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NUMERICAL REGULARITY MAP FOR FUNDAMENTAL ONE-DIMENSIONAL FRACTIONAL DIFFERENTIAL EQUATIONS WITH HÖLDER CONTINUOUS SOLUTIONS

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Abstract. In the paper, fundamental one-dimensional fractional differential equations in the sense of Caputo with Hölder continuous solutions are solved numerically. A large number of numerical experiments results in a numerical regularity map which visualizes relationship between the Hölder exponent of the solution, the convergence rate of schemes, and the order of the derivative. The map reveals that the rate of convergence changes depending on whether the derivative order is greater than or less than one.

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1 Introduction

We have proposed a high-accuracy numerical method using both the spectral method and the multiple-precision arithmetic [12] for linear partial differential equations and integral equations with analytic solutions. It is well known that the spectral method is useful for the arbitrary reduction of truncation errors [2, 3], and the multiple-precision arithmetic is useful for the arbitrary reduction of rounding errors. As for the spectral method, it is also well known that the convergence rate of errors is related to the regularity of the solution [2, 3]. Thus, we have also paid attention to the numerical simulation on the existence or the regularity of the solution to differential equations with the aim of contributing to theoretical analysis [7,11,16]. As a study along this aim, we have recently proposed the numerical regularity map of the solution, and made such maps for nonlinear differential equations with a blow-up phenomenon [14]. The numerical regularity map is the visualization of the regularity of functions or solutions. It contributes not only to theoretical analysis but also to guarantee reliability of numerical results.

On the other hand, fractional differential equations have attracted attention in many fields, and the progress of their research is remarkable [1, 4, 6, 13, 17]. In our recent papers [8,9], we have proposed a highly accurate numerical method by the use of Chebyshev spectral collocation method for fractional differential equations involving Caputo derivative [5];

$$
{}_{\mathrm{C}}D_0^{\alpha}u(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-s)^{n-\alpha-1} \frac{d^n u(s)}{ds^n}\, ds, & n-1 < \alpha < n \in \mathbb{N}, \\[2ex] \dfrac{d^n}{dt^n} u(t), & \alpha = n \in \mathbb{N}, \end{cases} \tag{1}
$$

where α and t are positive numbers.

In our earlier studies, we showed possibility to detect analyticity of solutions and degrees of solutions with a polynomial from numerical results by leveraging spectral collocation methods. Furthermore, for fractional differential equations, we found that convergence of errors behaves as a function for solutions with Hölder continuity [9].

In the paper, we consider the following fundamental initial value problem with the Hölder continuous solution.

Problem. Let α and β be positive numbers with $\alpha < \beta$. Find $u(t)$ satisfying the initial value problem

$$
{}_{\mathrm{C}}D_0^{\alpha}u(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \qquad 0 < t \leq 3, \tag{2a}
$$

$$
u(0) = 0, \quad u'(0) = 0, \quad \ldots, \quad u^{(n-1)}(0) = 0, \tag{2b}
$$

where $n = \lceil \alpha \rceil$ is the smallest integer more than or equal to α.

Obviously the function $u(t) = t^{\beta}$ satisfies the problem.

Our main purpose is to obtain a numerical regularity map which visualizes relationship between the Hölder exponent of the solution, the convergence rate of schemes, and the order of the derivative.

2 Discretization by spectral collocation method

In this section we shall summarize the Chebyshev-Gauss-Lobatto (CGL) collocation method [2] for the sake of application to fractional differential equations in accordance with [9]. Numerical schemes to (2) will be also presented.

Suppose that a function $u(t)$ is sufficiently smooth on the interval $[0,T]$. Let N be a positive integer. In order to apply CGL collocation method to the problem (2), we introduce

$$t_j = \frac{3}{2}(x_j+1), \quad u_j = u(t_j), \qquad j = 0,1,2,\dots,N, \tag{3}$$

where $x_j = \cos(j\pi/N)$ is called the CGL points in the reference interval $[-1,1]$. Let C be a square matrix of order $N+1$ satisfying

$$C \begin{pmatrix} 1 \\ \frac{T}{2}(x+1) \\ \vdots \\ \left(\frac{T}{2}\right)^N (x+1)^N \end{pmatrix} = \begin{pmatrix} T_0(x) \\ T_1(x) \\ \vdots \\ T_N(x) \end{pmatrix}, \quad \text{for any } x \in [-1,1].$$

It is shown that there uniquely exists such C independent of x [18]; readers should refer to [9] to find the entries of C explicitly. We also write $\tilde{T}$ for the square matrix of order $N+1$ whose (k,j)-entry is given by

$$\tilde{T}_{kj} = \frac{2}{N\bar{c}_k\bar{c}_j} T_k(x_j),$$

where $T_k(x) = \cos(k \arccos x)$ is the Chebyshev polynomial of order k, and

$$\bar{c}_j = \begin{cases} 2, & j = 0, N, \\ 1, & \text{otherwise.} \end{cases}$$

Then the approximation $U_N(t)$ to $u(t)$ by Chebyshev polynomials with respect to CGL points is given by

$$U_N(t) = \begin{pmatrix} 1 & t & \cdots & t^N \end{pmatrix} C^\top \tilde{T} \begin{pmatrix} u_0 \\ \vdots \\ u_N \end{pmatrix},$$

where $C^\top$ represents the transpose of C. Since C and $\tilde{T}$ are independent of t, derivatives of U_N are expressed as

$$
{}_{\mathrm{C}}D_0^{\alpha}\left[U_N\right](t) = \left({}_{\mathrm{C}}D_0^{\alpha}[1](t) \quad {}_{\mathrm{C}}D_0^{\alpha}[t](t) \quad \cdots \quad {}_{\mathrm{C}}D_0^{\alpha}\left[t^N\right](t)\right) C^\top \tilde{T} \begin{pmatrix} u_0 \\ \vdots \\ u_N \end{pmatrix},
$$

and the Caputo derivatives of monomials t^k for a nonnegative integer k appeared in the leading row matrix on the right hand side are

$$
{}_{\mathrm{C}}D_0^{\alpha}\left[t^k\right](t) = \begin{cases} 0, & 0 \le k < \alpha, \\[2ex] \dfrac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} t^{k-\alpha}, & \alpha \le k. \end{cases}
$$

In particular, for a non-negative integer m, we have

$$
U_N^{(m)}(0) = \big(\underbrace{0 \quad \cdots \quad 0}_{m} \quad m! \quad \underbrace{0 \quad \cdots \quad 0}_{N-m}\big) C^\top \tilde{T} \begin{pmatrix} u_0 \\ \vdots \\ u_N \end{pmatrix}.
$$

Using above notations, we propose following two numerical schemes to the problem (2).

Scheme 1. The discretization of the equation (2a) is satisfied at collocation points t_j, $n-1 \le j \le N-1$ as

$$
{}_{\mathrm{C}}D_0^{\alpha}[U_N](t_j) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t_j^{\beta-\alpha}, \qquad n-1 \le j \le N-1.
$$

The initial condition (2b) is approximated as

$$
U_N^{(m)}(0) = 0, \qquad\qquad 0 \le m \le n-1.
$$

Totally the system consists of $N+1$ linear equations with $N+1$ unknowns $\left(u_j\right)_{j=0}^{N}$.

Note that, in Scheme 1, the discretized equation is required to be satisfied at $N-n+1$ CGL points except the origin in order to equalize the number of equations with the number of unknowns. Since the problem (2) is the initial value problem, we ignore $n-1$ CGL points ($t = t_0$, t_1, …, and t_{n-2}) from the end point if n is greater than or equal to 2. Figure 1 illustrates the way to impose conditions at CGL points in Scheme 1.

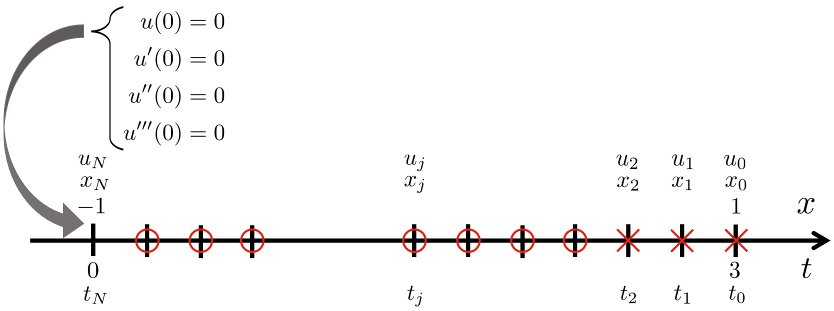


Figure 1: Conditions in Scheme 1 ($3 < \alpha \leq 4$, ○: diff. eq., ×: no condition).

Scheme 2. The fractional differential equation (2a) is discretized at the collocation points except the origin:

$$ {}_{\mathrm{C}}D_0^{\alpha}[U_N](t_j) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t_j^{\beta-\alpha}, \qquad 0 \leq j \leq N-1, $$

with the initial conditions (2b) posed as

$$ U_N^{(m)}(0) = 0, \qquad 0 \leq m \leq n-1. $$

Totally the system consists of $N+n$ linear equations with $N+1$ unknowns $\left(u_j\right)_{j=0}^{N}$. Since it is over-determined, we solve the corresponding normal equation to find the least square solution.

Since our discretization method for the Caputo derivative is concrete and explicit [9], derivation of discretized equations in Schemes 1 and 2 is straight forward.

3 Numerical results

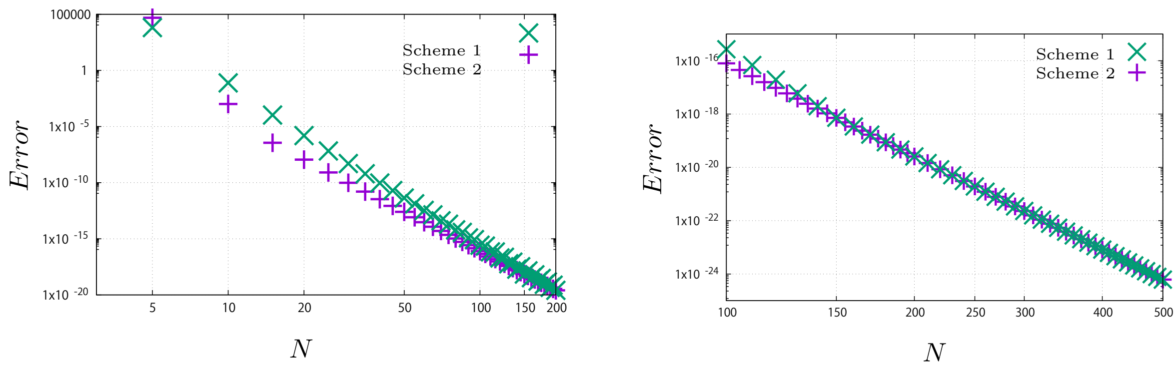
In this section, we discuss numerical results and some remarks. Numerical computations are performed in multiple precision using exflib [10]. The values of the gamma function are computed using the double exponential formula [15]. The discretized linear systems are solved by direct methods in order to clarify the behaviours of errors originated from discretization, disuse of CGL points $t = t_0, t_1, \dots, t_{n-2}$ in Scheme 1, and over-determined approximation in Scheme 2 by getting rid of residuals of iterative solvers. *Error* in below denotes the maximum absolute value among errors at the CGL points:

$$ Error = \max_{0 \leq j \leq N} \left| U_N(t_j) - t_j^{\beta} \right| $$

As can be seen from numerical results in below, the error decreases as N increases.

3.1 Comparison of the proposed schemes

Figure 2 shows decays of errors in Schemes 1 and 2, where the horizontal and vertical axis are the oder N and *Error*, respectively. Note that the both axis are in the logarithmic scale. It means that *Error* converges as polynomial orders [9]. When N is relatively small, Scheme 2 is more accurate, while Scheme 1 and 2 do not exhibit significant differences in their accuracy for larger N. It is considered that this is because collocation points of Scheme 1 to which the condition is imposed are distributed sufficiently widely in the domain as N increases, and the influence of collocation points to which the condition is not imposed decreases. Although both Scheme 1 and 2 achieve accurate numerical results, Scheme 1 is more convenient in order to save computational resources.



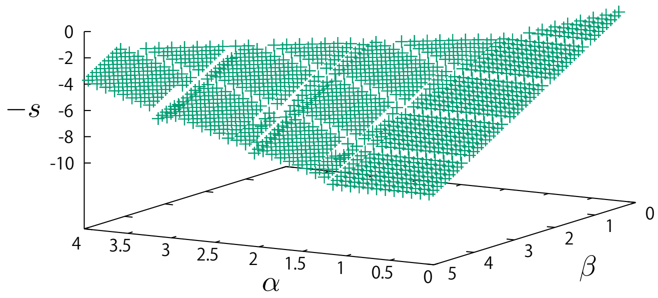
(1) $N \leq 200$.

(2) $100 \leq N \leq 500$.

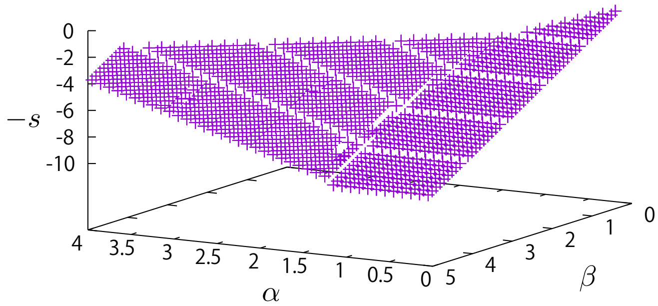
Figure 2: Comparison of Scheme 1 and Scheme 2 ($\alpha = 3.5$, $\beta = 8.3$, 500 digits).

3.2 Numerical regularity map

Under the assumption that the error decay rate is as $O(N^{-s})$, s is computed by linear regression. At the early stage of our numerical experiments, the relationship between α, β, and s was suggested. To be precise, as shown in Fig. 2, values of s seem to depend on the interval of N. This means that the relationship cannot be expressed by simple expressions. Thus, we consider the visualization of the relationship as is shown in Fig. 3, which is called the numerical regularity map of the solution [14]. When β is an integer, errors in the numerical solution U_N are only rounding errors. Hence the data is not appeared in the depicted range. Although there are some differences in the numerical results depending on Schemes 1 and 2, the overall appearance is the same. From Fig. 3, it seems to be that two different planes are connected at $\alpha = 1$. It should be noted that this interesting phenomenon is clarified by the visualization of the regularity map.



(1) Scheme 1.



(2) Scheme 2.

Figure 3: Numerical regularity map (intereval for the regression line : $300 \le N \le 500$, 500 digits).

4 Concluding remarks

In the paper, we investigate a fundamental one-dimensional initial value problem for the fractional differential equation with Hölder continuous solutions numerically. Two kinds of numerical algorithms are proposed and compared numerically. Both types attain high-accuracy, and convergence rates mutually agree. Numerical regularity maps which visualize the relationship between the order of the derivative, the Hölder exponent of the solution, and the convergence rate of errors are presented. Although maps are slightly different depending on proposed algorithms, the overall appearance is the same. Maps enable us to find an interesting phenomenon that rates of error convergence systematically change before and after the order of derivative is one.

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