Advances in Mathematical Sciences and Applications
Vol. 30, No. 2 (2021), pp. 415–427

GAKKOTOSHO
TOKYO JAPAN

REMARKS ON NON-UNIQUENESS FOR A TRANSPORT EQUATION WITH A FRACTIONAL DIFFUSION

Tomonori Tsuruhashi
Graduate School of Mathematical Sciences, The University of Tokyo
3-8-1 Komaba Meguro-ku, Tokyo 153-8914, Japan
(E-mail: tomonori@ms.u-tokyo.ac.jp)

Abstract. In this paper we extend the results on non-uniqueness for a transport equation to that with a fractional diffusion. We construct examples for which uniqueness of weak solutions fails when the velocity field is divergence-free but irregular. By using the convex integration method, it can be proved that there exist infinite pairs which satisfy such an equation for given initial data.

Communicated by Yoshikazu Giga; Received August 30, 2021.
AMS Subject Classification: 35F50, 35A02, 35Q35.
Keywords: weak solution, convex integration, non-uniqueness, transport equation, fractional diffusion.

1 Introduction

1.1 Main results

In this paper we extend the results on non-uniqueness for a transport equation to that with a fractional diffusion. A transport equation is fundamental in physics, for instance, related to fluid motion or conservation laws. By the linearity of a transport equation, existence of weak solutions is easily shown. However, it is difficult to prove whether uniqueness holds when the velocity field is not smooth. It has been investigated how much regularity is needed for velocity fields so that uniqueness for a transport equation holds. From this perspective, the results of DiPerna-Lions [14] are pioneering. They proved that uniqueness holds if velocity fields are in a suitable Sobolev space. The uniqueness results are gradually extended such as in the BV or more general frameworks [1, 3]. For an overview of the theory of flows associated with non-smooth vector fields, the reader is referred to [2]. On the other hand, progress on non-uniqueness has recently been made by using the convex integration method. Modena and Székelyhidi Jr. constructed examples for which uniqueness of solutions fails under less space-regularity of velocity fields [19]. They showed non-uniqueness of solutions to a transport equation, including the case with an ordinary diffusion. These days, such convex integration techniques have become popular in fluid dynamics. Starting with the groundbreaking works of De Lellis and Székelyhidi Jr. [9, 10, 11], we have a number of recent results on wild weak solutions of the incompressible Euler and Navier-Stokes equations. For a review of these results, see for instance [13, 7].

In order to improve the versatility of the convex integration methods, we show that the framework in [19] is valid for a transport equation with a fractional diffusion. We consider the following equation:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = (-\Delta)^s \rho \\ \\ \operatorname{div} u = 0 \end{cases} \quad \text{in} \quad [0,T] \times \mathbb{T}^d \tag{1.1}$$

and

$$\rho|_{t=0} = \rho_0 \qquad \text{in} \quad \mathbb{T}^d \tag{1.2}$$

for a real-valued function ρ and a vector-valued function u on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Here, we define $(-\Delta)^s$ $(s > 0)$ as follows:

$$(-\Delta)^s f(x) = \sum_{k \in \mathbb{Z}^d} |k|^{2s} \hat{f}(k) e^{2\pi i k \cdot x} \qquad \text{for} \quad f \in \mathcal{D}'(\mathbb{T}^d) \tag{1.3}$$

where $\hat{f}$ represents the Fourier coefficients of f and $\mathcal{D}'(\mathbb{T}^d)$ is the set of distributions on torus. We assume that ρ_0 is a smooth function with mean zero. This system is a model describing an anomalous diffusion for a non-integer s. When $s = 1$, the equation (1.1) is the standard transport-diffusion equation, which is also treated in [19]. The contribution of this paper is to extend the results of Modena and Székelyhidi Jr.[19]. Our main theorem is as follows:

Assumption (A).

$$\frac{1}{p} + \frac{1}{\tilde{p}} > 1 + \frac{1}{d-1} \tag{1.4}$$

where $p \in (1, \infty)$ and $\tilde{p} \in [1, \infty)$.

Assumption (B).

$$\frac{1}{p'} > \frac{2s-1}{d-1} \tag{1.5}$$

where $s > 0$.

Theorem 1.1. *Assume* (A) *and* (B). *Then there exist infinite pairs* (ρ, u) *such that*

(i) $\rho \in C([0,T]; L^p(\mathbb{T}^d))$, $u \in C([0,T]; L^{p'}(\mathbb{T}^d) \cap W^{1,\tilde{p}}(\mathbb{T}^d))$,

(ii) (ρ, u) *satisfies* (1.1) *and* (1.2) *in the sense of distributions.*

Here, we denote by p' the dual exponent of p. Our results include the results of Modena and Székelyhidi Jr. [19]. The case with $s = 1$ in (1.1) corresponds to their results on a transport-diffusion equation. In this paper, we show that their strategy is applicable to a transport equation with a fractional diffusion by refining their estimations. We adopt the Mikado density and flow as the building blocks and construct such pairs (ρ, u) by the convex integration method. The most significant difference from their proof is the estimation of the fractional viscosity term $(-\Delta)^s \rho$. Since this viscosity term is not in a divergent form, we use Fourier analysis to prepare estimations of the building blocks. Theorem 1.1 is a consequence of the following Theorem 1.2.

Theorem 1.2. (cf. [19, Theorem 1.2]) *Assume* (A) *and* (B). *Let* $\epsilon > 0$ *and* $\bar{\rho} \in C^\infty([0,T] \times \mathbb{T}^d)$ *with*

$$\int_{\mathbb{T}^d} \bar{\rho}(0,x)dx = \int_{\mathbb{T}^d} \bar{\rho}(t,x)dx$$

for every $t \in [0,T]$. *Then there exist* $\rho \in C([0,T]; L^p(\mathbb{T}^d))$, $u \in C([0,T]; L^{p'}(\mathbb{T}^d) \cap W^{1,\tilde{p}}(\mathbb{T}^d))$ *such that*

(i) (ρ, u) *satisfies* (1.1) *in the sense of distributions,*

(ii) $\rho|_{t=0} = \bar{\rho}|_{t=0}$ *and* $\rho|_{t=T} = \bar{\rho}|_{t=T}$,

(iii) $\sup_{t \in [0,T]} \|\rho(t,\cdot) - \bar{\rho}(t,\cdot)\|_{L^p(\mathbb{T}^d)} \le \varepsilon$.

In [19] Modena and Székelyhidi Jr. worked entirely in the physical space without using the Fourier space. However, we need to work in the Fourier space in order to prove our main theorem. Our main theorem can be shown by examining the effects of the fractional viscosity on "Mikado."

This paper is organized as follows. In Section 2 we recall the results of Modena and Székelyhidi Jr. in [19]. Since this work is the basis of our results, we present the main ideas of their paper. Section 3 is devoted to the estimations of the fractional viscosity term. These estimations are the most significant difference from the work [19] and discussed in detail. Fourier analysis of "Mikado" plays an essential role in the proof.

1.2 Recent developments on a transport equation by the convex integration method

Before starting the main discussion, we present a brief overview of recent developments based on the work of Modena and Székelyhidi Jr. [19]. Although the convex integration method, which originates from Nash's ideas on C^1 isometric embeddings, leads to the results on wild weak solutions for Euler or Navier-Stokes equation, we do not touch them here. See [9, 12, 13, 7, 6] for more information on this topic.

In this overview, we consider the transport equation on $[0,T]\times\mathbb{T}^d$:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 \\ \operatorname{div} u = 0 \\ \rho|_{t=0} = \rho_0. \end{cases} \tag{1.6}$$

There are important mathematical models such as the Boltzmann equation, in which we need to consider an unbounded scalar field. It is essential to understand the well-posedness of the Cauchy problem for such models. The investigation on uniqueness for the transport equation is also based on these motivations. DiPerna and Lions obtained uniqueness results by the use of renormalized solutions, which are introduced in the context of kinetic models.

Theorem 1.3. ([14]) *Assume that $p,\tilde{p}\in[0,\infty]$ and that u in* (1.6) *is in the class $L^1(0,T;W^{1,\tilde{p}}(\mathbb{T}^d))$. Then, for any $\rho_0\in L^p(\mathbb{T}^d)$ there exists a unique renormalized solution of* (1.6) *with $\rho\in C([0,T];L^p(\mathbb{T}^d))$. Moreover, the condition*

$$\frac{1}{p}+\frac{1}{\tilde{p}}\le 1 \tag{1.7}$$

implies that this solution is unique in the class $L^\infty(0,T;L^p(\mathbb{T}^d))$.

On the other hand, Modena and Székelyhidi Jr. showed in [19] that uniqueness fails under the assumption (A). More precisely, they constructed infinite pairs (ρ,u) satisfying (1.6) with $\rho\in C([0,T];L^p(\mathbb{T}^d))$ and $u\in C([0,T];L^{p'}(\mathbb{T}^d)\cap W^{1,\tilde{p}}(\mathbb{T}^d))$. The gap between the condition (1.4) and (1.7) has been filled by subsequent research. In [20], Modena and Székelyhidi Jr. extended the non-uniqueness results of [19] to the results for Sobolev and continuous vector fields including the case $p=1$. Although the building blocks "Mikado" are independent of time in [19], this extension of [20] was done by modifying the interaction and the oscillation of "Mikado" in the temporal direction. It is emphasized that the time dependence of the approximate sequence for a solution plays an important role in these regularity questions. By promoting this strategy, in [18] Modena and Satting improved the condition (1.4) to the following condition

$$\frac{1}{p}+\frac{1}{\tilde{p}}>1+\frac{1}{d} \tag{1.8}$$

for $p\in[1,\infty)$ and $\tilde{p}\in[1,\infty)$. Here, the class of ρ and u is the same as in [19]. Under the condition (1.8), it is possible to treat the case $d=2$. They revised the constructions of

"Mikado" which depends not only on space but also on time. In these results, the spatial intermittency is essential. However, it is not enough to reach the full complement of the condition in [14]. In order to attain the full complement, we need not only the spatial intermittency but also the temporal one. In [8] Cheskidov and Luo filled the gap between the condition (1.4) and (1.7) by improving the space regularity at the expense of the time regularity. By the use of the temporal intermittency, they refined the condition (1.4) as follows:

$$\frac{1}{p} + \frac{1}{\tilde{p}} > 1 \tag{1.9}$$

for $p \in (1, \infty)$ and $\tilde{p} \in [1, \infty)$. Here, they considered in the class $\rho \in L^1(0, T; L^p(\mathbb{T}^d))$ and $u \in L^\infty(0, T; L^{p'}(\mathbb{T}^d) \cap L^1(0, T; W^{1,\tilde{p}}(\mathbb{T}^d)))$. The question of whether the condition (1.9) implies non-uniqueness in the class $\rho \in L^\infty(0, T; L^p(\mathbb{T}^d))$ remains to be solved.

The convex integration scheme in [19] has an influence on the problem for the uniqueness of the regular Lagrangian flow. It is well known that a transport equation is closely related to the trajectories connected to the ordinary differential equation. For the relation between these aspects and the above results, see [5]. This paper [5] includes results not only on uniqueness but also on positivity of a solution. The problem on the existence of a positive solution is also treated in [21].

The author investigated the difference in mathematical structure between uniqueness and non-uniqueness by the kinetic theory. In the previous work [22], we treated the time-independent transport equation with an external force and constructed infinite irregular pairs which violate the chain rule of differentiation. We also showed that such a violation does not occur when we consider a more regular pair. It is emphasized in [22] that these differences of violation can be interpreted from a microscopic point of view. This microscopic point of view is based on [4, 15, 16, 17] and seems to be effective not only in our simplified settings but also in other settings such as those described above.

2 Existence of wild weak solutions for a transport equation

In the sequel, we assume without loss of generality that $T = 1$ and that $\mathbb{T}^d$ is the periodic extension of the unit cube $[0, 1]^d$. We give the ideas of the proof for the analogues of Theorem 1.2, which is provided in [19]. The content in this section is a rewrite of a brief overview in [20]. The reader is referred to [19, 20] for more details.

2.1 Convex integration scheme

In order to construct a pair (ρ, u) satisfying a transport equation, we consider a smooth approximate triplet (ρ_n, u_n, R_n) satisfying the following equation:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho u) = -\mathrm{div} R \\ \\ \mathrm{div}\ u = 0. \end{cases} \tag{2.1}$$

These triplets are constructed so that

$$
\begin{aligned}
\sum_n \|\rho_{n+1}-\rho_n\|_{C_tL^p_x} &< \infty,\\
\sum_n \|u_{n+1}-u_n\|_{C_tL^{p'}_x} &< \infty,\\
\sum_n \|u_{n+1}-u_n\|_{C_tW^{1,\tilde{p}}_x} &< \infty,\\
\|R_n\|_{C_tL^1_x} &\leq \delta_n,
\end{aligned}
$$

where $\{\delta_n\}_n$ is a suitable sequence which converges to 0 such as $\delta_n = 2^{-n}$. Once these estimations are established, we define ρ and u as a limit of ρ_n and u_n in a suitable norm, respectively. Then, (ρ, u) satisfies a transport equation. The most important point is how to construct such triplets in order for them to satisfy the desired properties. We settle this matter in an inductive method by using "Mikado." Explicitly, it suffices to show that for a smooth triplet (ρ_n, u_n, R_n) there exists another smooth triplet $(\rho_{n+1}, u_{n+1}, R_{n+1})$ satisfying

$$
\|\rho_{n+1}-\rho_n\|_{C_tL^p_x} \lesssim \|R_n^{1/p}\|_{C_tL^1_x}, \tag{2.2a}
$$
$$
\|u_{n+1}-u_n\|_{C_tL^{p'}_x} \lesssim \|R_n^{1/p'}\|_{C_tL^1_x}, \tag{2.2b}
$$
$$
\|u_{n+1}-u_n\|_{C_tW^{1,\tilde{p}}_x} \leq \delta_{n+1}, \tag{2.2c}
$$
$$
\|R_{n+1}\|_{C_tL^1_x} \leq \delta_{n+1}. \tag{2.2d}
$$

From the given triplet (ρ_n, u_n, R_n), we define $(\rho_{n+1}, u_{n+1}, R_{n+1})$ as follows:

$$
\begin{aligned}
\rho_{n+1} &= \rho_n + \theta_n,\\
u_{n+1} &= u_n + w_n,\\
\theta_n &= F(R_n)\Theta_{\mu_n}(\lambda_n x),\\
w_n &= G(R_n)W_{\mu_n}(\lambda_n x),
\end{aligned}
$$

where F and G are non-linear functions, $\{\Theta_\mu\}_\mu$ is a family of Mikado densities and $\{W_\mu\}_\mu$ is a family of Mikado flows. We choose an oscillation parameter λ_n and a concentration parameter μ_n, which will be fixed later so that the desired estimations hold. We give the precise definition of Mikado in Section 3. In addition to these constructions of ρ_n and u_n, we define R_n so that it satisfies the equation (2.1). By the direct calculation, R_n should be defined as follows:

$$
R_{n+1} = (R_n - \theta_n w_n) - \nabla\Delta^{-1}\partial_t\theta_n - \rho_n w_n - \theta_n u_n. \tag{2.3}
$$

The estimations (2.2a) to (2.2c) follow from the definition of the Mikado density and the Mikado flow. The estimation of (2.2d) is the most difficult part. The parameter λ_n is used in order to make the quadratic term $R_n - \theta_n w_n$ suitably small. The explicit form of F and G also contributes to this estimation. On the other hand, the parameter μ_n is used in order to make the linear term remaining in the right-hand side of (2.3) suitably small. The assumption (A) guarantees the suitable choice of these parameters. By the arguments on the oscillation and the concentration, we obtain the estimation (2.2d).

3 Non-uniqueness for a transport equation with a fractional diffusion

In this section, we prove Theorem 1.2. From Theorem 1.2, we can prove Theorem 1.1 in a similar way as [19]. Therefore, we omit the proof of Theorem 1.1. We only discuss the significant difference between our proof and that in [19].

3.1 Mikado density and Mikado flow

For the proof of the main theorem, we need calculations on the Mikado density in the Fourier space. We use these computations to control the fractional viscosity. To begin with, we recall the precise definition of the Mikado density and flow from [19].

Proposition 3.1. ([19, Proposition 4.1]) *Suppose that $\Psi \in C_0^\infty(\mathbb{R}^{d-1})$ with*

$$\mathrm{supp}\Psi \subseteq (0,1)^{d-1}, \quad \int_{\mathbb{R}^{d-1}} \Psi = 0 \quad \textit{and} \quad \int_{\mathbb{R}^{d-1}} \Psi^2 = 1.$$

Let $a, b \in \mathbb{R}$ with

$$a + b = d - 1.$$

For every $\mu > 2d$ and $j \in \{1, \dots, d\}$, there exist $\Theta_\mu^j \in C^\infty(\mathbb{T}^d)$ and $W_\mu^j \in C^\infty(\mathbb{T}^d; \mathbb{R}^d)$ with mean zero such that the following properties hold.

(i)
$$\mathrm{div} W_\mu^j = \mathrm{div}(\Theta_\mu^j W_\mu^j) = 0.$$

(ii)
$$\frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} \Theta_\mu^j W_\mu^j = e_j$$

where $\{e_1, \dots, e_d\}$ is the standard basis in $\mathbb{R}^d$.
(iii) *For every $k \in \mathbb{N}$ and $r \in [1, \infty]$,*

$$\begin{aligned} \|D^k \Theta_\mu^j\|_{L^r(\mathbb{T}^d)} \leq& \|\Psi\|_{L^r(\mathbb{R}^{d-1})} \mu^{a+k-(d-1)/r}, \\ \|D^k W_\mu^j\|_{L^r(\mathbb{T}^d)} \leq& \|\Psi\|_{L^r(\mathbb{R}^{d-1})} \mu^{b+k-(d-1)/r}. \end{aligned}$$

(iv) *For $j \neq k$,*

$$\mathrm{supp}\Theta_\mu^j = \mathrm{supp} W_\mu^j \quad \textit{and} \quad \mathrm{supp}\Theta_\mu^j \cap \mathrm{supp} W_\mu^k = \emptyset.$$

We call Θ_μ^j and W_μ^j Mikado density and Mikado flow, respectively.

Proof. We use these constructions for the proof of Theorem 1.2. So, we give the proof for the reader's convenience, although the proof is the same as [19].

After constructing the functions on $\mathbb{R}^d$, we define the Mikado density and the Mikado flow. For each j, we define $\tilde{\tilde{\Theta}}_\mu^j$ and $\tilde{\tilde{W}}_\mu^j$ as follows:

$$\begin{aligned} \tilde{\tilde{\Theta}}_\mu^j(x_1, \dots, x_d) =& \mu^a \Psi_\mu(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d), \\ \tilde{\tilde{W}}_\mu^j(x_1, \dots, x_d) =& \mu^b \Psi_\mu(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) e_j \end{aligned}$$

where $\Psi_\mu(x)=\Psi(\mu x)$. By this definition, it holds that

$$\begin{aligned}
&\mathrm{div}\tilde{\tilde{W}}^j_\mu=\mathrm{div}\left(\tilde{\tilde{\Theta}}^j_\mu\tilde{\tilde{W}}^j_\mu\right)=0,\\
&\int_{(0,1)^d}\tilde{\tilde{\Theta}}^j_\mu=0,\quad \int_{(0,1)^d}\tilde{\tilde{W}}^j_\mu=0,\\
&\frac{1}{|(0,1)^d|}\int_{(0,1)^d}\tilde{\tilde{\Theta}}^j_\mu\tilde{\tilde{W}}^j_\mu=e_j.
\end{aligned}$$

Notice that

$$\|D^k(\mu^a\Psi_\mu)\|_{L^r(\mathbb{R}^{d-1})}=\|D^k\Psi\|_{L^r(\mathbb{R}^{d-1})}\mu^{a+k-(d-1)/r}.$$

Therefore, it also holds that

$$\begin{aligned}
\left\|D^k\tilde{\tilde{\Theta}}^j_\mu\right\|_{L^r((0,1)^d)}&=\|D^k\Psi\|_{L^r(\mathbb{R}^{d-1})}\mu^{a+k-(d-1)/r},\\
\left\|D^k\tilde{\tilde{W}}^j_\mu\right\|_{L^r((0,1)^d)}&=\|D^k\Psi\|_{L^r(\mathbb{R}^{d-1})}\mu^{b+k-(d-1)/r}.
\end{aligned}$$

Recall that $\mathrm{supp}\Psi\subseteq(0,1)^{d-1}$. Since $\tilde{\tilde{\Theta}}^j_\mu$ and $\tilde{\tilde{W}}^j_\mu$ are independent of the j-th coordinate, we can define $\tilde{\Theta}^j_\mu$ and $\tilde{W}^j_\mu$ as the 1-periodic extension of $\tilde{\tilde{\Theta}}^j_\mu$ and $\tilde{\tilde{W}}^j_\mu$, respectively. The estimations of $\tilde{\Theta}^j_\mu$ and $\tilde{W}^j_\mu$ follow from that of $\tilde{\tilde{\Theta}}^j_\mu$ and $\tilde{\tilde{W}}^j_\mu$. Finally, since $\mu>2d$, we obtain Θ^j_μ and W^j_μ after a suitable translation that makes (iv) true. □

From these constructions of Mikado, we can calculate the Fourier components of the Mikado density. This is required for the estimations on the fractional viscosity term.

Proposition 3.2. *Let Θ^j_μ be the Mikado density in Proposition 3.1. Then*

$$\widehat{\Theta^j_\mu}(k)=\begin{cases}\mu^{a-(d-1)}\widehat{\Psi}(P_jk/\mu)e^{2\pi ik\cdot\tau^j} & \textit{if } k_j=0,\\ 0 & \textit{if } k_j\neq 0.\end{cases}$$

where $P_jk=(k_1,\dots,k_{j-1},k_{j+1},\dots,k_d)$ and τ^j is a suitable translation.

Proof. This is a direct calculation. Indeed, from the constructions of Mikado,

$$\begin{aligned}
\widehat{\Theta^j_\mu}(k)&=\int_{\mathbb{T}^d}\tilde{\tilde{\Theta}}^j_\mu(x)e^{-2\pi ik\cdot x}dx\\
&=\mu^a\int_{(0,1)^d}\Psi_\mu(x_1,\dots,x_{j-1},x_{j+1},\dots,x_d)e^{-2\pi ik\cdot(x-\tau^j)}dx.
\end{aligned}$$

By integrating with respect to x_j, it follows that

$$\widehat{\Theta^j_\mu}(k)=\begin{cases}\mu^a\displaystyle\int_{\mathbb{T}^d}\Psi(\mu y)e^{2\pi iP_jk\cdot(y-\tau^j)}dy & \text{if } k_j=0,\\ 0 & \text{if } k_j\neq 0.\end{cases}$$

Therefore, if $k_j=0$, we have

$$\widehat{\Theta^j_\mu}(k)=\mu^{a-(d-1)}\int_{\mathbb{R}^{d-1}}\widehat{\Psi}(z)e^{-2\pi i\frac{P_jk}{\mu}\cdot z}dz\ e^{2\pi ik\cdot\tau^j}.$$

This completes the proof. □

3.2 Estimations on the fractional viscosity

Theorem 1.2 follows from the following lemma. This plays an important role to construct pairs satisfying (1.1). The strategy of proof is the same as that of [19]. We consider the approximate equation of (1.1):

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho u) - (-\Delta)^s \rho = -\mathrm{div} R \\ \\ \mathrm{div}\ u = 0 \end{cases} \quad \text{in} \quad [0,T] \times \mathbb{T}^d. \tag{3.1}$$

By using the following lemma, we can construct the approximate sequences in an inductive way.

Lemma 3.3. (cf. [19, Proposition 3.1]) *Assume* (A) *and* (B). *Then there exists a constant* $M > 0$ *such that the following holds: If* η, δ, σ *are positive numbers and a smooth triplet* (ρ_0, u_0, R_0) *satisfies the equation* (3.1), *then there exists another smooth triplet* (ρ_1, u_1, R_1) *satisfying the equation* (3.1) *such that*

$$\|\rho_1(t) - \rho_0(t)\|_{L^p(\mathbb{T}^d)} \le \begin{cases} M\eta \|R_0(t)\|_{L^1(\mathbb{T}^d)}^{1/p} & \text{for} \quad t \in I_{\sigma/2}, \\ 0 & \text{for} \quad t \in [0,1] \setminus I_{\sigma/2}, \end{cases}$$

$$\|u_1(t) - u_0(t)\|_{L^{p'}(\mathbb{T}^d)} \le \begin{cases} M\eta^{-1} \|R_0(t)\|_{L^1(\mathbb{T}^d)}^{1/p'} & \text{for} \quad t \in I_{\sigma/2}, \\ 0 & \text{for} \quad t \in [0,1] \setminus I_{\sigma/2}, \end{cases}$$

$$\|u_1(t) - u_0(t)\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} \le \delta \qquad \text{for} \quad t \in [0,1],$$

$$\|R_1(t)\|_{L^1(\mathbb{T}^d)} \le \begin{cases} \delta & \text{for} \quad t \in I_\sigma, \\ \|R_0(t)\|_{L^1(\mathbb{T}^d)} + \delta & \text{for} \quad t \in I_{\sigma/2} \setminus I_\sigma, \\ \|R_0(t)\|_{L^1(\mathbb{T}^d)} & \text{for} \quad t \in [0,1] \setminus I_{\sigma/2}, \end{cases}$$

where $I_\sigma = (\sigma, 1-\sigma)$.

Proof. We can prove this key lemma in a very similar way to [19]. By applying Proposition 3.1 with $a = (d-1)/p$ and $b = (d-1)/p'$, we prepare the Mikado density and flow. In addition to that, we set

$$\begin{aligned} \rho_1 &= \rho_0 + \theta + \theta_c, \\ u_1 &= u_0 + w + w_c. \end{aligned} \tag{3.2}$$

Here,

$$\begin{aligned} \theta(t,x) &= \eta \sum_{j=1}^{d} \psi(t) h_j(t,x)\ \mathrm{sign}(R_{0,j}(t,x))\ |R_{0,j}(t,x)|^{1/p}\ (\Theta_\mu^j)_\lambda(x), \\ \theta_c(t,x) &= -\frac{1}{|\mathbb{T}^d|} \int_{\mathbb{T}^d} \theta(t,x) dx, \\ w(t,x) &= \eta^{-1} \sum_{j=1}^{d} \psi(t) h_j(t,x)\ |R_{0,j}(t,x)|^{1/p'}\ (W_\mu^j)_\lambda(x), \\ w_c(t,x) &= -\eta^{-1} \sum_{j=1}^{d} S\left(\nabla(\psi(t) h_j(t,x)\ |R_{0,j}(t,x)|^{1/p'}) \cdot\ (W_\mu^j)_\lambda(x) \right) \end{aligned} \tag{3.3}$$

where we write

$$(\Theta^j_\mu)_\lambda(x) = \Theta^j_\mu(\lambda x), \qquad (W^j_\mu)_\lambda(x) = W^j_\mu(\lambda x)$$

and

$$R_0 = \sum_{j=1}^{d} R_{0,j} e_j$$

using the standard basis in $\mathbb{R}^d$ and take the following cut-off functions $\psi \in C_0^\infty(0,1)$ and $h_j \in C^\infty(\mathbb{T}^d)$ with $|\psi|, |h_j| \le 1$ and

$$\psi = \begin{cases} 0 & \text{on } [0, \sigma/2] \cup [1-\sigma/2, 1], \\ 1 & \text{on } [\sigma, 1-\sigma], \end{cases}$$

$$h_j(x) = \begin{cases} 0 & \text{if } |R_{0,j}(t,x)| \le \delta/8d, \\ 1 & \text{if } |R_{0,j}(t,x)| \ge \delta/4d. \end{cases}$$

We denote the improved anti-divergence operator by S. The above representations of ρ_1 and u_1 and the operator S are the same as those of [19]. Furthermore, we define R_1 from (3.2) so that R_1 satisfies (3.1). The direct calculation gives the following form:

$$\begin{aligned} R_1 = {} & (R_0 - \theta w) - \nabla\Delta^{-1}\partial_t\theta - \rho_0 w - \theta u_0 \\ & + \nabla\Delta^{-1}(-\Delta)^s(\theta + \theta_{\mathrm{c}}) + (\text{the other term including } \theta_{\mathrm{c}} \text{ or } w_{\mathrm{c}}). \end{aligned}$$

For Ψ in Proposition 3.1, we set

$$M = 2d \max\{\|\Psi\|_{L^\infty(\mathbb{R}^{d-1})}, \|\Psi\|^2_{L^\infty(\mathbb{R}^{d-1})}, \|\nabla\Psi\|_{L^\infty(\mathbb{R}^{d-1})}\}$$

and

$$\gamma_0 = (d-1)\min\left\{1 - \frac{1}{p},\ 1 - \frac{1}{p'},\ \frac{1}{p} + \frac{1}{\tilde{p}} - \left(1 + \frac{1}{d-1}\right)\right\}.$$

Notice that $\gamma_0 > 0$ by Assumption (A). By using M and γ_0, we can estimate the R_1 in the same way as [19] except the term $\nabla\Delta^{-1}(-\Delta)^s(\theta + \theta_{\mathrm{c}})$. We only have to show the L^1-estimation of this term. The following lemma plays an essential role in the proof.

Lemma 3.4. *Under these settings, set $\tilde{\theta} = \theta + \theta_{\mathrm{c}}$. Then, it follows that*

$$\|\nabla\Delta^{-1}(-\Delta)^s\tilde{\theta}\|_{L^\infty(\mathbb{T}^d)} \le C\mu^{-\gamma_1}\lambda^s \qquad \text{for} \quad t \in [0,1]$$

where $\gamma_1 = -(a - (d-1) + s)$ and C is a constant independent of μ and λ.

Proof of Lemma 3.4. In the sequel, the exact value denoted by C may change from line to line. Notice that $a - (d-1) + s < 0$ from the assumption (B). Since $\nabla\Delta^{-1}(-\Delta)^s\tilde{\theta} = \Delta^{-1}(-\Delta)^s\nabla\tilde{\theta}$, we have

$$|\nabla\Delta^{-1}(-\Delta)^s\tilde{\theta}(t,x)| \le C\sum_{k\neq 0}\left|\frac{|k|}{|k|^2}|k|^{2s}\hat{\theta}(t,k)\right|.$$

Hence, it follows that

$$\|\nabla\Delta^{-1}(-\Delta)^s\tilde{\theta}(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} \le C\sum_{k\neq 0}\left||k|^{2s-1}\hat{\theta}(t,k)\right|.$$

Furthermore, we rewrite θ in (3.3) as follows:

$$\theta = \sum_j \zeta_j(\Theta_\mu^j)_\lambda,$$

where ζ_j is smooth. By using this notation, we have

$$\begin{aligned}\|\nabla\Delta^{-1}(-\Delta)^s\tilde{\theta}\|_{L^\infty(\mathbb{T}^d)} &\le C\sum_k |k|^{2s-1}\sum_j\left|(\widehat{\zeta_j(\Theta_\mu^j)_\lambda})(k)\right| \\ &\le C\sum_{k,j}|k|^{2s-1}\sum_m\left|\widehat{\zeta}_j(m)\right|\left|\widehat{(\Theta_\mu^j)_\lambda}(k-m)\right| \\ &\le C\sum_{k,j,m}|k|^{2s-1}\left|\widehat{\zeta}_j(m)\right|\left|\widehat{\Theta_\mu^j}\left(\frac{k-m}{\lambda}\right)\right|.\end{aligned}$$

By taking the supremum of ζ_j and using Proposition 3.2, we have

$$\|\nabla\Delta^{-1}(-\Delta)^s\tilde{\theta}\|_{L^\infty(\mathbb{T}^d)} \le C\mu^{a-(d-1)}\sum_{k,j,m}|k|^{2s-1}\left|\widehat{\Psi}\left(\frac{P_j(k-m)}{\mu\lambda}\right)\right|.$$

By the change of variables, it holds that

$$\begin{aligned}\|\nabla\Delta^{-1}(-\Delta)^s\tilde{\theta}\|_{L^\infty(\mathbb{T}^d)} &\le C\mu^{a-(d-1)}\sum_{\tilde{k},j,\tilde{m}}(\mu\lambda)^{2s-1}|\tilde{k}|^{2s-1}\left|\widehat{\Psi}\left(P_j(\tilde{k}-\tilde{m})\right)\right| \\ &\le C\mu^{a-(d-1)+s}\lambda^s\sum_{\tilde{k},j,\tilde{m}}|\tilde{k}|^{2s-1}\left|\widehat{\Psi}\left(P_j(\tilde{k}-\tilde{m})\right)\right| \\ &\le C\mu^{a-(d-1)+s}\lambda^s.\end{aligned}$$

This completes the proof of Lemma 3.4. □

Now, we continue the proof of Lemma 3.3. If we set the parameter $\mu = \lambda^c$ and increase the value λ for a suitable c, $\|\nabla\Delta^{-1}(-\Delta)^s\tilde{\theta}\|_{L^\infty(\mathbb{T}^d)}$ becomes small by Lemma 3.4. So, we set

$$\gamma = \min\{\gamma_0, \gamma_1\} > 0.$$

Then it follows that

$$\|\nabla\Delta^{-1}(-\Delta)^s\tilde{\theta}\|_{L^\infty(\mathbb{T}^d)} \le C\mu^{-\gamma}\lambda^s$$

for $t \in [0,1]$. By using this γ, the remaining part of the proof follows in a similar way of [19]. By setting $\mu = \lambda^c$ and increasing the value of λ for $c > 1/s$ in accordance with the remaining estimations, we obtain the desired properties. □

Acknowledgments

The author would like to thank his supervisor Professor Yoshikazu Giga for helpful discussions and comments on this work.

References

[1] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. *Invent. Math.*, 158(2):227–260, 2004.

[2] L. Ambrosio. Well posedness of ODE's and continuity equations with nonsmooth vector fields, and applications. *Rev. Mat. Complut.*, 30(3):427–450, 2017.

[3] S. Bianchini and P. Bonicatto. A uniqueness result for the decomposition of vector fields in $\mathbb{R}^d$. *Invent. Math.*, 220(1):255–393, 2020.

[4] Y. Brenier. Averaged multivalued solutions for scalar conservation laws. *SIAM J. Numer. Anal.*, 21(6):1013–1037, 1984.

[5] E. Brué, M. Colombo, and C. De Lellis. Positive solutions of transport equations and classical nonuniqueness of characteristic curves. *Arch. Ration. Mech. Anal.*, 240(2):1055–1090, 2021.

[6] T. Buckmaster, C. De Lellis, L. Székelyhidi, Jr., and V. Vicol. Onsager's conjecture for admissible weak solutions. *Comm. Pure Appl. Math.*, 72(2):229–274, 2019.

[7] T. Buckmaster and V. Vicol. Convex integration and phenomenologies in turbulence. *EMS Surv. Math. Sci.*, 6(1-2):173–263, 2019.

[8] A. Cheskidov and X. Luo. Nonuniqueness of weak solutions for the transport equation at critical space regularity. *Ann. PDE*, 7(1):Paper No. 2, 45, 2021.

[9] C. De Lellis and L. Székelyhidi, Jr. The h-principle and the equations of fluid dynamics. *Bull. Amer. Math. Soc. (N.S.)*, 49(3):347–375, 2012.

[10] C. De Lellis and L. Székelyhidi, Jr. Dissipative continuous Euler flows. *Invent. Math.*, 193(2):377–407, 2013.

[11] C. De Lellis and L. Székelyhidi, Jr. Dissipative Euler flows and Onsager's conjecture. *J. Eur. Math. Soc. (JEMS)*, 16(7):1467–1505, 2014.

[12] C. De Lellis and L. Székelyhidi, Jr. High dimensionality and h-principle in PDE. *Bull. Amer. Math. Soc. (N.S.)*, 54(2):247–282, 2017.

[13] C. De Lellis and L. Székelyhidi, Jr. On turbulence and geometry: from Nash to Onsager. *Notices Amer. Math. Soc.*, 66(5):677–685, 2019.

[14] R. J. DiPerna and P. L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[15] Y. Giga and T. Miyakawa. A kinetic construction of global solutions of first order quasilinear equations. *Duke Math. J.*, 50(2):505–515, 1983.

[16] Y. Giga, T. Miyakawa, and S. Oharu. A kinetic approach to general first order quasilinear equations. *Trans. Amer. Math. Soc.*, 287(2):723–743, 1985.

[17] T. Miyakawa. Construction of solutions of a semilinear parabolic equation with the aid of the linear Boltzmann equation. *Hiroshima Math. J.*, 14(2):299–310, 1984.

[18] S. Modena and G. Sattig. Convex integration solutions to the transport equation with full dimensional concentration. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 37(5):1075–1108, 2020.

[19] S. Modena and L. Székelyhidi, Jr. Non-uniqueness for the transport equation with Sobolev vector fields. *Ann. PDE*, 4(2):Art. 18, 38, 2018.

[20] S. Modena and L. Székelyhidi, Jr. Non-renormalized solutions to the continuity equation. *Calc. Var. Partial Differential Equations*, 58(6):Paper No. 208, 30, 2019.

[21] J. Pitcho and M. Sorella. Almost everywhere non-uniqueness of integral curves for divergence-free sobolev vector fields. arXiv:2108.03194, 2021.

[22] T. Tsuruhashi. On divergence of products of irregular scalar and solenoidal vector fields. *Adv. Math. Sci. Appl.* to appear.