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CONTINUOUS GALERKIN METHOD AND LANE-EMDEN EQUATIONS

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Abstract. Lane-Emden type equations arise various physical phenomena in mathematical and astrophysics like stellar structure, thermionic currents, thermal explosions, radiative cooling, CTC, etc. In this work, we consider a model by considering the equation

$$\begin{cases} \left(x^{\beta}y'(x)\right)' + x^{\beta}f(x,y) = 0, \quad 0 < x < 1, \\ y'(0) = 0, \quad b_1y(1) + a_1y'(1) = c_1. \end{cases} \tag{p}$$

For $\beta = 1$ and $\beta = 2$, the singular differential equation (p) has been considered by (Chambre ([9]), Chandrasekhar ([10]), Keller ([42]), Goodman and Duggan ([22]). Here, we apply continuous Galerkin finite element method (CGFEM) by choosing the Lagrange linear and quadratic polynomials as a test and trial functions. We approximate the integral value of nonlinear function by using trapezoidal and Simpson's $\frac{1}{3}$ rule. We compare the numerical results with other numerical results computed by various methods. Residue table shows that our proposed techniques are efficient, convenient and promising in the existing literature.

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1 Introduction

The theory of thermal explosion is based on the concept, that progressive heating raises the rate at which heat is released by the reaction until it exceeds the rate of heat loss from the area. Originally, it proposed by Frank-Kamenetzky ([38, 39]). Here, we discuss about few models which arises in the field of the thermal explosion.
P. L. Chambre's model on cylindrical vessel arising in Thermal explosion: In 1952, Chambre ([9]) produce a model arising in the thermal explosion which is given by

$$\lambda \Delta T = -QW. \tag{1}$$

The equation (1) has a clear geometrical meaning. The operator Δ^2 be the Laplacian operator, W be the reaction velocity, λ be the thermal conductivity, T be the gas temperature and Q be the heat of reaction. Chambre assumed that the reaction velocity W is monomolecular which follows Arrhenius Law, such that

$$W = ca \exp\left[-\frac{E}{RT}\right], \tag{2}$$

where c be the concentration of the reactant, a be the frequency factor and E be the energy factor of the reaction. Therefore the equation (1) takes into the account

$$\Delta T = -\frac{Qca}{\lambda} exp\left[-\frac{E}{RT}\right], \tag{3}$$

and which is also known as the Poisson-Boltzmann equation. After simplification as in [9], equations (3) becomes

$$\nabla^2 \theta = -\left[\frac{QEca}{\lambda RT_0^2} exp\left[-\frac{E}{RT_0}\right]\right] \exp(\theta), \tag{4}$$

where $\theta = \dfrac{E}{RT_0^2}(T - T_0)$ and T_0 is the temperature of the wall of the vessel.
J. B. Keller's model on ideal gas: In 1956, Keller consider the equilibrium of a charged ideal gas within a container. Such an ideal gas attains in equilibrium position when the pressure forces and electrostatic forces are balanced to each other. Under the certain assumption, he developed the following model

$$\nabla^2 w = \frac{4\pi a^2 m}{RT} \exp\left(\frac{m}{RT} w\right). \tag{5}$$

Again each variable of equation (5) has clear geometrical meaning. T is the constant temparature, R is the ideal gas constant, m is the average mass of molecules in the gas, $a = \dfrac{e}{m}$ and $w = \dfrac{RT}{m} log(p)$. The variable e be the charge of each molecule and p be the presure. Now Define $v = \dfrac{m}{RT} w + \log\left(4\pi \left[\dfrac{am}{RT}\right]^2\right)$. Then the equation (5) transformed into the following equation

$$\nabla^2 v = \exp(v). \tag{6}$$

Therefore, in veiw of equations (6) and (4) we can write the following equation

$$\nabla^2 u = f(u). \tag{7}$$

If a container is a sphere, a cylinder or a pain of parallel plane, then we can assume the solution u of equation (7) is a function from $\mathbb{R}^n$ to $\mathbb{R}$. Let $\tilde{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, and x is the distance from the axis of the cylinder, or the center of the sphere, or from the median of the plane in a different dimension. Therefore we define

$$x = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \text{ and } u = y(|\tilde{x}|). \tag{8}$$

By using the transformation (8), the radial equation of (7) leads to the following equation

$$y_{xx} + \frac{n-1}{x} y_x = f(y). \tag{9}$$

Equation (9) presents a singularity at the origin. Therefore, the regularity of y at the center requires $y_x(0) = 0$. In the view of equation (9), we have the following equation

$$\left(x^\beta y'(x)\right)' + x^\beta f(x, y) = 0, \tag{10}$$

where β is a function of n. Now, we define the boundary condition in the following form

$$y'(0) = 0, \quad b_1 y(1) + a_1 y'(1) = c_1, \quad 0 < x < 1. \tag{11}$$

We consider the problem (10) and (11) under the following assusmptions:
1. c_1, a_1 and b_1 are constant, and b_1, a_1 are not vanish simultaneously.
2.$\beta \geq 1$.
3.$f(x, y)$ is Lipschitz continuous function in $D = \{(x, y) : x \in [0, 1] \text{ and } y \in \mathbb{R}\}$ and $f(x, y) \geq 0$ in this domain.

Most useful form of $f(y)$ is $f(y) = y^m$, $m \in \mathbb{N} \cup \{0\}$. In this case, the equation (10) is known as Lane-Emden type equations. These equations arise various physical phenomena in astrophysics and mathematical physics like stellar structure, thermionic currents, thermal explosions, radiative cooling, CTC ([1, 10, 39, 38, 32, 3, 2, 62]). These also arise in chemical engineering, chemistry, physics and other branches to govern the attractions of spherical catalysts ([73]), the microbial flow particle ([82, 22]), polytropic models of astrophysics ([60]), etc. Many researchers had done their work based on equation (10) and (11) in the existing literature. We divide all these work in the literature survey into two parts. One part is theoretical and another one is numerical.

The theoretical picture for Lane-Emden type equations is well known. In 1964, Coffman ([17]) proved the theoretical results related to the existence and uniqueness of the solution of nonlinear boundary value problems. Aziz et. al. ([4]) derived the bounds of the solutions. In 1986, Dunninger et. al. ([24, 23]) proved the existence of solutions of nonlinear singular BVP of the form $f(x, y, py')$. Apart from these, many researchers followed the different approaches to prove the existence and uniqueness of Lane-Emden type equations. In the following, we mention few: L. E. Bobisud ([6]) proved the existence of a doubly singular nonlinear boundary value problem. Y. Zhang ([85, 86]) improved the results of [6]. Pandey et. al. ([53, 54, 57, 58, 59]) proved the existence and uniqueness by using monotone lower and upper solution technique. Cherpion et. al. ([16]) applied the monotone method on second-order BVP for the case $y'(1) = 0$. Ford et. al. ([28]) applied a general approach for proving the existence and uniqueness of solutions to the singular BVP of the form $f(x, y)$. They also found the solution's bound in finite region, where $\frac{\partial f}{\partial y} \geq 0$.

For many years, this area has been attracted researchers through different difficulties in the attempt to develop a new algorithm to find the numerical solution of second order SBVPs with desired accuracy. Here we mention few of them. In nineteenth century, many authors like Jamet ([35]), Gustafsson ([30]), Chawla et. al. ([11, 12, 14, 15, 13]), Jain et. al. ([34]), Pandey ([51, 52]), etc applied finite difference method to SBVPs. Russel et. al. ([68, 69]) developed collocation with piecewise polynomial functions as a method for solving two-point boundary value problems. In 2002, Danie et. al. ([18]) proved local and global existence of solutions. Apart from these techniques many authors applied different types of discretization method like rational Legendre approximation technique ([37]), methods based on splines polynomials ([40, 7, 66, 45]), different types of collocation approaches ([67, 5, 55, 84, 49, 44]), methods based on Legendre function ([83, 56, 8]), differential transform methods ([27, 48]), spectral method ([21, 25]), Haar wavelets and other orthonormal polynomial wavelets ([79, 80, 78]), etc. Attractively, the semi-analytical or iterative methods have been used by many researchers to find the approximate numerical solution. In 1980, Mahan et. al. ([50]) used a power series method to compute the numerical approximation. In 2003, J. H. He ([33]) applied a variational approach to find the approximate solution of Lane-Emden equations. Furthermore, many authors applied different iterative approach such as piecewise quasilinearization techniques ([65]), modified variational iterative method ([20, 41, 70, 82, 72, 19]), homotopy analysis method ([62]), iterative method based on Pade series ([76]), an approximation algorithm using Hermite functions collocation method ([61]), tau method ([74, 75, 31]), Adomian decomposition method ([64, 63]), homotopy perturbation method ([70]), optimized homotopy analysis method ([46, 71]), etc to compute the approximate solution of singular nonlinear boundary as well as initial value problem. In the above, we presented a few literature surveys corresponding to Lane-Emden type equations. To know more about the literature survey reader may see the references and the references are therein.

In this article, we propose a computational technique based on the finite element method to find the discrete solution of singular boundary value problem in different collocation points. It is not easy to handle the non-linearity in a singular boundary value problem by discrete method. Here, we use Trapezoidal and Simpson's $\frac{1}{3}$ rule to approximate the integrand of a nonlinear function. Our proposed technique is simple but efficient, which gives good accuracy. We also compare the results computed by our proposed technique with [77] and [70]. This technique can be popularized under the name of Galerkin finite element method ([36, 81, 1, 87]). In [43, 26], Thomee et. al. apply Galerkin finite element method on singular boundary value problem. Recently, Priyadarshi and R. Kumar ([29]) used Wavelet Galerkin method to find error bound, and numerical solution of fourth order linear and nonlinear differential equations. To know more about this method reader can read the references and the references therein.

We organized remaining of the paper in four sections. In section 2, we provide basic definition regarding our proposed technique. In section 3, we discuss about variational principle and derive the weak formulation. In section 4, We chose five real-life example, and solve by using our proposed technique. We also include numerical data, tables, and figures in the existing section. Finally, our conclusion is drawn in section 5.

2 Some Preliminaries

Here, we present some well-known results, which help us to derive our proposed methodology. We also provide an algorithm for our methodology.

Note 2.1: Usual inner product of two functions $F(x)$ and $G(x)$ on any arbitrary interval $[0, 1]$ is defined by

$$\langle F, G\rangle = \int_0^1 F(t)G(t)dt. \tag{12}$$

If $< F, G >= 0$, then $F(x)$ and $G(x)$ are orthogonal to each other and a norm associated with this inner product is defined by

$$||F||^2 = \langle F, F\rangle = \int_a^b |F(t)|^2 dt. \tag{13}$$

Note 2.2: Let $\tau_h : 0 = x_1 < x_2 < \cdots < x_n < x_{n+1} = 1$ be the partition of $[0, 1]$, and $h = (x_{i+1} - x_i)$ be the uniform length. Let $C_p[0, 1]$ be the space of continuous piecewise function on τ_h. Therefore, we define the finite element space V_h and $V_h^{'}$ in the following

$$V_h = \{v \in C_p[0, 1] : \int_0^1 v^2 dx < \infty, \ \int_0^1 (v')^2 dx < \infty \text{ with } v(1) = 0\},$$

and

$$V_h^{'} = \{v \in C_p[0, 1] : \int_0^1 v^2 dx < \infty, \ \int_0^1 (v')^2 dx < \infty\}.$$

To develop our proposed technique, we approximate the integrand $\int_0^1 F(t)dt$ associated with nonlinear function $F(t)$ by using composite Trapezoidal and Simpson's $\frac{1}{3}$ rule in the following manner

$$\int_0^1 F(t)dt \approx \frac{h}{2}\left[F(x_1) + F(x_{n+1}) + 2\left(F(x_2) + F(x_3) + \cdots + F(x_n)\right)\right], \tag{14}$$

and

$$\int_0^1 F(t)dt \approx \frac{h}{3}\left[F(x_1) + F(x_{n+1}) + 4\left(F(x_2) + \cdots + F(x_n)\right) + 2\left(F(x_3) + \cdots + F(x_{n-1})\right)\right], \tag{15}$$

n even, respectively.

Note 2.3: Now, to compute the finite element solution we use different basis function. Let ϕ_i be the nodal basis function corresponding to unkonwn nodes x_i on τ_h. Therefore, based of Lagrange linear interpolation function, we define ϕ_i

$$\phi_1(x) = \begin{cases} \dfrac{x - x_2}{x_1 - x_2} & \text{if } x_1 \leq x \leq x_2, \\ 0 & \text{otherwise,} \end{cases} \qquad \phi_{n+1}(x) = \begin{cases} \dfrac{x - x_n}{x_{n+1} - x_n} & \text{if } x_n \leq x \leq x_{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x_{i-1} \leq x \leq x_i, \\ \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & \text{if } x_i \leq x \leq x_{i+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

for $i = 2, 3, \cdots, n$.

Again, by using Lagrange quadratic interpolation function, we define the nodal basis function ϕ_i in the following form:

$$\phi_1 = \begin{cases} \dfrac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}, & x_1 \leq x \leq x_3, \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_{n+1} = \begin{cases} \dfrac{(x - x_n)(x - x_{n-1})}{(x_{n+1} - x_n)(x_{n+1} - x_{n-1})}, & x_{n-1} \leq x \leq x_{n+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_{i+1}(x) = \begin{cases} \dfrac{(x - x_{i-1})(x - x_i)}{(x_{i+1} - x_{i-1})(x_{i+1} - x_i)}, & \text{if } x_{i-1} \leq x \leq x_{i+1}, \\ \dfrac{(x - x_{i+2})(x - x_{i+3})}{(x_{i+1} - x_{i+2})(x_{i+1} - x_{i+3})}, & \text{if } x_{i+1} \leq x \leq x_{i+3}, \\ 0, & \text{otherwise,} \end{cases} \tag{17}$$

for $i = 2, 4, \cdots, n-2$, and

$$\phi_i(x) = \begin{cases} \dfrac{(x - x_{i-1})(x - x_{i+1})}{(x_i - x_{i-1})(x_i - x_{i+1})}, & \text{if } x_{i-1} \leq x \leq x_{i+1}, \\ 0, & \text{otherwise,} \end{cases} \tag{18}$$

for $i = 2, 4, \cdots, n$. Also, every basis function $\phi_i(x)$ satisfies the following property

$$\phi_i(x_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq \text{j}. \end{cases} \tag{19}$$

Note 2.4: Let

$$
\begin{aligned}
&f_1(x_1, x_2, \cdots, x_n) = 0,\\
&f_2(x_1, x_2, \cdots, x_n) = 0,\\
&\vdots\\
&f_n(x_1, x_2, \cdots, x_n) = 0,
\end{aligned}
$$

be the system of nonlinear algebric equations. We can solve above system of equations by using the following three steps algorithm

Step 1: Choose the initial point $x_0 = (x_1^0, x_2^0, \cdots, x_n^0)$.

Step 2: For $i = 0, 1, \cdots, m$
 i. Compute the jacobian matrix $J = Df(\vec{x_i})$,
 ii. Set $\vec{x_{i+1}} = \vec{x_i} - J^{-1}\vec{f}(\vec{x_i})$.

Step 3: Set stopping criteria $|\vec{x_{i+1}} - \vec{x_i}| < \epsilon$, where ϵ is the tolerance.

Note 2.5: If $Y(x)$ be the approximate numerical solution and $y(x)$ be the exact solution of a differential equation, then we can define the L_2 error and L_∞ error in the following manner:

$$
L_2 = \sqrt{h\sum_{i=0}^{n} |Y(x_i) - y(x_i)|^2} \text{ and } L_\infty = \max_i |Y(x_i) - y(x_i)|, \tag{20}
$$

where x_i are the spatial point of a partition on the interval $[0, 1]$.

3 The finite element method

Now, we derive the weak formulation of equation (10). Let $Y(x) = \sum_{i=1}^{n} C_i\phi_i(x)$ be the finite element approximation of $y(x)$ corresponding to Dirichlet boundary condition, and $Y(x) = \sum_{i=1}^{n+1} C_i\phi_i(x)$ be the finite element approximation of $y(x)$ corresponding to mixed boundary condition. $C_i,\ i = 0, 1, \cdots, n+1$ be the unknow constants, which are to be determined.

(a) Weak formulation of equation (10) corresponding to homogeneous Dirichlet boundary condition: To obtain the weak formulation, we multiply both sides of equation (10) by a test function $v(x) \in V_h$, and integrating by parts, we have

$$
\int_0^1 x^\beta f v\ dx = -\int_0^1 (x^\beta y')'v\ dx = \int_0^1 x^\beta y'v'\ dx. \tag{21}
$$

Therefore, the variational formulation represents to find $y(x) \in V_h$ such that

$$\int_0^1 x^\beta y' v' \, dx = \int_0^1 x^\beta f v \, dx \quad \text{for all } v \in V_h. \tag{22}$$

Here, v is called the test function and y is called the trial function. Now, from equation (22), we have

$$\left\langle x^\beta y', v' \right\rangle = \left\langle x^\beta f, v \right\rangle \text{ for all } v \in V_h. \tag{23}$$

Therefore, for Dirichlet boundary condition, the continuous Galerkin finite element method reads: find $Y(x) \in V_h$ such that

$$\left\langle x^\beta Y', v' \right\rangle = \left\langle x^\beta f(x, Y), v \right\rangle \quad \text{for all } v \in V_h. \tag{24}$$

Now, we chose $v = \phi_i, \; i = 1, 2, \cdots, n$. Therefore, the discrete version of equation (24) is

$$\left\langle x^\beta \sum_{i=1}^{n} C_i \phi_i'(x), \phi_j' \right\rangle = \left\langle x^\beta f\left(x, \sum_{i=1}^{n} C_i \phi_i\right), \phi_j \right\rangle \quad \text{for all } j = 1, 2, \cdots, n. \tag{25}$$

(b) Weak formulation of equation (10) corresponding to nonhomogeneous Dirichlet boundary condition: Let us consider $y(1) = c_1$. To convert the nonhomogeneous boundary condition into homogeneous boundary condition, we put $y(x) = w(x) + c_1$. Therefore the equation (10) becomes

$$\left(x^\beta w'(x)\right)' + x^\beta f(x, w(x) + c_1) = 0, \tag{26}$$

with the boudary condition is

$$w'(0) = 0 \text{ and } w(1) = 0. \tag{27}$$

Therefore, by using

$$W(x) = \sum_{i=1}^{n} C_i \phi_i(x) \text{ and } W'(x) = \sum_{i=1}^{n} C_i \phi_i'(x), \tag{28}$$

we have

$$\left\langle x^\beta W', v' \right\rangle = \left\langle x^\beta f(x, W + c_1), v \right\rangle \quad \text{for all } v \in V_h. \tag{29}$$

Similarly, we have

$$\left\langle x^\beta \sum_{i=1}^{n} C_i \phi_i', \phi_j' \right\rangle = \left\langle x^\beta f\left(x, \sum_{i=1}^{n} C_i \phi_i + c_1\right), \phi_j \right\rangle \quad \text{for all } j = 1, 2, \cdots, n. \tag{30}$$

(c) Weak formulation of equation (10) corresponding to mixed boundary condition: In this case, we consider $b_1 y(1) + a_1 y'(1) = c_1$. Therefore, we have

$$\int_0^1 x^\beta y' v' \, dx = \int_0^1 x^\beta f v \, dx + \frac{c_1 - b_1 y(1)}{a_1} v(1) \quad \text{for all } v \in V_h'. \tag{31}$$

So, the Galerkin finite element yields to find $Y(x) \in V_h'$ such that

$$\langle x^\beta Y', v\rangle = \langle x^\beta f(x,Y), v\rangle + \left\langle 1, \frac{c_1 - b_1 y(1)}{a_1} v(1) \right\rangle \quad \text{for all } v \in V_h'. \tag{32}$$

Since $Y(x_{n+1})$ is the unkown function value corresponding to nodal point x_{n+1}, therefore we put

$$Y(x) = \sum_{i=1}^{n+1} C_i \phi_i(x) \text{ and } Y'(x) = \sum_{i=1}^{n+1} C_i \phi_i'(x). \tag{33}$$

Hence, we have

$$\left\langle x^\beta \sum_{i=1}^{n+1} C_i \phi_i', \phi_j' \right\rangle = \left\langle x^\beta f\left(x, \sum_{i=1}^{n+1} C_i \phi_i\right), \phi_j \right\rangle + \left\langle 1, \frac{c_1 - b_1 \sum_{i=1}^{n+1} C_i \phi_i(1)}{a_1} \phi_j(1) \right\rangle, \tag{34}$$

for all $j = 1, 2, \cdots, n+1$.

4 Numerical Examples

To validate our proposed method, we consider five real life problems depending on β. We mainly focus on the problem when $\beta = 1$ and $\beta = 2$.

Example 4.1. *In [10], Chandrasekhar derived a singular boundary value problem which is a special form of equation (10). He has considered $\beta = 2$ and $f(x,y) = y^\gamma$, γ be the physical constant. For simplycity here we consider $\gamma = 5$. Therefore the two point singular boundary value problem is of the following form*

$$\left(x^2 y'\right)' + x^2 y^5 = 0, \;\; 0 < x < 1, \tag{35}$$

subject to the boundary condition

$$y'(0) = 0 \textit{ and } y(1) = \frac{\sqrt{3}}{2}. \tag{36}$$

The exact solution of equation (35) and (36) is given by

$$y(x) = \frac{1}{\sqrt{1 + \frac{x^2}{3}}}. \tag{37}$$

The boundary condition of equation (36) is of nonhomogeneous Dirichlet type. Therefore, from equation (30), we have

$$\left\langle x^2 \sum_{i=1}^{n} C_i \phi_i', \phi_j' \right\rangle = \left\langle x^2 \left(\sum_{i=1}^{n} C_i \phi_i + \frac{\sqrt{3}}{2}\right)^5, \phi_j \right\rangle, \quad \text{for all } j = 1, 2, \cdots, n. \tag{38}$$

Equivalently, we can write

$$\sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \, dx = \int_0^1 x^2 \left(\sum_{i=1}^{n} C_i \phi_i + \frac{\sqrt{3}}{2} \right)^5 \phi_j \, dx, \quad \text{for all } j = 1, 2, \cdots, n. \tag{39}$$

It is not easy to determine the values of C_i of equation (39). To avoid the difficulty, we approximate the right-hand side integrand in equation (39) by using composite Trapezoidal and composite Simpson's $\frac{1}{3}$ quadrature rule over the whole domain. Therefore, from equation (39), we have

(a) Trapezoidal rule:

$$\begin{cases} \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \, dx = \frac{h}{2} x_j^2 \left(C_j + \frac{\sqrt{3}}{2} \right)^5, & \text{for } j = 1, \\ \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \, dx = h x_j^2 \left(C_j + \frac{\sqrt{3}}{2} \right)^5, & \text{for } \ j = 2, \cdots, n. \end{cases} \tag{40}$$

(b) Simpson's $\frac{1}{3}$ rule:

$$\begin{cases} \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \, dx = \frac{h}{3} x_j^2 \left(C_j + \frac{\sqrt{3}}{2} \right)^5, & \text{for } j = 1, \\ \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \, dx = \frac{4h}{3} x_j^2 \left(C_j + \frac{\sqrt{3}}{2} \right)^5, & \text{for even } j, \\ \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \, dx = \frac{2h}{3} x_j^2 \left(C_j + \frac{\sqrt{3}}{2} \right)^5, & \text{for odd } j. \end{cases} \tag{41}$$

Now, equations (40) and (41) are in the following form:

$$A\vec{C} = \vec{B}, \tag{42}$$

where A is the $n \times n$ matrix, $\vec{C}$ is the $n \times 1$ column vector and $\vec{B}$ is the $n \times 1$ column vector. In equation (40), we use Lagrange linear interpolation, and corresponding to equation (41) we use Lagrange quadratic basis function. Now, by using Newton Raphson method, we can easily compute the finite element solution of equations (35) and (36). We present the numerical data of approximations of equations (35) and (36) in table 1.

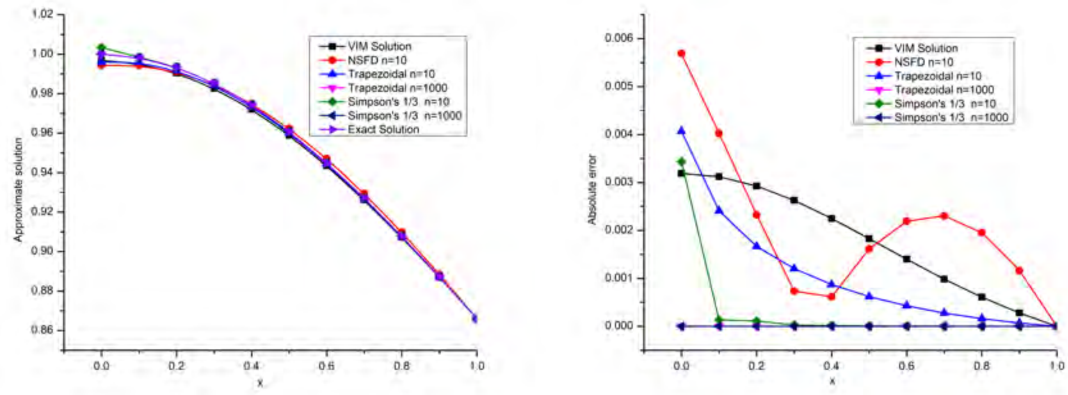
Table 1: Finite element solution of example 4.1 for some values of n:

	Solution in [70]	Solution in [77]	Trapezoidal rule		Simpson's $\frac{1}{3}$ rule		Exact solution
x	u_6	$n = 10$	$n = 10$	$n = 10^3$	$n = 10$	$n = 10^3$	
0	0.996812	0.994312521	0.995929513	0.999999078	1.003433017	0.999998633	1
0.1	0.995216	0.994312521	0.995929513	0.998337239	0.998471122	0.998335789	0.998337488
0.2	0.990472	0.991072907	0.991730316	0.993399098	0.993509227	0.993399135	0.993399268
0.3	0.982703	0.984593678	0.984124319	0.985329156	0.985302732	0.985329191	0.985329278
0.4	0.972106	0.974969408	0.973482384	0.974354616	0.974370933	0.974354647	0.974354704
0.5	0.95894	0.962384523	0.96014802	0.96076886	0.960762622	0.960768887	0.960768923
0.6	0.943513	0.947102766	0.944484312	0.94491114	0.944913428	0.944911161	0.944911183
0.7	0.926161	0.929449782	0.926869344	0.927145513	0.927145312	0.92714553	0.927145541
0.8	0.907234	0.909792535	0.907681974	0.907841283	0.907841262	0.907841294	0.907841299
0.9	0.887079	0.888517674	0.88728739	0.887356502	0.887357917	0.887356508	0.887356509
1	0.866025	0.8660254	0.866025404	0.866025404	0.866025404	0.866025404	0.866025404

Table 2: Finite element solution error of example 4.1 for different values of n:

	Solution in [70]	Solution in [77]	Trapezoidal rule		Simpson's $\frac{1}{3}$ rule	
	u_6	$n = 10$	$n = 10$	$n = 10^3$	$n = 10$	$n = 10^3$
L_∞ error	3.18E-03	5.68E-03	4.07E-03	9.22E-07	3.43E-03	1.71E-07
L_2 error	2.17E-03	2.69E-03	1.67E-03	1.52E-07	1.00E-03	1.25E-07

In table 1 and 2, we compare the approximate solution and error with the data computed by combined variational and homotopy perturbation method as well as non-standard finite difference (NSFD) method. Because of table 2, we see that for $n = 10$ our proposed techniques give better accuracy than the NSFD method. Also, we observe that, if we increase the value of n, then the absolute error is decreasing, which indicates that our proposed techniques converge numerically. Below, we provide the graph [See: figure 1] of the finite element solution, and absolute error corresponding to different methods.



(a) Approximate Solution (b) Absolute error

Figure 1: Approximate solutions $y(x)$ and absolute error corresponding to equations (35) and (36).

Example 4.2. *In [9], Chambre derive the following SBVP. He has considered* $\beta = 1$ *and* $f(x, y) = e^y$*. Therefore the equation (10) becomes*

$$(xy')' + xe^y = 0, \;\; 0 < x < 1, \tag{43}$$

subject to the boundary condition

$$y'(0) = 0 \text{ and } y(1) = 0. \tag{44}$$

The exact solution of the singular ordinary diffrential equation is given by

$$y(x) = 2\ln\left(\frac{A+1}{Ax^2+1}\right), \tag{45}$$

where $A = 3-2\sqrt{2}$. From equation (25), we have the variational formulation of equations (43) and (44)

$$\sum_{i=1}^{n} C_i \int_0^1 x\phi_i'\phi_j' \; dx = \int_0^1 x \; exp\left(\sum_{i=1}^{n} C_i\phi_i\right) \phi_j \; dx, \quad \text{for all } j = 1, 2, \cdots, n. \tag{46}$$

Similarly, we can approximate the right-hand side integral in equation (46) by using Trapezoidal and Simpson's $\frac{1}{3}$ rule. Hence we have

(a) Trapezoidal rule:

$$\begin{cases} \sum_{i=1}^{n} C_i \int_0^1 x\phi_i'\phi_j' \; dx = \frac{h}{2} x_j \; e^{C_j}, & \text{for } j = 1, \\ \sum_{i=1}^{n} C_i \int_0^1 x\phi_i'\phi_j' \; dx = h x_j \; e^{C_j}, & \text{for } j = 2, \cdots, n. \end{cases} \tag{47}$$

(b) Simpson's $\frac{1}{3}$ rule:

$$\begin{cases} \sum_{i=1}^{n} C_i \int_0^1 x\phi_i'\phi_j' \, dx = \dfrac{h}{3} x_j \, e^{C_j}, & \text{for } j = 1, \\ \sum_{i=1}^{n} C_i \int_0^1 x\phi_i'\phi_j' \, dx = \dfrac{4h}{3} x_j \, e^{C_j}, & \text{for even } j, \\ \sum_{i=1}^{n} C_i \int_0^1 x\phi_i'\phi_j' \, dx = \dfrac{2h}{3} x_j \, e^{C_j}, & \text{for odd } j. \end{cases} \tag{48}$$

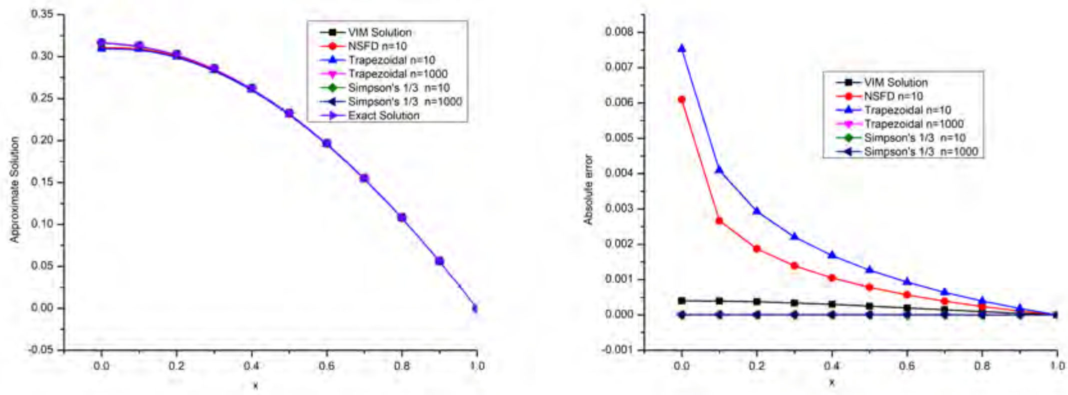
In equation (47), we use Lagrange linear basis function, and in equation (48) we use Lagrange quadratic basis function. By similar manner, we can determine the values of $C_i,\ i = 1, 2, \cdots, n$. We have listed bellow [See: table 3, figure 2] the finite element solution and corresponding solution graph of example 4.2.

Table 3: The finite element solution of example 4.2 for some values of n:

	Solution in [70]	Solution in [77]	Trapezoidal rule		Simpson's $\frac{1}{3}$ rule		Exact solution
x	u_6	$n = 10$	$n = 10$	$n = 10^3$	$n = 10$	$n = 10^3$	
0	0.316294	0.310600377	0.309170157	0.316692821	0.31669415	0.316689311	0.316694368
0.1	0.312872	0.310600377	0.309170157	0.313265433	0.313274408	0.313260812	0.31326585
0.2	0.302642	0.301144158	0.300088196	0.303015127	0.303015182	0.303010445	0.303015423
0.3	0.285707	0.284653686	0.283839196	0.286047043	0.286055223	0.286042427	0.286047265
0.4	0.262232	0.261481022	0.260848031	0.262530958	0.262530973	0.262526553	0.262531127
0.5	0.232446	0.231913687	0.231427965	0.232696656	0.232703691	0.232692606	0.232696784
0.6	0.196628	0.196259942	0.195898834	0.196826712	0.196826756	0.196823076	0.196826806
0.7	0.155103	0.154860498	0.15460738	0.155248042	0.155253683	0.155244921	0.155248107
0.8	0.108229	0.108085861	0.10792759	0.108322724	0.108322779	0.108320514	0.108322763
0.9	0.0563936	0.056329533	0.056255144	0.056438584	0.056442754	0.056437454	0.056438602
1	4.16334E-17	0	0	0	0	0	0

Table 4: Finite element solution error of example 4.2 for different values of n:

	Solution in [70]	Solution in [77]	Trapezoidal rule		Simpson's $\frac{1}{3}$ rule	
	u_6	$n = 10$	$n = 10$	$n = 10^3$	$n = 10$	$n = 10^3$
L_∞ error	4.00E-04	6.09E-03	7.52E-03	1.55E-06	8.56E-06	3.94E-06
L_2 error	2.81E-04	2.20E-03	3.04E-03	2.57E-07	4.82E-06	5.05E-06



(a) Approximate Solution (b) Absolute error

Figure 2: Approximate solutions $y(x)$ and absolute error corresponding to equations (43) and (44).

Example 4.3. *Here, we consider $\beta = 2$ and $f(x, y) = y$. Therefore from equation (10), we can easily obtain the following two-point boundary value problem. The governing equation first introduced by Chandrasekhar in [10], which arises in hydrostatic theory, and given by*

$$\left(x^2 y'\right)' + x^2 y = 0, \;\; 0 < x < 1, \tag{49}$$

subject to the boundary condition

$$y'(0) = 0 \; and \; y(1) = \sin(1). \tag{50}$$

The exact solution of equations (49) and (50) is given by

$$y(x) = \frac{\sin(x)}{x}. \tag{51}$$

We can easily convert non homogeneous boundary condition into homogeneous boundary condition by using the transformation $w(x) + \sin(1) = y(x)$. Let $W(x) = \sum_{i=1}^{n} C_i \phi_i$ be the finite element solution. Therefore from equation (30), we get

$$\sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \; dx = \int_0^1 x^2 \left(\sum_{i=1}^{n} C_i \phi_i + \sin(1) \right) \phi_j \; dx, \quad \text{for all } j = 1, 2, \cdots, n. \tag{52}$$

By similar analysis, we get the following schemes

(a) Trapezoidal rule:

$$\begin{cases} \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \; dx = \dfrac{h}{2} x_j^2 \left(C_j + \sin(1) \right), & \text{for } j = 1, \\ \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \; dx = h x_j^2 \left(C_j + \sin(1) \right), & \text{for } j = 2, \cdots, n. \end{cases} \tag{53}$$

(b) Simpson's $\frac{1}{3}$ rule:

$$\begin{cases} \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \; dx = \dfrac{h}{3} x_j^2 \left(C_j + \sin(1)\right), & \text{for} j = 1, \\ \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \; dx = \dfrac{4h}{3} x_j^2 \left(C_j + \sin(1)\right), & \text{for even } j, \\ \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \; dx = \dfrac{2h}{3} x_j^2 \left(C_j + \sin(1)\right), & \text{for odd } j. \end{cases} \tag{54}$$

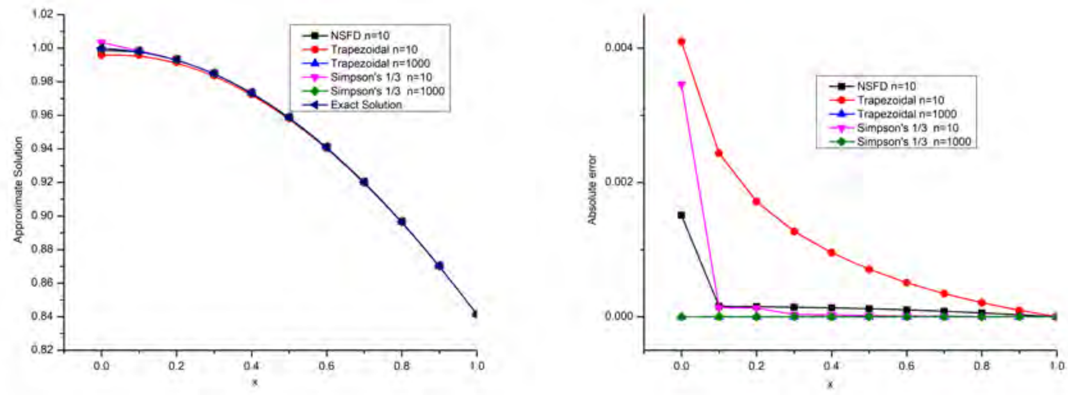
Now, by similar calculation as in example 4.1, we can compute the finite element solutions of equations (49) and (50). In table 5 and figure 3, we present numerical data and graph of approximate numerical solution corresponding to different values of n.

Table 5: Finite element solution of example 4.3 for some values of n:

	Solution in [77]	Trapezoidal rule		Simpson's $\frac{1}{3}$ rule		Exact solution
x	$n = 10$	$n = 10$	$n = 10^3$	$n = 10$	$n = 10^3$	
0	0.998488547	0.995899523	0.999999078	1.003462347	0.999999127	1
0.1	0.998488547	0.995899523	0.998333915	0.998469997	0.998332948	0.998334166
0.2	0.993496105	0.991631382	0.99334648	0.993477647	0.993345444	0.993346654
0.3	0.98520865	0.983795974	0.985067228	0.985034132	0.985066172	0.985067356
0.4	0.973678043	0.972593335	0.97354576	0.97357538	0.973544728	0.973545856
0.5	0.958970255	0.958145098	0.958851007	0.95883616	0.958850037	0.958851077
0.6	0.941173644	0.940563215	0.941070738	0.941080049	0.941069857	0.941070789
0.7	0.92039222	0.919966683	0.920310947	0.920302927	0.920310195	0.920310982
0.8	0.896750362	0.896486739	0.896695093	0.896697736	0.896694546	0.896695114
0.9	0.870391135	0.870268466	0.870363223	0.870358562	0.870362936	0.870363233
1	0.841470984	0.841470985	0.841470985	0.841470985	0.841470985	0.841470985

Table 6: Finite element solution error of example 4.3 for some values of n:

	Solution in [77]	Trapezoidal rule		Simpson's $\frac{1}{3}$ rule	
	$n = 10$	$n = 10$	$n = 10^3$	$n = 10$	$n = 10^3$
L_∞ error	1.15E-03	4.10E-03	7.55E-07	3.46E-03	1.22E-06
L_2 error	4.90E-04	1.70E-03	1.52E-07	1.09E-03	9.67E-07



(a) Approximate Solution (b) Absolute error

Figure 3: Approximate solutions $y(x)$ and absolute error corresponding to equations (49) and (50).

Example 4.4. *Here we consider $\beta = 2$ and $\gamma = 0$, which is described by Chandrasekhar in [10]. Therefore the two-point boundary value problem reduces to*

$$\left(x^2 y'\right)' + x^2 = 0, \quad 0 < x < 1, \tag{55}$$

subject to the boundary condition

$$y'(0) = 0 \text{ and } y(1) = \frac{1}{6}. \tag{56}$$

The exact solution of this problem is given by

$$y(x) = \left(\frac{1}{3} - \frac{x^2}{6}\right). \tag{57}$$

Let us consider $w(x) + \dfrac{1}{6} = y(x)$. Therefore by similar analysis as in example 4.3, we have

(a) Trapezoidal rule:

$$\begin{cases} \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \, dx = \dfrac{h}{2} x_j^2, & \text{for } j = 1, \\ \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \, dx = h x_j^2, & \text{for } j = 2, \cdots, n. \end{cases} \tag{58}$$

(b) Simpson' $\dfrac{1}{3}$ rule:

$$\begin{cases} \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \, dx = \dfrac{h}{3} x_j^2, & \text{for } j = 1, \\ \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \, dx = \dfrac{4h}{3} x_j^2, & \text{for even } j, \\ \sum_{i=1}^{n} C_i \int_0^1 x^2 \phi_i' \phi_j' \, dx = \dfrac{2h}{3} x_j^2, & \text{for odd } j. \end{cases} \tag{59}$$

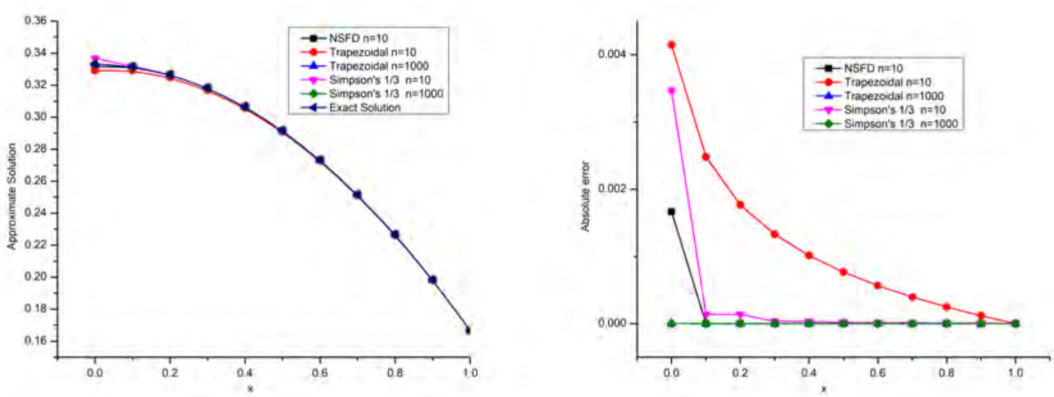
Here also we chose the respective basis function corresponding to equations (58) and (59) as in example 4.1. Again, by using Newton Raphson method, from equations (58) and (59), we can easily compute the values of C_i. The finite element solution of example 4.4 are listed in table 7, and plotted in Figure 4. We have also listed below the error table [See: table 8], and plotted the graph [See: figure 4].

Table 7: Finite element solution of example 4.4 for few values of n:

	Solution in [77]	Trapezoidal rule		Simpson's $\frac{1}{3}$ rule		
x	$n = 10$	$n = 10$	$n = 10^3$	$n = 10$	$n = 10^3$	Exact solution
0	0.331667827	0.32918515	0.333332407	0.336802947	0.333332542	0.333333333
0.1	0.331667827	0.32918515	0.331666411	0.331802947	0.331665529	0.331666667
0.2	0.326667827	0.324899436	0.326666488	0.326802947	0.326665531	0.326666667
0.3	0.318334493	0.317004699	0.3183332	0.318297792	0.318332215	0.318333333
0.4	0.306668512	0.305653348	0.306666565	0.306699854	0.306665593	0.306666667
0.5	0.291668923	0.290899249	0.29166659	0.291647838	0.291665667	0.291666667
0.6	0.273335863	0.272767381	0.273333277	0.273344847	0.273332427	0.273333333
0.7	0.251668757	0.251271318	0.251666627	0.251653547	0.251665892	0.251666667
0.8	0.226667826	0.226419247	0.226666642	0.226670343	0.226666101	0.226666667
0.9	0.198333769	0.198216482	0.198333322	0.198322714	0.198333035	0.198333333
1	0.166666667	0.166666667	0.166666667	0.166666667	0.166666667	0.166666667

Table 8: Finite element solution error of example 4.4 for few values of n:

	Solution in [77]	Trapezoidal rule		Simpson's $\frac{1}{3}$ rule	
	$n = 10$	$n = 10$	$n = 10^3$	$n = 10$	$n = 10^3$
L_∞ error	1.67E-03	4.14E-03	9.26E-07	3.46E-03	1.14E-06
L_2 error	5.26E-04	1.70E-03	1.58E-07	1.09E-03	9.20E-07



(a) Approximate Solution (b) Absolute error

Figure 4: Approximate solutions $y(x)$ and absolute error corresponding to equations (55) and (56).

Example 4.5. *In [22], Duggan and Goodman consider a model of the form (10), where $\beta = 2$ and $f(x,y) = e^{-y}$. The stationary version of the model leads to a two-point boundary value problem*

$$\left(x^2y'\right)' + x^2e^{-y} = 0,\ 0 < x < 1, \tag{60}$$

and

$$y'(0) = 0 \text{ and } 2y(1) + y'(1) = 0. \tag{61}$$

The equations (60) and (61) has no exact solution. Also, the boundary condition (61) is of mixed type. Now, from equation (34), we have

$$\left\langle x^2\sum_{i=1}^{n+1} C_i\phi_i', \phi_j' \right\rangle = \left\langle x^2exp\left(-\sum_{i=1}^{n+1} C_i\phi_i\right), \phi_j \right\rangle + \left\langle 1, \frac{-2\sum_{i=1}^{n+1} C_i\phi_i(1)}{1}\phi_j(1) \right\rangle, \tag{62}$$

for all $j = 1, 2, \cdots, n+1$. Equivalently, we have

$$\sum_{i=1}^{n+1} C_i \int_0^1 x^2\phi_i'\phi_j'\ dx = \int_0^1 x^2exp\left(-\sum_{i=1}^{n+1} C_i\phi_i\right)\phi_j\ dx - 2c_{n+1}\phi_j \tag{63}$$

for all $j = 1, 2, \cdots, n+1$. Hence, by similar analysis as in example 4.1, we have

(a) Trapezoidal rule:

$$\begin{cases} \sum_{i=1}^{n+1} C_i \int_0^1 x^2\phi_i'\phi_j'\ dx = \dfrac{h}{2}x_j^2e^{-C_j}, & \text{for } j = 1, \\ \sum_{i=1}^{n+1} C_i \int_0^1 x^2\phi_i'\phi_j'\ dx = hx_j^2e^{-C_j}, & \text{for } j = 2, \cdots, n, \\ \sum_{i=1}^{n+1} C_i \int_0^1 x^2\phi_i'\phi_j'\ dx = \dfrac{h}{2}x_j^2e^{-C_j} - 2c_{n+1}, & \text{for } j = n. \end{cases} \tag{64}$$

(b) Simpson's $\frac{1}{3}$ rule:

$$\begin{cases} \sum_{i=1}^{n+1} C_i \int_0^1 x^2\phi_i'\phi_j'\ dx = \dfrac{h}{3}x_j^2e^{-C_j}, & \text{for } j = 1, \\ \sum_{i=1}^{n+1} C_i \int_0^1 x^2\phi_i'\phi_j'\ dx = \dfrac{4h}{3}x_j^2e^{-C_j}, & \text{for even } j, \\ \sum_{i=1}^{n+1} C_i \int_0^1 x^2\phi_i'\phi_j'\ dx = \dfrac{2h}{3}x_j^2e^{-C_j}, & \text{for odd } j, \\ \sum_{i=1}^{n+1} C_i \int_0^1 x^2\phi_i'\phi_j'\ dx = \dfrac{h}{3}x_j^2e^{-C_j} - 2C_{n+1}, & \text{for } j = n+1. \end{cases} \tag{65}$$

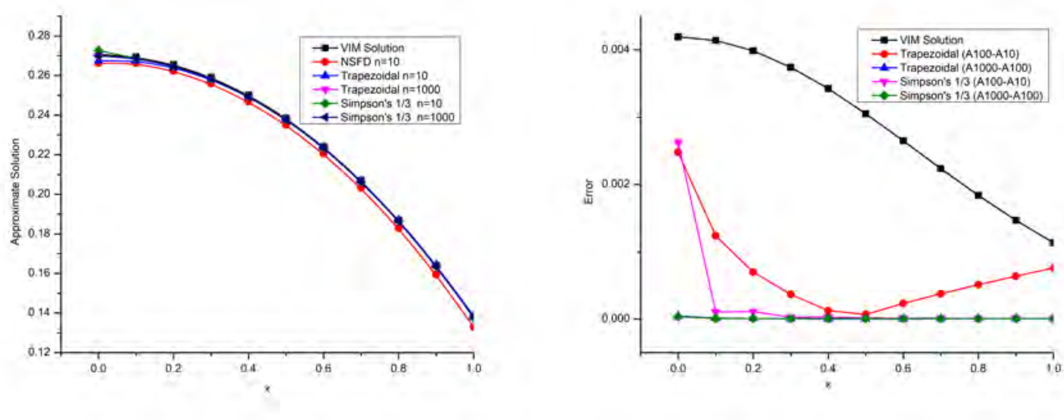
The Lagrange linear and quadratic basis function are chosen to compute $C_i,\ i = 1, 2, \cdots, n+1$ in the equations (64) and (65) respectively. Here also we provide the finite element solution in table 9 and figure 5. Now, we denote A_n be the approximate numerical solution corresponding to $n = 10,\ 100,\ 1000$. Since the equations (60) and (61) has no exact solution, therefore we provide absolute error of $(A_{100} - A_{10})$ and $(A_{1000} - A_{100})$. Since $|A_{1000} - A_{100}| < |A_{100} - A_{10}|$, therefore from table 10 we have seen that our proposed techniques converge numerically to the exact solution.

Table 9: Finite element solution of example 4.5 for some values of n:

	Solution in [70]	Solution in [77]	Trapezoidal rule		Simpson's $\frac{1}{3}$ rule	
x	u_7	$n = 10$	$n = 10$	$n = 10^3$	$n = 10$	$n = 10^3$
0	0.270736	0.266263465	0.267507244	0.270029006	0.272683919	0.270022442
0.1	0.269457	0.266263465	0.267507244	0.26875677	0.268862678	0.26874942
0.2	0.265615	0.262417516	0.264227452	0.264932746	0.265041437	0.264925313
0.3	0.259194	0.255974454	0.258169852	0.258539752	0.258512735	0.258532266
0.4	0.250163	0.246894301	0.249422293	0.249548167	0.24957723	0.249540665
0.5	0.238484	0.235122416	0.237984586	0.237915894	0.237900487	0.237908415
0.6	0.224104	0.220590786	0.223821276	0.223587731	0.223599467	0.223580301
0.7	0.206954	0.203215373	0.206874222	0.206494521	0.206482434	0.206487206
0.8	0.186953	0.182895848	0.187066144	0.186552066	0.186557144	0.186545066
0.9	0.164001	0.159513476	0.164301206	0.163659746	0.163648492	0.163653258
1	0.137982	0.132927843	0.138464255	0.137698823	0.137700469	0.137693191

Table 10: Finite element solution error of example 4.5 for some values of n:

	Residue error in [70]	Trapezoidal rule		Simpson's $\frac{1}{3}$ rule	
	u_7	$\lvert A_{100} - A_{10}\rvert$	$\lvert A_{1000} - A_{100}\rvert$	$\lvert A_{100} - A_{10}\rvert$	$\lvert A_{1000} - A_{100}\rvert$
L_∞ error	4.19E-03	2.40E-03	4.04E-05	2.62E-03	3.37E-05
L_2 error	3.23E-03	9.86E-04	1.58E-05	8.32E-04	1.28E-05



(a) Approximate Solution (b) Absolute error

Figure 5: Approximate solutions $y(x)$ and absolute error corresponding equations (60) and (61).

5 Conclusion

In this article, we developed numerical techniques based on Galerkin finite element method. We also successfully applied these proposed techniques to different real-life problems. Based on the comparison, we can conclude that our proposed technique gives the desired accuracy, and solutions are reliable. We have also noticed that for a large number of partition the finite element solution converge numerically very fast. Finally, we conclude that these techniques shall be useful to compute the numerical solution of real-life problems arising in physical science, chemical science, and engineering science.

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