Advances in Mathematical Sciences and Applications
Vol. 30, No. 2 (2021), pp. 555–570

GAKKOTOSHO
TOKYO JAPAN

INTERFACE VANISHING FOR NONSTATIONARY MAXWELL EQUATION

Takashi Suzuki
Center for Mathematical Modeling and Data Science
Unit of Mathematical Science
Osaka University
(E-mail: suzuki@sigmath.es.osaka-u.ac.jp)

Abstract. We study Maxwell's equation involved by interface in space-time. Some components of electric magnetic fields do not suffer the effect of interface and their singularity propagates under the light speed.

Communicated by Editors; Received October 7, 2021
This work is supported by JSPS Grand-in-Aid Scientific Research 19H01799
AMS Subject Classification: 35B65, 36L40, 35Q61
Keywords: Maxwell equation, interface regularity, differential forms

1 Introduction

Maxwell's equation is concerned on the electric field E, the magnetic field B, the current density J, and the electric charge density ρ, depending on the space-time variables (x,t). Using the gradient operator

$$\nabla = \nabla_x \equiv \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}, \quad x = (x_1, x_2, x_3),$$

and the outer and the inner products in $\mathbf{R}^3$ denoted by $\times$ and $\cdot$, respectively, it is given by

$$\nabla \times H - \frac{\partial D}{\partial t} = J, \quad \nabla \cdot D = \rho, \quad \nabla \times E + \frac{\partial B}{\partial t} = 0, \quad \nabla \cdot B = 0$$

with

$$D = \varepsilon E, \quad B = \mu H, \quad J = \sigma E,$$

where ε, μ, and σ denote the permittivity, the permeability, and the conductivity, respectively. Assuming the the first two physical constants ε and μ are independent of the media, we put them to be 1, to reach

$$\begin{aligned} \nabla \times B - \frac{\partial E}{\partial t} &= J, \quad \nabla \cdot E = \rho \\ \nabla \times E + \frac{\partial B}{\partial t} &= 0, \quad \nabla \cdot B = 0 \qquad \text{in } \Omega, \end{aligned} \tag{1}$$

with the normalized light speed $c = \mu\varepsilon = 1$, where $\Omega \subset \mathbf{R}^4$ is a domain in space-time.

We take the case that this Ω is composed of two media, with the interface $\mathcal{M}$ forming a part of smooth hyper-surface, and that the variables J and ρ are only piecewise regular. We refer to [2, 3] for such a situation of the secondary current in magnet-encephalography, because of the discontinuity of the conductivity σ and that of the primary current evoked by the neuron activity.

Here we study (1), viewing J and ρ as given data provided with the piecewise regularity. Then, our purpose is to clarify the components of B and E of which singularities propagate accross the interface without suffering any effects from it. So far, such a property is known to the stationary problem. To provide a uniform description for stationary and nonstationary problems, we formulate the above described geometric profile of Ω in a general setting as follows.

Definition 1. *The domain $\Omega \subset \mathbf{R}^n$ is said to have interface, denoted by $\mathcal{M}$, if this $\mathcal{M}$ is a non-compact smooth hyper-surface in $\mathbf{R}^n$ without boundary, and it holds that $\Gamma \equiv \mathcal{M} \cap \Omega \neq \emptyset$.*

In this case the domain Ω is divided into two parts by $\mathcal{M}$, denoted by $\Omega_\pm$, and then $\Gamma = \Gamma_\pm$ is distinguished as a subset of $\partial\Omega_\pm$:

$$\Omega = \Omega_+ \cup \Gamma \cup \Omega_-, \quad \Gamma_\pm = \partial\Omega_\pm \setminus \partial\Omega (= \Gamma).$$

We suppose the connectivity of $\Omega_\pm$ for simplicity. Hence they form domains in $\mathbf{R}^n$ by themselves.

Our assumption is the piecewise regularity of $J \in L^2(\Omega)$ and $\rho \in L^2(\Omega)$ in (1), precisely,

$$\nabla \times J \in L^2(\Omega_\pm)^3, \quad \frac{\partial J}{\partial t} + \nabla\rho \in L^2(\Omega_\pm)^3, \tag{2}$$

where the differentiations are taken in the sense of distributions in $\Omega_\pm$.

Theorem 1. *Let $\Omega \subset \mathbf{R}^4$ be a domain with the interface. Let ν be the outer unit normal vector on Γ_-, extended smoothly on Ω. Let $B, E \in H^1(\Omega)^3$ be the solution to (1), for $J \in L^2(\Omega)^3$ and $\rho \in L^2(\Omega)$ satisfying (2). Then it holds that*

$$\Box(\nu^0 B + \tilde{\nu} \times E) \in L^2(\Omega)^3, \quad \Box(\tilde{\nu} \cdot B) \in L^2(\Omega), \tag{3}$$

where

$$\nu = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \\ \nu^0 \end{pmatrix}, \quad \tilde{\nu} = \begin{pmatrix} \nu^1 \\ \nu^2 \\ \nu^3 \end{pmatrix}, \quad \Box = -\frac{\partial^2}{\partial t^2} + \Delta_x.$$

Since the system (1) implies

$$-\Delta B - \frac{\partial}{\partial t}\nabla \times E = \nabla \times J, \quad \nabla\rho - \Delta E + \frac{\partial}{\partial t}\nabla \times B = 0,$$

there arises that

$$\Box B = -\nabla \times J, \quad \Box E = \nabla\rho + \frac{\partial J}{\partial t}.$$

The assumption (2) thus implies

$$\Box B,\ \Box E \in L^2(\Omega_\pm)^3,$$

which, however, does not mean $\Box B,\ \Box E \in L^2(\Omega)^3$. The conclusion (3), therefore, assures that H^2-singularities of $\nu^0 B + \tilde{\nu} \times E$ and $\tilde{\nu} \cdot B$ propagate through the interface under the light speed, without suffering any effects from it.

In the stationary state, $E_t = B_t = 0$, the system (1) sprits into

$$\nabla \times B = J, \quad \nabla \cdot B = 0 \qquad \text{in } \Omega \tag{4}$$

and

$$\nabla \times E = 0, \quad \nabla \cdot E = \rho \qquad \text{in } \Omega \tag{5}$$

where $\Omega \subset \mathbf{R}^3$ is a three-dimensional domain. By Theorem 1, if this Ω is provided with interface, we obtain the following facts because $\nu_0 = 0$ under the notation of Theorem 1:

1. If $B \in H^1(\Omega)^3$ solves (4) with $J \in L^2(\Omega)$ satisfying $\nabla \times J \in L^2(\Omega_\pm)^3$, it follows that $\Delta(\nu \cdot B) \in L^2(\Omega)$, and hence $\nu \cdot B \in H^2_{loc}(\Omega)$.

2. If $E \in H^1(\Omega)^3$ solves (5) with $\rho \in L^2(\Omega)$ satisfying $\nabla\rho \in L^2(\Omega_\pm)^3$, it follows that $\Delta(\nu \times E) \in L^2(\Omega)$, and hence $\nu \times E \in H^2_{loc}(\Omega)^3$.

These results on stationary solutions are obtained by [7, 8] and [5, 6], respectively. In the latter work, especially, the method of differential forms is developed and the results are extended to arbitrary space dimension n.

To state details, we use several notations in the theory of harmonic integration [10], recalling that $\Omega \subset \mathbf{R}^n$ is a domain with interface. First, let $\Lambda^p = \Lambda^p(\Omega)$ be the set of p-forms on Ω. Second, the outer derivative

$$d : \Lambda^p \to \Lambda^{p+1}$$

and the co-differentiation

$$\delta : \Lambda^p \to \Lambda^{p-1}$$

are defined usually. Then it holds that

$$-\Delta = \delta d + d\delta : \Lambda^p \to \Lambda^p.$$

Analogous result is valid piecewisely to $\Lambda^p_\pm = \Lambda^p(\Omega_\pm)$, the set of p-forms on $\Omega_\pm$.

Let $H^q(\Lambda^p(\Omega))$, $q = 0, 1, 2$, be the set of p-forms of which coefficients are H^q functions on Ω. We write L^2 for H^0. Hence it holds that $L^2(\Lambda^0(\Omega)) = L^2(\Omega)$, and we say, for example, that

$$B = B^1 dx_1 + \cdots + B^n dx_n \in L^2(\Lambda^1(\Omega)), \tag{6}$$

if and only if $B^i \in L^2(\Omega)$ for $1 \leq i \leq n$. For the moment we put

$$H^q(\Lambda^p) = H^q(\Lambda^p(\Omega)), \quad H^q(\Lambda^p_\pm) = H^q(\Lambda^p(\Omega_\pm)).$$

Given $B \in L^2(\Lambda^1)$ as in (6), next, we put

$$B^\nu = (B, \nu)\nu, \quad B^\tau = B - B^\nu, \tag{7}$$

where $(\ ,\ )$ denotes the $\mathbf{R}^n$ inner product. Here, the vector field $\nu = (\nu^i)$ on Ω is identified with the 1-form

$$\nu = \nu^1 dx_1 + \cdots + \nu^n dx_n, \tag{8}$$

and therefore, it follows that

$$(B, \nu) = \sum_{i=1}^n B^i \nu^i.$$

Theorem 2 ([5])**.** *For $B \in H^1(\Lambda^1)$ the following facts hold true.*

1. *Suppose that $dB = J$ and $\delta B = 0$ in Ω, and that $J \in L^2(\Lambda^2)$ satisfies $\delta J \in L^2(\Lambda^1_\pm)$. Then it follows that $\Delta B^\nu \in L^2(\Lambda^1)$ and hence $B^\nu \in H^2_{loc}(\Lambda^1)$.*

2. *Suppose that $dB = 0$ and $\delta B = g$ in Ω, and that $g \in L^2(\Lambda^0)$ satisfies $dg \in L^2(\Lambda^1_\pm)$. Then it follows that $\Delta B^\tau \in L^2(\Lambda^1)$ and hence $B^\tau \in H^2_{loc}(\Lambda^1)$.*

Remark 1. *In both cases of the above theorem, the 1-form $B \in H^1(\Lambda^1)$ actually satisfies $\Delta B \in L^2(\Lambda^1_\pm)$ by $-\Delta = d\delta + \delta d$. This property, however, implies only $B \in H^2_{loc}(\Lambda^1_\pm)$, which does not mean $B \in H^2_{loc}(\Lambda^1(\Omega))$. Since the trace of the component of B to $\Gamma_\pm$ is not defined as an element in $H^{1/2}(\Gamma_\pm)$, we use the paring between $H^{1/2}(\Gamma_\pm)$ and $H^{-1/2}(\Gamma_\pm)$ to justify the proof of the above theorem. See §5.*

It is classical that (1) takes a representation of differential forms in the Minkowski space. The unknown, however, is 2-form in this case, and therefore, we take the stratagy of extending Theorem 2 on $B \in H^1(\Lambda^1)$ to that on $\omega \in H^1(\Lambda^2)$, to prove Theorem 1. The essential difference here is that this 2-form ω does not take any natural normal or tangential components as in B^ν and B^τ defined by (7), respectively, for the 1-form B. We have to formulate, therefore, the component of 2-form, provided with analogous properties to Theorem 2 on 1-forms.

Given a 2-form on Ω, denoted by

$$\omega = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j,$$

thus we define the 1-forms $\hat{\omega}^i$ by

$$\hat{\omega}^i = \sum_{\ell} \tilde{\omega}^{\ell i} dx_i, \quad 1 \leq i \leq n,$$

where

$$\tilde{\omega}^{ij} = \begin{cases} \omega^{ij}, & i < j \\ 0, & i = j \\ -\omega^{ji}, & i > j. \end{cases} \tag{9}$$

Recall that ν is identified with the 1-form as in (8). Then the result on 2-forms, comparable to 1-forms on Theorem 2, is given as follows.

Theorem 3. *Let $\Omega \subset \mathbf{R}^n$ be a domain with interface and assume that $\omega \in H^1(\Lambda^2)$ solves*

$$d\omega = \theta, \ \delta\omega = 0 \quad \text{in } \Omega$$

with $\theta \in L^2(\Lambda^3)$ satisfying $\delta\theta \in L^2(\Lambda^2_\pm)$, Then it holds that

$$\Delta(\nu, \hat{\omega}^i) \in L^2(\Lambda^0), \quad 1 \leq i \leq n.$$

Once Theorem 3 is obtained, Theorem 1 is proven by replacing the Euclidean metric to the Minkowski metric.

This paper is composed of five sections. Taking preliminaries in §2, we show Theorem 3 in §3. Then Theorem 1 is proven in §4. Section 5 is an appendix, where a short proof of Theorem 2 is provided.

2 Preliminaries

Henceforth $D \subset \mathbf{R}^n$ denotes Lipschitz domain, standing for $\Omega_\pm$ or Ω in the previous section. The trace operator $\gamma : H^1(D) \to H^{1/2}(\partial D)$ is then well-defined. It holds that

$$H^{1/2}(\partial D) \cong H^1(D)/H^1_0(D)$$

by this homomorphism and $C^\infty(\overline{D})$ is dense in $H^1(D)$. See [4, 9, 11] for the proof of these facts. Then we write

$$\varphi|_{\partial D} = \gamma\varphi$$

for $\varphi \in H^1(D)$.

We confirm several fundamental facts on differential forms on the Euclidean space. See [1, 10], for example, for the detailed proof. First, the Euclidean inner product of the 1-forms $\alpha = \sum_\ell \alpha^\ell dx_\ell$ and $\beta = \sum_\ell \beta^\ell dx_\ell$ is given by

$$(\alpha, \beta) = \sum_\ell \alpha^\ell \beta^\ell.$$

Second, if $\lambda = \alpha_1 \wedge \cdots \wedge \alpha_p$ and $\mu = \beta_1 \wedge \cdots \wedge \beta_p$ are p-forms made by the 1-forms α_i and β_i for $1 \leq i \leq p$, we put

$$(\lambda, \mu) = \det \left((\alpha_i, \beta_j) \right)_{i,j}. \tag{10}$$

Let $\Lambda^p(D)$ be the set of p-forms on D. The outer derivative and the wedge product of these differential forms are denoted by

$$d : \Lambda^p(D) \to \Lambda^{p+1}(D) \tag{11}$$

and $\wedge$, respectively. The Hodge operator $* : \Lambda^p(D) \to \Lambda^{n-p}(D)$ is defined by

$$\omega \wedge \tau = (*\omega, \tau)\ dx_1 \wedge \cdots \wedge dx_n \tag{12}$$

for $\omega \in \Lambda^p(D)$ and $\tau \in \Lambda^{n-p}(D)$, and then it follows that

$$*(dx_{j_1} \wedge \cdots \wedge dx_{j_p}) = \mathrm{sgn}\ \sigma \cdot dx_{j_{p+1}} \wedge \cdots \wedge dx_{j_n},$$

where $\sigma : (1, \cdots, n) \mapsto (j_1, \cdots, j_n)$. Then we put

$$\delta = (-1)^p *^{-1} d* : \Lambda^p(D) \to \Lambda^{p-1}(D), \tag{13}$$

to obtain

$$\delta d + d\delta = -\Delta : \Lambda^p(D) \to \Lambda^p(D).$$

By the definition, if

$$B = \sum_i B^i dx_i$$

and

$$\omega = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j$$

are 1- and 2-forms, respectively, it holds that

$$\delta B = -\sum_i B_i^i \tag{14}$$

and

$$\delta\omega = -\sum_{i,\ell} \tilde{\omega}_\ell^{\ell i} dx_i, \tag{15}$$

for $\tilde{\omega}^{ij}$ defined by (9). Here and henceforth, we put

$$B_i^i = \frac{\partial B^i}{\partial x_i}, \quad \omega_\ell^{\ell i} = \frac{\partial \omega^{\ell i}}{\partial x_\ell}.$$

Given $\omega \in \Lambda^p(D)$ and $\theta \in \Lambda^{p-1}(D)$, we have

$$(d\theta, \omega)\ dx_1 \wedge \cdots \wedge dx_n = d\theta \wedge *\omega = d(\theta \wedge *\omega) + (-1)^p \theta \wedge d * \omega$$

and hence

$$\int_D (d\theta, \omega)\ dx_1 \wedge \cdots \wedge dx_n = \int_{\partial D} \theta \wedge *\omega + \int_D (\theta, \delta\omega)\ dx_1 \wedge \cdots \wedge dx_n. \tag{16}$$

The area element ds on ∂D, on the other hand, is defined by

$$\nu^i ds = *dx_i, \quad 1 \le i \le n.$$

Lemma 1. *It holds that*

$$*B = (B, \nu)\ ds, \quad B \wedge *C = (\nu \wedge B, C)\ ds \tag{17}$$

for $B \in \Lambda^1(D)$ and $C \in \Lambda^2(D)$.

Proof: Let

$$B = \sum_i B^i dx_i, \quad C = \sum_{i<j} \omega^{ij} dx_i \wedge dx_j, \quad \nu = \sum_i \nu^i dx_i.$$

First, a direct calculation implies

$$(B, \nu)\ ds = \sum_i B^i \nu^i ds = \sum_i B^i * dx_i = * \sum_i B^i dx_i = *B.$$

Second, it holds that

$$\nu \wedge B = \sum_{i,j} (\nu^i dx_i) \wedge (B^j dx_j) = \sum_{i<j} (\nu^i B^j - \nu^j B^i) dx_i \wedge dx_j$$

and hence

$$(\nu \wedge B, C)\ ds = \sum_{i<j} (\nu^i B^j - \nu^j B^i) C^{ij} ds = \sum_{i<j} C^{ij} (B^j * dx_i - B^i * dx_j).$$

Here, we have

$$*C = \sum_{i<j} C^{ij} * (dx_i \wedge dx_j),$$

which implies

$$\begin{aligned} B \wedge *C &= \sum_k \sum_{i<j} B^k C^{ij} dx_k \wedge *(dx_i \wedge dx_j) \\ &= \sum_{i<j} C^{ij} (B^j * dx_i - B^i * dx_j). \end{aligned}$$

Then it follows that

$$B \wedge *C = (\nu \wedge B, C)\ ds.$$

$\square$

Henceforth, we write

$$\int_D \cdots \ dx_1 \wedge \cdots \wedge dx_n = \int_D \cdots , \qquad \int_{\partial D} \cdots \ ds = \int_{\partial D} \cdots ,$$

in short. Given $\theta \in H^1(\Lambda^p(D))$, we say, simply, that this p-form is in $H^1(D)$. Then its trace $\theta|_{\partial D}$ of this p-form is said to belong to $H^{1/2}(\partial D)$. The following fact is well-known. We, however, present the proof, noticing its validity for differential forms on the Minkowski space.

Proposition 1. *Let φ, B, and J be 0-, 1-, and 2-forms in $H^1(D)$, respectively. Then it holds that*

$$\int_D (\delta B, \varphi) = \int_D (B, d\varphi) - \int_{\partial D} (B, \nu)\varphi \tag{18}$$

$$\int_D (dB, J) = \int_D (B, \delta J) + \int_{\partial D} (\nu \wedge B, J). \tag{19}$$

Proof: Having $(\varphi, B, J)|_{\partial D} \in H^{1/2}(\partial D)$, we apply (16) for $\omega = B$ and $\theta = \varphi$. Since

$$\theta \wedge *\omega = \varphi \cdot *B = \varphi(B, \nu) \ ds$$

it holds that (18). For (19) we put $\omega = J$ and $\theta = B$ in (16). There arises

$$\theta \wedge *\omega = B \wedge *J = (\nu \wedge B, C) \ ds$$

and hence the conclusion. $\square$

Henceforth, X' denotes the dual space of the Banach space X over $\mathbf{R}$, and $\langle \ , \ \rangle$ denotes the paring between X and X'. We put $H^{-1/2}(\partial D) = H^{1/2}(\partial D)'$.

Proposition 2. *Let p be a 0-form in $H^1(D)$.*

1. *If $\Delta p \in H^1(D)'$, then $(dp, \nu)|_{\partial D} \in H^{-1/2}(\partial D)$ is well-defined, and it holds that*

$$\langle (dp, \nu), \varphi \rangle = \int_D (dp, d\varphi) + \langle \Delta p, \varphi \rangle, \quad \forall \varphi \in H^1(D). \tag{20}$$

2. *The 2-form $\nu \wedge dp|_{\partial D} \in H^{-1/2}(\partial D)$ is well-defined, and it holds that*

$$\langle \nu \wedge dp, J \rangle = -\int_D (dp, \delta J), \quad \forall \text{2-form } J \in H^1(D). \tag{21}$$

Proof: Equality (20) follows if $p \in H^2(D)$ is the case from (18) for $B = dp$, because $\delta d = -\Delta$ on $\Lambda^0(D)$. To extend this identity for $p \in H^1(\Omega)$, we regard its right-hand side as a functional on $H^{1/2}(\partial D) \cong H^1(D)/H^1_0(D)$, noting $\Delta p \in H^1(D)'$. Well-posedness of this functional holds if its value is independent of the choice of $\varphi \in H^1(D)$ for given

$\varphi|_{\partial D} \in H^{1/2}(\partial D)$. This property is reduced to confirmirming that the right-hand side becomes 0 for $\varphi \in H^1_0(D)$. Finally, this identidy is obvious again if $p \in H^2(D)$, and then we can extend it to $p \in H^1(D)$ because $C^\infty(\overline{D})$ is dense in $H^1(D)$.

The proof of the second case is the same. First, equality (21) for $p \in H^2(D)$ follows from (19) applied to $B = dp$ by $d^2 = 0$. Then, given $p \in H^1(D)$, we regard its right-hand side as a functional on 2-forms in $H^1(D)$. This functional is actually regarded as an element in $H^{-1/2}(\partial D)$ becasue $H^{1/2}(\partial D) \cong H^1(D)/H^1_0(D)$ and the right-hand side is 0 for $J \in H^1_0(D)$. Finally, this property is obvious for $p \in H^2(D)$ and hence extends to $p \in H^2(\Omega)$ because $C^\infty(\overline{D})$ is dense in $H^1(D)$. □

3 Proof of Theorem 3

We recall the key identity used in [7]. If $\Omega \subset \mathbf{R}^n$ is a domain with interface and $p \in H^1(\Omega)$ is a 0-form, we have $\nu \wedge dp|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)$ by Proposition 2. Then we define

$$[\nu \wedge dp]^+_- = \nu \wedge dp|_{\Gamma_+} - \nu \wedge dp|_{\Gamma_-}$$

as an element in $H^{-1/2}(\Gamma)$.

Lemma 2. *It holds that* $[\nu \wedge dp]^+_- = 0$.

Proof: Recall that ν is the outer unit normal vector on Γ_- extended smoothly on Ω. Given the 2-form $J \in C^\infty_0(\Omega)$, therefore, we obtain

$$\pm\langle \nu \wedge dp, J\rangle = \int_{\Omega_\pm} (dp, \delta J).$$

Then it follows that

$$[\langle \nu \wedge dp, J\rangle]^+_- = \int_\Omega (dp, \delta J),$$

of which right-hand side is equal to 0 if $p \in H^2(\Omega)$ by $J \in C^\infty_0(\Omega)$ and (19) for $B = dp$ and $D = \Omega$. Then this equality is extended to $p \in H^1(\Omega)$ because $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$. □

Proposition 3. *If $p \in H^1(\Omega)$ is a 0-form, it holds that*

$$\left[\nu^i p_j - \nu^j p_i\right]^+_- = 0 \ \textit{in}\ H^{-1/2}(\Gamma), \quad 1 \leq \forall i, j \leq n,$$

where $p_i = \frac{\partial p}{\partial x_i}$.

Proof: The result is a direct consequence of Lemma 2 because

$$dp = \sum_i p_i dx_i$$

and

$$\nu \wedge dp = \sum_{i<j} (\nu^i p_j - \nu^j p_i) dx_i \wedge dx_j.$$

$\square$

Given a 0-form f, let

$$(\nu,d)f=\sum_i \nu^i f_i=\frac{\partial f}{\partial \nu}, \tag{22}$$

recalling $f_i=\frac{\partial f}{\partial x_i}$. Given a 2-form ω, recall also $\hat{\omega}^i$ defined in Theorem 3.

Proposition 4. *If $\omega\in H^1(\Omega)$ is a 2-form, it holds that*

$$\left[\delta\omega+\sum_i[(\nu,d)(\nu,\hat{\omega}^i)dx_i\right]_-^+=0 \quad \textit{in } H^{-1/2}(\Gamma).$$

Proof: Since (15) holds for

$$\omega=\sum_{i<j}\omega^{ij}dx_i\wedge dx_j,$$

it suffices to confirm

$$\left[\sum_\ell \tilde{\omega}_\ell^{\ell i}-\frac{\partial}{\partial\nu}(\nu,\hat{\omega}^i)\right]_-^+=0 \quad \text{in } H^{-1/2}(\Gamma),\ 1\le i\le n. \tag{23}$$

Henceforth, we write $A\sim B$ if $A-B\in H^1(\Omega)$. It then follows that

$$[A-B]_-^+=0 \quad \text{in } H^{1/2}(\Gamma).$$

Fix i, put $B^\ell=\tilde{\omega}^{\ell i}$, and define the one form

$$B=\sum_\ell B^\ell dx_\ell (=\hat{\omega}^i).$$

It follows that

$$\sum_\ell \tilde{\omega}_\ell^{\ell i}-\frac{\partial}{\partial\nu}(\nu,\hat{\omega}^i)=\sum_\ell\left\{B_\ell^\ell-\nu^\ell(\nu,B)_\ell\right\}$$

with

$$(\nu,B)=\sum_k \nu^k B^k.$$

We thus obtain

$$\begin{aligned}\sum_\ell \tilde{\omega}_\ell^{\ell i}-\frac{\partial}{\partial\nu}(\nu,\hat{\omega}^i) &\sim \sum_\ell B_\ell^\ell-\sum_{\ell,k}\nu^\ell\nu^k B_\ell^k\\ &= \sum_\ell B_\ell^\ell-\sum_{k,\ell}\nu^k\nu^\ell B_k^\ell\\ &= \sum_\ell\left\{B_\ell^\ell-\nu^\ell(\nu,d)B^\ell\right\}.\end{aligned}$$

For $p = B^\ell$ we have

$$\begin{aligned} B^\ell_\ell - \nu^\ell(\nu, d)B^\ell &= p_\ell - \nu^\ell(\nu, d)p \\ &= \sum_k \left\{(\nu^k)^2 p_\ell - \nu^\ell \nu^k p_k\right\} \\ &= \sum_k \nu^k(\nu^k p_\ell - \nu^\ell p_k) \end{aligned}$$

Then (23) follows from Proposition 3 as

$$\left[B^\ell_\ell - \nu^\ell(\nu, d)B^\ell\right]^+_- = \sum_k \nu^k \left[\nu^k p_\ell - \nu^\ell p_k\right]^+_- = 0 \quad \text{on } H^{-1/2}(\Gamma).$$

□

Proof of Theorem 3: Since $\omega \in H^1(\Omega)$ is 2-form satisfying $\delta\omega = 0$, it holds that

$$\left[\frac{\partial}{\partial \nu}(\nu, \hat{\omega}^i)\right]^+_- = 0 \quad \text{in } H^{-1/2}(\Gamma),\ 1 \le i \le n \tag{24}$$

by Proposition 4. Using $d\omega = \theta \in L^2(\Omega)$ and $\delta\theta \in L^2(\Omega_\pm)$, we obtain

$$-\Delta\omega = (d\delta + \delta d)\omega = \delta\theta \in L^2(\Omega_\pm)$$

and hence there exists $-\Delta(\nu, \hat{\omega}^i) \in L^2(\Omega_\pm)$, denoted by $h^i_\pm$ for $1 \le i \le n$:

$$-\Delta(\nu, \hat{\omega}^i) = h^i_\pm \quad \text{in } L^2(\Omega_\pm).$$

Define $h^i \in L^2(\Omega)$ by $h^i = h^i_\pm$ in $\Omega_\pm$. Equality (24) then implies

$$\int_\Omega h^i \varphi = \int_\Omega (-\Delta\varphi) \cdot (\nu, \hat{\omega}^i), \quad \forall \varphi \in C^\infty_0(\Omega)$$

by (18), and hence

$$-\Delta(\nu, \hat{\omega}^i) = h^i \in L^2(\Omega), \quad 1 \le i \le n$$

in the sense of distributions. □

4 Proof of Theorem 1

We confirm the structure of differential forms in the Minkowski space $\mathbf{R}^4 \cong \mathbf{R}^{3,1}$, using $x = (x_1, x_2, x_3)$ and $t = x_0$. Let $D \subset \mathbf{R}^4$ be a domain in this space and $\Lambda^p(D)$ be the set of p-forms on D. The outer derivative and the wedge product are denoted similarly by d in (11) and $\wedge$, respectively.

Given 1-forms

$$\alpha = \sum_{i=1}^3 \alpha^i dx_i + \alpha^0 dx_0, \quad \beta = \sum_{i=1}^3 \beta^i dx_i + \beta^0 dx_0,$$

first, the Minkowski inner product is defined by

$$(\alpha, \beta) = -\sum_{i=1}^{3} \alpha^i \beta^i + \alpha^0 \beta^0. \tag{25}$$

Then the Minkowski inner product for p-froms on D is defined similarly by (10).

The Hodge operator $*$ is now defined by (12), where $(\ ,\ )$ is replaced by the above described Minkowski inner product. It then holds that

$$\begin{aligned} &*(dx_0 \wedge dx_i \wedge dx_j) = -dx_k,\ (i,j,k) = (1,2,3) \\ &*(dx_1 \wedge dx_2 \wedge dx_3) = dx_0 \end{aligned}$$

and

$$\begin{aligned} &*(dx_0 \wedge dx_j) = -dx_k \wedge dx_\ell \\ &*(dx_j \wedge dx_k) = dx_0 \wedge dx_\ell, \quad (j,k,\ell) = (1,2,3). \end{aligned}$$

Putting δ as in (13), this time we obtain

$$d\delta + \delta d = \Box \equiv -\frac{\partial^2}{\partial t^2} + \Delta_x : \Lambda^p(D) \to \Lambda^p(D).$$

Then Maxwell's equation (1) takes the form

$$d\omega = 0,\ d * \omega = -j \quad \text{in } \Omega, \tag{26}$$

for

$$\begin{aligned} \omega &= E^2 dx_0 \wedge dx_1 + E^2 dx_0 \wedge dx_2 + E^3 dx_0 \wedge dx_3 \\ &\quad -B^1 dx_2 \wedge dx_3 - B^2 dx_3 \wedge dx_1 - B^3 dx_1 \wedge dx_2 \\ j &= J^1 dx_0 \wedge dx_2 \wedge dx_3 + J^2 dx_0 \wedge dx_3 \wedge dx_1 + J^3 dx_0 \wedge dx_1 \wedge dx_2 \\ &\quad +\rho dx_1 \wedge dx_2 \wedge dx_3 \end{aligned}$$

and

$$E = \begin{pmatrix} E^1 \\ E^2 \\ E^3 \end{pmatrix}, \quad B = \begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}, \quad J = \begin{pmatrix} J^1 \\ J^2 \\ J^3 \end{pmatrix}.$$

This (26) is equivalent to

$$d\theta = j,\ \delta\theta = 0 \quad \text{in } \Omega$$

for $\theta = - * \omega$.

Henceforth we write ω and θ for j and θ, respectively. Thus we handle with

$$d\omega = \theta,\ \delta\omega = 0 \quad \text{in } \Omega \tag{27}$$

for

$$\begin{aligned} \omega &= E^2 dx_0 \wedge dx_1 + E^2 dx_0 \wedge dx_2 + E^3 dx_0 \wedge dx_3 \\ &\quad +B^1 dx_2 \wedge dx_3 + B^2 dx_3 \wedge dx_1 + B^3 dx_1 \wedge dx_2 \\ \theta &= J^1 dx_0 \wedge dx_2 \wedge dx_3 + J^2 dx_0 \wedge dx_3 \wedge dx_1 + J^3 dx_0 \wedge dx_1 \wedge dx_2 \\ &\quad +\rho dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

In this setting of Theorem 1, therefore, the domain $D \subset \mathbf{R}^4$ is provided with interface and $\omega \in H^1(\Omega)$ is 2-form satisfying (27). As in Theorem 3, therefore, if

$$\delta\theta \in L^2(\Omega_\pm), \tag{28}$$

it holds that

$$(\delta d + d\delta)(\nu, \hat{\omega}^i) \in L^2(\Omega), \quad 0 \le i \le 3 \tag{29}$$

in accordance with the Minkowski inner product (25).

First, since

$$\begin{aligned}\delta\theta = &-\left(\frac{\partial J^2}{\partial x_1} - \frac{\partial J^1}{\partial x_2}\right) dx_0 \wedge dx_3 - \left(\frac{\partial J^1}{\partial x_3} - \frac{\partial J^3}{\partial x_1}\right) dx_0 \wedge dx_2 \\ &- \left(\frac{\partial J^3}{\partial x_2} - \frac{\partial J^2}{\partial x_3}\right) dx_0 \wedge dx_1 + \left(\frac{\partial J^1}{\partial x_3} + \frac{\partial \rho}{\partial x_1}\right) dx_2 \wedge dx_3 \\ &+ \left(\frac{\partial J^2}{\partial x_0} + \frac{\partial \rho}{\partial x_2}\right) dx_3 \wedge dx_1 + \left(\frac{\partial J^3}{\partial x_0} + \frac{\partial \rho}{\partial x_3}\right) dx_1 \wedge dx_2.\end{aligned}$$

the condition (28) is equivalent to (2).

Second, we obtain

$$\hat{\omega}^0 = \tilde{\omega}^{01} dx_0 + \tilde{\omega}^{20} dx_2 + \tilde{\omega}^{30} dx_3 = -B^1 dx_1 - B^2 dx_2 - B^3 dx_3$$

and hence

$$(\nu, \hat{\omega}^0) = \nu^1 B^1 + \nu^2 B^2 + \nu^3 B^3.$$

It holds also that

$$\hat{\omega}^1 = \tilde{\omega}^{01} dx_0 + \tilde{\omega}^{21} dx_2 + \tilde{\omega}^{31} dx_3 = B^1 dx_0 - E^3 dx_2 + E^2 dx_3$$

and therefore,

$$(\nu, \hat{\omega}^1) = \nu^0 B^1 + \nu^2 E^3 - \nu^3 E^2.$$

We obtain, similarly,

$$\begin{aligned}(\nu, \hat{\omega}^2) &= \nu^0 B^2 + \nu^3 E^1 - \nu^1 E^3 \\ (\nu, \hat{\omega}^3) &= \nu^0 B^3 + \nu^1 E^2 - \nu^2 E^1.\end{aligned}$$

Hence (29) is equivalent to (3). The proof is complete by

$$-(\delta d + d\delta) = \Box = -\frac{\partial^2}{\partial t^2} + \Delta_x.$$

$\square$

5 A short proof of Theorem 2

This section is an appendix. We give a short proof of Theorem 2, taking regards to the trace to $\Gamma = \Gamma_\pm$ carefully. Recall that $\Omega \subset \mathbf{R}^n$ is a domain with interface,

$$B = \sum_i B^i dx_i \in H^1(\Omega)$$

is a 1-form, and B^ν and B^τ are defined by (7). Recall (14) and (22) also,

$$\delta B = -\sum_i B^i_i, \quad \frac{\partial}{\partial \nu} = \sum_k \nu^k \frac{\partial}{\partial x_k} = (\nu, d),$$

and put

$$\begin{aligned} \delta_\nu B &= -\sum_i \nu^i \frac{\partial B^i}{\partial \nu} \\ \delta_\tau B &= \delta B - \delta_\nu B = -\sum_i \left(B^i_i - \nu^i (\nu, d) B^i \right). \end{aligned} \tag{30}$$

Lemma 3. *If $B \in H^1(\Omega)$ is 1-form, it holds that*

$$[\delta_\tau B]^+_- = 0 \quad \textit{in } H^{-1/2}(\Gamma).$$

Proof: In (30) we have

$$B^i_i - \nu^i (\nu, d) B^i = \sum_k \nu^k (\nu^k B^i_i - \nu^i B^i_k)$$

Then it holds that

$$\left[\nu^k B^i_i - \nu^i B^i_k \right]^+_- = 0 \quad \text{in } H^{-1/2}(\Gamma)$$

by Proposition 3. □

Lemma 4. *If $B \in H^1(\Omega)$ is a 1-form, it holds that*

$$[\delta B^\tau]^+_- = 0 \quad \textit{in } H^{-1/2}(\Gamma). \tag{31}$$

Proof: We have

$$\delta B^\tau = \delta(B - \nu(\nu, B)) = \sum_i \left(-B^i_i + (\nu^i (\nu, B))_i \right).$$

As in the proof of Proposition 4, therefore, it holds that

$$\begin{aligned} \delta B^\tau &\sim \sum_i \left(-B^i_i + \nu^i (\nu, B)_i \right) \\ &\sim \sum_i (-B^i_i) + \sum_{i,\ell} \nu^i \nu^\ell B^\ell_i = \sum_i (-B^i_i) + \sum_{i,\ell} \nu^i \nu^\ell B^i_\ell \\ &= -\sum_i \left(B^i_i - \nu^i (\nu, d) B^i \right) = \delta_\tau B, \end{aligned}$$

and the result follows from the previous lemma. □

Lemma 5. *If $B \in H^1(\Omega)$ is a 1-form, it holds that*

$$[dB^\nu]_-^+ = 0 \quad in\ H^{-1/2}(\Gamma).$$

Proof: Since

$$B^\nu = \sum_i (B,\nu)\nu^i dx_i$$

it holds that

$$\begin{aligned} dB^\nu &= \sum_{i<j} \left[((B,\nu)\nu^j)_i - ((B,\nu)\nu^i)_j\right] dx_i \wedge dx_j \\ &\sim \sum_{i<j} \left\{(B,\nu)_i\nu^j - (B,\nu)_j\nu^i\right\} dx_i \wedge dx_j. \end{aligned}$$

Then we obtain the result by Proposition 3. □

Proof of the first case of Theorem 2: Given the 1-form $B \in H^1(\Omega)$, we have

$$\delta B = \delta B^\tau + \delta B^\nu \quad \text{in } \Omega,$$

where equality (31) is applicable for the first term on the right-hand side. For the second term we use

$$\delta B^\nu = \delta_\nu B^\nu + \delta_\tau B^\nu$$

and Lemma 3 for the second term on the right-hand side. Since

$$\begin{aligned} \delta_\nu B^\nu &= -\sum_{i,k} \nu^i\nu^k((B.\nu)\nu^i)_k \sim -\sum_{i,k}(\nu^i)^2\nu^k(B,\nu)_k \\ &= (\nu,d)(B,\nu) = \frac{\partial}{\partial\nu}(B,\nu) \end{aligned}$$

we see that $\delta B = 0$ in Ω implies

$$\left[\frac{\partial}{\partial\nu}(B,\nu)\right]_-^+ = 0 \quad \text{in } H^{-1/2}(\Gamma).$$

Then we obtain

$$(d\delta + \delta d)B \in L^2(\Omega)$$

similarly to the proof of Theorem 3. □

Proof of the second case of Theorem 2: Since

$$0 = dB = dB^\tau + dB^\nu \quad \text{in } \Omega$$

we obtain

$$[dB^\tau]_-^+ = 0 \quad \text{in } H^{-1/2}(\Gamma) \tag{32}$$

by Lemma 5.

Given 2-form $J \in C_0^\infty(\Omega)$, we have

$$\int_\Omega (d\delta B^\tau, C) = \int_\Omega (B^\tau, d\delta C)$$

by Lemma 4, while equality (32) implies

$$\int_\Omega (\delta d B^\tau, C) = \int_\Omega (B^\tau, \delta d C).$$

Hence we obtain $-\Delta B^\tau \in L^2(\Omega)$. □

References

[1] H. Flanders, *Differential Forms with Applications to the Physical Sciences*, Academic Press, New York 1963.

[2] D.G. Geselowitz, *On bioelastic potentials in an inhomogeneous volume conductor*, Biophys. J. 7 (1967) 1-11.

[3] D.B. Geselowitz, *On the magnetic field generated outside on inhomogeneous volume conductor by integral current sources*, IEEE Tras. Magn. MAG-6 (1970) 911-925,

[4] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, Berlin, 1986.

[5] M. Kanou, T. Sato, and K. Watanabe, *Interface retularity of the solutions for the rotation free and divergence free system in Euclidian space*, Tokyo J Math. 36 (2013) 473-482.

[6] M. Kanou, T. Sato, and K. Watanabe, *Interface regularity of the solutions to the systems on Riemannian manifold*, Tokyo J. Math. 39 (2016) 83-100.

[7] T. Kobayashi, T. Suzui, and K. Watanabe, *Interface regularity for Maxwell and Stokes system*, Osaka J. Math. 40 (2003) 925-943.

[8] T. Kobayashi, T. Suzuki, and K. Watanabe, *Interface vanising for solutions to Maxwell and Stokes sytems*, J. Math. Fluid. Mech. 8 (2006) 382-397.

[9] A. Kufner, O. John, and S. Fučik, *Function Spaces*, Academia, Prague 1977.

[10] S. Morita, *Geometry of Differential Forms*, AMS, Providence, RI, 2001.

[11] J. Nečas, *Le Methodes Directres en Théorie des Equations Elliptiques*, Masson, Paris, 1967.

[12] T. Suzuki, *Mean Field Theories and Dual Variation, 2nd edition, Mathematial Structures of the Mesoscopic Model*, Atlantis Press, Paris, 2015.