \leq \Big\{\frac{1}{2} + C\left(\frac{1}{\nu}+\frac{1}{\nu\beta^2}+\frac{1}{\nu^2\beta^2}\right)\Big\}D_0 + \frac{C}{\beta^2}\Big(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 + \|\partial_{x_3}\phi\|_{L^2}^2\Big) + N_1.$$

Proof. We take the inner product of (3.3) with ζ

$$(\partial_t w, \zeta) - \nu(\Delta w, \zeta) - \tilde{\nu}(\nabla \mathrm{div} w, \zeta) + \gamma^2(\nabla\phi, \zeta) - \beta^2(\mathrm{div} G, \zeta)$$

$$+ (\overline{v}^1\partial_{x_1}w,\zeta) + (w^3\partial_{x_3}\overline{v}^1,\zeta^1) + \nu(\phi\partial_{x_3}^2\overline{v}^1,\zeta^1) - \beta^2(\mathrm{div}(G^\top\overline{E}),\zeta) - \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1,\zeta^1)$$
$$= (\tilde{f}_2,\zeta).$$

By integration by parts, we have

$$\begin{aligned}
-(\mathrm{div}G,\zeta) &= (G,\nabla\zeta)\\
&= \|\nabla\zeta\|_{L^2}^2 + (\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E},\nabla\zeta) + (h_1,\nabla\zeta)\\
&\geq \|\nabla\zeta\|_{L^2}^2 - C\left(\frac{1}{\beta^2}+\frac{1}{\beta^4}\right)\|\nabla\zeta\|_{L^2}^2 - C\|\nabla\zeta\|_{H^2}\|\nabla\zeta\|_{L^2}\\
&\geq \|\nabla\zeta\|_{L^2}^2 - \frac{1}{2}\|\nabla\zeta\|_{L^2}^2 = \frac{1}{2}\|\nabla\zeta\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
&\nu(\Delta w,\zeta) + \tilde{\nu}(\nabla\mathrm{div}w,\zeta)\\
&\quad= -\nu(\nabla w,\nabla\zeta) - \tilde{\nu}(\mathrm{div}w,\mathrm{div}\zeta)\\
&\quad\leq \frac{\nu}{2}\|\nabla w\|_{L^2}^2 + \frac{\nu}{2}\|\nabla\zeta\|_{L^2}^2 + \frac{\tilde{\nu}}{2}\|\mathrm{div}w\|_{L^2}^2 + \frac{\tilde{\nu}}{2}\|\mathrm{div}\zeta\|_{L^2}^2\\
&\quad= \frac{1}{2}D_0^0 + \frac{\nu}{2}\|\nabla\zeta\|_{L^2}^2 + \frac{\tilde{\nu}}{2}\|\mathrm{div}\zeta\|_{L^2}^2.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
&(\partial_t w,\zeta) + \frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2 + \gamma^2(\nabla\phi,\zeta)\\
&\leq \frac{1}{2}D_0 + \frac{\nu}{2}\|\nabla\zeta\|_{L^2}^2 + \frac{\tilde{\nu}}{2}\|\mathrm{div}\zeta\|_{L^2}^2\\
&\quad -(\overline{v}^1\partial_{x_1}w,\zeta) - (w^3\partial_{x_3}\overline{v}^1,\zeta^1) - \nu(\phi\partial_{x_3}^2\overline{v}^1,\zeta^1)\\
&\quad +\beta^2(\mathrm{div}(G^\top\overline{E}),\zeta) + \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1,\zeta^1) + (\tilde{f}_2,\zeta).
\end{aligned} \tag{4.7}$$

It follows from (3.5) that

$$\nabla\phi = -({}^\top\overline{F})^{-1}\mathrm{div}{}^\top G + \tilde{f}_4,$$

where $\tilde{f}_4 = ({}^\top\overline{F})^{-1}f_4$. Since $({}^\top\overline{F})^{-1} = I - {}^\top\overline{E}$, we have

$$\nabla\phi = -\mathrm{div}{}^\top G + {}^\top\overline{E}\mathrm{div}{}^\top G + \tilde{f}_4. \tag{4.8}$$

We take the inner product of (4.8) with $-\zeta$ to obtain

$$-(\nabla\phi,\zeta) = (\mathrm{div}{}^\top G,\zeta) - ({}^\top\overline{E}\mathrm{div}{}^\top G,\zeta) - (\tilde{f}_4,\zeta). \tag{4.9}$$

Since $G = \nabla\zeta + \overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E} + h_1$, we have

$$(\mathrm{div}{}^\top G,\zeta) = (\mathrm{div}{}^\top(\nabla\zeta),\zeta) + (\mathrm{div}{}^\top(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}),\zeta) + (\mathrm{div}{}^\top h_1,\zeta).$$

By integration by parts, we have

$$(\mathrm{div}{}^\top(\nabla\zeta),\zeta) = (\nabla\mathrm{div}\zeta,\zeta) = -\|\mathrm{div}\zeta\|_{L^2}^2.$$

Hence we see from (4.9) that

$$\begin{aligned}
&-(\nabla\phi,\zeta)+\|\mathrm{div}\zeta\|_{L^2}^2\\
&\quad=(\mathrm{div}^\top(\overline{E}\nabla\zeta+\nabla\zeta\overline{E}+\overline{E}\nabla\zeta\overline{E}),\zeta)-({}^\top\overline{E}\mathrm{div}^\top G,\zeta)\\
&\quad+(\mathrm{div}^\top h_1,\zeta)-(\tilde{f}_4,\zeta).
\end{aligned}\tag{4.10}$$

We take the inner product of (4.5) with w to obtain

$$(\partial_t\zeta,w)+(\overline{v}^1\partial_{x_1}\zeta,w)-\|w\|_{L^2}^2+(w^3\partial_{x_3}\overline{\psi}^1,w^1)=(f_5,w).$$

Since $\|w\|_{L^2}^2\le\frac{1}{\nu}D_0^0$, we see from this identity that

$$(\partial_t\zeta,w)\le\frac{1}{\nu}D_0^0-(\overline{v}^1\partial_{x_1}\zeta,w)-(w^3\partial_{x_3}\overline{\psi}^1,w^1)+(f_5,w).\tag{4.11}$$

It follows from (4.7) + $\gamma^2\times$ (4.10) + (4.11) and $\frac{d}{dt}(w,\zeta)=(\partial_t\zeta,w)+(\zeta,\partial_t w)$ that

$$\begin{aligned}
&\frac{d}{dt}(w,\zeta)+\frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2+\gamma^2\|\mathrm{div}\zeta\|_{L^2}^2\\
&\le\left(\frac{1}{2}+\frac{C}{\nu}\right)D_0^0+\frac{\nu}{2}\|\nabla\zeta\|_{L^2}^2+\frac{\tilde{\nu}}{2}\|\mathrm{div}\zeta\|_{L^2}^2\\
&\quad-(\overline{v}^1\partial_{x_1}w,\zeta)-(w^3\partial_{x_3}\overline{v}^1,\zeta^1)-\nu(\phi\partial_{x_3}^2\overline{v}^1,\zeta^1)\\
&\quad+\beta^2(\mathrm{div}(G^\top\overline{E}),\zeta)+\beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1,\zeta^1)+\gamma^2(\mathrm{div}^\top(\overline{E}\nabla\zeta+\nabla\zeta\overline{E}+\overline{E}\nabla\zeta\overline{E}),\zeta)\\
&\quad-\gamma^2({}^\top\overline{E}\mathrm{div}^\top G,\zeta)-(\overline{v}^1\partial_{x_1}\zeta,w)-(w^3\partial_{x_3}\overline{\psi}^1,w^1)\\
&\quad+(\tilde{f}_2,\zeta)-\gamma^2(\tilde{f}_4,\zeta)+(f_5,w)+\gamma^2(\mathrm{div}^\top h_1,\zeta).
\end{aligned}\tag{4.12}$$

We set

$$\begin{aligned}
R_1:=&-(\overline{v}^1\partial_{x_1}w,\zeta)-(w^3\partial_{x_3}\overline{v}^1,\zeta^1)-\nu(\phi\partial_{x_3}^2\overline{v}^1,\zeta)+\beta^2(\mathrm{div}(G^\top\overline{E}),\zeta)\\
&+\beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1,\zeta^1)+\gamma^2(\mathrm{div}^\top(\overline{E}\nabla\zeta+\nabla\zeta\overline{E}+\overline{E}\nabla\zeta\overline{E}),\zeta)\\
&-\gamma^2({}^\top\overline{E}\mathrm{div}^\top G,\zeta)-(\overline{v}^1\partial_{x_1}\zeta,w)-(w^3\partial_{x_3}\overline{\psi}^1,w^1),\\
N_1^0:=&(\tilde{f}_2,\zeta)-\gamma^2(\tilde{f}_4,\zeta)+(f_5,w)+\gamma^2(\mathrm{div}^\top h_1,\zeta).
\end{aligned}$$

Then we see from (4.12) that

$$\begin{aligned}
&\frac{d}{dt}(w,\zeta)+\frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2+\gamma^2\|\mathrm{div}\zeta\|_{L^2}^2\\
&\le\left(\frac{1}{2}+\frac{C}{\nu}\right)D_0^0+\frac{\nu}{2}\|\nabla\zeta\|_{L^2}^2+\frac{\tilde{\nu}}{2}\|\mathrm{div}\zeta\|_{L^2}^2+R_1+N_1^0.
\end{aligned}\tag{4.13}$$

By using the Schwartz inequality, we obtain

$$R_1\le\left\{4\epsilon+C\left(\frac{1}{\beta^2}+\frac{\gamma^2}{\beta^4}+\frac{\gamma^2}{\beta^6}\right)\right\}\beta^2\|\nabla\zeta\|_{L^2}^2+\frac{C}{\epsilon}\left(\frac{1}{\nu^2\beta^2}+\frac{1}{\nu\beta^2}\right)D_0^0+\frac{C}{\epsilon\beta^2}\|\nabla\phi\|_{L^2}^2$$

for any $\epsilon>0$. This, together with (4.13), yields

$$\frac{d}{dt}(w,\zeta)+\frac{\beta^2}{2}\|\nabla\zeta\|_{L^2}^2+\gamma^2\|\mathrm{div}\zeta\|_{L^2}^2$$

$$
\begin{aligned}
&\le \left\{\frac{1}{2} + C\left(\frac{1}{\nu} + \frac{1}{\nu\beta^2} + \frac{1}{\nu^2\beta^2}\right)\right\} D_0^0 \\
&+ \{4\beta^2\epsilon + C\beta^2(\frac{1}{\beta^2} + \frac{\gamma^2}{\beta^4} + \frac{\gamma^2}{\beta^6}) + \frac{\nu + 9\tilde{\nu}}{2}\}\|\nabla\zeta\|_{L^2}^2 \\
&\quad + \frac{C}{\beta^2}\|\nabla\phi\|_{L^2}^2 + N_1^0.
\end{aligned}
$$

We set

$$
I := \Big\{4\beta^2\epsilon + C\beta^2\Big(\frac{1}{\beta^2} + \frac{\gamma^2}{\beta^4} + \frac{\gamma^2}{\beta^6}\Big) + \frac{\nu + 9\tilde{\nu}}{2}\Big\}\|\nabla\zeta\|_{L^2}^2
$$

and choose $\epsilon = \frac{1}{32}$. Then we see that

$$
I = \frac{\beta^2}{8}\|\nabla\zeta\|_{L^2}^2 + C\beta^2\Big(\frac{1}{\beta^2} + \frac{\gamma^2}{\beta^4} + \frac{\gamma^2}{\beta^6} + \frac{\nu + 9\tilde{\nu}}{2\beta^2}\Big)\|\nabla\zeta\|_{L^2}^2.
$$

We take $\beta^2, \gamma^2, \nu, \tilde{\nu}$ so that $C(\frac{1}{\beta^2} + \frac{\gamma^2}{\beta^4} + \frac{\gamma^2}{\beta^6} + \frac{\nu+9\tilde{\nu}}{2\beta^2}) \le \frac{1}{8}$, then we have $I \le \frac{\beta^2}{4}\|\nabla\zeta\|_{L^2}^2$. Thus we obtain

$$
\frac{d}{dt}E_1^0 + \frac{1}{4}D_1^0 \le \Big\{\frac{1}{2} + C\Big(\frac{1}{\nu} + \frac{1}{\nu\beta^2} + \frac{1}{\nu^2\beta^2}\Big)\Big\}D_0^0 + \frac{C}{\beta^2}\|\nabla\phi\|_{L^2}^2 + N_1^0.
$$

Similarily, we can show the following estimate for $j = 1, 2$:

$$
\frac{d}{dt}E_1^j + \frac{1}{4}D_1^j \le \Big\{\frac{1}{2} + C\Big(\frac{1}{\nu} + \frac{1}{\nu\beta^2} + \frac{1}{\nu^2\beta^2}\Big)\Big\}D_0^j + \frac{C}{\beta^2}\|\partial^j\phi\|_{L^2}^2 + N_1^j.
$$

This completes the proof. $\square$

We set

$$
\begin{aligned}
&E_2 := E_0 + E_1, \\
&D_2 := D_0 + D_1 + \frac{\beta^2 + \gamma^2}{\nu + \tilde{\nu}}\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2, \\
&N_2 := N_0 + N_1 + \sum_{j=1}^{2}(\|\partial^{j-1}f_4\|_{L^2}^2 + \|\mathrm{div}\partial^{j-1\top}h_1\|_{L^2}^2),
\end{aligned}
$$

Proposition 4.4. *The following estimate holds true:*

$$
\frac{d}{dt}E_2 + \frac{1}{32}D_2 \le C\Big(\frac{1}{\nu} + \frac{1}{\beta^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2 + CN_2. \tag{4.14}
$$

Proof. We first show that

$$
\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 \le C\Big(\frac{1}{\beta^2} + \frac{1}{\beta^6}\Big)D_1 + C\sum_{j=1}^{2}\Big(\|\partial^{j-1}f_4\|_{L^2}^2 + \|\mathrm{div}\partial^{j-1\top}h_1\|_{L^2}^2\Big). \tag{4.15}
$$

We see from (4.9) that

$$\partial_{x_j}\phi = -\partial_{x_j}(\mathrm{div}\zeta) - (\mathrm{div}^\top(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}))^j - (\mathrm{div}^\top h_1)^j + (\tilde{f}_4)^j,\ j = 1, 2. \tag{4.16}$$

Therefore we obtain

$$\|\partial\phi\|_{L^2} \le \|\nabla\partial\zeta\|_{L^2} + \sum_{j=1}^{2}\|(\mathrm{div}^\top(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}))^j\|_{L^2} + \|\mathrm{div}^\top h_1\|_{L^2} + \|\tilde{f}_4\|_{L^2}.$$

Since

$$\mathrm{div}^\top(\overline{E}\nabla\zeta) = \mathrm{div}\begin{pmatrix} \partial_{x_1}\zeta^3\partial_{x_3}\overline{\psi}^1 & 0 & 0 \\ \partial_{x_2}\zeta^3\partial_{x_3}\overline{\psi}^1 & 0 & 0 \\ \partial_{x_3}\zeta^3\partial_{x_3}\overline{\psi}^1 & 0 & 0 \end{pmatrix} = \partial_{x_3}\overline{\psi}^1\begin{pmatrix} \partial_{x_1}^2\zeta^3 \\ \partial_{x_1}\partial_{x_2}\zeta^3 \\ \partial_{x_1}\partial_{x_3}\zeta^3 \end{pmatrix},$$

$$\begin{aligned}\mathrm{div}^\top(\nabla\zeta\overline{E}) &= \mathrm{div}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_{x_3}\overline{\psi}^1\partial_{x_1}\zeta^1 & \partial_{x_3}\overline{\psi}^1\partial_{x_1}\zeta^2 & \partial_{x_3}\overline{\psi}^1\partial_{x_1}\zeta^3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \partial_{x_3}\overline{\psi}^1\partial_{x_1}^2\zeta^1 + \partial_{x_3}\overline{\psi}^1\partial_{x_1}\partial_{x_2}\zeta^2 + \partial_{x_3}(\partial_{x_3}\overline{\psi}^1\partial_{x_1}\zeta^3) \end{pmatrix},\end{aligned}$$

$$\mathrm{div}^\top(\overline{E}\nabla\zeta\overline{E}) = \mathrm{div}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\partial_{x_3}\overline{\psi}^1)^2\partial_{x_1}\zeta^3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (\partial_{x_3}\overline{\psi}^1)^2\partial_{x_1}^2\zeta^3 \end{pmatrix},$$

we have $\displaystyle\sum_{j=1}^{2}\|(\mathrm{div}^\top(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}))^j\|_{L^2}^2 \le \frac{C}{\beta^4}\|\nabla\partial\zeta\|_{L^2}^2$. It then follows

$$\|\partial\phi\|_{L^2}^2 \le C\Big(1 + \frac{1}{\beta^4}\Big)\|\nabla\partial\zeta\|_{L^2}^2 + \|\mathrm{div}^\top h_1\|_{L^2}^2 + \|\tilde{f}_4\|_{L^2}^2.$$

Similarily, the following inequality holds :

$$\|\partial^2\phi\|_{L^2}^2 \le C\Big\{\Big(1 + \frac{1}{\beta^4}\Big)\|\nabla\partial^2\zeta\|_{L^2}^2 + \|\partial f_4\|_{L^2}^2 + \|\mathrm{div}\partial^\top h_1\|_{L^2}^2\Big\}.$$

Therefore we obtain

$$\begin{aligned}\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 &\le C\Big(1 + \frac{1}{\beta^4}\Big)\sum_{j=1}^{2}\|\nabla\partial^j\zeta\|_{L^2}^2 + C\sum_{j=1}^{2}\Big(\|\partial^{j-1}f_4\|_{L^2}^2 + \|\mathrm{div}\partial^{j-1\top}h_1\|_{L^2}^2\Big) \\ &\le C\Big(\frac{1}{\beta^2} + \frac{1}{\beta^6}\Big)D_1 + C\sum_{j=1}^{2}\Big(\|\partial^{j-1}f_4\|_{L^2}^2 + \|\mathrm{div}\partial^{j-1\top}h_1\|_{L^2}^2\Big).\end{aligned}$$

By combining Proposition 4.1 and Proposition 4.2, we have

$$\begin{aligned}
&\frac{d}{dt}E_0 + D_0 \\
&\leq \frac{C}{\nu}(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 + \|\partial_{x_3}\phi\|_{L^2}^2) + C\left(\frac{1}{\nu}+\frac{\beta^2}{\nu}\right)\sum_{j=0}^{2}\|\nabla\partial^j\zeta\|_{L^2}^2 + CN_0 \\
&= \frac{C}{\nu}(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 + \|\partial_{x_3}\phi\|_{L^2}^2) + C\left(\frac{1}{\nu}+\frac{1}{\nu\beta^2}\right)D_1 + CN_0.
\end{aligned} \tag{4.17}$$

By (4.17) and Proposition 4.3, we obtain

$$\begin{aligned}
&\frac{d}{dt}(E_0+E_1) + D_0 + \frac{1}{4}D_1 \\
&\leq \Big\{\frac{1}{2} + C\Big(\frac{1}{\nu}+\frac{1}{\nu\beta^2}+\frac{1}{\nu^2\beta^2}\Big)\Big\}D_0 + C\Big(\frac{1}{\nu}+\frac{1}{\nu\beta^2}\Big)D_1 \\
&\quad + C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}\Big)\Big(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 + \|\partial_{x_3}\phi\|_{L^2}^2\Big) + C(N_0+N_1).
\end{aligned}$$

We take ν, β^2 so that $\frac{1}{2} + C(\frac{1}{\nu}+\frac{1}{\nu\beta^2}+\frac{1}{\nu^2\beta^2}) \leq \frac{15}{16}, C(\frac{1}{\nu}+\frac{1}{\nu\beta^2}) \leq \frac{3}{16}$, and then we have

$$\frac{d}{dt}(E_0+E_1)+\frac{1}{16}(D_0+D_1) \leq C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}\Big)\Big(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2+\|\partial_{x_3}\phi\|_{L^2}^2\Big)+C(N_0+N_1). \tag{4.18}$$

It follows from $\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}} \times (4.15) + (4.18)$ that

$$\begin{aligned}
&\frac{d}{dt}(E_0+E_1) + \frac{1}{16}(D_0+D_1) + \frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 \\
&\leq C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}\Big)\Big(\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2 + \|\partial_{x_3}\phi\|_{L^2}^2\Big) \\
&\quad + C\Big\{\frac{1}{\nu+\tilde{\nu}}\Big(1+\frac{\gamma^2}{\beta^2}\Big) + \frac{1}{(\nu+\tilde{\nu})\beta^4}\Big(1+\frac{\gamma^2}{\beta^2}\Big)\Big\}D_1 \\
&\quad + C\Big\{N_0 + N_1 + \sum_{j=1}^{2}(\|\partial^{j-1}f_4\|_{L^2}^2 + \|\mathrm{div}\partial^{j-1\top}h_1\|_{L^2}^2\Big\}.
\end{aligned}$$

We take $\nu+\tilde{\nu}, \gamma^2, \beta^2$ so that

$$C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}\Big) \leq \frac{1}{2}\cdot\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}} \cdot C\Big\{\frac{1}{\nu+\tilde{\nu}}\Big(1+\frac{\gamma^2}{\beta^2}\Big) + \frac{1}{(\nu+\tilde{\nu})\beta^4}\Big(1+\frac{\gamma^2}{\beta^2}\Big)\Big\} \leq \frac{1}{32}$$

. It then follows

$$\frac{d}{dt}(E_0+E_1) + \frac{1}{16}D_0 + \frac{1}{32}D_1 + \frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2$$

$$\leq C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2 + C\Big\{N_0+N_1+\sum_{j=1}^{2}(\|\partial^{j-1}f_4\|_{L^2}^2+\|\mathrm{div}\partial^{j-1\top}h_1\|_{L^2}^2)\Big\},$$

This completes the proof. □

We set

$$E_3 := \sum_{j=0}^{1} E_3^j, D_3 := \sum_{j=0}^{1} D_3^j, N_3 := \sum_{j=0}^{1} N_3^j,$$

where

$$\begin{aligned}
E_3^j &:= \nu\|\nabla\partial^j w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial^j w\|_{L^2}^2 - 2\gamma^2(\partial^j\phi, \mathrm{div}\partial^j w) + 2\beta^2(\partial^j G, \nabla\partial^j w),\\
D_3^j &:= \|\partial_t\partial^j w\|_{L^2}^2,\\
N_3^j &:= -\gamma^2(\partial^1 f_1, \mathrm{div}\partial^j w) + (\partial^j f_2, \partial_t\partial^j w) + \beta^2(\partial^j G, \nabla\partial^j w).
\end{aligned}$$

Proposition 4.5. *The following inequality holds true:*

$$\begin{aligned}
\frac{d}{dt}E_3 + D_3 \leq C\|\partial_{x_3}\phi\|_{L^2}^2 + C\Big(&\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} + \frac{\gamma^4(\nu+\tilde{\nu})}{\nu^2\tilde{\nu}(\beta^2+\gamma^2)}\\
&+\frac{\beta^2}{\nu}+\frac{\gamma^2}{\tilde{\nu}}+1+\frac{1}{\nu}+\frac{1}{\nu^3}+\frac{1}{\beta^2}+\frac{\beta^2}{\nu^3}\Big)D_2 + CN_3.
\end{aligned}$$

Proof. We take the inner product of (3.2) with $\mathrm{div}w$ to obtain

$$(\partial_t\phi, \mathrm{div}w) + (\overline{v}^1\partial_{x_1}\phi, \mathrm{div}w) + \|\mathrm{div}w\|_{L^2}^2 = (f_1, \mathrm{div}w).$$

Since $(\partial_t\phi, \mathrm{div}w) = \frac{d}{dt}(\phi, \mathrm{div}w) - (\phi, \mathrm{div}\partial_t w)$, we have

$$\frac{d}{dt}(\phi, \mathrm{div}w) - (\phi, \mathrm{div}\partial_t w) + (\overline{v}^1\partial_{x_1}\phi, \mathrm{div}w) + \|\mathrm{div}w\|_{L^2}^2 = (f_1, \mathrm{div}w). \tag{4.19}$$

We take the inner product of (3.3) with $\partial_t w$ to obtain

$$\begin{aligned}
\|\partial_t w\|_{L^2}^2 &- \nu(\Delta w, \partial_t w) - \tilde{\nu}(\nabla\mathrm{div}w, \partial_t w) + \gamma^2(\nabla\phi, \partial_t w) - \beta^2(\mathrm{div}G, \partial_t w)\\
&+ (w^3\partial_{x_3}\overline{v}^1, \partial_t w^1) + (\overline{v}^1\partial_{x_1}w, \partial_t w) + \nu(\phi\partial_{x_3}^2\overline{v}^1, \partial_t w^1) - \beta^2(\mathrm{div}(G^\top\overline{E}), \partial_t w)\\
&- \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1, \partial_t w^1) = (f_2, \partial_t w).
\end{aligned}$$

We set

$$\begin{aligned}
\tilde{R}_2 := -(\overline{v}^1\partial_{x_1}w, \partial_t w) &- (w^3\partial_{x_3}\overline{v}^1, \partial_t w^1) - \nu(\phi\partial_{x_3}^2\overline{v}^1, \partial_t w^1)\\
&+ \beta^2(\mathrm{div}(G^\top\overline{E}), \partial_t w) + \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1, \partial_t w^1).
\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}
-\nu(\Delta w, \partial_t w) - \tilde{\nu}(\nabla\mathrm{div}w, \partial_t w) &= \nu(\nabla w, \nabla\partial_t w) + \tilde{\nu}(\mathrm{div}w, \mathrm{div}\partial_t w)\\
&= \frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2),
\end{aligned}$$

$$(\nabla\phi, \partial_t w) = -(\phi, \mathrm{div}\partial_t w),\ (\mathrm{div}G, \partial_t w) = -(G, \nabla\partial_t w).$$

We then obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2) + \|\partial_t w\|_{L^2}^2 \\ &\quad -\gamma^2(\phi, \mathrm{div}\partial_t w) + \beta^2(G, \nabla\partial_t w) = \tilde{R}_2 + (f_2, \partial_t w).\end{aligned} \tag{4.20}$$

We take the inner product of (3.4) with ∇w to obtain

$$\begin{aligned}&(\partial_t G, \nabla w) - \|\nabla w\|_{L^2}^2 + (\overline{v}^1\partial_{x_1}G, \nabla w) \\ &\quad +(w^3\partial_{x_3}\overline{\psi}, \partial_{x_3}w^1) - (\nabla w\overline{E}, \nabla w) - (\nabla\overline{v}G, \nabla w) = (f_3, \nabla w).\end{aligned}$$

Since $(\partial_t G, w) = \frac{d}{dt}(G, \nabla w) - (G, \nabla\partial_t w)$, we have

$$\begin{aligned}&(\partial_t G, \nabla w) - \|\nabla w\|_{L^2}^2 + (\overline{v}^1\partial_{x_1}G, \nabla w) \\ &\quad +(w^3\partial_{x_3}^2\overline{\psi}^1, \partial_{x_3}w^1) - (\nabla w\overline{E}, \nabla w) - (\nabla\overline{v}G, \nabla w) = (f_3, \nabla w).\end{aligned} \tag{4.21}$$

It follows from $-\gamma^2 \times (4.19) + (4.20) + \beta^2 \times (4.21)$ that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 - 2\gamma^2(\phi, \mathrm{div}w) + 2\beta^2(G, \nabla w)) + \|\partial_t w\|_{L^2}^2 \\ &\quad = \beta^2\|\nabla w\|_{L^2}^2 + \gamma^2\|\mathrm{div}w\|_{L^2}^2 + \tilde{R}_2 + \gamma^2(\overline{v}^1\partial_{x_1}\phi, \mathrm{div}w) \\ &\qquad +\beta^2(w^3\partial_{x_3}^2\overline{\psi}, \partial_{x_3}w^1) + \beta^2(\overline{v}^1\partial_{x_1}G, \nabla w) - \beta^2(\nabla w\overline{E}, \nabla w) - \beta^2(\nabla\overline{v}G, \nabla w) \\ &\qquad -\gamma^2(f_1, \mathrm{div}w) + (f_2, \partial_t w) + \beta^2(f_3, \nabla w).\end{aligned} \tag{4.22}$$

We set

$$\begin{aligned}R_2 := \tilde{R}_2 &+ \gamma^2(\overline{v}^1\partial_{x_1}\phi, \mathrm{div}w) - \beta^2(w^3\partial_{x_3}^2\overline{\psi}^1, \partial_{x_3}w^1) - \beta^2(\overline{v}^1\partial_{x_1}G, \nabla w) \\ &+ \beta^2(\nabla\overline{v}G, \nabla w) + \beta^2(\nabla w\overline{E}, \nabla w).\end{aligned}$$

We then see that (4.22) is rewritten as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 - 2\gamma^2(\phi, \mathrm{div}w) + 2\beta^2(G, \nabla w)) + \|\partial_t w\|_{L^2}^2 \\ &= \gamma^2\|\mathrm{div}w\|_{L^2}^2 + \beta^2\|\nabla w\|_{L^2}^2 + R_2 + N_3^0.\end{aligned}$$

Since $\gamma^2\|\mathrm{div}w\|_{L^2}^2 + \beta^2\|\nabla w\|_{L^2}^2 \le \left(\frac{\beta^2}{\nu} + \frac{\gamma^2}{\tilde{\nu}}\right)D_0^0$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}w\|_{L^2}^2 - 2\gamma^2(\phi, \mathrm{div}w) + 2\beta^2(G, \nabla w)) + \|\partial_t w\|_{L^2}^2 \\ &\le \left(\frac{\beta^2}{\nu} + \frac{\gamma^2}{\tilde{\nu}}\right)D_0^0 + R_2 + N_3^0.\end{aligned} \tag{4.23}$$

By using the Schwartz inequality, R_2 is estimated as

$$\begin{aligned}R_2 \le{}& \frac{1}{2}\|\partial_t w\|_{L^2}^2 + \left(\frac{1}{2} + \frac{C}{\nu} + \frac{C}{\nu^3}\right)D_0^0 \\ &+C\left(1 + \frac{\gamma^4}{\nu^2\tilde{\nu}}\right)\|\partial\phi\|_{L^2}^2 + C\|\partial_{x_3}\phi\|_{L^2}^2 + C\left(1 + \frac{\beta^4}{\nu^3}\right)(\|G\|_{L^2}^2 + \|\partial G\|_{L^2}^2).\end{aligned} \tag{4.24}$$

By combining (4.23) and (4.24), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\nu\|\nabla w\|_{L^2}^2+\tilde{\nu}\|\mathrm{div}w\|_{L^2}^2-2\gamma^2(\phi,\mathrm{div}w)+2\beta^2(G,\nabla w))+\frac{1}{2}\|\partial_t w\|_{L^2}^2\\
&\le C\Big(\frac{\beta^2}{\nu}+\frac{\gamma^2}{\tilde{\nu}}+1+\frac{1}{\nu}+\frac{1}{\nu^3}\Big)D_0^0\\
&\quad+C\left(1+\frac{\gamma^4}{\nu^2\tilde{\nu}}\right)\|\partial\phi\|_{L^2}^2+C\|\partial_{x_3}\phi\|_{L^2}^2+C\Big(1+\frac{\beta^4}{\nu^3}\Big)(\|G\|_{L^2}^2+\|\partial G\|_{L^2}^2)+N_3^0.
\end{aligned}
$$

Similarly the following estimate holds

$$
\begin{aligned}
&\frac{d}{dt}(\nu\|\nabla\partial w\|_{L^2}^2+\tilde{\nu}\|\mathrm{div}\partial w\|_{L^2}^2-2\gamma^2(\partial\phi,\mathrm{div}\partial w)+2\beta^2(\partial G,\nabla\partial w))+\|\partial_t\partial w\|_{L^2}^2\\
&\le C\Big(\|\partial\phi\|_{L^2}^2+\frac{\gamma^4}{\nu^2\tilde{\nu}}\|\partial^2\phi\|_{L^2}^2\Big)+C\Big(\frac{\beta^2}{\nu}+\frac{\gamma^2}{\tilde{\nu}}+1+\frac{1}{\nu}+\frac{1}{\nu^3}\Big)D_0^1\\
&\quad+C\Big(1+\frac{\beta^4}{\nu^3}\Big)(\|\partial G\|_{L^2}^2+\|\partial^2 G\|_{L^2}^2)+N_3^1.
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
&\frac{d}{dt}E_3+D_3\\
&\le C\Big(\|\partial_{x_3}\phi\|_{L^2}^2+\|\partial\phi\|_{L^2}^2\Big)+\frac{C\gamma^4}{\nu^2\tilde{\nu}}\sum_{j=1}^{2}\|\partial^j\phi\|_{L^2}^2+C\Big(\frac{\beta^2}{\nu}+\frac{\gamma^2}{\tilde{\nu}}+1+\frac{1}{\nu}+\frac{1}{\nu^3}\Big)D_0\\
&\quad+C\Big(1+\frac{\beta^4}{\nu^3}\Big)\cdot\frac{1}{\beta^2}D_1+CN_3\\
&\le C\|\partial_{x_3}\phi\|_{L^2}^2+C\Big(\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}+\frac{\gamma^4(\nu+\tilde{\nu})}{\nu^2\tilde{\nu}(\beta^2+\gamma^2)}\\
&\quad+\frac{\beta^2}{\nu}+\frac{\gamma^2}{\tilde{\nu}}+1+\frac{1}{\nu}+\frac{1}{\nu^3}+\frac{1}{\beta^2}+\frac{\beta^2}{\nu^3}\Big)D_2+CN_3.
\end{aligned}
$$

This completes the proof. □

Now we estimate $\partial_{x_3}\phi$. We define $\dot{\phi}$ by

$$\dot{\phi}:=\partial_t\phi+(\overline{v}+w)\cdot\nabla\phi.$$

We see from (3.2) that

$$\dot{\phi}=-\mathrm{div}w+\tilde{f}_1, \tag{4.25}$$

where $\tilde{f}_1=-\phi\mathrm{div}w$. We next set

$$E_4:=\sum_{j=0}^{1}E_4^j, D_4:=\sum_{j=0}^{1}D_4^j, N_4:=\sum_{j=0}^{1}N_4^j,$$

where

$$\tilde{D_3}=\frac{1}{64}D_2+\frac{1}{\beta^2+\gamma^2}D_3,$$

$$
\begin{aligned}
E_4^j &:= \|\partial^j \partial_{x_3}\phi\|_{L^2}^2, \\
D_4^j &:= \frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial^j \partial_{x_3}\phi\|_{L^2}^2 + b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}(\|\partial^j \dot{\phi}\|_{L^2}^2 + \|\partial^j \partial_{x_3}\dot{\phi}\|_{L^2}^2), \\
N_4^j &:= |(\partial^j \tilde{f}_6, \partial^j \partial_{x_3}\phi)| + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial^j \tilde{f}_1\|_{L^2}^2 + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial^j \tilde{f}_1\|_{L^2}^2 + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial^j f_6\|_{L^2}^2.
\end{aligned}
$$

Here $b_0 > 0$ is a constant independent of $\nu,\ \tilde{\nu},\ \gamma^2,\ \beta^2$, and $\tilde{f}_6 := \frac{1}{\nu+\tilde{\nu}} f_6 - \partial_{x_3}((\overline{v}+w)\cdot\nabla\phi)$, $f_6 := (\tilde{f}_2)^3 + \beta^2(f_4)^3 + (\nu+\tilde{\nu})\partial_{x_3}\tilde{f}_1$.

Proposition 4.6. *The following inequality holds true:*

$$
\frac{d}{dt}E_4 + D_4 \le \frac{1}{2}\tilde{D}_3 + CN_4.
$$

Proof. It follows from (4.25) that

$$
\sum_{j=0}^{1}\|\partial^j \dot{\phi}\|_{L^2}^2 \le \frac{C}{\tilde{\nu}}D_0 + C\|\tilde{f}_1\|_{H^2}^2. \tag{4.26}
$$

and

$$
\partial_{x_3}^2 w^3 = -\partial_{x_3}\dot{\phi} - \partial_{x_3}(\nabla'\cdot w') + \partial_{x_3}\tilde{f}_1. \tag{4.27}
$$

We see from the 3rd equation of (3.3) that

$$
\begin{aligned}
&-(\nu+\tilde{\nu})\partial_{x_3}^2 w^3 + \gamma^2\partial_{x_3}\phi - \beta^2\partial_{x_3}G^{33} \\
&\quad = -\partial_t w^3 + \nu\Delta' w^3 + \tilde{\nu}\partial_{x_3}(\nabla'\cdot w') + \beta^2(\partial_{x_1}G^{31} + \partial_{x_2}G^{32}) \\
&\qquad -\overline{v}^1\partial_{x_1}w^3 + \beta^2\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33} + (f_2)^3.
\end{aligned} \tag{4.28}
$$

We see from (3.5) that

$$
\partial_{x_3}\phi + \partial_{x_3}G^{33} = -\partial_{x_1}G^{13} - \partial_{x_2}G^{23} - \partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi + (f_4)^3. \tag{4.29}
$$

By $(\nu+\tilde{\nu})\times(4.27) + (4.28) + \beta^2\times(4.29)$ to eliminate $\partial_{x_3}^2 w^3$ and $\partial_{x_3}G^{33}$, we obtain

$$
\begin{aligned}
&(\gamma^2+\beta^2)\partial_{x_3}\phi + (\nu+\tilde{\nu})\partial_{x_3}\dot{\phi} \\
&= -\partial_t w^3 + \nu\Delta' w^3 + \tilde{\nu}\partial_{x_3}(\nabla'\cdot w') + \beta^2(\partial_{x_1}(G^{31}-G^{13}) + \partial_{x_2}(G^{32}-G^{23})) \\
&\quad -\beta^2\partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi + \beta^2\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33} - \overline{v}^1\partial_{x_1}w^3 + f_6.
\end{aligned} \tag{4.30}
$$

We take the inner product of (4.30) with $\partial_{x_3}\phi$ to obtain

$$
\begin{aligned}
&(\gamma^2+\beta^2)\|\partial_{x_3}\phi\|_{L^2}^2 + (\nu+\tilde{\nu})(\partial_{x_3}\dot{\phi}, \partial_{x_3}\phi) \\
&= -(\partial_t w^3, \partial_{x_3}\phi) + \nu(\Delta' w^3, \partial_{x_3}\phi) + \tilde{\nu}(\partial_{x_3}(\nabla'\cdot w'), \partial_{x_3}\phi) - \beta^2(\partial_{x_3}\tilde{\psi}^1\partial_{x_1}\phi, \partial_{x_3}\phi) \\
&\quad +\beta^2(\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33}, \partial_{x_3}\phi) - (\overline{v}\partial_{x_1}w^3, \partial_{x_3}\phi) \\
&\quad +\beta^2(\partial_{x_1}(G^{31}-G^{13}) + \partial_{x_2}(G^{32}-G^{23}), \partial_{x_3}\phi) + (f_6, \partial_{x_3}\phi).
\end{aligned} \tag{4.31}
$$

By the definition of $\dot{\phi}$, the left hand side of (4.31) is calculated as

$$
(\gamma^2+\beta^2)\|\partial_{x_3}\phi\|_{L^2}^2 + (\nu+\tilde{\nu})(\partial_{x_3}\dot{\phi}, \partial_{x_3}\phi)
$$

$$= \frac{1}{2}(\nu+\tilde{\nu})\frac{d}{dt}\|\partial_{x_3}\phi\|_{L^2}^2 + (\gamma^2+\beta^2)\|\partial_{x_3}\phi\|_{L^2}^2 + (\nu+\tilde{\nu})(\partial_{x_3}((\overline{v}+w)\cdot\nabla\phi), \partial_{x_3}\phi).$$

On the other hand, the right hand side of (4.31) is estimated as

$$\begin{aligned}
&-(\partial_t w^3, \partial_{x_3}\phi) + \nu(\Delta' w^3, \partial_{x_3}\phi) + \tilde{\nu}(\partial_{x_3}(\nabla'\cdot w'), \partial_{x_3}\phi) - \beta^2(\partial_{x_3}\tilde{\psi}^1\partial_{x_1}\phi, \partial_{x_3}\phi) \\
&\quad +\beta^2(\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33}, \partial_{x_3}\phi) - (\overline{v}\partial_{x_1}w^3, \partial_{x_3}\phi) \\
&\quad +\beta^2(\partial_{x_1}(G^{31}-G^{13}) + \partial_{x_2}(G^{32}-G^{23}), \partial_{x_3}\phi) + (f_6, \partial_{x_3}\phi) \\
&\le \frac{\beta^2+\gamma^2}{2}\|\partial_{x_3}\phi\|_{L^2}^2 + \frac{4}{\beta^2+\gamma^2}\|\partial_t w^3\|_{L^2}^2 + \frac{C\nu^2}{\beta^2+\gamma^2}\|\nabla\partial w\|_{L^2}^2 + \frac{C}{\beta^2+\gamma^2}\|\partial\phi\|_{L^2}^2 \\
&\quad +\frac{C}{\beta^2+\gamma^2}\|\partial G\|_{L^2}^2 + \frac{C}{\nu^2(\beta^2+\gamma^2)}\|\nabla w\|_{L^2}^2 + \frac{C\beta^2}{\beta^2+\gamma^2}\|\partial G\|_{L^2}^2 + (f_6, \partial_{x_3}\phi).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\frac{1}{2}(\nu+\tilde{\nu})\frac{d}{dt}\|\partial_{x_3}\phi\|_{L^2}^2 + \frac{\beta^2+\gamma}{2}\|\partial_{x_3}\phi\|_{L^2}^2 + (\nu+\tilde{\nu})(\partial_{x_3}((\overline{v}+w)\cdot\nabla\phi), \partial_{x_3}\phi) \\
&\quad \le +\frac{4}{\beta^2+\gamma^2}\|\partial_t w^3\|_{L^2}^2 + \frac{C\nu^2}{\beta^2+\gamma^2}\|\nabla\partial w\|_{L^2}^2 + \frac{C}{\beta^2+\gamma^2}\|\partial\phi\|_{L^2}^2 \\
&\qquad + \frac{C}{\beta^2+\gamma^2}\|\partial G\|_{L^2}^2 + \frac{C}{\nu^2(\beta^2+\gamma^2)}\|\nabla w\|_{L^2}^2 + \frac{C\beta^2}{\beta^2+\gamma^2}\|\partial G\|_{L^2}^2 + (f_6, \partial_{x_3}\phi).
\end{aligned}$$

Dividing this inequality by $\nu+\tilde{\nu}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_{x_3}\phi\|_{L^2}^2 + \frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}\phi\|_{L^2}^2 \\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|\partial_t w^3\|_{L^2}^2 + \nu^2\|\nabla\partial w\|_{L^2}^2 + \|\partial\phi\|_{L^2}^2 + \|\partial G\|_{L^2}^2 \\
&\qquad + \beta^2\|\partial G\|_{L^2}^2 + \frac{1}{\nu^2}\|\nabla w\|_{L^2}^2) + (\tilde{f}_6, \partial_{x_3}\phi),
\end{aligned}$$

Similarly, we obtain the following inequality

$$\begin{aligned}
&\frac{d}{dt}\|\partial\partial_{x_3}\phi\|_{L^2}^2 + \frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial\partial_{x_3}\phi\|_{L^2}^2 \\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}(\|\partial_t\partial w\|_{L^2}^2 + \nu^2\|\nabla\partial^2 w\|_{L^2}^2 + \frac{1}{\nu^2}\|\nabla\partial w\|_{L^2}^2 + \|\partial^2\phi\|_{L^2}^2 \\
&\qquad + (\beta^2+1)\|\nabla\partial^2\zeta\|_{L^2}^2) + (\partial\tilde{f}_6, \partial\partial_{x_3}\phi).
\end{aligned}$$

By $\frac{1}{\sqrt{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}} \times$ (4.30), we have

$$\begin{aligned}
&\sqrt{\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}}\partial_{x_3}\dot{\phi} + \sqrt{\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}}\partial_{x_3}\phi \\
&= \frac{1}{\sqrt{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}}(-\partial_t w^3 + \nu\Delta' w^3 + \tilde{\nu}\partial_{x_3}(\nabla'\cdot w') \\
&\quad +\beta^2(\partial_{x_1}(G^{31}-G^{13}) + \partial_{x_2}(G^{32}-G^{23})) \\
&\quad -\beta^2\partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi + \beta^2\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33} - \overline{v}^1\partial_{x_1}w^3 + f_6).
\end{aligned}$$

We thus obtain

$$
\begin{aligned}
&\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}\dot{\phi}\|_{L^2}^2 \\
&\le C\Bigg(\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial_{x_3}\phi\|_{L^2}^2+\frac{1}{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}\|\partial_t w\|_{L^2}^2 \\
&+\left(1+\frac{1}{\nu^2(\nu+\tilde{\nu})^2}\right)\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla\partial w\|_{L^2}^2 \\
&\quad+\frac{\beta^4+1}{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}\|\partial G\|_{L^2}^2+\frac{1}{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}(\|\partial\phi\|_{L^2}^2 \\
&\quad\|\nabla w\|_{L^2}^2+\frac{1}{(\nu+\tilde{\nu})(\beta^2+\gamma^2)}\|f_6\|_{L^2}^2\Bigg).
\end{aligned}
\tag{4.32}
$$

By adding (4.32), the following estimate holds true:

$$
\begin{aligned}
\frac{d}{dt}E_4+D_4 &\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\sum_{j=0}^{1}(\|\partial_t\partial^j w\|_{L^2}^2+\nu^2\|\nabla\partial^{j+1}w\|_{L^2}^2 \\
&\quad+\frac{1}{\nu^2}\|\nabla\partial^j w\|_{L^2}^2+\|\partial^{j+1}\phi\|_{L^2}^2+(\beta^2+1)\|\nabla\partial^{j+1}\zeta\|_{L^2}^2)+N_4 \\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\left((\beta^2+\gamma^2)+\nu+\frac{1}{\nu^3}+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}+\left(1+\frac{1}{\beta^2}\right)\right)\tilde{D}_3+CN_4 \\
&\le \frac{1}{2}\tilde{D}_3+CN_4.
\end{aligned}
$$

This completes the proof. □

We estimate $\partial_{x_3}^2\phi$. We set $\tilde{D}_4:=\tilde{D}_3+D_4$.

Proposition 4.7. *The following inequality holds true:*

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_{x_3}^2\phi\|_{L^2}^2+\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2+b_0\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\partial_{x_3}^2\dot{\phi}\|_{L^2}^2 \\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial_t\nabla w\|_{L^2}^2+C\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla^2\partial_{x_3}w\|_{L^2}^2+C\frac{\beta^4+1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial\partial_{x_3}G\|_{L^2}^2 \\
&\quad+C\left(\frac{1}{\nu+\tilde{\nu}}+\frac{1}{\beta^2+\gamma^2}\right)\tilde{D}_4+|(\partial_{x_3}\tilde{f}_6,\partial_{x_3}^2\phi)|.
\end{aligned}
$$

Proof. By applying $\frac{1}{\nu+\tilde{\nu}}\partial_{x_3}$ to (4.30), we have

$$
\begin{aligned}
&\partial_{x_3}^2\dot{\phi}+\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\partial_{x_3}^2\phi \\
&=\frac{1}{\nu+\tilde{\nu}}\big(-\partial_t\partial_{x_3}w^3+\nu\partial_{x_3}\Delta' w^3+\tilde{\nu}\partial_{x_3}^2(\nabla'\cdot w') \\
&\quad+\beta^2(\partial_{x_1}\partial_{x_3}(G^{31}-G^{13})+\partial_{x_2}\partial_{x_3}(G^{32}-G^{23})) \\
&\quad-\beta^2\partial_{x_3}(\partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi)+\beta^2\partial_{x_3}(\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33})-\partial_{x_3}(\overline{v}^1\partial_{x_1}w^3)+\partial_{x_3}f_6),
\end{aligned}
\tag{4.33}
$$

We take the inner product of $\partial_{x_3}^2\phi$ with (4.33) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_{x_3}^2\phi\|_{L^2}^2 + \frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2 + (\partial_{x_3}^2((\overline{v}+w)\cdot\nabla\phi), \partial_{x_3}^2\phi)\\
&= \frac{1}{\nu+\tilde{\nu}}\big(-\partial_t\partial_{x_3}w^3 + \nu\partial_{x_3}\Delta' w^3 + \tilde{\nu}\partial_{x_3}^2(\nabla'\cdot w')\\
&\quad +\beta^2(\partial_{x_1}\partial_{x_3}(G^{31}-G^{13}) + \partial_{x_2}\partial_{x_3}(G^{32}-G^{23}))\\
&\quad -\beta^2\partial_{x_3}(\partial_{x_3}\overline{\psi}^1\partial_{x_1}\phi) + \beta^2\partial_{x_3}(\partial_{x_3}\overline{\psi}^1\partial_{x_1}G^{33}) - \partial_{x_3}(\overline{v}^1\partial_{x_1}w^3) + \partial_{x_3}f_6, \partial_{x_3}^2\phi).
\end{aligned}
\tag{4.34}
$$

We have the estimate for the right hand side of (4.34) in a similar manner to Proposition 4.7

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_{x_3}^2\phi\|_{L^2}^2 + \frac{1}{2}\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2 + (\partial_{x_3}^2((\overline{v}+w)\cdot\nabla\phi), \partial_{x_3}^2\phi)\\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial_t\nabla w\|_{L^2}^2 + C\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla^2\partial_{x_3}w\|_{L^2}^2 + C\frac{\beta^4+1}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial\partial_{x_3}G\|_{L^2}^2\\
&\quad + C\left(\frac{1}{\nu+\tilde{\nu}} + \frac{1}{\beta^2+\gamma^2}\right)\tilde{D}_4 + |(\partial_{x_3}\tilde{f}_6, \partial_{x_3}^2\phi)|.
\end{aligned}
$$

This completes the proof. □

We next estimate higher order derivatives of w and G. We set

$$
\begin{aligned}
E_5 &:= E_2 + \frac{1}{\beta^2+\gamma^2}E_3 + E_4 + \|\partial_{x_3}^2\phi\|_{L^2}^2,\\
D_5 &:= \tilde{D}_4 + \frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2,\\
N_5 &:= N_2 + \frac{1}{\beta^2+\gamma^2}N_3 + N_4 + |(\partial_{x_3}\tilde{f}_6, \partial_{x_3}^2\phi)|,\\
\tilde{D}_5 &:= D_5 + \frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}}\|\partial_{x_3}^2\phi\|_{L^2}^2 + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla\dot{\phi}\|_{H^1}^2.
\end{aligned}
$$

Proposition 4.8. *There exists a positive number $\beta_4^2 > 0$ such that if $\beta^2 > \beta_4^2$, then the following inequality holds true:*

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_{x_3}G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\partial_{x_3}G\|_{L^2}^2 + b_1\frac{\nu}{\beta^2}\|\nabla^2 w\|_{L^2}^2\\
&\le \frac{C}{\nu\beta^2}\|\nabla q\|_{H^1}^2 + C\left(\frac{1}{\nu\beta^4} + \frac{1}{\beta^6} + \frac{1}{\nu\beta^8}\right)\tilde{D}_5 + \left|\left(\partial_{x_3}f_3 + \frac{\beta^2}{\nu}\partial_{x_3}h_1, \partial_{x_3}G\right)\right|.
\end{aligned}
$$

Proof. We first introduce the following quantity

$$
q = \nu w + \beta^2\zeta.
$$

Since $w = \frac{1}{\nu}q - \frac{\beta^2}{\nu}\zeta$, we have

$$
\nabla w = \frac{1}{\nu}\nabla q - \frac{\beta^2}{\nu}\nabla\zeta
$$

$$= \frac{1}{\nu}\nabla q - \frac{\beta^2}{\nu}G + \frac{\beta^2}{\nu}(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}) + \frac{\beta^2}{\nu}h_1.$$

By inserting this to (3.4), we obtain

$$\begin{aligned} \partial_t G + \frac{\beta^2}{\nu}G = \frac{1}{\nu}\nabla q - w^3\partial_{x_3}\overline{E} - \overline{v}^1\partial_{x_1}G + \nabla w\overline{E} + \nabla\overline{v}G \\ + \frac{\beta^2}{\nu}(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}) + \frac{\beta^2}{\nu}h_1 + f_3. \end{aligned} \tag{4.35}$$

By applying ∂_{x_3} to (4.35) and taking the inner product of $\partial_{x_3}G$, it holds the following identity:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial_{x_3}G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\partial_{x_3}G\|_{L^2}^2 \\ &= \frac{1}{\nu}(\nabla\partial_{x_3}q, \partial_{x_3}G) - (\partial_{x_3}(w^3\partial_{x_3}^2\overline{\psi}^1), \partial_{x_3}G^{13}) - (\partial_{x_3}(\overline{v}^1\partial_{x_1}G), \partial_{x_3}G) \\ &\quad + (\partial_{x_3}(\nabla w\overline{E}), \partial_{x_3}G) + (\partial_{x_3}(\nabla\overline{v}G), \partial_{x_3}G) + \frac{\beta^2}{\nu}(\partial_{x_3}(\overline{E}\nabla\zeta), \partial_{x_3}G) \\ &\quad + \frac{\beta^2}{\nu}(\partial_{x_3}(\nabla\zeta\overline{E}), \partial_{x_3}G) + \frac{\beta^2}{\nu}(\partial_{x_3}(\overline{E}\nabla\zeta\overline{E}), \partial_{x_3}G) + \left(\partial_{x_3}f_3 + \frac{\beta^2}{\nu}\partial_{x_3}h_1, \partial_{x_3}G\right). \end{aligned} \tag{4.36}$$

The right-hand side of (4.36) is estimated by

$$\begin{aligned} &\frac{\beta^2}{\nu}\left(\frac{1}{4} + \frac{C}{\beta^2}\right)\|\partial_{x_3}G\|_{L^2}^2 + \frac{C}{\nu\beta^2}\|\nabla q\|_{H^1}^2 \\ &+ C\Big(\frac{1}{\nu\beta^4} + \frac{1}{\beta^6} + \frac{1}{\nu\beta^8}\Big)\tilde{D}_5 + \left|\left(\partial_{x_3}f_3 + \frac{\beta^2}{\nu}\partial_{x_3}h_1, \partial_{x_3}G\right)\right|. \end{aligned}$$

We thus obtain

$$\begin{aligned} &\frac{d}{dt}\|\partial_{x_3}G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\partial_{x_3}G\|_{L^2}^2 \\ &\le \frac{C}{\nu\beta^2}\|\nabla q\|_{H^1}^2 + C\Big(\frac{1}{\nu\beta^4} + \frac{1}{\beta^6} + \frac{1}{\nu\beta^8}\Big)\tilde{D}_5 + \left|\left(\partial_{x_3}f_3 + \frac{\beta^2}{\nu}\partial_{x_3}h_1, \partial_{x_3}G\right)\right|. \end{aligned}$$

This completes the proof. □

Proposition 4.9. *The following inequality holds true:*

$$\begin{aligned} \|\nabla q\|_{H^1}^2 \le C\Big(&\beta^2 + \gamma^2 + \frac{(\beta^2+\gamma^2)\tilde{\nu}^2}{\nu+\tilde{\nu}} + \frac{\gamma^2(\nu+\tilde{\nu})}{\beta^2} \\ &+ \nu + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} + \frac{1}{\nu^3} + \frac{1}{\beta^2}\Big)\tilde{D}_5 + C\|\partial_{x_3}G\|_{L^2}^2 \\ &+ C(\tilde{\nu}^2\|\tilde{f}_1\|_{H^1}^2 + \|f_2\|_{L^2}^2 + \beta^2\|h_1\|_{H^1}^2). \end{aligned}$$

Proof. By simple calculation, we have

$$\|\nabla q\|_{H^1}^2 = \|\nabla q\|_{L^2}^2 + \|\nabla\partial q\|_{L^2}^2 + \|\partial_{x_3}^2 q\|_{L^2}^2 \le C(\nu+\beta^2)D_6 + \|\partial_{x_3}^2 q\|_{L^2}^2.$$

By using $\dot{\phi}$ and q, it follows from (3.3) that

$$\begin{aligned}&\partial_t w - \Delta q + \tilde{\nu}\nabla\dot{\phi} + \gamma^2\nabla\phi + \overline{v}^1\partial_{x_1} w + (w^3\partial_{x_3}\overline{v}^1)\boldsymbol{e}_1\\ &-\nu(\phi\partial_{x_3}^2\overline{v}^1)\boldsymbol{e}_1 - \beta^2\mathrm{div}(G^\top\overline{E}) - \beta^2 G^{33}(\partial_{x_3}\overline{\psi}^1)\boldsymbol{e}_1\\ &-\beta^2\mathrm{div}(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E})\\ &= -\tilde{\nu}\nabla\tilde{f}_1 + f_2 + \beta^2\mathrm{div}h_1,\end{aligned} \tag{4.37}$$

which gives

$$\begin{aligned}\|\partial_{x_3}^2 q\|_{L^2}^2 &\leq C(\|\partial_t w\|_{L^2}^2 + \tilde{\nu}^2\|\nabla\dot{\phi}\|_{L^2}^2 + (\gamma^4+1)\|\nabla\phi\|_{L^2}^2 + \|\partial^2 q\|_{L^2}^2 + \frac{1}{\nu^2}\|\nabla w\|_{L^2}^2\\ &\quad + \|G\|_{L^2}^2 + \|\partial G\|_{L^2}^2 + \|\nabla\zeta\|_{H^1}^2 + \tilde{\nu}^2\|\tilde{f}_1\|_{H^1}^2 + \|f_2\|_{L^2}^2 + \beta^2\|h_1\|_{H^1}^2)\\ &\leq C\Big(\beta^2 + \gamma^2 + \frac{(\beta^2+\gamma^2)\tilde{\nu}^2}{\nu+\tilde{\nu}} + \frac{\gamma^2(\nu+\tilde{\nu})}{\beta^2} + \nu + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} + \frac{1}{\nu^3} + \frac{1}{\beta^2}\Big)\tilde{D}_5\\ &\quad + C\|\partial_{x_3} G\|_{L^2}^2 + C(\tilde{\nu}^2\|\tilde{f}_1\|_{H^1}^2 + \|f_2\|_{L^2}^2 + \beta^2\|h_1\|_{H^1}^2).\end{aligned}$$

This completes the proof. □

We introduce the following quantities to estimate $\nabla^2 G$

$$\begin{aligned}E_6 &:= E_5 + \frac{1}{\tilde{\nu}(\beta^2+\gamma^2)}\|\partial_{x_3} G\|_{L^2}^2,\\ D_6 &:= \tilde{D}_5 + \frac{\beta^2}{\nu\tilde{\nu}(\beta^2+\gamma^2)}\|\partial_{x_3} G\|_{L^2}^2,\\ N_6 &:= N_5 + \frac{1}{\tilde{\nu}(\beta^2+\gamma^2)}\Big(\frac{\tilde{\nu}^2}{\nu\beta^2}\|\tilde{f}_1\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|f_2\|_{L^2}^2 + \frac{1}{\nu}\|h_1\|_{H^1}^2\\ &\qquad + \left|\left(\nabla^2\left(f_3 + \frac{\beta^2}{\nu}h_1\right), \nabla^2 G\right)\right|\Big).\end{aligned}$$

Proposition 4.10. *The following inequality holds true:*

$$\begin{aligned}&\frac{d}{dt}\|\nabla^2 G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\nabla^2 G\|_{L^2}^2\\ &\leq \frac{C}{\nu\beta^2}\|\partial_t\nabla w\|_{L^2}^2 + \frac{C\nu}{\beta^6}\|\nabla^2 w\|_{H^1}^2 + C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{\tilde{\nu}}{\beta^2} + 1 + \frac{\tilde{\nu}}{\nu}\Big)D_6\\ &\quad + C\Big(\left|\left(\nabla^2\left(f_3 + \frac{\beta^2}{\nu}h_1\right), \nabla^2 G\right)\right| + \frac{\nu+\tilde{\nu}}{\nu\beta^2}\|\tilde{f}_1\|_{H^2}^2 + \frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2 + \frac{\beta^2}{\nu}\|f_4\|_{H^1}^2 + \frac{\beta^2}{\nu}\|h_1\|_{H^2}^2\Big).\end{aligned}$$

Proof. By applying ∇^2 to (4.35) and taking the inner product of $\nabla^2 G$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla^2 G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\nabla^2 G\|_{L^2}^2\\ &\leq \frac{\beta^2}{2\nu}\|\nabla^2 G\|_{L^2}^2 + C\Big(\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2 + \frac{1}{\nu\beta^2}\|G\|_{H^2}^2 + \frac{\nu}{\beta^6}\|\nabla w\|_{H^2}^2\Big)\\ &\quad + |(\nabla^2(f_3 + \frac{\beta^2}{\nu}h_1), \nabla^2 G)|.\end{aligned} \tag{4.38}$$

The estimates of each terms are given as follows

$$\begin{aligned}\frac{1}{\nu\beta^2}\|G\|_{H^2}^2 &= \frac{1}{\nu\beta^2}\|\nabla^2 G\|_{L^2}^2 + \frac{1}{\nu\beta^2}(\|\partial_{x_3}G\|_{L^2}^2 + \|\partial G\|_{L^2}^2) + \frac{1}{\nu\beta^2}\|G\|_{L^2}^2 \\ &\le \frac{1}{\nu\beta^2}\|\nabla^2 G\|_{L^2}^2 + C\Big(\frac{1}{\nu\beta^4} + \frac{\tilde\nu}{\beta}\cdot\frac{\beta^2+\gamma^2}{\beta^2}\Big)D_6 \\ &\le \frac{1}{\nu\beta^2}\|\nabla^2 G\|_{L^2}^2 + C\Big(\frac{1}{\nu\beta^4} + \frac{\tilde\nu}{\beta^2}\Big)D_6.\end{aligned}$$

Since

$$\begin{aligned}\|\nabla w\|_{H^2}^2 &= \|\nabla^3 w\|_{L^2}^2 + \|\nabla^2 w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \\ &\le \frac{C}{\nu}D_6 + \|\nabla^2 w\|_{L^2}^2 + \|\nabla^3 w\|_{L^2}^2.\end{aligned}$$

we need the estimate for $\|\nabla^k w\|_{L^2}^2$, $k = 2, 3$ to complete the proof. To derive this, we show the following estimate for $\|\nabla^3 q\|_{L^2}^2$.

Lemma 4.11. *The following inequality holds true:*

$$\begin{aligned}&\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2 \\ &\le \frac{C}{\nu\beta^2}(\|\partial_t\nabla w\|_{L^2}^2 + \|\nabla^2 G\|_{L^2}^2) \\ &+ C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \Big(1+\frac{\gamma^2}{\beta^2}\Big)\Big(1+\frac{\tilde\nu}{\nu}\Big) \\ &+ \frac{\nu+\tilde\nu}{\beta^2+\gamma^2}(\beta^4+\gamma^2)\cdot\frac{1}{\nu\beta^2} + \frac{\tilde\nu(\beta^2+\gamma^2)}{\beta^4}\Big)D_6 \\ &+ \frac{(\nu+\tilde\nu)^2}{\nu\beta^2}\|\tilde f_1\|_{H^2}^2 + \frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2 + \frac{\beta^2}{\nu}\|f_4\|_{H^1}^2 + \frac{\beta^2}{\nu}\|h_1\|_{H^2}^2.\end{aligned}$$

Before starting the proof of Lemma 4.11, we introduce the following estimate for the solution to the Stokes system.

Lemma 4.12. *Let $r \in H^k(\Omega)$ and $s \in H^{k+1}(\Omega)$ with $\int_\Omega r\,\mathrm{d}x = 0$. If (u, p) satisfies the Stokes system*

$$\begin{cases}\mathrm{div} u = r \text{ in } \Omega, \\ -\Delta u + \nabla p = s \text{ in } \Omega, \\ u = 0 \text{ on } \{x_3 = 0, 1\}.\end{cases}$$

Then there exists a positive constant C independent of (u, p) such that

$$\|\nabla^{k+2}u\|_{L^2} + \|\nabla^{k+1}p\|_{L^2} \le C(\|r\|_{H^{k+1}} + \|s\|_{H^k} + \|\nabla u\|_{L^2}). \tag{4.39}$$

Proof. It follows from an easy calculation that

$$\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2 = \frac{1}{\nu\beta^2}(\|\nabla^2\partial q\|_{L^2}^2 + \|\partial_{x_3}^3 q\|_{L^2}^2).$$

We see from (4.16), (4.25) and (4.37) that $(\partial_{x_j}\phi, \partial_{x_j}q)$ $(j = 1, 2)$ satisfies the following problem

$$\begin{cases} \mathrm{div}\partial_{x_j}q = r_j \text{ in } \Omega, \\ -\Delta\partial_{x_j}q + \gamma^2\nabla\partial_{x_j}\phi = s_j \text{ in } \Omega, \\ \partial_{x_j}q = 0 \text{ on } \{x_3 = 0, 1\}, \end{cases}$$

where

$$\begin{aligned} r_j = &-\nu\partial_{x_j}\dot{\phi} - \beta^2\partial_{x_j}\phi - \beta^2\partial_{x_3}\overline{\psi}^1\partial_{x_j}\partial_{x_1}\zeta^3 - \nu\partial_{x_j}\tilde{f}_1 - \beta^2(\mathrm{div}^\top h_1)^j + \beta^2(f_4)^j, \\ s_j = &-\partial_t\partial_{x_j}w - \overline{v}^1\partial_{x_1}\partial_{x_j}w - (\partial_{x_j}w^3\partial_{x_3}\overline{v}^1)\boldsymbol{e}_1 - \tilde{\nu}\nabla\partial_{x_j}\dot{\phi} \\ &+\nu(\partial_{x_j}\phi\partial_{x_3}^2\overline{v}^1)\boldsymbol{e}_1 + \beta^2\mathrm{div}(\partial_{x_j}G^\top\overline{E}) + \beta^2(\partial_{x_j}G^{33}\partial_{x_3}\overline{\psi}^1)\boldsymbol{e}_1 \\ &+\beta^2\mathrm{div}(\overline{E}\nabla\partial_{x_j}\zeta + \nabla\partial_{x_j}\zeta\overline{E} + \overline{E}\nabla\partial_{x_j}\zeta\overline{E}) \\ &-\tilde{\nu}\nabla\partial_{x_j}\tilde{f}_1 + \partial_{x_j}f_2 + \beta^2\mathrm{div}\partial_{x_j}h_1. \end{aligned}$$

It follows from Lemma 4.12 that

$$\begin{aligned} \|\nabla^2\partial q\|_{L^2}^2 \leq C\Big(&(\beta^2+\gamma^2)(1+\nu+\tilde{\nu}) + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}(\beta^4+1) \\ &+ \frac{1}{\nu^3} + \frac{1}{\beta^2} + \frac{\nu\tilde{\nu}(\beta^2+\gamma^2)}{\beta^2}\Big)D_6 \\ &+ C((\nu+\tilde{\nu})^2\|\tilde{f}_1\|_{H^2}^2 + \|f_2\|_{H^1}^2 + \beta^4\|f_4\|_{H^1}^2 + \beta^2\|h_1\|_{H^2}^2). \end{aligned}$$

We see from (4.37) that

$$\begin{aligned} \partial_{x_3}^2 q = &\partial_t w - \Delta' q + \tilde{\nu}\nabla\dot{\phi} + \gamma^2\nabla\phi + \overline{v}^1\partial_{x_1}w + (w^3\partial_{x_3}\overline{v}^1)\boldsymbol{e}_1 - \nu(\phi\partial_{x_3}^2\overline{v}^1)\boldsymbol{e}_1 \\ &- \beta^2\mathrm{div}(G^\top\overline{E}) - \beta^2(G^{33}\partial_{x_3}^2\overline{\psi}^1)\boldsymbol{e}_1 - \beta^2\mathrm{div}(\overline{E}\nabla\zeta + \nabla\zeta\overline{E} + \overline{E}\nabla\zeta\overline{E}) \\ &+ \tilde{\nu}\nabla\tilde{f}_1 - f_2 - \beta^2\mathrm{div}h_1. \end{aligned}$$

By differentiating in x_3 and taking L^2-norm to this equation, we have

$$\begin{aligned} \|\partial_{x_3}^3 q\|_{L^2}^2 \leq\ & C\Big\{\|\partial_t\nabla w\|_{L^2}^2 + \|\nabla\partial^2 q\|_{L^2}^2 + \tilde{\nu}^2\|\nabla\dot{\phi}\|_{H^1}^2 + (\gamma^4+1)\|\nabla\phi\|_{H^1}^2 \\ &+ \frac{1}{\nu^2}(\|\nabla w\|_{L^2}^2 + \|\nabla\partial w\|_{L^2}^2) + \|G\|_{H^2}^2\Big\} \\ &+ C(\tilde{\nu}^2\|\tilde{f}_1\|_{H^2}^2 + \|f_2\|_{H^1}^2 + \beta^4\|h_1\|_{H^2}^2) \\ \leq\ & C(\|\partial_t\nabla w\|_{L^2}^2 + \|\nabla^2 G\|_{L^2}^2) \\ &+ C\Big(\beta^2 + \nu + \tilde{\nu}^2\cdot\frac{\beta^2+\gamma^2}{\nu+\tilde{\nu}} \\ &+ (\gamma^4+1)\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} + \frac{1}{\nu^3} + \frac{1}{\beta^2}\Big)\tilde{D}_5 \\ &+ C((\nu+\tilde{\nu})^2\|\tilde{f}_1\|_{H^2}^2 + \|f_2\|_{H^1}^2 + \beta^4\|h_1\|_{H^2}^2). \end{aligned}$$

Hence we obtain

$$
\begin{aligned}
&\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2 \\
&\le \frac{C}{\nu\beta^2}(\|\partial_t\nabla w\|_{L^2}^2+\|\nabla^2 G\|_{L^2}^2) \\
&\quad + C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}+\Big(1+\frac{\gamma^2}{\beta^2}\Big)\Big(1+\frac{\tilde{\nu}}{\nu}\Big) \\
&\quad + \frac{(\nu+\tilde{\nu})(\beta^4+\gamma^4)}{\nu\beta^2(\beta^2+\gamma^2)}\Big)\tilde{D}_5 \\
&\quad + \frac{(\nu+\tilde{\nu})^2}{\nu\beta^2}\|\tilde{f}_1\|_{H^2}^2+\frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2+\frac{\beta^2}{\nu}\|f_4\|_{H^1}^2+\frac{\beta^2}{\nu}\|h_1\|_{H^2}^2.
\end{aligned}
\tag{4.40}
$$

□

Proof of Proposition 4.10 (continued)
We see from (4.38) and (4.40) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^2 G\|_{L^2}^2+\frac{\beta^2}{\nu}\|\nabla^2 G\|_{L^2}^2 \\
&\le \frac{\beta^2}{2\nu}\|\nabla^2 G\|_{L^2}^2 + C\Big(\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2+\frac{1}{\nu\beta^2}\|G\|_{H^2}^2+\frac{\nu}{\beta^6}\|\nabla w\|_{H^2}^2\Big) \\
&\quad + \left|\left(\nabla^2\left(f_3+\frac{\beta^2}{\nu}h_1\right),\nabla^2 G\right)\right| \\
&\le \frac{\beta^2}{4\nu}\|\nabla^2 G\|_{L^2}^2+\frac{C}{\nu\beta^2}\|\nabla^2 G\|_{L^2}^2+\frac{C}{\nu\beta^2}\|\partial_t\nabla w\|_{L^2}^2+\frac{C\nu}{\beta^6}(\|\nabla^2 w\|_{L^2}^2+\|\nabla^3 w\|_{L^2}^2) \\
&\quad + C\Big(\frac{1}{\nu\beta^4}+\frac{\tilde{\nu}}{\beta^2}+\frac{1}{\nu}+\frac{1}{\beta^2}+\Big(1+\frac{\gamma^2}{\beta^2}\Big)\Big(1+\frac{\tilde{\nu}}{\nu}\Big)+\frac{\tilde{\nu}(\beta^2+\gamma^2)}{\beta^4}\Big)D_6 \\
&\quad + C\Big(\left|\left(\nabla^2\left(f_3+\frac{\beta^2}{\nu}h_1\right),\nabla^2 G\right)\right|+\frac{\nu+\tilde{\nu}}{\nu\beta^2}\|\tilde{f}_1\|_{H^2}^2+\frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2+\frac{\beta^2}{\nu}\|f_4\|_{H^1}^2+\frac{\beta^2}{\nu}\|h_1\|_{H^2}^2\Big).
\end{aligned}
$$

We take ν,β so that $\frac{C}{\nu\beta^2}\le 1\le\frac{\beta^2}{4\nu}$. Then we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla^2 G\|_{L^2}^2+\frac{\beta^2}{\nu}\|\nabla^2 G\|_{L^2}^2 \\
&\le \frac{C}{\nu\beta^2}\|\partial_t\nabla w\|_{L^2}^2+\frac{C\nu}{\beta^6}\|\nabla^2 w\|_{H^1}^2+C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}+\frac{\tilde{\nu}}{\beta^2}+1+\frac{\tilde{\nu}}{\nu}\Big)D_6 \\
&\quad + C\Big(\left|\left(\nabla^2\left(f_3+\frac{\beta^2}{\nu}h_1\right),\nabla^2 G\right)\right|+\frac{\nu+\tilde{\nu}}{\nu\beta^2}\|\tilde{f}_1\|_{H^2}^2 \\
&\quad + \frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2+\frac{\beta^2}{\nu}\|f_4\|_{H^1}^2+\frac{\beta^2}{\nu}\|h_1\|_{H^2}^2\Big).
\end{aligned}
$$

This completes the proof of of Proposition 4.10. □

We set

$$E_7 := \frac{\nu(\beta^2+\gamma^2)^2}{\beta^2}E_6+\|\nabla^2 G\|_{L^2}^2,$$

$$
\begin{aligned}
D_7 &:= \frac{\nu(\beta^2+\gamma^2)^2}{\beta^4} D_6 + \frac{\nu}{\beta^6}\|\nabla^2 w\|_{L^2}^2 + \frac{\beta^2}{C\nu}\|\nabla^2 G\|_{L^2}^2,\\
N_7 &:= \frac{\nu(\beta^2+\gamma^2)^2}{\beta^4} N_6 + |(\nabla^2(f_3 + \frac{\beta^2}{\nu}h_1), \nabla^2 G)| + \frac{(\nu+\tilde{\nu})}{\nu\beta^2}\|\tilde{f}_1\|_{H^2}^2\\
&\quad + \frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2 + \frac{\beta^2}{\nu}\|f_4\|_{H^1}^2 + \frac{\beta^2}{\nu}\|h_1\|_{H^2}^2.
\end{aligned}
$$

Proposition 4.13. *The following inequality holds true:*

$$
\begin{aligned}
\frac{\nu}{\beta^2}\|\nabla^3 w\|_{L^2}^2 &\le \frac{C}{\nu\beta^2}\|\partial_t\nabla w\|_{L^2}^2 + C\Big(\frac{1}{\beta^2} + \frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\Big) D_7\\
&\quad + C\Big\{\frac{(\nu+\tilde{\nu})^2}{\nu\beta^6}\|\tilde{f}_1\|_{H^2}^2 + \frac{1}{\nu\beta^6}\|f_2\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|f_4\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|h_1\|_{H^2}^2\Big\}.
\end{aligned}
$$

Proof. It follows from $\frac{\sqrt{\nu}}{\beta}w = \frac{1}{\sqrt{\nu}\beta}q - \frac{\beta}{\sqrt{\nu}}\zeta$ that

$$
\begin{aligned}
&\frac{\nu}{\beta^2}\|\nabla^3 w\|_{L^2}^2\\
&\le C\Big(\frac{1}{\nu\beta^2}\|\nabla^3 q\|_{L^2}^2 + \frac{\beta^2}{\nu}\|G\|_{H^2}^2\Big)\\
&\le C\Big(\frac{C}{\nu\beta^2}(\|\partial_t\nabla w\|_{L^2}^2 + \|\nabla^2 G\|_{L^2}^2) + \frac{C}{\beta^4}\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \Big(1+\frac{\gamma^2}{\beta^2}\Big)\Big(1+\frac{\tilde{\nu}}{\nu}\Big)\\
&\quad + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}(\beta^4+\gamma^2)\cdot\frac{1}{\nu\beta^2} + \frac{\tilde{\nu}(\beta^2+\gamma^2)}{\beta^4}\Big) D_5\\
&\quad + \frac{(\nu+\tilde{\nu})^2}{\nu\beta^6}\|\tilde{f}_1\|_{H^2}^2 + \frac{1}{\nu\beta^6}\|f_2\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|f_4\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|h_1\|_{H^2}^2 + \frac{1}{\nu\beta^2}\|G\|_{H^2}^2\Big\}\\
&\le \frac{C}{\nu\beta^2}\|\partial_t\nabla w\|_{L^2}^2 + \Big(\frac{1}{\beta^2} + \frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\Big) D_7\\
&\quad + C\Big\{\frac{(\nu+\tilde{\nu})^2}{\nu\beta^6}\|\tilde{f}_1\|_{H^2}^2 + \frac{1}{\nu\beta^6}\|f_2\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|f_4\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|h_1\|_{H^2}^2\Big\}.
\end{aligned}
$$

This completes the proof. $\square$

We set

$$
\tilde{D_7} := \frac{1}{8}D_7 + \frac{C\nu}{\beta^6}\|\nabla^3 w\|_{L^2}^2.
$$

Proposition 4.14. *The following inequalities hold true:*

$$
\begin{aligned}
\|\partial_t\phi\|_{L^2}^2 &\le C\Big(\frac{1}{\nu\tilde{\nu}}\cdot\frac{\beta^4}{(\beta^2+\gamma^2)^2} + \frac{1}{\nu^3}\cdot\frac{\nu+\tilde{\nu}}{(\beta^2+\gamma^2)^2}\Big)\tilde{D_7} + C\|f_1\|_{L^2}^2.\\
\|\partial_t G\|_{L^2}^2 &\le C\Big(\frac{\beta^4}{\nu^2(\beta^2+\gamma^2)^2} + \frac{1}{\nu^2(\beta^2+\gamma^2)^2} + \frac{\beta^2}{\nu^3(\beta^2+\gamma^2)^2}\Big)\tilde{D_7} + C\|f_3\|_{L^2}^2.
\end{aligned}
$$

Proof. We see from (3.2) that

$$
\|\partial_t\phi\|_{L^2}^2 \le C(\|\mathrm{div} w\|_{L^2}^2 + \|\overline{v}^1\partial_{x_1}\phi\|_{L^2}^2 + \|f_1\|_{L^2}^2)
$$

$$
\begin{aligned}
&\le C(\frac{1}{\tilde{\nu}} + \frac{1}{\nu^2} \cdot \frac{\nu + \tilde{\nu}}{\beta^2 + \gamma^2}) D_6 + C\|f_1\|_{L^2}^2 \\
&\le C\left(\frac{1}{\nu\tilde{\nu}} \frac{\beta^4}{(\beta^2+\gamma^2)^2} + \frac{1}{\nu^3} \frac{\nu + \tilde{\nu}}{(\beta^2+\gamma^2)^3}\right) \tilde{D}_7 + C\|f_1\|_{L^2}^2.
\end{aligned}
$$

Similarly it follows from (3.4) that

$$
\begin{aligned}
\|\partial_t G\|_{L^2}^2 &\le C(\|\overline{v}^1 \partial_{x_1} G\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla w \overline{E}\|_{L^2}^2 \\
&\quad + \|w^3 \partial_{x_3}^2 \overline{\psi}^1\|_{L^2}^2 + \|\nabla \overline{v} G\|_{L^2}^2 + \|f_3\|_{L^2}^2 \\
&\le C\Big(\frac{1}{\nu} + \frac{1}{\nu\beta^4} + \frac{1}{\nu^2\beta^2}\Big) D_6 + C\|f_3\|_{L^2}^2 \\
&\le C\Big(\frac{\beta^4}{\nu^2(\beta^2+\gamma^2)^2} + \frac{1}{\nu^2(\beta^2+\gamma^2)^2} + \frac{\beta^2}{\nu^3(\beta^2+\gamma^2)^2}\Big) \tilde{D}_7 + C\|f_3\|_{L^2}^2.
\end{aligned}
$$

This completes the proof. □

Proposition 4.15. *The following inequality holds true:*

$$
\begin{aligned}
&\frac{1}{\beta^2+\gamma^2}\Big\{\frac{1}{2}\frac{d}{dt}(\gamma^2\|\partial_t\phi\|_{L^2}^2 + \|\partial_t w\|_{L^2}^2 + \beta^2\|\partial_t G\|_{L^2}^2) + \nu\|\nabla\partial_t w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial_t w\|_{L^2}^2\Big\} \\
&\le \Big(\frac{1}{4}\cdot\frac{\nu}{\beta^2+\gamma^2} + \frac{C}{\nu(\beta^2+\gamma^2)}\Big)\|\nabla\partial_t w\|_{L^2}^2 \\
&\quad + C\Big(\frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\Big) D_6 + \frac{C}{\nu}(\|\partial_t\phi\|_{L^2}^2 + \|\partial_t G\|_{L^2}^2) \\
&\quad + \frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1, \partial_t\phi)| + \frac{1}{\beta^2+\gamma^2}|(\partial_t f_2, \partial_t w)| + \frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3, \partial_t G)|.
\end{aligned}
$$

Proof. We obtain the following estimate by similar argument to the proof of Proposition 4.1

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\gamma^2\|\partial_t\phi\|_{L^2}^2 + \|\partial_t w\|_{L^2}^2 + \beta^2\|\partial_t G\|_{L^2}^2) + \nu\|\partial_t\nabla w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial_t w\|_{L^2}^2 \\
&\le \gamma^2|(\partial_t(\overline{v}^1\partial_{x_1}\phi), \partial_t\phi)| + |(\partial_t(\overline{v}^1\partial_{x_1}w), \partial_t w)| + |(\partial_t(w^3\partial_{x_3}\overline{v}^1), \partial_t w^1)| \\
&\quad + \nu|(\partial_t(\phi\partial_{x_3}^2\overline{v}^1), \partial_t w^1)| + \beta^2|(\partial_t(\mathrm{div}(G^\top\overline{E})), \partial_t w)| \\
&\quad + \beta^2|(\partial_t(G^{33}\partial_{x_3}^2\overline{\psi}^1), \partial_t w^1)| + \beta^2|(\partial_t(\overline{v}^1\partial_{x_1}G), \partial_t G)| + \beta^2|(\partial_t(\nabla w\overline{E}), \partial_t G)| \\
&\quad + \beta^2|(\partial_t(w^3\partial_{x_3}^2\overline{\psi}^1), \partial_t G^{13})| + \beta^2|(\partial_t(\nabla\overline{v}G), \partial_t G)| \\
&\quad + \gamma^2|(\partial_t f_1, \partial_t\phi)| + |(\partial_t f_2, \partial_t w)| + \beta^2|(\partial_t f_3, \partial_t G)|.
\end{aligned}
$$

The right-hand side is estimated by

$$
\begin{aligned}
&\gamma^2|(\partial_t(\overline{v}^1\partial_{x_1}\phi), \partial_t\phi)| + |(\partial_t(\overline{v}^1\partial_{x_1}w), \partial_t w)| + |(\partial_t(w^3\partial_{x_3}\overline{v}^1), \partial_t w^1)| \\
&\quad + \nu|(\partial_t(\phi\partial_{x_3}^2\overline{v}^1), \partial_t w^1)| + \beta^2|(\partial_t(\mathrm{div}(G^\top\overline{E})), \partial_t w)| \\
&\quad + \beta^2|(\partial_t(G^{33}\partial_{x_3}^2\overline{\psi}^1), \partial_t w^1)| + \beta^2|(\partial_t(\overline{v}^1\partial_{x_1}G), \partial_t G)| + \beta^2|(\partial_t(\nabla w\overline{E}), \partial_t G)| \\
&\quad + \beta^2|(\partial_t(w^3\partial_{x_3}^2\overline{\psi}^1), \partial_t G^{13})| + \beta^2|(\partial_t(\nabla\overline{v}G), \partial_t G)|
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma^2|(\partial_t f_1, \partial_t\phi)| + |(\partial_t f_2, \partial_t w)| + \beta^2|(\partial_t f_3, \partial_t G)| \\
\le& \left(\frac{\nu}{4} + \frac{C}{\nu}\right)\|\nabla\partial_t w\|_{L^2}^2 \\
&+ C\Big(\frac{\beta^2+\gamma^2}{\nu} + \nu + \tilde{\nu}\Big)D_6 + C\frac{\beta^2+\gamma^2}{\nu}(\|\partial_t\phi\|_{L^2}^2 + \|\partial_t G\|_{L^2}^2) \\
&+\gamma^2|(\partial_t f_1, \partial_t\phi)| + |(\partial_t f_2, \partial_t w)| + \beta^2|(\partial_t f_3, \partial_t G)|.
\end{aligned}
$$

We thus obtain

$$
\begin{aligned}
&\frac{1}{\beta^2+\gamma^2}\Big\{\frac{1}{2}\frac{d}{dt}(\gamma^2\|\partial_t\phi\|_{L^2}^2 + \|\partial_t w\|_{L^2}^2 + \beta^2\|\partial_t G\|_{L^2}^2) + \nu\|\nabla\partial_t w\|_{L^2}^2 + \tilde{\nu}\|\mathrm{div}\partial_t w\|_{L^2}^2\Big\} \\
\le& \Big(\frac{1}{4}\cdot\frac{\nu}{\beta^2+\gamma^2} + \frac{C}{\nu(\beta^2+\gamma^2)}\Big)\|\nabla\partial_t w\|_{L^2}^2 \\
&+ C\Big(\frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\Big)D_6 + \frac{C}{\nu}(\|\partial_t\phi\|_{L^2}^2 + \|\partial_t G\|_{L^2}^2) \\
&+ \frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1, \partial_t\phi)| + \frac{1}{\beta^2+\gamma^2}|(\partial_t f_2, \partial_t w)| + \frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3, \partial_t G)|.
\end{aligned}
$$

This completes the proof. □

5 Proof of Proposition 3.4

We see from (4.14) and Proposition 4.5 that

$$
\begin{aligned}
&\frac{d}{dt}\Big(E_2 + \frac{1}{\beta^2+\gamma^2}E_3\Big) + \frac{1}{32}D_2 + \frac{1}{\beta^2+\gamma^2}D_3 \\
&\le C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{1}{\beta^2+\gamma^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2 \\
&\quad +C\Big\{\frac{\nu+\tilde{\nu}}{(\beta^2+\gamma^2)^2} + \Big(\frac{\gamma^2}{\beta^2+\gamma^2}\Big)^2\left(\frac{1}{\nu\tilde{\nu}} + \frac{1}{\nu^2}\right) \\
&\quad +\Big(\frac{1}{\nu} + \frac{1}{\nu^3}\Big)\frac{\beta^2}{\beta^2+\gamma^2} + \frac{1}{\tilde{\nu}}\frac{\gamma^2}{\beta^2+\gamma^2} + \frac{1}{\beta^2+\gamma^2}\Big(1 + \frac{1}{\nu} + \frac{1}{\nu^3} + \frac{1}{\beta^2}\Big)\Big\}D_2 \\
&\quad +C\Big(N_2 + \frac{1}{\beta^2+\gamma^2}N_3\Big).
\end{aligned} \tag{5.1}
$$

If $\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} \ll 1, \frac{1}{\nu}, \frac{1}{\beta^2} \ll 1$, then the right-hand side of (5.1) can be estimated as

$$
\begin{aligned}
&C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{1}{\beta^2+\gamma^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2 \\
&\quad +C\Big\{\frac{\nu+\tilde{\nu}}{(\beta^2+\gamma^2)^2} + \Big(\frac{\gamma^2}{\beta^2+\gamma^2}\Big)^2\left(\frac{1}{\nu\tilde{\nu}} + \frac{1}{\nu^2}\right) \\
&\quad +\Big(\frac{1}{\nu} + \frac{1}{\nu^3}\Big)\frac{\beta^2}{\beta^2+\gamma^2} + \frac{1}{\tilde{\nu}}\frac{\gamma^2}{\beta^2+\gamma^2} + \frac{1}{\beta^2+\gamma^2}\Big(1 + \frac{1}{\nu} + \frac{1}{\nu^3} + \frac{1}{\beta^2}\Big)\Big\}D_2 \\
&\quad +C\Big(N_2 + \frac{1}{\beta^2+\gamma^2}N_3\Big) \\
&\le C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{1}{\beta^2+\gamma^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2 + \frac{1}{64}D_2 + C\Big(N_2 + \frac{1}{\beta^2+\gamma^2}N_3\Big),
\end{aligned}
$$

and hence,

$$\begin{aligned}
&\frac{d}{dt}\Big(E_2 + \frac{1}{\beta^2+\gamma^2}E_3\Big) + \frac{1}{64}D_2 + \frac{1}{\beta^2+\gamma^2}D_3 \\
&\le C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{1}{\beta^2+\gamma^2}\Big)\|\partial_{x_3}\phi\|_{L^2}^2 + C\Big(N_2 + \frac{1}{\beta^2+\gamma^2}N_3\Big).
\end{aligned} \tag{5.2}$$

It follows from (5.2) and Proposition 4.6 that

$$\begin{aligned}
&\frac{d}{dt}\Big(E_2 + \frac{1}{\beta^2+\gamma^2}E_3 + E_4\Big) + \tilde{D}_3 + D_4 \\
&\le \frac{1}{2}\tilde{D}_3 + C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{1}{\beta^2+\gamma^2}\Big)\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}D_4 \\
&\quad + C\Big(N_2 + \frac{1}{\beta^2+\gamma^2}N_3 + N_4\Big).
\end{aligned}$$

Moreover by taking $\nu, \tilde{\nu}, \beta^2, \gamma^2$ so that $C(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{1}{\beta^2+\gamma^2})\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2} \le \frac{1}{2}$, we have

$$\frac{d}{dt}\Big(E_2 + \frac{1}{\beta^2+\gamma^2}E_3 + E_4\Big) + \frac{1}{2}(\tilde{D}_3 + D_4) \le C\Big(N_2 + \frac{1}{\beta^2+\gamma^2}N_3 + N_4\Big). \tag{5.3}$$

By combining (5.3) and Proposition 4.7, we obtain

$$\begin{aligned}
\frac{d}{dt}E_5 + \frac{1}{2}D_5 \le{}& \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial_t\nabla w\|_{L^2}^2 + C\frac{\beta^2}{\nu}\|\partial\partial_{x_3}G\|_{L^2}^2 \\
&+ C\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\|\nabla^2\partial w\|_{L^2}^2 + CN_5.
\end{aligned} \tag{5.4}$$

It follows from Proposition 4.8 and Proposition 4.9 that

$$\begin{aligned}
&\frac{d}{dt}\|\partial_{x_3}G\|_{L^2}^2 + \frac{\beta^2}{\nu}\|\partial_{x_3}G\|_{L^2}^2 \\
&\le C\Big(\frac{1}{\nu} + \frac{\gamma^2}{\nu\beta^2} + \frac{\beta^2+\gamma^2}{\beta^2}\cdot\frac{\tilde{\nu}^2}{\nu(\nu+\tilde{\nu})} + \frac{\nu+\tilde{\nu}}{\nu}\cdot\frac{1+\gamma^2}{\beta^2}\cdot\frac{1+\gamma^2}{\beta^2+\gamma^2} \\
&\quad + \frac{1}{\beta^2} + \frac{1}{\nu^4\beta^2} + \frac{1}{\nu^2\beta^2} + \frac{1}{\nu\beta^4} + \frac{1}{\nu\beta^6} + \frac{1}{\nu\beta^8}\Big)\tilde{D}_5 + \frac{C}{\nu\beta^2}\|\partial_{x_3}G\|_{L^2}^2 \\
&\quad + C\Big(\frac{\tilde{\nu}^2}{\nu\beta^2}\|\tilde{f}_1\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|f_2\|_{L^2}^2 + \frac{1}{\nu}\|h_1\|_{H^1}^2\Big) + \Big|\Big(\partial_{x_3}f_3 + \frac{\beta^2}{\nu}\partial_{x_3}h_1, \partial_{x_3}G\Big)\Big| \\
&\le C\Big(\frac{1}{\nu} + \frac{1}{\beta^2} + \frac{\nu+\tilde{\nu}}{\nu} + \frac{\tilde{\nu}^2}{\nu(\nu+\tilde{\nu})}\Big)\tilde{D}_5 + \frac{C}{\nu\beta^2}\|\partial_{x_3}G\|_{L^2}^2 \\
&\quad + C\Big(\frac{\tilde{\nu}^2}{\nu\beta^2}\|\tilde{f}_1\|_{H^1}^2 + \frac{1}{\nu\beta^2}\|f_2\|_{L^2}^2 + \frac{1}{\nu}\|h_1\|_{H^1}^2\Big) + \Big|\Big(\partial_{x_3}f_3 + \frac{\beta^2}{\nu}\partial_{x_3}h_1, \partial_{x_3}G\Big)\Big|.
\end{aligned} \tag{5.5}$$

By combining (5.4) and (5.5), we have

$$\frac{d}{dt}E_6 + \frac{1}{4}D_6 \le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial_t\nabla w\|_{L^2}^2 + CN_6. \tag{5.6}$$

We next estimate $\|\nabla^2 w\|_{L^2}^2$. We see from Proposition 4.9 that if $C\left(\frac{5}{\nu}+\frac{\gamma^2}{\beta^2}\right)\le\frac{1}{8}$, then

$$\begin{aligned}
&\frac{1}{\beta^2(\beta^2+\gamma^2)^2}\|\nabla^2 w\|_{L^2}^2\\
&\le \frac{C}{\nu^2\beta^2(\beta^2+\gamma^2)^2}\|\nabla^2 q\|_{L^2}^2+\frac{C\beta^2}{\nu^2(\beta^2+\gamma^2)^2}\|G\|_{H^1}^2\\
&\le \frac{1}{8}D_6+\frac{C}{\nu^2}(\tilde{\nu}^2\|\tilde{f}_1\|_{H^1}^2+\|f_2\|_{L^2}^2+\beta^2\|h_1\|_{H^1}^2).
\end{aligned}\tag{5.7}$$

It follows from (5.6) and (5.7) that

$$\begin{aligned}
&\frac{d}{dt}E_6+\frac{1}{8}D_6+\frac{1}{\beta^2(\beta^2+\gamma^2)^2}\|\nabla^2 w\|_{L^2}^2\\
&\le \frac{C}{(\beta^2+\gamma^2)(\nu+\tilde{\nu})}\|\partial_t\nabla w\|_{L^2}^2+CN_6+\frac{C}{\nu^2}(\tilde{\nu}^2\|\tilde{f}_1\|_{H^1}^2+\|f_2\|_{L^2}^2+\beta^2\|h_1\|_{H^1}^2).
\end{aligned}\tag{5.8}$$

By coupling Proposition 4.10 and (5.8), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{\nu(\beta^2+\gamma^2)^2}{\beta^4}E_6+\|\nabla^2 G\|_{L^2}^2\Big)+\frac{\nu(\beta^2+\gamma^2)^2}{8\beta^4}D_6+\frac{\nu}{\beta^6}\|\nabla^2 w\|_{L^2}^2+\frac{\beta^2}{\nu}\|\nabla^2 G\|_{L^2}^2\\
&\le C\Big(\frac{\nu(\beta^2+\gamma^2)}{\beta^4(\nu+\tilde{\nu})}+\frac{1}{\nu\beta^2}\Big)\|\partial_t\nabla w\|_{L^2}^2+\frac{C\nu}{\beta^6}\|\nabla^3 w\|_{L^2}^2+C\Big(\frac{1}{\nu}+\frac{1}{\beta^2}+\frac{\tilde{\nu}}{\beta^2}+1+\frac{\tilde{\nu}}{\nu}\Big)D_6\\
&\quad+C\Big(\frac{\nu(\beta^2+\gamma^2)^2}{\beta^4}N_6+|(\nabla^2(f_3+\frac{\beta^2}{\nu}h_1),\nabla^2 G)|+\frac{\nu+\tilde{\nu}}{\nu\beta^2}\|\tilde{f}_1\|_{H^2}^2\\
&\quad+\frac{1}{\nu\beta^2}\|f_2\|_{H^1}^2+\frac{\beta^2}{\nu}\|f_4\|_{H^1}^2+\frac{\beta^2}{\nu}\|h_1\|_{H^2}^2\Big).
\end{aligned}$$

Hence we have

$$\frac{d}{dt}E_7+\frac{1}{16}D_7\le C\Big(\frac{1}{\beta^2}+\frac{1}{\nu\beta^2}\Big)\|\partial_t\nabla w\|_{L^2}^2+\frac{C\nu}{\beta^6}\|\nabla^3 w\|_{L^2}^2+CN_7.\tag{5.9}$$

We see from Proposition 4.13 and (5.9) that

$$\begin{aligned}
&\frac{d}{dt}E_7+\frac{1}{16}D_7+\frac{C\nu}{\beta^2}\|\nabla^3 w\|_{L^2}^2\\
&\le C\Big(\frac{1}{\beta^2}+\frac{1}{\nu\beta^2}+\frac{1}{\nu\beta^6}\Big)\|\partial_t\nabla w\|_{L^2}^2+C\Big(\frac{1}{\beta^2}+\frac{1}{\nu}+\frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\Big)D_7+CN_7.
\end{aligned}$$

Hence we obtain

$$\frac{d}{dt}E_7+\frac{1}{2}\tilde{D}_7\le C\Big(\frac{1}{\beta^2}+\frac{1}{\nu\beta^2}+\frac{1}{\nu\beta^6}\Big)\|\partial_t\nabla w\|_{L^2}^2+CN_7.\tag{5.10}$$

We next estimate for time derivatives of (ϕ,w,G). By combining Proposition 4.14 and Proposition 4.15, we have

$$\frac{d}{dt}\Big(\frac{\gamma^2}{\beta^2+\gamma^2}\|\partial_t\phi\|_{L^2}^2+\frac{1}{\beta^2+\gamma^2}\|\partial_t w\|_{L^2}^2+\frac{\beta^2}{\beta^2+\gamma^2}\|\partial_t G\|_{L^2}^2\Big)$$

$$
\begin{aligned}
&+ \|\partial_t \phi\|_{L^2}^2 + \frac{\nu}{\beta^2+\gamma^2}\|\nabla \partial_t w\|_{L^2}^2 + \frac{\tilde{\nu}}{\beta^2+\gamma^2}\|\mathrm{div}\partial_t w\|_{L^2}^2 + \|\partial_t G\|_{L^2}^2 \\
&\le C\Big(\frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\Big) D_6 + C\Big(\|f_1\|_{L^2}^2 + \frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1, \partial_t \phi)| \\
&+ \frac{1}{\beta^2+\gamma^2}|(\partial_t f_2, \partial_t w)| + \frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3, \partial_t G)| + \|f_3\|_{L^2}^2\Big). \qquad (5.11)
\end{aligned}
$$

We see from (5.10) + (5.11) that

$$
\begin{aligned}
&\frac{d}{dt}\Big(E_7 + \frac{\gamma^2}{\beta^2+\gamma^2}\|\partial_t \phi\|_{L^2}^2 + \frac{1}{\beta^2+\gamma^2}\|\partial_t w\|_{L^2}^2 + \frac{\beta^2}{\beta^2+\gamma^2}\|\partial_t G\|_{L^2}^2\Big) \\
&\quad + \tilde{D}_7 + \|\partial_t \phi\|_{L^2}^2 + \frac{\nu}{4\beta^2}\|\partial_t \nabla w\|_{L^2}^2 + \|\partial_t G\|_{L^2}^2 \\
&\le C\Big(\frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\Big) D_6 + C\Big(N_7 + \|f_1\|_{L^2}^2 + \frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1, \partial_t \phi)| + \frac{1}{\beta^2+\gamma^2}|(\partial_t f_2, \partial_t w)| \\
&\quad + \frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3, \partial_t G)| + \|f_3\|_{L^2}^2\Big) \\
&\le C\Big(\frac{1}{\nu} + \frac{\nu+\tilde{\nu}}{\beta^2+\gamma^2}\Big)\frac{\beta^4}{\nu(\beta^2+\gamma^2)}\tilde{D}_7 + C\Big(N_7 + \|f_1\|_{L^2}^2 + \frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1, \partial_t \phi)| \\
&\quad + \frac{1}{\beta^2+\gamma^2}|(\partial_t f_2, \partial_t w)| + \frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3, \partial_t G)| + \|f_3\|_{L^2}^2\Big).
\end{aligned}
$$

We set

$$
\begin{aligned}
E_8 &:= E_7 + \frac{\gamma^2}{\beta^2+\gamma^2}\|\partial_t \phi\|_{L^2}^2 + \frac{1}{\beta^2+\gamma^2}\|\partial_t w\|_{L^2}^2 + \frac{\beta^2}{\beta^2+\gamma^2}\|\partial_t G\|_{L^2}^2 \\
D_8 &:= \tilde{D}_7 + \|\partial_t \phi\|_{L^2}^2 + \frac{\nu}{\beta^2}\|\nabla \partial_t w\|_{L^2}^2 + \|\partial_t G\|_{L^2}^2 \\
N_8 &:= N_7 + \|f_1\|_{L^2}^2 + \frac{\gamma^2}{\beta^2+\gamma^2}|(\partial_t f_1, \partial_t \phi)| \\
&\qquad + \frac{1}{\beta^2+\gamma^2}|(\partial_t f_2, \partial_t w)| + \frac{\beta^2}{\beta^2+\gamma^2}|(\partial_t f_3, \partial_t G)| + \|f_3\|_{L^2}^2.
\end{aligned}
$$

It then follows

$$\frac{d}{dt}E_8 + D_8 \le C N_8.$$

We note that there exists a positive number $C_1 > 0$ such that $E_8 \le C_1 D_8$. This leads to

$$\frac{d}{dt}E_8 + C_1(E_8 + D_8) \le C N_8,$$

and hence,

$$E_8(t) + C_1\int_0^t e^{-C_1(t-s)} D_8(s)ds \le Ce^{-C_1 t}E_8(0) + C\int_0^t e^{-C_1(t-s)} N_8(s)ds.$$

We see from the third equation of (3.3) that

$$\|\partial_{x_3}^2 w\|_{L^2}^2 \le C\Big(\frac{1}{\nu^2}\|\partial_t w\|_{L^2}^2 + \|\nabla \partial w\|_{L^2}^2 + \frac{\gamma^4+1}{\nu^2}\|\nabla \phi\|_{L^2}^2 + \frac{\beta^4}{\nu^2}\|\nabla G\|_{L^2}^2$$

$$+\frac{1}{\nu^4}\|\nabla w\|_{L^2}^2+\frac{1}{\nu^2}(\|G\|_{L^2}^2+\|\partial G\|_{L^2}^2)+\frac{1}{\nu^2}\|f_2\|_{L^2}^2\Big)$$
$$\le C\frac{\beta^4+\gamma^4}{\nu^2}E_8+\frac{C}{\nu^2}E_8^2\le C\frac{\beta^4+\gamma^4}{\nu^2}E_8.$$

We finally introduce the following quantities

$$\begin{aligned} E(t) &:= 2CE_8(t)+\frac{\nu^2}{\beta^4+\gamma^4}\|\partial_{x_3}^2 w(t)\|_{L^2}^2\\ D(t) &:= 2CD_8(t)\\ N(t) &:= 2CN_8(t). \end{aligned}$$

Then the following inequality holds true:

$$E(t)+\int_0^t e^{-C_1(t-s)}D(s)ds\le Ce^{-C_1t}E(0)+C\int_0^t e^{-C_1(t-s)}N(s)ds. \tag{5.12}$$

To complete the proof of Proposition 3.4, it remains to estimate $N(t)$. We shall show that $N(t)$ is estimated in the following way.

Proposition 5.1. *There exists a positive constant $\delta_1>0$ such that if $E(t)\le\delta$, then it holds the following estimate*

$$N(t)\le C(E(t)^{\frac{1}{2}}+E(t))D(t)\qquad (t\ge 0).$$

Direct applications of Lemmata 2.1-2.3 yield Proposition 5.1. The proof of Proposition 3.4 now follows from combining (5.12) and Proposition 5.1.

□

6 Appendix: Proof of Proposition 3.2.

In this section, we will give a proof of Proposition 3.2.

Proof of Proposition 3.2. We set $(\rho,v,F)=(\bar\rho,\bar v,\bar F)=(1,\bar v^1(x_3,t)e_1,\big(\nabla(x-\bar\psi^1(x_3,t)e_1)\big)^{-1})$ in (1.1). Since

$$\bar F=\nabla(x+\bar\psi^1e_1)=\begin{pmatrix}1&0&\partial_{x_3}\bar\psi^1\\0&1&0\\0&0&1\end{pmatrix},$$

$$\rho(\partial_t v+v\cdot\nabla v)-\nu\Delta v-(\nu+\nu')\nabla\mathrm{div}v+\nabla p(\rho)=(\partial_t\bar v^1-\nu\partial_{x_3}^2\bar v^1)e_1,$$
$$\beta^2\mathrm{div}(\rho F^\top F)+\rho g=(\beta^2\partial_{x_3}^2\bar\psi^1+g^1)e_1,$$

we see from the 2nd equation of (1.1) that $\bar\psi^1$ should satisfy the following time-periodic problem:

$$\begin{cases}\partial_t^2\bar\psi^1-\beta^2\partial_{x_3}^2\bar\psi^1-\nu\partial_t\partial_{x_3}^2\bar\psi^1=g^1, & (6.1)\\ \bar\psi^1(0,t)=\bar\psi^1(1,t)=0, & (6.2)\\ \bar\psi^1(x_3,t+T)=\bar\psi^1(x_3,t). & (6.3)\end{cases}$$

For simplicity we assume that $\frac{2\beta}{\pi\nu}$ is not integer.

We set

$$A = \begin{pmatrix} 0 & -1 \\ -\beta^2\partial_{x_3}^2 & -\nu\partial_{x_3}^2 \end{pmatrix}.$$

It follows that the problem (6.1)–(6.3) is written as

$$\partial_t \begin{pmatrix} \bar\psi^1 \\ \bar v^1 \end{pmatrix} + A \begin{pmatrix} \bar\psi^1 \\ \bar v^1 \end{pmatrix} = \begin{pmatrix} 0 \\ g^1 \end{pmatrix}, \ \begin{pmatrix} \bar\psi^1(x_3, t+T) \\ \bar v^1(x_3, t+T) \end{pmatrix} = \begin{pmatrix} \bar\psi^1(x_3, t) \\ \bar v^1(x_3, t) \end{pmatrix}. \tag{6.4}$$

To solve (6.4), we consider the Fourier-sine expansions;

$$\bar\psi^1 = \sum_{k=1}^{\infty} \hat\psi_k(t)\sin(k\pi x_3), \ \bar v^1 = \sum_{k=1}^{\infty} \hat v_k(t)\sin(k\pi x_3),$$
$$g^1 = \sum_{k=1}^{\infty} \hat g_k^1(t)\sin(k\pi x_3).$$

We see from (6.4) that $(\hat v_k, \hat\psi_k)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \hat\psi_k \\ \hat v_k \end{pmatrix} + \hat A \begin{pmatrix} \hat\psi_k \\ \hat v_k \end{pmatrix} = \begin{pmatrix} 0 \\ \hat g_k^1 \end{pmatrix}, \ \begin{pmatrix} \hat\psi_k(t+T) \\ \hat v_k(t+T) \end{pmatrix} = \begin{pmatrix} \hat\psi_k(t) \\ \hat v_k(t) \end{pmatrix}. \tag{6.5}$$

where

$$\hat A = \begin{pmatrix} 0 & -1 \\ \beta^2 k^2\pi^2 & \nu k^2\pi^2 \end{pmatrix}.$$

The solution of (6.5) is given by

$$\begin{pmatrix} \hat\psi_k(t) \\ \hat v_k(t) \end{pmatrix} = \int_{-\infty}^{t} e^{-(t-s)\hat A}\begin{pmatrix} 0 \\ \hat g_k^1(s) \end{pmatrix}\mathrm{d}s. \tag{6.6}$$

From [9], the solution semigroup $e^{-t\hat A}$ is represented as

$$e^{-t\hat A} = \frac{1}{\lambda_+ - \lambda_-}\left(e^{t\lambda_+}\begin{pmatrix} -\lambda_- & 1 \\ -\lambda_+\lambda_- & \lambda_+ \end{pmatrix} + e^{t\lambda_-}\begin{pmatrix} \lambda_+ & -1 \\ \lambda_+\lambda_- & -\lambda_- \end{pmatrix}\right),$$

where

$$\lambda_\pm = \frac{-\nu k^2\pi^2 \pm \sqrt{\nu^2k^4\pi^4 - 4\beta^2k^2\pi^2}}{2}.$$

We note that $\lambda_\pm$ are the characteristic roots of $-\hat A$ satisfying the following properties

$$\lambda_+ = -\frac{\beta^2}{\nu} + O\left(\frac{1}{k^2}\right), \ \lambda_- = -\nu k^2\pi^2 + O(1) \quad \text{for } k \gg 1,$$
$$\lambda_+\lambda_- = \beta^2k^2\pi^2.$$

It then follows that $\hat\psi_k(t)$ and $\hat v_k(t)$ are written as

$$\hat\psi_k(t) = \int_{-\infty}^{t} \frac{e^{\lambda_+(t-s)} - e^{\lambda_-(t-s)}}{\lambda_+ - \lambda_-}\hat g_k^1(s)\mathrm{d}s,$$

$$\hat{v}_k(t) = \int_{-\infty}^{t} \frac{\lambda_+ e^{\lambda_+(t-s)} - \lambda_- e^{\lambda_-(t-s)}}{\lambda_+ - \lambda_-} \hat{g}_k^1(s)\mathrm{d}s.$$

By integration by parts, we see that $\hat{\psi}_k(t)$ and $\hat{v}_k(t)$ are rewritten in the following forms:

$$\hat{\psi}_k(t) = \frac{1}{\lambda_+\lambda_-}\hat{g}_k^1(t) + \int_{-\infty}^{t} \frac{\lambda_- e^{\lambda_+(t-s)} - \lambda_+ e^{\lambda_-(t-s)}}{\lambda_+\lambda_-(\lambda_+ - \lambda_-)} (\hat{g}_k^1)'(s)\mathrm{d}s, \tag{6.7}$$

$$\hat{v}_k(t) = \int_{-\infty}^{t} \frac{e^{\lambda_+(t-s)} - e^{\lambda_-(t-s)}}{\lambda_+ - \lambda_-} (\hat{g}_k^1)'(s)\mathrm{d}s. \tag{6.8}$$

Let us estimate $\hat{\psi}_k$ and $\hat{v}_k$ by using (6.7) and (6.8).

We first prepare the following inequalities shown in [9] to estimate (6.7) and (6.8):

$$\left|\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \le C\frac{1}{\beta k\pi + \nu k^2\pi^2} e^{-c\kappa t}, \tag{6.9}$$

$$\left|\frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \le Ce^{-c\kappa t}. \tag{6.10}$$

Based on (6.7) and (6.10), we have

$$\begin{aligned}
|\hat{\psi}_k(t)|^2 &\le \frac{1}{\beta^4 k^4\pi^4}\left(|\hat{g}_k^1(t)| + \int_{-\infty}^{t} e^{-c\kappa(t-s)}|(\hat{g}_k^1)'(s)|\mathrm{d}s\right)^2 \\
&\le \frac{C}{\beta^4 k^4\pi^4}|\hat{g}_k^1(t)|^2 + \frac{C}{\beta^4 k^4\pi^4}\left(\int_{-\infty}^{t} e^{-c\kappa(t-s)}|(\hat{g}_k^1)'(s)|\mathrm{d}s\right)^2 \\
&\le \frac{C}{\beta^4 k^4\pi^4}|\hat{g}_k^1(t)|^2 + \frac{C}{\beta^4 k^4\pi^4}\left(\int_{-\infty}^{t} e^{-c\kappa(t-s)}\mathrm{d}s\right)\left(\int_{-\infty}^{t} e^{-c\kappa(t-s)}|(\hat{g}_k^1)'(s)|^2\mathrm{d}s\right) \\
&\le \frac{C}{\beta^4 k^4\pi^4}|\hat{g}_k^1(t)|^2 + \frac{C}{\kappa\beta^4 k^4\pi^4}\int_{-\infty}^{t} e^{-c\kappa(t-s)}|(\hat{g}_k^1)'(s)|^2\mathrm{d}s.
\end{aligned}$$

In the case that $\frac{2\beta}{\pi\nu}$ is a positive integer, we can deduce above estimates by using the following forms with $k = \frac{2\beta}{\pi\nu}$:

$$\begin{aligned}
\hat{\psi}_k(t) =& \frac{1}{\beta^2 k^2\pi^2}\hat{g}_k^1(t) + \frac{1}{\beta^2 k^2\pi^2}\int_{-\infty}^{t} e^{-\frac{2\beta^2}{\nu}(t-s)}(\hat{g}_k^1)'(s)\mathrm{d}s \\
&+ \frac{2}{\nu k^2\pi^2}\int_{-\infty}^{t} (t-s)e^{-\frac{2\beta^2}{\nu}(t-s)}(\hat{g}_k^1)'(s)\mathrm{d}s, \\
\hat{v}_k(t) =& -\int_{-\infty}^{t} (t-s)e^{-\frac{2\beta^2}{\nu}(t-s)}(\hat{g}_k^1)'(s)\mathrm{d}s - \frac{4}{\nu k^2\pi^2}\int_{-\infty}^{t} e^{-\frac{2\beta^2}{\nu}(t-s)}(\hat{g}_k^1)'(s)\mathrm{d}s.
\end{aligned}$$

Hence, we obtain the following estimate for $0 \le l \le 6$

$$\|\partial_{x_3}^l \bar{\psi}^1(t)\|_{L^2(0,1)}^2 \le \frac{C}{\beta^4}\|\partial_{x_3}^{l-2} g^1(t)\|_{L^2(0,1)}^2 + \frac{C}{\kappa\beta^4}\int_{-\infty}^{t} e^{-c\kappa(t-s)}\|\partial_t\partial_{x_3}^{l-2} g^1(s)\|_{L^2(0,1)}^2\mathrm{d}s$$

$$\leq \frac{C}{\beta^4}\|g^1(t)\|^2_{H^4(0,1)} + \frac{C}{\kappa^2\beta^4}\sup_{t\in[0,T]}\|\partial_t g^1(t)\|^2_{H^4(0,1)}$$
$$\leq \frac{C}{\beta^4}\left(1+\frac{1}{\kappa^2}\right)\|g^1\|^2_{H^1(0,T,H^4(0,1))}.$$

As a result, $\bar{\psi}^1(t)$ satisfies

$$\sup_{t\in[0,T]}\|\bar{\psi}^1(t)\|^2_{H^6(0,1)} \leq \frac{C}{\beta^4}\left(1+\frac{1}{\kappa^2}\right)\|g^1\|^2_{H^1(0,T,H^4(0,1))}.$$

Similarly, we obtain the estimate of $\sup_{t\in[0,T]}\|\bar{v}^1(t)\|^2_{H^4(0,1)}$ by using (6.8) and (6.9).

The estimate of $\sup_{t\in[0,T]}\|\partial_t\bar{v}^1(t)\|^2_{H^2(0,1)}$ is immediately proved by using (6.1). This completes the proof of Proposition 3.2. ■

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