Advances in Mathematical Sciences and Applications
Vol. 30, No. 2 (2021), pp. 507–520

GAKKOTOSHO
TOKYO JAPAN

RANDOM DISCRETIZATION OF O'HARA KNOT ENERGY

Jun Okamoto *
Graduate School of Mathematical Sciences
The University of Tokyo
Komaba 3-8-1 Meguro Tokyo 153-8914, Japan.
(E-mail: okamoto@ms.u-tokyo.ac.jp)

Abstract. In this paper, we consider the random discrete approximation of O'Hara energy. O'Hara energy is the energy defined for a knot, and O'Hara energy was introduced for defining the standard shape for each knot class (equivalence class by ambient isotopy) by variational method. If the exponent is taken so that the energy is invariant Möbius transformation, O'Hara energy is called Möbius energy. Although discretization for various Möbius energies has been defined to analyze the shape of the minimizer so far, only Γ-convergence to the original energy has been shown for a conventional discretization. In this study, we are successful to show locally uniform convergence and compactness of discrete energy in a space involving the optimal transport theory, by introducing random discrete approximation of O'Hara energy using random variable and we can show convergence from the minimizer to the minimizer.

Communicated by Yoshikazu Giga; Received September 20, 2021
AMS Subject Classification: 49J45, 57M25, 49Q10, 53A04.
Keywords: Knot energy, Γ-convergence, Optimal transport theory.

1 Introduction

Let $\mathcal{A}$ be the set of all closed regular curves that is parametrized by arc length in $\mathbb{R}^d$, with no self-intersections and with total length $\mathcal{L}$ i.e. $\mathcal{A} := \{\gamma \in C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \mid |\gamma'(x)| = 1 \text{ a.e. } x \in \mathbb{R}/\mathcal{L}\mathbb{Z}\}$. For $\alpha, p \in (0, \infty)$, the O'Hara (α, p)-energy $\mathcal{E}^{\alpha,p} : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ is defined as follows:

$$\mathcal{E}^{\alpha,p}(\gamma) := \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathcal{M}^{\alpha,p}(\gamma) dxdy, \tag{1}$$

where

$$\mathcal{M}^{\alpha,p}(\gamma)(x,y) = \left(\frac{1}{|\gamma(x) - \gamma(y)|^\alpha} - \frac{1}{\mathcal{D}(\gamma(x), \gamma(y))^\alpha}\right)^p, \qquad (x,y) \in (\mathbb{R}/\mathcal{L}\mathbb{Z})^2 \tag{2}$$

and $\mathcal{D}$ is the length of a shortest arc of the curve γ connecting the two points $\gamma(x)$ and $\gamma(y)$, i.e.

$$\mathcal{D}(\gamma(x), \gamma(y)) = \min\{\mathcal{L} - |x - y|, |x - y|\}. \tag{3}$$

This energy was introduced and investigated by O'Hara in [6]-[9] for defining the standard shape for each knot class by variational method. In the case of $\alpha = 2, p = 1$, due to energy invariance under Möbius transformation, this energy is called "Möbius energy". It is possible to show the existence of minimizer in the "prime knot". R. Kusner and J. Sullivan conjectured the minimizer in composite knot class may not exist [13]. This conjecture was established by numerical calculation with discretization of Möbius energy.

In this paper, we introduce the weighted O'Hara energy $\mathcal{E}_\rho^{\alpha,p} : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ with weight $\rho : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}_{\geq 0}$ defined

$$\mathcal{E}_\rho^{\alpha,p}(\gamma) := \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathcal{M}^{\alpha,p}(\gamma)\rho(x)\rho(y) dxdy. \tag{4}$$

Our main goal is to construct random discretization of this O'Hara energy and to show Γ-convergence in a space involving the optimal transport theory. We even show **locally uniform convergence** and **compactness**.

1.1 Known results

Various discretizations of O'Hara energy are defined not only for numerical calculation but also for shape analysis of minimizer.

First, D. Kim and R. Kusner constructed a Möbius energy for polygonal knots in [3].

This energy, defined on the class of arc length parametrizations of polygons of length $\mathcal{L}$ with n segments, is given by

$$\mathcal{E}_n(P) := \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{1}{|P(a_j) - P(a_i)|^2} - \frac{1}{d(a_j, a_i)^2}\right) d(a_{i+1}, a_i) d(a_{j+1}, a_j),$$

where a_i's are consecutive points on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, or interval $[0, \mathcal{L}]$ if we consider the polygon parametrized over an interval. This energy scales invariant. A slight variant would be to take $2^{-1}(d(a_{k-1}, a_k) + d(a_k, a_{k+1}))$ instead of $d(a_{k+1}, a_k)$.

S. Scholtes proved that this discretization Γ-converges to the Möbius energy in [17]. He furthermore showed that this energy is minimized by regular n-gons.

Second, J. Simon [5] defined the so-called *minimal distance energy* for a polygon P by

$$\mathcal{E}_s^m(P) = \tilde{\mathcal{E}}_s^m(P) - \tilde{\mathcal{E}}_s^m(R_m) + 4$$

with

$$\tilde{\mathcal{E}}_s^m(P) = \sum_{|i-j|>1} \frac{|X_i||X_j|}{\operatorname{dist}(X_i, X_j)^2},$$

where R_n is the regular n-gon. Note, that this energy scales invariant. Third, Möbius invariant discrete energy is introduced in [15] and show Γ-convergence is established in $W^{1,q}$-metric sense. The definition of that energy is as follows:

$$\mathcal{E}_{\cos}^m(P) = \sum_{d_m(i,j)>1} \frac{|\Delta_i P||\Delta_j P|}{|\Delta_i^j P||\Delta_{i+1}^{j+1} P|} \left(1 - \frac{1}{2}(\cos(\alpha_{ij}) + \cos(\tilde{\alpha}_{ij}))\right),$$

where $P(\theta_i)$ is a vertex of a closed polygon, $i = 1, 2, ..., m$ and $\Delta_i^j P := P(\theta_j) - P(\theta_i)$, $\Delta_i P = \Delta_i^{i+1} P$, α_{ij} be the angle of the crossing of the circles through at the points $P(\theta_i), P(\theta_{i+1}), P(\theta_j)$ and $P(\theta_j), P(\theta_{j+1}), P(\theta_i)$ and $\tilde{\alpha}_{ij}$ be the angle of the crossing of the circles through at the points $P(\theta_i), P(\theta_{i+1}), P(\theta_{j+1})$ and $P(\theta_j), P(\theta_{i+1}), P(\theta_{j+1})$.

1.2 Main results

We introduce a new discretization of O'Hara (α, p)-energy using random variables on $\mathbb{R}/\mathcal{L}\mathbb{Z}$.

Definition 1.1 (Random O'Hara Energy)**.** Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables on $\mathbb{R}/\mathcal{L}\mathbb{Z}$ with probability density function ρ.

Random O'hara energy $R_{n,\rho}\mathcal{E}^{\alpha,p} : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ is defined as follow:

$$R_{n,\rho}\mathcal{E}^{\alpha,p}(\gamma) := \frac{1}{n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{1}{|\gamma(X_i) - \gamma(X_j)|^\alpha} - \frac{1}{\mathcal{D}(\gamma(X_i), \gamma(X_j))^\alpha} \right)^p.$$

Remark 1. Since $\{X_i\}_{i \in \mathbb{N}}$ has the probability density function ρ, we always have $\mathbb{P}(X_i = X_j) = 0$ for any $i \neq j$. Therefore $R_{n,\rho}\mathcal{E}^{\alpha,p}$ is well-defined almost surely.

Then we introduce the space for comparing continuous model and discrete model as follows.

Definition 1.2 (The TL^q metric space [10])**.** TL^q metric is defined on particular spaces of the family

$$TL^q(\mathbb{R}/\mathcal{L}\mathbb{N}) := \{(\mu, f) \mid \mu \in \mathcal{P}(\mathbb{R}/\mathcal{L}\mathbb{N}), f \in L^q(\mathbb{R}/\mathcal{L}\mathbb{Z}; \mu)\},$$

where $1 \le q < \infty$ and $\mathcal{P}(\mathbb{R}/\mathcal{L}\mathbb{N})$ denotes the set of Borel probability measure on $\mathbb{R}/\mathcal{L}\mathbb{N}$. For (μ, f) and (ν, g) in TL^q we define the distance

$$d_{TL^q}\big((\mu, f), (\nu, g)\big) := \inf_{\pi \in \Gamma(\mu,\nu)} \left(\int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} (|x-y|^q + |f(x) - g(y)|^q)\, d\pi(x,y) \right)^{1/q},$$

where $\Gamma(\mu, \nu)$ is the set of all coupling (or transportation plans) between μ and ν, that is, the set of all Borel probability measures on $(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$ for which the marginal on the first variable is μ and the marginal on the second variable is ν. It is shown in [11] that d_{TL^q} is actually a metric. The distance d_{TL^q} is called a transportation distance between functions defined on graph. The TL^q topology provides a general and versatile way to compare functions in a discrete setting with functions in a continuum setting. It is a generalization of the weak convergence of measures and L^q convergence of functions.

Definition 1.3. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables and let us denote by ν_n the *empirical measure* of $\{X_i\}_{i\in\mathbb{N}}$:

$$\nu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i},$$

where δ_X is the Dirac measure of X.

Definition 1.4. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables on $\mathbb{R}/\mathcal{L}\mathbb{Z}$ and ν_n be an empirical measure of $\{X_i\}_{i\in\mathbb{N}}$ and ν be a distribution measure of $\{X_i\}_{i\in\mathbb{N}}$. We use a slight abuse of notation: $\gamma_n \xrightarrow{TL^q} \gamma$ instead of $(\nu_n, \gamma_n) \xrightarrow{TL^q} (\nu, \gamma)$. (Note that $\mathcal{A} \subset L^q(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$.) Thus $\mathcal{A}$ is not a metric space with the metric d_{TL^q} but only convergence is defined.

The main results of the paper are

Theorem 1.1 (Γ-convergence and locally uniform convergence)**.** *Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with probability density function ρ on $\mathbb{R}/\mathcal{L}\mathbb{Z}$. Then $R_{n,\rho}\mathcal{E}^{\alpha,p}$ Γ-converge to $\mathcal{E}^{\alpha,p}_\rho$ as $n \to \infty$ in the TL^1 sense.*

Moreover, we set $\mathcal{F} := \{\gamma \in \mathcal{A} \mid \mathcal{E}^{\alpha,p}_\rho(\gamma) < \infty\}$, then $R_{n,\rho}\mathcal{E}^{\alpha,p}\lfloor_{\mathcal{F}}$ locally uniformly converges to $\mathcal{E}^{\alpha,p}_\rho\lfloor_{\mathcal{F}}$ as $n \to \infty$ in the TL^1 sense a.s. $\omega \in \Omega$.

Theorem 1.2 (Compactness)**.** *Let ρ be bounded from below by a positive constant and let $2 \le \alpha p < 2p+1$ and $1 \le q < \infty$,*

Assume $\{\gamma_n\}_{n\in\mathbb{N}} \subset TL^q(\mathbb{R}/\mathcal{L}\mathbb{Z})$ satisfying

$$\sup_{n\in\mathbb{N}} R_{n,\rho}\mathcal{E}^{\alpha,p}(\gamma_n) < \infty.$$

and we assume that there exists $x \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ and $C \in \mathbb{R}^d$ such that $\gamma_n(x) = C$ for all $n \in \mathbb{N}$.

Then $\{\gamma_n\}_{n\in\mathbb{N}}$ is relatively compact in the $TL^q(\mathbb{R}/\mathcal{L}\mathbb{Z})$ sense a.s. $\omega \in \Omega$.

The metric used in Γ-convergence is the TL^1 sense. Locally uniform convergence is also in the TL^1 sense, which will be defined in Section 2.

Corollary 1 (Minimizer to minimizer)**.** Under the assumption of Theorem 1.1 and Theorem 1.2, let $\{\gamma_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ be minimizers of $R_{n,\rho}\mathcal{E}^{\alpha,p}$ then there exists $\tilde{\gamma} \in \mathcal{A}$ and subsequence $\{\gamma_{n_k}\}_{k\in\mathbb{N}}$ with $\gamma_{n_k} \xrightarrow{TL^q} \tilde{\gamma}$ such that $\tilde{\gamma}$ is a minimizer of $\mathcal{E}^{\alpha,p}$.

2 Preliminaries

2.1 Γ-convergence and locally uniformly convergence in the TL^q sense

We recall notation of general Γ-converge and locally uniform convergence in the TL^q sense.

Definition 2.1 (Γ-convergence in the TL^q sense)**.** Let $F_n : \mathcal{A} \to \mathbb{R}\cup\{+\infty\}$ be a sequence of functionals. The sequence $\{F_n\}_{n\in\mathbb{N}}$ Γ-converges to the functional $F : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ as $n \to \infty$ in the TL^q sense if the following inequality hold:

i) For every $x \in X$ and every sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \xrightarrow{TL^q} x$

$$\liminf_{n\to\infty} F_n(x_n) \geq F(x),$$

ii) For every $x \in X$ there exists a sequence $\{x_n\}_{n\in N}$ and $x_n \xrightarrow{TL^q} x$ satisfying

$$\limsup_{n\to\infty} F_n(x_n) \leq F(x).$$

Definition 2.2 (Locally uniform convergence in the TL^q sense)**.** A set $K \subset \mathcal{A}$ *is sequentially compact in the TL^q sense* if it satisfies the following conditions. For all sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset K$, there is a subsequence $\{\gamma_{n_k}\}_{k\in\mathbb{N}}$ and a $\gamma \in K$ such that $\gamma_{n_k} \xrightarrow{TL^q} \gamma$ as $k \to \infty$. Let X be a space containing $L^q(\nu_n)$ and $L^q(\nu)$, and let $F_n : X \to \mathbb{R}$ and $F : X \to \mathbb{R}$. The sequence $\{F_n\}_{n\in\mathbb{N}}$ *locally uniformly converges to F in the TL^q sense* if it satisfies the following conditions. For any sequentially compact set $K \subset X$ in the TL^q sense,

$$\lim_{n\to\infty} \sup_{\gamma\in K} |F_n(\gamma) - F(\gamma)| = 0.$$

We first discuss an equivalent condition for locally uniform convergence in the TL^q sense.

Proposition 2.1. *Let $F : X \to \mathbb{R}$ is continuous in the TL^q sense, (i.e. $\gamma_n \xrightarrow{TL^q} \gamma$ then $\lim_{n\to\infty} F(\gamma_n) = F(\gamma)$.) and a sequence $\{F_n\}_{n\in\mathbb{N}}$ locally uniformly converge to F in the TL^q sense.*

if and only if

For any sequence $\{\gamma_n\}_{n\in\mathbb{N}} \subset X$ with $\gamma_n \xrightarrow{TL^q} \gamma$,

$$\lim_{n\to\infty} F_n(\gamma_n) = F(\gamma).$$

Proof. This can be proved in a similar way to prove Ascoli-Arzelà theorem [18, Theorem 7.25.]. For $\gamma \in \mathcal{A}$ and $r > 0$, we set $\mathcal{B}(\gamma, r) := \{\tilde{\gamma} \in X \mid \text{There exists an } n \in \mathbb{N} \text{ such that } d_{TL^q}\big((\nu_n, \gamma), (\nu, \tilde{\gamma})\big) < r\}$.

First, we show that suppose $K \subset X$ be a sequentially compact set in the TL^q sense, then for any $\varepsilon > 0$, there is a sequence $\{\gamma_i\}_{i=1}^{N_\varepsilon} \subset K$, such that $\displaystyle\bigcup_{i=1}^{N_\varepsilon} \mathcal{B}(\gamma_i, \varepsilon) \supset K$. If not, there is a $r > 0$ such that for all $\{\gamma_i\}_{i=1}^{m} \subset K$, $K \setminus \displaystyle\bigcup_{i=1}^{m} \mathcal{B}(\gamma_i, r) \neq \emptyset$. We choose $\gamma_1 \in K$ and inductively choose $\gamma_n \in K \setminus \displaystyle\bigcup_{i=1}^{n-1} \mathcal{B}(\gamma_i, r)$, then for all $m, n \in \mathbb{N}$,

$$\begin{aligned} r &\leq d_{TL^q}\big((\nu_n, \gamma_n), (\nu, \gamma_m)\big) \leq d_{TL^q}\big((\nu_n, \gamma_n), (\nu_m, \gamma_m)\big) + d_{TL^q}\big((\nu_m, \gamma_m), (\nu, \gamma_m)\big) \\ &= d_{TL^q}\big((\nu_n, \gamma_n), (\nu_m, \gamma_m)\big). \end{aligned}$$

This is a contradiction to the fact that K is sequentially compact set in the TL^q sense.

Let $\varepsilon_m > 0$ with $\varepsilon_m \searrow 0$. and we set

$$K_0 := \left\{ \{\gamma_i^m\}_{m=1}^{N_m} \in K \;\middle|\; \bigcup_{i=1}^{N_m} \mathcal{B}(\gamma_i^m, \varepsilon_m) \supset K \right\}. \tag{5}$$

Second, we show that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\gamma_1, \gamma_2 \in K$ and all $n \in \mathbb{N}$, if $d_{TL^q}\big((\nu_n, \gamma_1), (\nu, \gamma_2)\big) < \delta$ then

$$|F_n(\gamma_1) - F_n(\gamma_2)| < \varepsilon. \tag{6}$$

If not, there exists an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $\gamma_1^n, \gamma_2^n \in K$ and $m_n \in \mathbb{N}$ such that $d_{TL^q}\big((\nu_n, \gamma_1^n), (\nu, \gamma_2^n)\big) < 1/n$, and $|F_{m_n}(\gamma_1^n) - F_{m_n}(\gamma_2^n)| \geq \varepsilon$. By the K is sequentially compact, there exists a subsequence $\{\gamma_1^{n_k}\}_{k\in\mathbb{N}}, \{\gamma_2^{n_k}\}_{k\in\mathbb{N}}$, and $\gamma \in K$ such that $\gamma_1^{n_k} \xrightarrow{TL^q} \gamma$, $\gamma_2^{n_k} \xrightarrow{TL^q} \gamma$. Then

$$\varepsilon < |F_{m_{n_k}}(\gamma_1^{n_k}) - F_{m_{n_k}}(\gamma_2^{n_k})| \leq |F_{m_{n_k}}(\gamma_1^{n_k}) - F(\gamma)| + |F(\gamma) - F_{m_{n_k}}(\gamma_2^{n_k})| \xrightarrow{k\to\infty} 0.$$

This is a contradiction.

Third, we show that there exists a subsequence $\{F_{n_k}\}_{k\in\mathbb{N}}$ such that for all j , $\{F_{n_k}(\gamma_j)\}_{k\in\mathbb{N}}$ is convergence sequence by a diagonal argument. By $\lim_{n\to\infty} F_n(\gamma_1) = F(\gamma_1)$, $\{F_n(\gamma_1)\}_{n\in N}$ is bounded sequence on $\mathbb{R}$. Therefore there exists a subsequence $\{F_{n(1,k)}\}_{k\in\mathbb{N}}$ such that $\{F_{n(1,k)}(\gamma_1)\}_{k\in\mathbb{N}}$ is a convergence sequence in $\mathbb{R}$. In the same way, there exists a subsequence $\{F_{n(2,k)}\}_{k\in\mathbb{N}}$ such that $\{F_{n(2,k)}(\gamma_2)\}_{k\in\mathbb{N}}$ is a convergence sequence on $\mathbb{R}$. Further, in the same way, we construct subsequence $\{F_{n(p,k)}\}_{k\in\mathbb{N}}, p = 3, 4, ...,$ and we set $F_{n_k} = F_{n(k,k)}$.

Finally, let any $\eta > 0$, for sufficiently large m such that for all $n \in \mathbb{N}$, if $d_{TL^q}\big((\nu_n, \gamma_1), (\nu, \gamma_2)\big) < \varepsilon_m$ then

$$|F_n(\gamma) - F_n(\gamma_i^m)| < \eta/3. \tag{7}$$

Since $\{F_{n_k}(\gamma_i^m)\}_{k\in\mathbb{N}}$ is a convergence sequence on $\mathbb{R}$, there exists a number N such that if $k,l > N$ then $|F_{n_k}(\gamma_i^m) - F_{n_l}(\gamma_i^m)| < \eta/2$ for $i = 1,2,...,N_m$. Now, let $\gamma \in K$, by (5) there exist an i and n such that

$$d_{TL^q}\big((\nu_n,\gamma_i^m),(\nu,\gamma)\big) < \varepsilon_m.$$

By (6) and (7) if $k,l > N$ then

$$\begin{aligned}|F_{n_k}(\gamma) - F_{n_l}(\gamma)| &\le |F_{n_k}(\gamma) - F_{n_k}(\gamma_i^m)| + |F_{n_k}(\gamma_i^m) - F_{n_l}(\gamma_i^m)| + |F_{n_l}(\gamma) - F_{n_l}(\gamma_i^m)| \quad (8)\\ &\le \eta. \quad (9)\end{aligned}$$

Therefore $\{F_{n_k}(\gamma)\}_{k\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of $\mathbb{R}$, there exists a $F(\gamma)$ such that $\lim_{k\to\infty} F_{n_k}(\gamma) = F(\gamma)$. In (8) and (9), suppose that l goes to infinity, then we get

$$|F_{n_k}(\gamma) - F(\gamma)| < \eta. \tag{10}$$

This indicates that $\{F_{n_k}\}_{k\in\mathbb{N}}$ uniformly converges to F on K.

Assume that F is continuous and that F_n locally uniformly converges to F in the TL^q sense and $\gamma_n \xrightarrow{TL^q} \gamma$.

Clearly, $\{\gamma_n \mid n \in \mathbb{N}\} \cup \{\gamma\} \subset X$ is sequentially compact in the TL^q sense.

Therefore

$$\begin{aligned}|F_n(\gamma_n) - F(\gamma)| &\le |F_n(\gamma_n) - F(\gamma_n)| + |F(\gamma_n) - F(\gamma)|\\ &\le \sup_{y\in\{\gamma_n\}_{n\in\mathbb{N}}\cup\{\gamma\}} \big(|F_n(y) - F(y)|\big) + |F(\gamma_n) - F(\gamma)| \xrightarrow{n\to\infty} 0.\end{aligned}$$

□

2.2 The property of TL^q space and empirical measure

In this subsection, we consider the space based on optimal transport theory to compare discrete and continuous models.

Given a Borel map $T : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}/\mathcal{L}\mathbb{Z}$ and $\mu \in \mathcal{P}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ the *push-forward* of μ by T, denoted by $T_\#\mu \in \mathcal{P}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ is given by:

$$T_\#\mu(A) := \mu(T^{-1}(A)), A \in \mathcal{B}(\mathbb{R}/\mathcal{L}\mathbb{Z}).$$

Definition 2.3 ([10])**.** We say that a sequence of transportation plans $\{\pi_n\}_{n\in\mathbb{N}} \subset \Gamma(\mu,\mu_n)$ is *stagnating* if it satisfies

$$\lim_{n\to\infty} \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} |x-y|^q d\pi_n(x,y) = 0.$$

$\mu,\mu_n \in \mathcal{P}(\mathbb{R}/\mathcal{L}\mathbb{Z}), T_n : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}/\mathcal{L}\mathbb{Z}$: transportation maps with $T_{n\#}\mu = \mu_n$.

We say that a sequence of transportation maps $\{T_n\}_{n\in\mathbb{N}}$ is *stagnating* if it satisfies

$$\lim_{n\to\infty} \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} |x - T_n(x)|^q d\mu(x) = 0.$$

Accept the following proposition introduced by [10]

Proposition 2.2 ([10]). *Let $(\mu,\gamma) \in TL^q$ and let $\{(\mu_n,\gamma_n)\}_{n\in\mathbb{N}} \subset TL^q$ The following statements are equivalent;*

i) $(\mu_n,\gamma_n) \to (\mu,\gamma)$ *in the* TL^q.

ii) $\mu_n \rightharpoonup \mu$ *and for every stagnating sequence of transportation plans* $\{\pi_n\}_{n\in\mathbb{N}} \subset \Gamma(\mu,\mu_n)$

$$\int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} |\gamma(x)-\gamma_n(x)|^q d\pi_n(x,y) \to 0. \tag{11}$$

iii) $\mu_n \rightharpoonup \mu$ *and there exists a stagnating sequence of transportation plans* $\{\pi_n\}_{n\in\mathbb{N}} \subset \Gamma(\mu,\mu_n)$ *such that*

$$\int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} |\gamma(x)-\gamma_n(x)|^q d\pi_n(x,y) \to 0. \tag{12}$$

Moreover, if the measure μ is absolutely continuous with respect to the Lebesgue measure, the following are equivalent to the previous statement:

iv) $\mu_n \rightharpoonup \mu$ *and there exists a stagnating sequence of transportation maps* $\{T_n\}_{n\in\mathbb{N}}$ *(with* $T_{n\#}\mu = \mu_n$*) such that*

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} |\gamma(x)-\gamma_n(T_n(x))|^q d\mu(x) \to 0. \tag{13}$$

v) $\mu_n \rightharpoonup \mu$ *and for any stagnating sequence of transportation maps* $\{T_n\}_{n\in\mathbb{N}}$ *with* $T_{n\#}\mu = \mu_n$*)*

$$\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} |\gamma(x)-\gamma_n(T_n(x))|^q d\mu(x) \to 0. \tag{14}$$

Remark 2. Thanks to Proposition 2.2 when μ is absolutely continuous with respect to the Lebesgue measure $\gamma_n \xrightarrow{TL^p} \gamma$ as $n \to \infty$ if and only if for every (or one) stagnating sequence $\{T_n\}_{n\in\mathbb{N}}$ of transportation maps (with $T_{n\#}\mu = \mu_n$) $\gamma_n \circ T_n \xrightarrow{L^p(\mu)} \gamma$ as $n \to \infty$. Also, $\{u_n\}_{n\in\mathbb{N}}$ is relatively compact in TL^p if and only if for every (or one) stagnating sequence $\{T_n\}_{n\in N}$ of transportation maps (with $T_{\#}\mu = \mu_n$) $\{u_n \circ T_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^p(\mu)$.

We recall the following proposition.

Proposition 2.3 (**Glivenko-Cantelli's Theorem**). *Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, and let ν is the distribution measure of $\{X_i\}_{i\in\mathbb{N}}$, and ν_n is the empirical measure of $\{X_i\}_{i\in\mathbb{N}}$. $F_n(x) := \nu_n((-\infty,x])$ and F is the distribution function of $\{X_i\}_{i\in\mathbb{N}}$, then,*

$$\lim_{n\to\infty} \|F_n - F\|_\infty = 0 \qquad a.s.\ \omega \in \Omega.$$

Theorem 2.1 ([12]). *Let $D \subset \mathbb{R}^d$ be a bounded, connected, open set with Lipschitz boundary. Let ν be a probability measure on D with density $\rho : D \to (0, \infty)$ which is bounded from below and from above by positive constants. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent random points distributed on D according to measure ν and let ν_n be the associated empirical measures. Then there is a constant $C > 0$ such that for a.s. $\omega \in \Omega$ there exists a sequence of transportation maps $\{T_n\}_{n\in\mathbb{N}}$ from ν to ν_n ($T_{n\#}\nu = \nu_n$) and such that*

if $d = 2$ then

$$\limsup_{n\to\infty} \frac{n^{1/2}\|Id - T_n\|_\infty}{(\log n)^{3/4}} \leq C$$

and if $d \geq 3$ then

$$\limsup_{n\to\infty} \frac{n^{1/d}\|Id - T_n\|_\infty}{(\log n)^{1/d}} \leq C.$$

3 Γ-convergence and locally uniformly convergence

Proof of Theorem 1.1

Proof. Let ν_n be an empirical measure of $\mathcal{L}^1\lfloor\rho$ and $T_n : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}/\mathcal{L}\mathbb{Z}$ be transportation maps with $T_{n\#}\mathcal{L}^1\lfloor\rho = \nu_n$.

liminf inequality

Assume that $\gamma_n \to \gamma$ in TL^1 as $n \to \infty$. Since $T_{n\#}\mathcal{L}^1\lfloor\rho = \nu_n$, we change variables to get

$$R_{n,\rho}\mathcal{E}^{\alpha,p}(\gamma_n) = \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathcal{M}^{\alpha,p}(\gamma_n)d\nu_n(x)d\nu_n(y) \tag{15}$$

$$= \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathcal{M}^{\alpha,p}(\gamma_n \circ T_n)\rho(x)\rho(y)dxdy. \tag{16}$$

By the way, we notice that

$$|\gamma_n(T_n(x)) - \gamma_n(T_n(y))| \leq |\gamma_n(T_n(x)) - \gamma(x)| + |\gamma(x) - \gamma(y)| + |\gamma(y) - \gamma_n(T_n(y))|.$$

and

$$\begin{aligned}\mathcal{D}(\gamma(x), \gamma(y)) &\leq \mathcal{D}(\gamma(x), \gamma_n(T_n(x))) + \mathcal{D}(\gamma_n(T_n(x)), \gamma_n(T_n(y))) + \mathcal{D}(\gamma_n(T_n(y)), \gamma(y)) \\ &\leq |x - T_n(x)| + \mathcal{D}(\gamma_n(T_n(x)), \gamma_n(T_n(y))) + |T_n(y) - y| \\ &\leq 2\|T_n - \mathrm{Id}\|_\infty + \mathcal{D}(\gamma_n(T_n(x)), \gamma_n(T_n(y))).\end{aligned}$$

By Proposition 2.2 we deduce $\gamma_n \circ T_n \to \gamma$ in $L^1(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Thus, by taking an appropriate subsequence $\{\gamma_{n_k} \circ T_{n_k}\}_{k\in\mathbb{N}}$ of $\{\gamma_n \circ T_n\}_{n\in\mathbb{N}}$, we deduce $\gamma_{n_k}(T_{n_k}(x)) \to \gamma(x)$ a.e. $x \in \mathbb{R}/\mathcal{L}\mathbb{Z}$, and therefore

$$\liminf_{n\to\infty} \mathcal{M}^{\alpha,p}(\gamma_n \circ T_n) \geq \mathcal{M}^{\alpha,p}(\gamma) \quad \text{a.e. } (x, y) \in (\mathbb{R}/\mathcal{L}\mathbb{Z})^2.$$

So that $\mathcal{M}^{\alpha,p}(\gamma_n \circ T_n) > 0$, using Fatou's lemma we get

$$\liminf_{n\to\infty} R_{n,\rho}\mathcal{E}^{\alpha,p}_n(\gamma_n) \geq \mathcal{E}^{\alpha,p}_\rho(\gamma).$$

limsup inequality

Assume that $\gamma_n \to \gamma$ in TL^1 as $n \to \infty$. If $\mathcal{E}^{\alpha,p}_\rho(\gamma) = \infty$, we are done, and so we henceforth assume $\mathcal{E}^{\alpha,p}_\rho(\gamma) < \infty$. We observe that

$$|\gamma(x) - \gamma(y)| \leq |\gamma(x) - \gamma_n(T_n(x))| + |\gamma_n(T_n(x)) - \gamma_n(T_n(y))| + |\gamma_n(T_n(y)) - \gamma(y)|,$$

and

$$\mathcal{D}(\gamma_n(T_n(x)), \gamma_n(T_n(y))) \leq \mathcal{D}(\gamma_n(T_n(x)), \gamma(x)) + \mathcal{D}(\gamma(x), \gamma(y)) + \mathcal{D}(\gamma(y), \gamma_n(T_n(y))).$$

In the same way as liminf inequality, we get

$\limsup_{n\to\infty} \mathcal{M}^{\alpha,p}(\gamma_n \circ T_n) \leq \mathcal{M}^{\alpha,p}(\gamma)$ a.e. $(x, y) \in (\mathbb{R}/\mathbb{Z})^2$. Since $\mathcal{E}^{\alpha,p}_\rho(\gamma) < \infty$, using Fatou's lemma to $\mathcal{M}^{\alpha,p}(\gamma) - \mathcal{M}^{\alpha,p}(\gamma_n \circ T_n)$, we get

$$\limsup_{n\to\infty} R_{n,\rho}\mathcal{E}^{\alpha,p}(\gamma_n) \leq \mathcal{E}^{\alpha,p}_\rho(\gamma).$$

By Proposition 2.1, the proof is now complete.

□

4 Compactness

In this section, we would like to prove Theorem 1.2. We first recall several function spaces.

4.1 Function spaces

Definition 4.1 (**Sobolev-Slobodeckij spaces).** For $s \in (0, 1)$ and $q \in [1, \infty)$ we set

$$[\gamma]_{W^{s,p}} := \left(\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \int_{-\mathcal{L}/2}^{\mathcal{L}/2} \frac{|\gamma(u+w) - \gamma(u)|^p}{|w|^{1+ps}} dw du \right)^{1/p}$$

$$W^{s,p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) := \left\{ \gamma \in L^q(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \mid [\gamma]_{W^{s,p}} < \infty \right\}$$

and equip this space with the norm

$$\|\gamma\|_{W^{s,q}} := \|\gamma\|_{L^q} + [\gamma]_{W^{s,p}}.$$

Furthermore, we let

$$W^{1+s,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) := \left\{ \gamma \in W^{1,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \mid \gamma' \in W^{s,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \right\}$$

and

$$\|\gamma\|_{W^{1+s,q}} := \left(\|\gamma\|^q_{W^{s,q}} + \|\gamma\|^q_{W^{1,q}} \right)^{1/q}.$$

Definition 4.2 (Besov spaces). For $s \in \mathbb{R}$, $1 \le p, q \le \infty$ and $\mathcal{S}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$ is the Schwartz space.

We set

$$\psi(\xi) = \begin{cases} 1 & (|\xi| \le 4) \\ 0 & (|\xi| \ge 8) \end{cases}, \quad \varphi(\xi) = \begin{cases} 1 & (2 \le |\xi| \le 4) \\ 0 & (|\xi| \le 1 \text{ or } |\xi| \ge 8) \end{cases}.$$

We set $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and $\tau(D)f := \mathcal{F}^{-1}[\tau \cdot \mathcal{F}f]$ for $\tau \in \mathcal{S}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$. We recall Besov norms:

$$\|f\|_{B^s_{p,q}} := \begin{cases} \|\psi(D)f\|_{L^p} + \left(\sum_{j=1}^{\infty} 2^{jqs} \|\varphi_j(D)f\|^q_{L^p} \right)^{1/q} & (1 \le q < \infty) \\ \|\psi(D)f\|_{L^p} + \sup_{j \in \mathbb{N}} 2^{js} \|\varphi_j(D)f\|_{L^p} & (q = \infty) \end{cases}$$

$$B^s_{p,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) := \{ f \in \mathcal{S}'(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \mid \|f\|_{B^s_{p,q}} < \infty \}.$$

Note that $W^{s,p}$ agrees with $B^s_{p,p}$.

We recall the following embedding results.

Proposition 4.1 (Embedding Besov spaces). *Let $0 < p_0, p_1 \le \infty$ and $0 < q_0, q_1 \le \infty$ and $-\infty < s_1 < s_0 < \infty$. Assume that $s_0 - \frac{1}{p_0} > s_1 - \frac{1}{p_1}$, then*

$$B^{s_0}_{p_0,q_0}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d).$$

Here $\hookrightarrow$ denotes a continuous embedding.

Theorem 4.1 (Kondrachov embedding theorem). *Let $1 \le p, q < \infty$ and $1 < k, s$. Assume that $k - \frac{1}{p} > s - \frac{1}{q}$, then the Sobolev embedding*

$$W^{k,p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \hookrightarrow W^{s,q}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d)$$

is completely continuous.

Proposition 4.2 (Embedding L^1 space for Besov space). *Let $s \in \mathbb{R}$ and $0 < q \le \infty$, then*

$$L^1 \hookrightarrow B^0_{1,q} \text{ if and only if } q = \infty.$$

The following theorem is based on [14] and explains conditions for which O'Hara energy becomes finite.

Theorem 4.2 ([14])**.** *Let $\gamma \in \mathcal{A}$ and $\alpha, p \in (0, \infty)$ with $\alpha p \ge 2$ and $s := \frac{\alpha p - 1}{2p} < 1$ and $p \ge 1$, then $\mathcal{E}^{\alpha,p}(\gamma) < \infty$ if and only if $\gamma \in W^{1+s,2p}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Moreover, there is a $C = C(\alpha, p)$ such that*

$$\|\gamma'\|^{2p}_{W^{s,2p}} \le C(\mathcal{E}^{\alpha,p}(\gamma) + \|\gamma'\|^{2p}_{L^{2p}}).$$

Lemma 4.3. *Let $\{\gamma_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ and we assume that there exists a $x \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ and $C \in \mathbb{R}^d$ such that $\gamma_n(x) = C$ for all $n \in \mathbb{N}$.*

Then

$$\sup_{n \in \mathbb{N}} \|\gamma_n\|_{L^1(\nu_n)} = \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=1}^{n} |\gamma_n(X_i)| < \infty.$$

Proof. First, we show that there exists a $M > 0$ such that

$$\max_{i=1,2,\ldots,n} |\gamma_n(X_i) - C| \le M.$$

If not, there exists a $n \in \mathbb{N}$ and $i \in \{1, 2, ..., n\}$ such that

$$|\gamma_n(X_i) - C| > \mathcal{L}.$$

Then a length of γ_n is more than $\mathcal{L}$. This is a contradiction.
Thus

$$\begin{aligned}\|\gamma_n\|_{L^1(\nu_n)} &= \frac{1}{n}\sum_{i=1}^{n} |\gamma_n(X_i)| \\ &\le \frac{1}{n}\sum_{i=1}^{n} |\gamma_n(X_i) - C| + \frac{1}{n}\sum_{i=1}^{n} |C| \le M + |C|.\end{aligned}$$

□

4.2 Proof of Theorem 1.2.

Proof. Let $s := \frac{\alpha p - 1}{2p}$. By Theorem 4.2 we see

$$\|\gamma\|^{2p}_{W^{1+s,2p}} \le C\left(\mathcal{E}^{\alpha,p}(\gamma) + \|\gamma\|^{2p}_{W^{1,2p}}\right).$$

By the Gagliardo-Nirenberg interpolation inequality, we choose $\Theta > 0$ such that $\Theta < \frac{(2p-1)q+2p}{(2p+\alpha p-2)q+2p} \le 1$, and t with $\frac{1}{2p} - 1 > t$, to get

$$\begin{aligned}\|\gamma\|_{W^{1,2p}} &\le \|\gamma\|_{W^{1+s/2,2p}} &(17)\\ &= \|\gamma\|_{B^{1+s/2}_{2p,2p}} \le C\|\gamma\|^{\Theta}_{B^{1+s}_{2p,2p}} \|\gamma\|^{1-\Theta}_{B^{t}_{2p,2p}}. &(18)\end{aligned}$$

Since $L^1 \hookrightarrow B^0_{1,\infty} \hookrightarrow B^t_{2p,2p}$, this yields

$$\|\gamma\|_{W^{1,2p}} \le C\|\gamma\|^{\Theta}_{B^{1+s}_{2p,2p}} \|\gamma\|^{1-\Theta}_{L^1}.$$

Using Young's inequality, for all $\varepsilon > 0$, we get

$$\begin{aligned}\|\gamma\|^{2p}_{W^{1,2p}} &\le C\|\gamma\|^{2p\theta}_{W^{1+s,2p}} \|\gamma\|^{2p(1-\Theta)}_{L^1} &(19)\\ &\le C\varepsilon^{1/\Theta}\Theta\|\gamma\|^{2p}_{W^{1+s,2p}} + C(1-\Theta)\frac{\|\gamma\|^{2p}_{L^1}}{\varepsilon^{1/(1-\Theta)}}. &(20)\end{aligned}$$

Therefore, for sufficient small $\varepsilon > 0$, we conclude that

$$\|\gamma\|^{2p}_{W^{1+s,2p}} \le C'\left(\mathcal{E}^{\alpha,p}(\gamma) + \|\gamma\|^{2p}_{L^1}\right). \tag{21}$$

Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of $TL^q(\mathbb{R}/\mathbb{Z})$ with

$$\sup_{n\in\mathbb{N}} R_{n,\rho}\mathcal{E}^{\alpha,p}(\gamma_n) < \infty,$$

and let $T_n : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}/\mathcal{L}\mathbb{Z}$ be transportation maps with $T_{n\#}\mu = \mu_n$. Then

$$\sup_{n\in\mathbb{N}} \mathcal{E}_\rho^{\alpha,p}(\gamma_n \circ T_n) < \infty.$$

Since ρ is bounded from below by a positive constant we deduce by Lemma 4.2 that

$$\sup_{n\in\mathbb{N}} (\mathcal{E}^{\alpha,p}(\gamma_n \circ T_n) + \|\gamma_n \circ T_n\|_{L^1}) < \infty.$$

Therefore by (21) and Lemma 4.3, we see

$$\sup_{n\in\mathbb{N}} \|\gamma_n \circ T_n\|_{W^{1+s,2p}} < \infty.$$

Since $1 + s - \frac{1}{2p} = \frac{2p+\alpha p-2}{2p} \geq 0 > -\frac{1}{q}$, Theorem 4.1 yields a compact embedding

$$\iota : W^{1+s,2p}(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d) \hookrightarrow\hookrightarrow L^q(\mathbb{R}/\mathcal{L}\mathbb{Z}, \mathbb{R}^d).$$

Therefore there exists an $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ and $\gamma \in L^q(\mathbb{R}/\mathcal{L}\mathbb{Z}, \rho)$ such that $\gamma_{n_k} \circ T_{n_k} \to \gamma$ in $L^q(\mathbb{R}/\mathbb{Z}, \rho)$. By Proposition 2.2 we see $\gamma_{n_k} \to \gamma$ in $TL^q(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

$\square$

References

[1] A. Braides, Γ-convergence for Beginners. Oxford Lecture Series in Mathematics and Its Applications Series. Oxford University Press, Incoporated, (2002).

[2] A. Lunardi, Interpolation theory. Edizioni della Normale, (2009).

[3] D. Kim, R. Kusner, Torus knots extremizing the Möbius energy, *Experiment. Math.* **2** (1993) 1-9.

[4] H. Triebel, Theory of Function Spaces. Modern Birkhäuser Classics, (1983).

[5] J. K. Simon, Energy functions for polygonal knots, *Knot Theory Ramifications* **3**(3) (1994) 299-320.

[6] J. O'Hara, Energy of a knot, *Topology* **30**(2) (1991) 241-247.

[7] J. O'Hara, Energy functionals of knots, in *Topology Hwaii* (Honolulu, HI ,1990) 147-161. (World Scientific Publishing, River Edge, NJ, 1992).

[8] J. O'Hara, Family of energy functionals of knots, *Topology Appl.* **48**(2) (1992) 147-161.

[9] J. O'Hara, Energy functionals of knots Ⅱ, *Topology Appl.* **56**(1) (1994) 45-61.

[10] N. G. Trillos, D. Slepčev, A variational approach to the consistency of spectral clustering, *Appl. Comput. Harmon. Anal.* (2016) 239-281.

[11] N. G. Trillos, D. Slepčev, Continuum Limit of Total Variation on Point Clouds, *Arch. Ratonal. Mach. Anal.* **220** (2016) 193-241.

[12] N. G. Trillos, D. Slepčev, On the rate of convergence of empirical measures in ∞-transportation distance, *Canad. J. Math.* (2015) 1358-1383.

[13] R. Kusner and J. Sullivan, Möbius energies for knots and links, surfaces and submanifolds, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., vol. 2, Amer. Math. Soc., Providence, RI, (1997) 570-604.

[14] S. Blatt, Boundedness and regularizing effects of O'Hara's knot energies, *Journal of Knot Theory and Its Ramifications* **21** (2012) 1250010, 9.

[15] S. Blatt, A. Ishizeki, T. Nagasawa, A Möbius invariant discretization of O'Hara's Möbius energy, arXiv:1809.07984.(2018)

[16] S. Blatt, P. Reiter, A. Schikorra, Harmonic Analysis meets Critical Knots.Critical Point of The Möbius Energy are smooth, *Trans. Amer. Math. Soc.* **368**(9) (2016) 6391-6438.

[17] S. Scholtes, Discrete Möbius energy, *J. Knot Theory Ramifications,* **23**(9):1450045, 16, (2014).

[18] W. Rudin. The Principles of Mathematical Analysis. McGraw-Hill Publishing Company, (2006).