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NUMERICAL RESULTS FOR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS DESCRIBING MOTIONS OF ELASTIC MATERIALS

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Abstract. We discuss an ordinary differential equation system proposed in [1] as a mathematical model for shrinking and stretching motions of elastic materials. Also, a numerical scheme due to the structure-preserving numerical method was constructed. Our aim of this paper is to compare the numerical results for periodic solutions by several methods in order to investigate their accuracy. We note that a proof for existence of periodic solutions of the ODE system is given. Finally, we derive a partial differential equation model from the ODE system and show numerical results for the PDE model.

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1 Introduction

In order to deal with the dynamics of an elastic material, we consider the following system of nonlinear second ordinary differential equations with some nonlinear function $f : (-1, \infty) \to \mathbb{R}$ (see Figure 1.1) as the follows:

$$m\frac{d^2 X_i}{dt^2} = f(\varepsilon_i)\frac{X_{i+1} - X_i}{l_i} - f(\varepsilon_{i-1})\frac{X_i - X_{i-1}}{l_{i-1}} \text{ on } [0, T] \text{ for } i = 1, 2, \ldots, N,$$

where

$$f(\varepsilon_i) = \frac{\kappa}{2}\left(\varepsilon_i + \frac{1}{2} - \frac{1}{2(1+\varepsilon_i)^2}\right), \quad \varepsilon_i = \frac{|X_{i+1} - X_i| - l_{N*}}{l_{N*}},$$

m is the mass of the material of each side concentrated at the vertex, ε_i is the strain of each side, $l_i = |X_{i+1} - X_i|$ is the length of each side and l_{N*} is the natural length of each side for $i = 1, 2, \ldots, N$, $N \in \mathbb{Z}_{>0} := \{n \in \mathbb{Z} \mid n > 0\}$ and $T > 0$. We note that $X_1, X_2, \ldots, X_N$ are the unknown functions from $[0, T]$ to $\mathbb{R}^2$ and denote the positions of vertexes of the polygon at time t (see Figure 1.2).

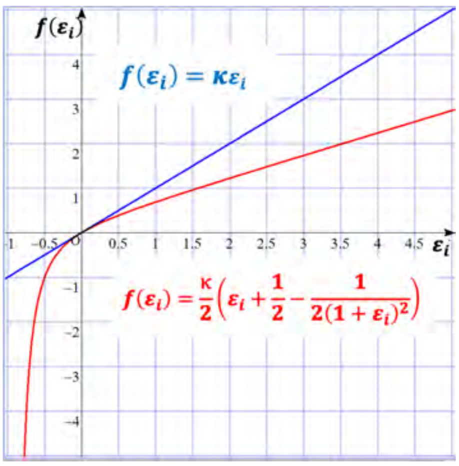

Figure 1.1: The stress function f

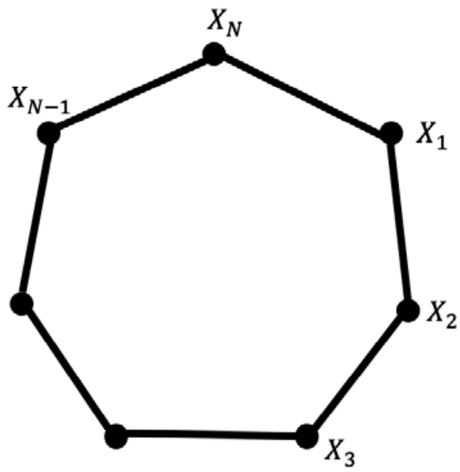

Figure 1.2: Graph of X_i in (OP)

By imposing the initial conditions, we get the following initial value problem (OP) consisting of (1.1)–(1.3) with $X_N = X_0$ and $X_{N+1} = X_1$. This problem is describing shrinking and stretching motions of elastic materials and it was already proposed and the precise physical background was discussed in [1].

Problem (OP): For each $i = 1, 2, \ldots, N$, X_i satisfies

$$m\frac{d^2 X_i}{dt^2} = f(\varepsilon_i)\frac{X_{i+1} - X_i}{l_i} - f(\varepsilon_{i-1})\frac{X_i - X_{i-1}}{l_{i-1}} \quad \text{on } [0, T], \tag{1.1}$$

$$\frac{dX_i}{dt} = V_i \quad \text{on } [0, T], \tag{1.2}$$

$$X_i(0) = X_{0i}, \quad \frac{dX_{0i}(0)}{dt} = V_{0i} \quad \text{for all } i = 1, 2, \ldots, N, \tag{1.3}$$

X_{0i} is the initial position and V_{0i} is the initial velocity for $i = 1, 2, \ldots, N$. Here, f is the stress function and κ is a positive constant. The stress function having singularity was already studied in research for large deformation isotropic elasticity, for instance [7, 12, 14]. We have already established the following theorem concerned with (OP) in [1].

Theorem 1.1. *(cf. [1]). Let $T > 0$, $X_{0i} \in \mathbb{R}^2$ and $V_{0i} \in \mathbb{R}^2$ for $i = 1, 2, \ldots, N$. If $X_{0i} \neq X_{0j}$ for $i \neq j$, then (OP) has a unique solution $X \in C^2([0,T])^{2N}$. Moreover, $F(X, V) = F(X_0, V_0)$, where $X_0 = (X_{01}, X_{02}, \ldots, X_{0N})$, $V_0 = (V_{01}, V_{02}, \ldots, V_{0N})$,*

$$F(X, V) = \sum_{i=1}^{N} \left\{ \frac{m}{2}|V_i|^2 + l_{N*}\hat{f}(\varepsilon_i) \right\},$$

for $X = (X_1, X_2, \ldots, X_N), V = (V_1, V_2, \ldots, V_N) \in \mathbb{R}^{2N}$ and $\hat{f}$ is primitive of f.

Moreover, in [1] we provided a numerical scheme to obtain approximate solutions of (OP) by applying the discrete variational derivative method (DVDM) which enables us to derive the structure-preserving scheme systematically (see for instance [6]). The idea of applying DVDM to analyze elastic materials was found in [15, 16, 17, 18].

Here, we remark that the nonlinear implicit formula obtained by DVDM leads to expensive calculation costs, since some iterative schemes such as the Newton method are used. In order to overcome this difficulty, Matsuo and Furihata [10] have constructed linearly implicit structure-preserving schemes for complex-valued nonlinear PDEs by introducing extra time steps of numerical schemes, and Furihata has designed explicit structure-preserving schemes for nonlinear wave equations in [4]. Also, Matsuo [9] has constructed new implicit and explicit structure-preserving schemes for second-order PDEs by representing the second-order ones as a system of first-order ones and discretizing the system using DVDM. Furthermore, other results of linearly implicit structure-preserving ones have been obtained in [5] and [8]. Besides, a similar framework by Furihata, Matsuo, and collaborators has also been developed by Dahlby and Owren [3]. For these results, we note that a linearly implicit scheme obtained by DVDM can be unstable depending on the discretization of the energy. Recently, for this issue, Matsuo and Furihata [11] and Sato et al. [13] have obtained the results of the analysis of the asymptotic behavior of the dissipative multi-steps linearly implicit schemes for the gradient system and the Duffing equation. By applying this idea, we propose a new multi-steps explicit scheme for (OP) in Section 2.3.

One of the purposes of this paper is to compare the accuracies of the numerical scheme (NS) obtained in [1] and the multi-steps explicit scheme (MS) for (OP). The exact formulation of (NS) and (MS) are given in Sections 2.1 and 2.3, respectively. In order to compare the accuracies of these methods we observe the periodic behaviors for the solutions of (OP). Moreover, in Section 2.1 we prove existence of a periodic solution of (OP) in time as Proposition 2.1, when initial functions X_0 and V_0 are spherically symmetric. In the proof we introduce the initial value problem $\mathrm{P}(R_0,\ v_0)$ for the one-dimensional ordinary differential equation:

$$m\frac{d^2R}{dt} = -\beta g(R) \text{ on } [0,T], R(0) = R_0 \text{ and } \frac{dR}{dt}(0) = v_0,$$

where $g(R) = f(\alpha R - 1)$, α and β are constants, R_0 is a positive constant and $v_0 \in \mathbb{R}$. The numerical results obtained by (NS) and (MS) are shown in Sections 2.2 and 2.3, respectively.

Furthermore, in this paper we derive a partial differential equation by letting $N \to \infty$ in (OP), and by adding the fourth derivative term, we provide the initial and boundary value problem (PP) for the partial differential equation called the beam equation in Section 3.1. Also, in order to clarify the role of the fourth derivative term we observe the numerical results in varying the values of its coefficient in Section 3.2. At the end of this paper, we mention about open problems and a conjecture as a conclusion.

2 Numerical results for ODE system

First, we recall the algorithm for the numerical scheme shown in [1] by applying the structure-preserving numerical method.

2.1 Algorithm

Let $\Delta t = T/K$ for $K \in \mathbb{Z}_{>0}$, $X^{(n)} = (X_1^{(n)}, X_2^{(n)}, \dots, X_N^{(n)})$ and $V^{(n)} = (V_1^{(n)}, V_2^{(n)}, \dots, V_N^{(n)})$ for $n = 0, 1, \dots, K$. In [1], we have constructed the following numerical scheme (NS) to obtain approximate solutions of (OP).

(NS):= **(NS)**$(\Delta t, X^{(n)}, V^{(n)})$**:** For given $\Delta t > 0$, $X^{(n)}$, $V^{(n)} \in \mathbb{R}^{2N}$, this scheme (NS) is to find $X^{(n+1)}$, $V^{(n+1)} \in \mathbb{R}^{2N}$ such that

$$\begin{aligned}
&\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t} \\
&= -\frac{\kappa}{4m}\left\{\varepsilon_{i-1}^{(n+1)} + \varepsilon_{i-1}^{(n)} + 1 - \frac{1}{\left(1+\varepsilon_{i-1}^{(n+1)}\right)\left(1+\varepsilon_{i-1}^{(n)}\right)}\right\} \frac{X_i^{(n+1)} - X_{i-1}^{(n+1)} + X_i^{(n)} - X_{i-1}^{(n)}}{\left|X_i^{(n+1)} - X_{i-1}^{(n+1)}\right| + \left|X_i^{(n)} - X_{i-1}^{(n)}\right|} \\
&\quad + \frac{\kappa}{4m}\left\{\varepsilon_{i}^{(n+1)} + \varepsilon_{i+1}^{(n)} + 1 - \frac{1}{\left(1+\varepsilon_{i}^{(n+1)}\right)\left(1+\varepsilon_{i}^{(n)}\right)}\right\} \frac{X_{i+1}^{(n+1)} - X_{i}^{(n+1)} + X_{i+1}^{(n)} - X_{i}^{(n)}}{\left|X_{i+1}^{(n+1)} - X_{i}^{(n+1)}\right| + \left|X_{i+1}^{(n)} - X_{i}^{(n)}\right|},
\end{aligned}$$

$$\frac{X_i^{(n+1)} - X_i^{(n)}}{\Delta t} = \frac{V_i^{(n+1)} + V_i^{(n)}}{2} \qquad \text{for all } n = 0, 1, \dots, K-1 \text{ and } i = 1, 2, \dots, N,$$

where $\varepsilon_i^{(n)} = l_i^{(n)}/l_{N*}$ and $l_i^{(n)} = \left|X_{i+1}^{(n)} - X_i^{(n)}\right|$.

Theorem 2.1. *(cf.[1]). Let $X_{0i} \in \mathbb{R}^2$ and $V_{0i} \in \mathbb{R}^2$ for $i = 1, 2, \dots, N$. If $X_{0i} \neq X_{0j}$ for $i \neq j$, then there exists $K_0 \in \mathbb{Z}_{>0}$ such that (NS) has a unique solution $(X^{(n+1)}, V^{(n+1)}) \in \mathbb{R}^4$ for $n = 0, 1, \dots, K-1$ and $K \geq K_0$, where $\Delta t = T/K$ and $X^{(0)} = (X_{01}, X_{02}, \dots, X_{0N})$, $V^{(0)} = (V_{01}, V_{02}, \dots, V_{0N})$. Moreover, $F(X^{(n)}, V^{(n)}) = F(X^{(0)}, V^{(0)})$ for $n = 0, 1, \dots, K$, where*

$$F(X^{(n)}, V^{(n)}) = \sum_{i=1}^{N} \left\{ \frac{m}{2}\left|V_i^{(n)}\right|^2 + l_{N*}\hat{f}\left(\varepsilon_i^{(n)}\right)\right\},$$

for $X^{(n)} = (X_1^{(n)}, X_2^{(n)}, \ldots, X_N^{(n)}), V^{(n)} = (V_1^{(n)}, V_2^{(n)}, \ldots, V_N^{(n)}) \in \mathbb{R}^{2N}$, $n = 0, 1, \ldots, K$ *and* $\hat{f}$ *is primitive of* f.

Theorem 2.2. *(cf.[1]). Assume* $X_{0i} \in \mathbb{R}^2$, $V_{0i} \in \mathbb{R}^2$ *for* $i = 1, 2, \ldots, N$, *and* $X_{0i} \neq X_{0j}$ *for* $i \neq j$. *Let* K_0 *be a positive integer defined in Theorem 2.1 and* $(X_K^{(n+1)}, V_K^{(n+1)})$ *be a solution of (NS)* $(\Delta t, X_K^{(n)}, V_K^{(n)})$ *for* $n = 0, 1, \ldots, K-1$ *and* $K \geq K_0$, *where* $\Delta t = \dfrac{T}{K}$, $X_K^{(0)} = (X_{01}, X_{02}, \ldots, X_{0N})$ *and* $V_K^{(0)} = (V_{01}, V_{02}, \ldots, V_{0N})$. *Moreover, put*

$$\begin{aligned} X^K(t) &= V_K^{(n)}\left(t - \frac{n}{K}\right) + X_K^{(n)} \quad \text{for } \frac{nT}{K} < t \leq \frac{(n+1)T}{K} \text{ and } n = 0, 1, \ldots, K-1, \\ X^K(0) &= X_0. \end{aligned}$$

There exists a positive constant C *such that*

$$\left|X(t) - X^K(t)\right| \leq C\left|\Delta t\right| \quad \text{for } 0 \leq t \leq T \text{ and } K \geq K_0,$$

where X *is a solution of (OP).*

In order to discuss the accuracy of the scheme (NS), we calculate approximations of periodic solutions in time. From now on, we consider (OP) in case the initial state is spherically symmetric such that (OP) has a periodic solution. For $\theta = \dfrac{2\pi}{N}$ let A be the rotation matrix of angle θ and choose the initial values (see Figure 2.1) as follows:

$$X_{0i} = A^i \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad V_{0i} = v_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } i = 1, 2, \ldots, N. \tag{2.1}$$

Also, we denote by $\mathbf{P}(R_0, v_0)$ the following initial value problem for the 1-dimensional ordinary differential equation, where $R_0 > 0$ and $v_0 \in \mathbb{R}$.

Problem $\mathbf{P}(R_0, v_0)$: Find a function R on $[0, T]$ satisfying

$$m\frac{d^2R(t)}{dt^2} = -\beta g(R(t)) \text{ for } t \in [0, T], R(0) = R_0, \frac{dR}{dt}(0) = v_0,$$

where R indicates the distance between the vertex X_i at time t, $\alpha = \dfrac{2}{l_{N*}} \sin\dfrac{\theta}{2}$, $\beta = 2\sin\dfrac{\theta}{2}$ and $g(R) = f(\alpha R - 1)$.

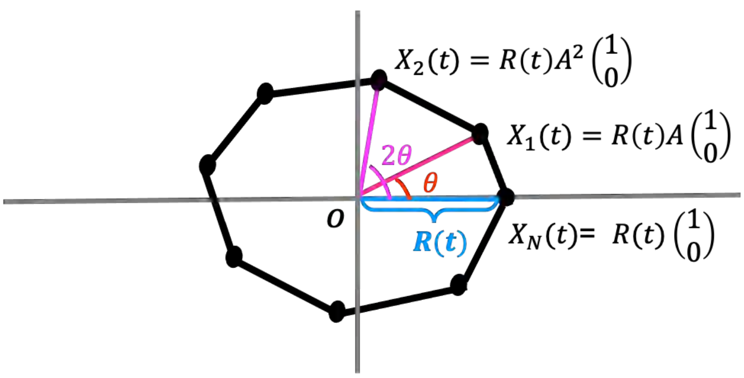

Figure 2.1: Spherically symmetric configuration

Here, let R be a solution of $\mathrm{P}(R_0, v_0)$ on $[0, T]$ and put

$$X_i(t) = R(t)A^i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{for all } i = 1, 2, \dots, N \text{ and } t \in [0, T].$$

Under the condition (2.1) by putting $\varepsilon = (\beta R - l_{N*})/l_{N*}$ on $[0, T]$ it is easy to see that $X = (X_1, X_2, \dots, X_N)$ is a solution of (OP) on $[0, T]$. As seen in Figures 2.2 and 2.3 obtained by numerical simulations with $N = 12$, $m = 5/6$, the time step $\Delta t = 0.001$ and $R_0 = 0.6$, the solution of $\mathrm{P}(R_0, 0)$ seems to be periodic in time.

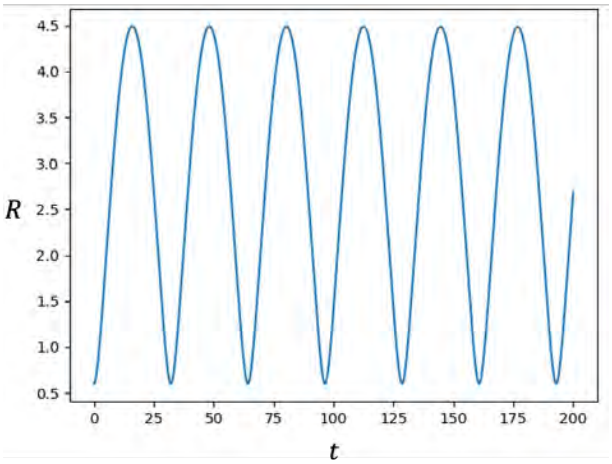

Figure 2.2: Time variation of R with $N = 12$ and $\Delta t = 0.001$

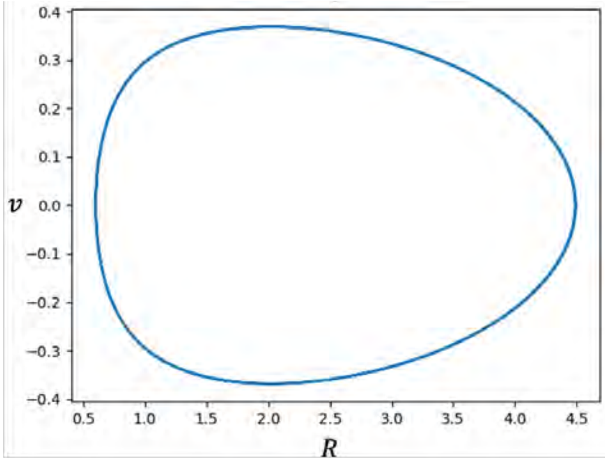

Figure 2.3: Phase diagram between R and $v := R'$ with $N = 12$ and $\Delta t = 0.001$

Moreover, we can prove this property, mathematically, as follows.

Proposition 2.1. *If $R_0 > 0$, $v_0 \in \mathbb{R}$, then the solution of $P(R_0, v_0)$ is periodic in time.*

By using the following Lemmas 2.1–2.3, we can prove Proposition 2.1. For simplicity, we put

$$a = \frac{\sqrt{\alpha\beta\kappa}}{2}, b = \frac{1}{2\alpha}, c = \frac{m}{2}, k = \frac{\beta\kappa}{4\alpha^2}, l = \frac{\beta\kappa}{2}\left(\frac{6}{\alpha} + \frac{1}{2\alpha^2 R_0}\right) + \frac{m}{2}|v_0|^2 \text{ and } v_0 \in \mathbb{R}.$$

Here, we consider the following equation:

$$a^2(x - b)^2 + \frac{k}{x} + c^2 y^2 = l \qquad \text{for } x > 0 \text{ and } y \in \mathbb{R}. \tag{2.2}$$

Lemma 2.1. *If $R_0 > 0$, then there exist positive constants α_1 and α_2 with $\alpha_1 < \alpha_2$ satisfying*

(i) For all $x \in (\alpha_1, \alpha_2)$ there exists $y > 0$ such that (2.2) holds.

(ii) $(\alpha_1, 0)$ and $(\alpha_2, 0)$ satisfy (2.2).

Proof. Put $\varphi(x) = a^2(x-b)^2 + \dfrac{k}{x}$ for all $x > 0$. We shall show existence of an interval $[\alpha_1, \alpha_2]$ of x such that $\varphi(x) \le l$ hold for $x \in [\alpha_1, \alpha_2]$. We have

$$\begin{aligned}\varphi'(x) &= \frac{\beta\kappa}{4\alpha^2 x^2}\left(2\alpha^3 x^3 - \alpha^2 x^2 - 1\right)\\ &= \frac{\beta\kappa}{4\alpha^2 x^2}\left(x - \frac{1}{\alpha}\right)\left(2\alpha^3 x^2 + \alpha^2 x + \alpha\right) \qquad \text{for } x > 0,\end{aligned}$$

and

$$\varphi'(x)\begin{cases} < 0 & \text{if } 0 < x < \dfrac{1}{\alpha},\\ = 0 & \text{if } x = \dfrac{1}{\alpha},\\ > 0 & \text{if } x > \dfrac{1}{\alpha}.\end{cases} \tag{2.3}$$

Also, we have

$$\lim_{x\downarrow 0}\varphi(x) = +\infty, \qquad \lim_{x\to+\infty}\varphi(x) = +\infty,$$

and

$$\varphi\left(\frac{1}{\alpha}\right) = \frac{5\beta\kappa}{16\alpha}, \qquad l - \varphi\left(\frac{1}{\alpha}\right) = \frac{43\beta\kappa}{16\alpha} + \frac{\beta\kappa}{4\alpha^2 R_0} + \frac{m}{2}|v_0|^2 > 0.$$

Therefore, since φ is continuous on $(0,\infty)$ we see that there exist α_1, $\alpha_2 > 0$ such that

$$\alpha_1 < \alpha < \alpha_2 \quad \text{and} \quad \varphi(\alpha_1) = \varphi(\alpha_2) = l.$$

Furthermore, by using (2.3), $\varphi(x) < l$ holds for $\alpha_1 < x < \alpha_2$. Thus, this lemma has been proved. □

Lemma 2.2. *$P(R_0, v_0)$ has a unique solution $R \in C^2([0,\infty))$ and there exists a positive constant M such that $\dfrac{1}{M} \le R(t) \le M$ for all $t \in [0,\infty)$.*

Proof. Since g is locally Lipschitz continuous on $(0,\infty)$, there exist $T_1 > 0$ and a unique solution $R_1 \in C^2([0,T_1])$ of $\mathrm{P}(R_0, v_0)$ on $[0,T_1]$, that is,

$$\begin{aligned}&mR_1'' = -\beta g(R_1) \text{ on } [0,T_1],\\ &R_1(0) = R_0, \quad R_1'(0) = v_0.\end{aligned} \tag{2.4}$$

By multiplying (2.4) with R_1', we see that the following energy is preserved:

$$\frac{d}{dt}\left(\frac{m}{2}|R_1'|^2 + \beta G(R_1)\right) = 0 \text{ on } [0,T_1], \tag{2.5}$$

where $G(R_1) = \dfrac{\kappa}{2}\left(\dfrac{\alpha}{2}R_1^2 - \dfrac{R_1}{2} + \dfrac{1}{2\alpha^2 R_1}\right)$.

Here, we can show $G(R) > 0$ for $R > 0$. Indeed, since it holds

$$G(R) = \frac{\kappa}{2}\left\{\frac{\alpha}{2}\left(R - \frac{1}{2\alpha}\right)^2 + \frac{1}{2\alpha}\left(\frac{1}{\alpha R} - \frac{1}{4}\right)\right\} \quad \text{for } R > 0,$$

we see that $G(R) > 0$ holds for all $0 < R < \dfrac{4}{\alpha}$. If $R \geq \dfrac{4}{\alpha}$, then

$$\left(R - \frac{1}{2\alpha}\right)^2 \geq \left(\frac{4}{\alpha} - \frac{1}{2\alpha}\right)^2, \text{ and } G(R) > 0.$$

Accordingly, by putting $d_0 = \dfrac{m}{2}|v_0|^2 + \beta g(G(R_0))$, we have $d_0 > 0$. From (2.5) and $d_0 > 0$, it follows that

$$R_1(t) \geq \frac{1}{2\alpha^2\left(\frac{2d_0}{\beta\kappa} + \frac{1}{8\alpha}\right)} > 0 \quad \text{for all } t \in [0, T_1]. \tag{2.6}$$

Since (R_1, R_1') satisfies (2.2), Lemma 2.1 implies that (R_1, R_1') is on the bounded closed curve which depends only on the initial data and the given values. Clearly, by combining (2.6), there exists a positive constant M independent of T_1 such that $\dfrac{1}{M} \leq R_1(t) \leq M$ for $t \in [0, T_1]$. Hence, we can extend the solution R_1 beyond T_1. Therefore, we obtain a unique solution $R \in C^2([0, \infty])$ of $\mathrm{P}(R_0, v_0)$ on $[0, \infty)$. Thus, this lemma is true. □

We can easily show the following lemma. So, we omit its proof.

Lemma 2.3. *$g\left(\dfrac{1}{\alpha}\right) = 0$, $g(R) > 0$ for $0 < R < \dfrac{1}{\alpha}$, and $g(R) > 0$ for $R > \dfrac{1}{\alpha}$ holds. Moreover, $g'(R) > 0$ for $R > 0$.*

Here, we give a proof of Proposition 2.1 by applying Lemmas 2.1, 2.2 and 2.3.

Proof of Proposition 2.1. Put $v = R'$ on $[0, \infty)$. For the sake of simplicity, we assume that $v_0 = 0$ and $0 < R_0 < \dfrac{1}{2\alpha}$. By lemma 2.2, $\mathrm{P}(R_0, v_0)$ has the unique solution $R \in C^2([0, \infty))$ on $[0, \infty)$, then we obtain

$$v'(0) = \frac{\beta\kappa}{2m}\left\{\frac{1}{2\alpha^2 R_0^2} + \alpha\left(\frac{1}{2\alpha} - R_0\right)\right\} > 0.$$

Now, since v' is continuous at $t = 0$, there exist $\delta_1 > 0$ and $T_1 > 0$ such that

$$v' \geq \delta_1 \text{ on } [0, T_1] \text{ and } v(t) \geq \delta_1 t \text{ for } t \in [0, T_1], \text{ and } R(t) \geq R_0 + \frac{\delta_1}{2}t \text{ for } t \in [0, T_1].$$

Here, we suppose that $v' > 0$ on $[T_1, \infty)$. It is easy to see that

$$v(t) \geq v(T_1) \geq \delta_1 T_1 > 0 \quad \text{for all } t \in [T_1, \infty).$$

Accordingly, we have

$$R(t) \geq R(T_1) + \delta_1 T_1 (t - T_1) \to +\infty \quad \text{as } t \to +\infty.$$

This is a contradiction for the boundedness of R shown in Lemma 2.2. Therefore, there exists $\widehat{T}_2 > T_1$ such that $v'(\widehat{T}_2) = 0$. This shows that the set $E_1 = \{t > T_1 \mid v'(t) = 0\}$ is not empty. Thus, we can put $T_2 = \inf E_1$. Since v' is continuous on $[0, \infty)$, it holds that

$$v'(T_2) = 0 \text{ and } v' > 0 \text{ on } [T_1, T_2), \tag{2.7}$$

and

$$v(T_2) > v(T_1) > \delta_1 T_1 > 0. \tag{2.8}$$

By (2.7), we have $g(R(T_2)) = 0$. On account of Lemma 2.3, we get $R(T_2) = \dfrac{1}{\alpha}$. The continuity of v at $t = T_2$ and (2.8) guarantee existence of $\delta_2 > 0$ and $b > 0$ such that

$$v \geq b \quad \text{on } [T_2, T_2 + \delta_2] \quad \text{and} \quad R(T_2 + b\delta_2) > \frac{1}{\alpha}.$$

Therefore, by applying Lemma 2.3 again, we have $g(R(T_2 + \delta_2)) > 0$.

Here, we assume that $v > 0$ on $[T_2 + \delta_2, \infty)$. Easily, we get

$$R(t) \geq R(T_2 + \delta_2) > \frac{1}{\alpha} \quad \text{for } t \in [T_2 + \delta_2, \infty).$$

Moreover, by using Lemma 2.3 again, we obtain

$$mv'(t) = -\beta g(R(t)) \leq -\beta g(R(T_1 + \delta_2)) < 0 \text{ for } t \in [T_2 + \delta_2, \infty),$$

and there exists a positive constant q such that $v'(t) \leq -q$ for all $t \in [T_2 + \delta_2, \infty)$ and $v(t) \leq v(T_2 + \delta_2) - q\{t - (T_2 + \delta_2)\}$ for all $t \in [T_2 + \delta_2, \infty)$. Hence, the convergence

$$v(T_2 + \delta_2) - q\{t - (T_2 + \delta_2)\} \to -\infty \quad \text{as } t \to \infty,$$

leads to a contradiction. Accordingly, there exists $\widehat{T}_3 > T_2 + \delta_2$ such that $v(\widehat{T}_3) = 0$. Namely, the set $E_2 = \{t > T_2 + \delta_2 | v(t) = 0\}$ is not empty. Thus, we can put $T_3 = \inf E_2$. As mentioned above, $v(T_3) = 0$ and $v > 0$ on $[T_2 + \delta_2, T_3)$, and $R(T_3) > R(T_2 + \delta_2) > \dfrac{1}{\alpha}$. This means that (R, v) arrives at $(\alpha_2, 0)$ at $t = T_3$.

Similarly, we can show that (R, v) reaches $(R_0, 0)$ in a finite time. Hence, there exists $T_* > 0$ such that $R(T_*) = R_0$ and $v(T_*) = 0$.

The uniqueness of solutions to $\mathrm{P}(R_0, 0)$ implies that R is periodic in time. Thus, this proposition has been proved. □

2.2 Numerical results

In this section, we shall show numerical results for (OP) by applying (NS) (see Section 2.1) and Euler's method (EM) with

$$N = 12, \Delta t = 0.001, \kappa = 0.1, m = \frac{5}{6}, l_* = 1,$$

$$X_{0i} = R_0 A^i \begin{pmatrix} 1 \\ 0 \end{pmatrix}, R_0 = 0.6 \text{ and } V_{0i} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } i = 1, 2, \ldots, N.$$

Figure 2.4 and Figure 2.6 indicate $R_i = |X_i|$ for $i = 1, 2, \ldots, N$, by (EM) and (NS), respectively. Also, Figures 2.5 and 2.7 represent time variation of the energy $F(X, V)$ (see Theorem 2.1) calculated by (EM) and (NS), respectively.

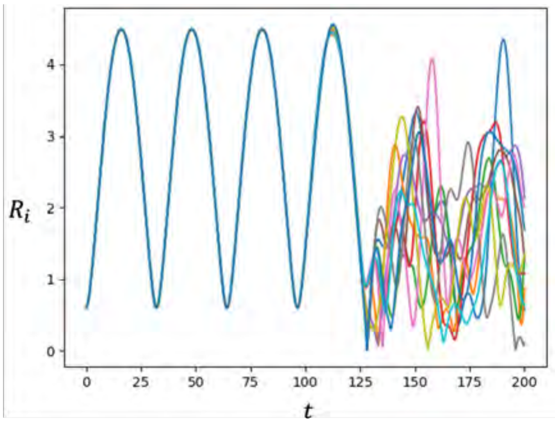

Figure 2.4: Time variation of radius for $N = 12$ and $\Delta t = 0.001$ by Euler's method

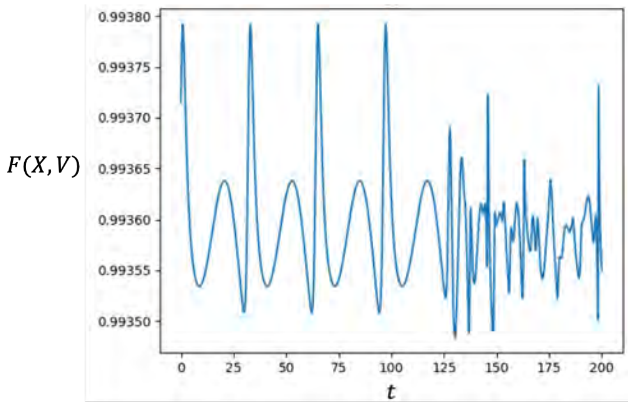

Figure 2.5: Time variation of the energy for $N = 12$ and $\Delta t = 0.001$ by Euler's method

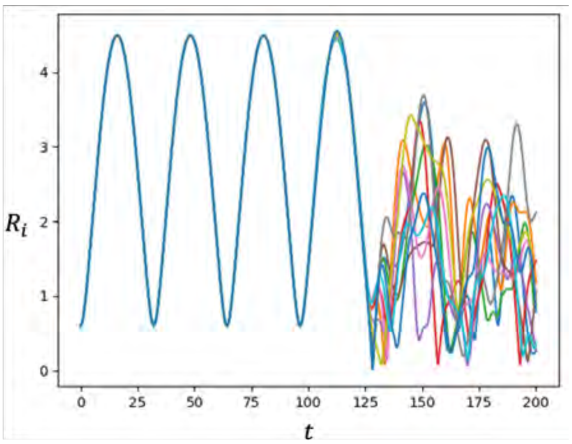

Figure 2.6: Time variation of radius for $N = 12$ and $\Delta t = 0.001$ by using (NS)

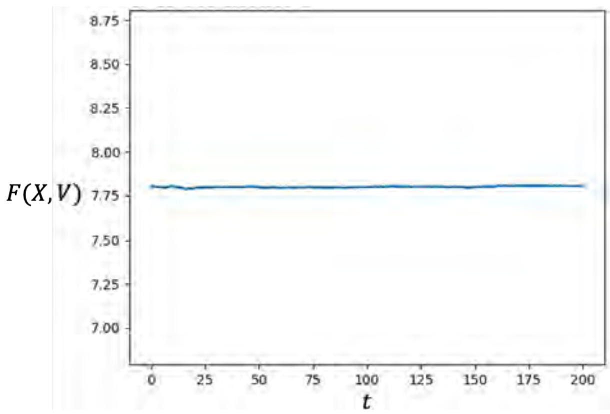

Figure 2.7: Time variation of the energy for $N = 12$ by using (NS), where the value of the vertical axis indicates $F(X, V) =$ (the value) $\times 10^{-12} + 0.99371558636$

From observation to these numerical results we infer the following remarks.

- The scheme (NS) conserves the energy F much better than (EM).

- By Proposition 2.1 the solution of (OP) must be periodic in time for spherically symmetric initial conditions. Up to time $t = 100$, the numerical solutions seem to be periodic in both cases. However, after passing $t = 125$, the solutions do not keep periodicity, in these simulations.

- From the behaviors of R_i, for $i = 1, 2, \dots, N$, it is difficult to find differences in the accuracy of the two methods, (NS) and (EM).

2.3 Multi-steps structure-preserving scheme

As mentioned in the introduction, here we give a multi-steps structure-preserving scheme (MS) with the staggered time mesh $(n + 1/2)\Delta t$ ($n = 0, 1, \dots, K$) for discretizing the variable V.

(MS): Find $X^{(n+1)}$ and $V^{(n+3/2)}$ such that

$$\frac{X_i^{(n+1)} - X_i^{(n)}}{\Delta t} = V_i^{\left(n+\frac{1}{2}\right)}, \tag{2.9}$$

$$\begin{aligned}
&\frac{V_i^{\left(n+\frac{3}{2}\right)} - V_i^{\left(n-\frac{1}{2}\right)}}{2\Delta t} \\
&= -\frac{\kappa}{4m}\left\{\varepsilon_{i-1}^{(n+1)} + \varepsilon_{i-1}^{(n)} + 1 - \frac{1}{\left(1-\varepsilon_{i-1}^{(n+1)}\right)\left(1-\varepsilon_{i-1}^{(n)}\right)}\right\} \frac{X_i^{(n+1)} - X_{i-1}^{(n+1)} + X_i^{(n)} - X_{i-1}^{(n)}}{\left|X_i^{(n+1)} - X_{i-1}^{(n+1)}\right| + \left|X_i^{(n)} - X_{i-1}^{(n)}\right|} \\
&\quad + \frac{\kappa}{4m}\left\{\varepsilon_i^{(n+1)} + \varepsilon_i^{(n)} + 1 - \frac{1}{\left(1-\varepsilon_i^{(n+1)}\right)\left(1-\varepsilon_i^{(n)}\right)}\right\} \frac{X_{i+1}^{(n+1)} - X_i^{(n+1)} + X_{i+1}^{(n)} - X_i^{(n)}}{\left|X_{i+1}^{(n+1)} - X_i^{(n+1)}\right| + \left|X_{i+1}^{(n)} - X_i^{(n)}\right|}
\end{aligned} \tag{2.10}$$

for all $n = 0, 1, \dots, K-1$ and $i = 1, 2, \dots, N$, where $X^{(n)}$, $V^{(n+1/2)}$, $V^{(n-1/2)}$, $\varepsilon^{(n)} = l_i^{(n)}/l_{N*}$, and $l_i^{(n)} = |X_{i+1}^{(n)} - X_i^{(n)}|$ are given.

Proposition 2.2. The solution of the scheme (2.9)–(2.10) satisfies

$$F\left(X^{(n)}, V^{\left(n+\frac{1}{2}\right)}, V^{\left(n-\frac{1}{2}\right)}\right) = F\left(X^{(0)}, V^{\left(\frac{1}{2}\right)}, V^{\left(-\frac{1}{2}\right)}\right) \quad (n = 0, 1, \dots, K).$$

where

$$F\left(X^{(n)}, V^{\left(n+\frac{1}{2}\right)}, V^{\left(n-\frac{1}{2}\right)}\right) := \sum_{i=1}^{N} \left\{\frac{m}{2} V_i^{\left(n+\frac{1}{2}\right)} V_i^{\left(n-\frac{1}{2}\right)} + l_{N*}\hat{f}\left(\varepsilon_i^{(n)}\right)\right\} \quad (n = 0, 1, \dots, K).$$

We can prove this proposition, easily. We omit its proof.

Remark. The above scheme is "explicit" if we compute $X^{(1)}$, $V^{(3/2)}$, $X^{(2)}$, $V^{(5/2)}$, ... in this order. By defining the above multi-step discrete energy and taking its variation based on DVDM, we have designed the explicit structure-preserving scheme for (OP).

Numerical results. In this section, we show computation examples by our multi-steps structure-preserving scheme (MS) and demonstrate that the scheme inherits the conservative property from (OP) in a discrete sense. Also, we compare (MS) with (NS).

As initial data, we consider

$$X_i^{(0)} = X_{0i} = R_0 A^i \begin{pmatrix} 1 \\ 0 \end{pmatrix} (i = 1, 2, \dots, N), \quad R_0 = 0.6,$$

$$\begin{aligned} V_i^{\left(\frac{1}{2}\right)} = V_i^{(0)} &- \frac{\kappa \Delta t}{8m} \left\{ 2\varepsilon_{i-1}^{(0)} + 1 - \frac{1}{\left(1 - \varepsilon_{i-1}^{(0)}\right)^2} \right\} \frac{X_i^{(0)} - X_{i-1}^{(0)}}{|X_i^{(0)} - X_{i-1}^{(0)}|} \\ &+ \frac{\kappa \Delta t}{8m} \left\{ 2\varepsilon_i^{(0)} + 1 - \frac{1}{\left(1 - \varepsilon_i^{(0)}\right)^2} \right\} \frac{X_{i+1}^{(0)} - X_i^{(0)}}{|X_{i+1}^{(0)} - X_i^{(0)}|} \quad (i = 1, 2, \dots, N), \end{aligned}$$

$$V_i^{(0)} = V_{0i} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (i = 1, 2, \dots, N), \text{ and } V_i^{(-1/2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (i = 1, 2, \dots, N).$$

Also, we choose $N = 12$, $\Delta t = 0.001$, $\kappa = 0.1$, $m = 5/6$, and $l_* = 4\pi$. We note that the computer language used in this section is Julia (Version 1.5.3). Figures 2.8 and 2.9 show the time variation of the radius $R_i = |X_i|$ for $i = 1, 2, \dots, N$, by (NS) and (MS), respectively.

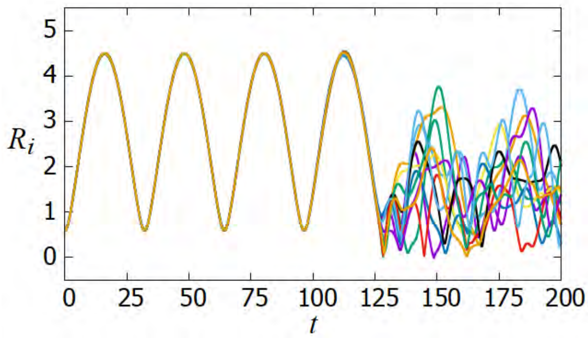

Figure 2.8: Time variation of radius for $N = 12$ by using (NS)

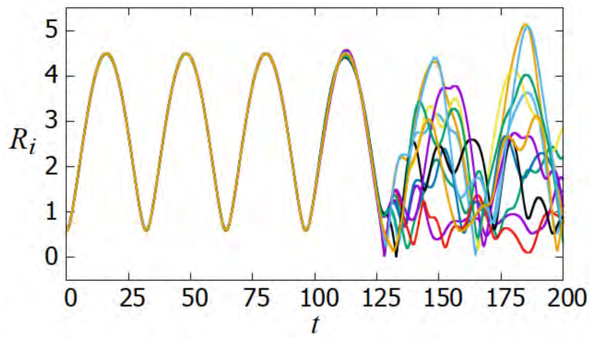

Figure 2.9: Time variation of radius for $N = 12$ by using (MS)

These figures show that the numerical solution obtained by (MS) is similar to the one by (NS) up to $t = 125$.

We note that the computation time by (NS) is 5463 seconds, i.e., about one and a half hours. On the other hand, the computation time by (MS) is 5 seconds. That is, the computation time by (MS) is much faster than that by (NS).

Next, we confirm the conservative property. Figure 2.10 shows the time variation of $M_{1\mathrm{d}}^{(n)} := F(X^{(n)}, V^{(n)}) - F(X^{(0)}, V^{(0)})$ $(n = 0, 1, \dots, K)$ calculated by (NS). Moreover, Figure 2.11 represents the time variation of $M_{2\mathrm{d}}^{(n)} := F(X^{(n)}, V^{(n+1/2)}, V^{(n-1/2)}) - F(X^{(0)}, V^{(1/2)}, V^{(-1/2)})$ $(n = 0, 1, \dots, K)$ calculated by (MS). From Theorems 2.1 and 2.2, we have $M_{1\mathrm{d}}^{(n)} = 0$ and $M_{2\mathrm{d}}^{(n)} = 0$ for all $n = 0, 1, \dots, K$.

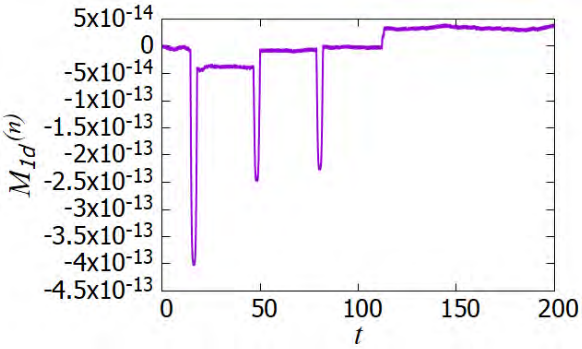

Figure 2.10: Time variation of $M_{1\mathrm{d}}^{(n)}$ for $N = 12$ by using (NS): $M_{1\mathrm{d}}^{(n)}$ does not change by about 12 orders of magnitude

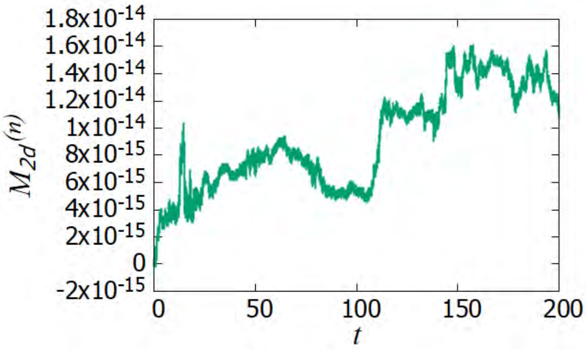

Figure 2.11: Time variation of $M_{2\mathrm{d}}^{(n)}$ for $N = 12$ by using (MS): $M_{2\mathrm{d}}^{(n)}$ does not change by about 13 orders of magnitude

These graphs show that the quantities $M_{1\mathrm{d}}^{(n)}$ and $M_{2\mathrm{d}}^{(n)}$ are conserved numerically. Besides, the scheme (MS) conserves the energy F much better than (NS).

Finally, we compare the relative errors up to $t = 100$. We denote the radii obtained by (NS) and (MS) by $R_{NS,i}$ and $R_{MS,i}$ for $i = 1, 2, \ldots, N$, respectively. Also, we denote the numerical solution to the problem $P(R_0, 0)$ by Heun's method by $R_{H,i}$ $(i = 1, 2, \ldots, N)$. Figures 2.12 and 2.13 indicate the relative errors $|R_{NS,10} - R_{H,10}|/|R_{H,10}|$ and $|R_{MS,10} - R_{H,10}|/|R_{H,10}|$, respectively.

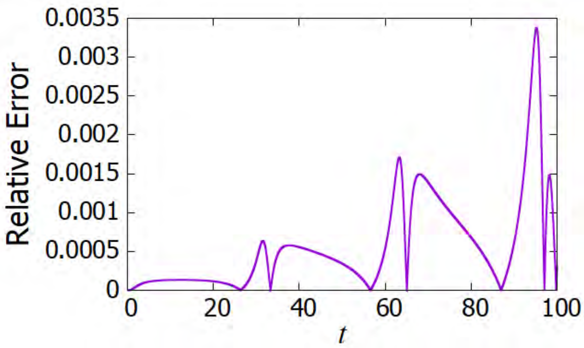

Figure 2.12: Time variation of relative error $|R_{NS,10} - R_{H,10}|/|R_{H,10}|$ by (NS)

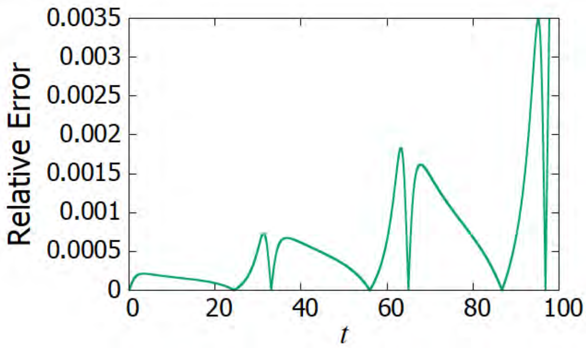

Figure 2.13: Time variation of relative error $|R_{MS,10} - R_{H,10}|/|R_{H,10}|$ by (MS)

From these figures, we can see that the relative error by (MS) is slightly larger than that by (NS) around $t = 100$, but there is no significant difference other than that. Even in the cases of $i = 0$ and $i = 5$, the relative error by (MS) is slightly larger than that by (NS) around $t = 100$. In the other cases, there is no major difference in the relative errors.

3 Numerical results for the PDE model

In this section, we derive the partial differential equation model from (OP) (see Section 1) and show the numerical results for the model.

3.1 Derivation of the PDE model

Let $M > 0$ be the total mass of the 1-dimensional elastic material occupied in the interval $[0,1]$ as a natural state. Namely, the natural length of the material l_* is given by 1. From now on, we consider the motion of the closed ring made by the material (see Figure 3.1). We denote by $u(t,x) \in \mathbb{R}^2$ the position of the point which is at $x \in [0,1]$ in the natural state for $t \in [0,T]$, $T > 0$.

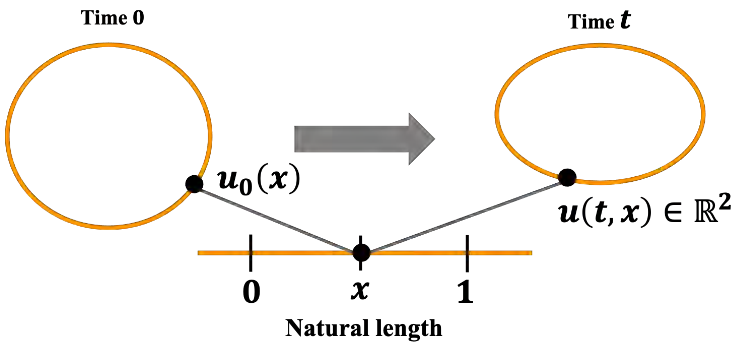

Figure 3.1: Description of the closed ring

The aim of Section 3.1 is to derive the following hyperbolic equation from (3.2) in formal calculations:

$$\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(f(\varepsilon) \frac{\frac{\partial u}{\partial x}}{\left|\frac{\partial u}{\partial x}\right|} \right) = 0, \qquad \varepsilon = \left|\frac{\partial u}{\partial x}\right| - 1, \qquad \text{in } Q(T), \tag{3.1}$$

where $\rho = \dfrac{M}{l_*}$ and $Q(T) = (0,T) \times (0,1)$.

Let $x \in (0,1)$, $t \in (0,T)$ and $N \in \mathbb{Z}_{>0}$. First, we suppose that the kinetic motion of the small part $(x - l_{N*}, x + l_{N*})$ is approximated by the similar ODE to (3.2) as follows:

$$\begin{aligned} m_N \frac{\partial^2 u(t,x)}{\partial t^2} &= f(\varepsilon_N(t,x)) \frac{u(t,x+l_{N*}) - u(t,x)}{l_N(t,x)} \\ &\quad - f(\varepsilon_N(t,x-l_{N*})) \frac{u(t,x) - u(t,x-l_{N*})}{l_N(t,x-l_{N*})} + o\left(\frac{1}{N}\right) \quad \text{for } t \in [0,T], \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &m_N = \frac{M}{N}, \quad l_N(t,x) = |u(t,x+l_{N*}) - u(t,x)|, \\ &\varepsilon_N(t,x) = \frac{l_N(t,x) - l_{N*}}{l_{N*}} \text{ and } \lim_{N\to\infty} o\left(\frac{1}{N}\right) N = 0. \end{aligned} \tag{3.3}$$

Moreover, in the argument below, we suppose that

- u is sufficiently smooth on $\overline{Q(T)}$.
- $u_x \neq 0$ on $\overline{Q(T)}$, $l_N > 0$ on $\overline{Q(T)}$ for each $N \in \mathbb{Z}_{>0}$.

- $f \in C^2((-1,\infty))$ and $\lim_{\varepsilon\downarrow -1} f(\varepsilon) = -\infty$.

The derivation for (3.1) is rather long. So, we divide it into several steps in the following. We fix $(t,x) \in Q(T)$ and sometimes omit it.

1st step. $\varepsilon_N \to \varepsilon$ as $N \to \infty$.

Proof of 1st step. Put $u = (u_1, u_2)$. Easily, we have

$$\begin{aligned}\varepsilon_N(t,x) &= \sqrt{\left(\frac{u_1(t,x+l_{N*}) - u_1(t,x)}{l_{N*}}\right)^2 + \left(\frac{u_2(t,x+l_{N*}) - u_2(t,x)}{l_{N*}}\right)^2} - 1 \\ &\to \sqrt{\left(\frac{\partial u_1(t,x)}{\partial x}\right)^2 + \left(\frac{\partial u_2(t,x)}{\partial x}\right)^2} - 1 = \left|\frac{\partial u(t,x)}{\partial x}\right| - 1 \\ &\qquad\qquad = \varepsilon(t,x) \quad \text{as } N \to \infty. \quad \square\end{aligned}$$

By dividing (3.2) by l_{N*}, we have

$$\begin{aligned}\rho\frac{\partial^2 u(t,x)}{\partial t^2} &= \frac{1}{l_{N*}}\left\{f(\varepsilon_N(t,x))\frac{u(t,x+l_{N*}) - u(t,x)}{l_N(t,x)}\right. \\ &\qquad \left. -f(\varepsilon_N(t,x-l_{N*}))\frac{u(t,x) - u(t,x-l_{N*})}{l_N(t,x-l_{N*})}\right\} + o\left(\frac{1}{N}\right)\frac{1}{l_{N*}},\end{aligned}$$

and for each $N \in \mathbb{Z}_{>0}$, we put

$$\begin{aligned}\mathrm{I}_N = \frac{1}{l_{N*}}&\left\{f(\varepsilon_N(t,x))\frac{u(t,x+l_{N*}) - u(t,x)}{l_N(t,x)}\right. \\ &\left. -f(\varepsilon_N(t,x-l_{N*}))\frac{u(t,x) - u(t,x-l_{N*})}{l_N(t,x-l_{N*})}\right\} + o\left(\frac{1}{N}\right)\frac{N}{l_*}.\end{aligned}$$

Clearly, it is sufficient to show the following convergence:

$$\mathrm{I}_N \to \frac{\partial}{\partial x}\left(f(\varepsilon)\frac{\frac{\partial u}{\partial x}}{\left|\frac{\partial u}{\partial x}\right|}\right) \quad \text{as } N \to \infty. \tag{3.4}$$

For proving (3.4) we see that

$$\begin{aligned}&\left|\mathrm{I}_N - \frac{\partial}{\partial x}\left(f(\varepsilon(t,x))\frac{\frac{\partial u}{\partial x}}{\left|\frac{\partial u}{\partial x}\right|}\right)\right| \\ &\le \left|\mathrm{I}_N - \left(\frac{f(\varepsilon(t,x+l_{N*}))\frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|} - f(\varepsilon(t,x))\frac{u_x(t,x)}{|u_x(t,x)|}}{l_{N*}}\right)\right| \\ &\quad + \left|\left(\frac{f(\varepsilon(t,x+l_{N*}))\frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|} - f(\varepsilon(t,x))\frac{u_x(t,x)}{|u_x(t,x)|}}{l_{N*}}\right) - \frac{\partial}{\partial x}\left(f(\varepsilon(t,x))\frac{u_x(t,x)}{|u_x(t,x)|}\right)\right| \\ &=: \mathrm{I}_{1N} + \mathrm{I}_{2N} \qquad \text{for } N \in \mathbb{Z}_{>0}.\end{aligned}$$

By the assumptions for differentiability of ε and $u_x(t,x) \neq 0$, we easily get

$$\mathrm{I}_{2N} \to 0 \quad \text{as } N \to \infty.$$

Hence, we have to prove the following convergence:

$$\mathrm{I}_{1N} \to 0 \quad \text{as } N \to \infty.$$

By elementary calculations, we have

$$\begin{aligned}
\mathrm{I}_{1N} \;\leq\; & \frac{1}{l_{N*}} \left| f(\varepsilon_N(t,x)) \left\{ \frac{u(t,x+l_{N*}) - u(t,x)}{l_N(t,x)} - \frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|} \right\} \right. \\
& \left. - \left\{ f(\varepsilon_N(t,x-l_{N*})) \left\{ \frac{u(t,x) - u(t,x-l_{N*})}{l_N(t,x-l_{N*})} + \frac{u_x(t,x)}{|u_x(t,x)|} \right\} \right\} \right| \\
& + \frac{1}{l_{N*}} \left| \{ f(\varepsilon_N(t,x)) - f(\varepsilon(t,x+l_{N*})) \} \frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|} \right. \\
& \left. - \{ f(\varepsilon_N(t,x-l_{N*})) - f(\varepsilon(t,x)) \} \frac{u_x(t,x)}{|u_x(t,x)|} \right| + o\left(\frac{1}{N}\right) \frac{N}{l_*}.
\end{aligned}$$

By (3.3), we see that $o\left(\frac{1}{N}\right)\frac{N}{l_*} \to 0$ as $N \to \infty$. We put

$$\begin{aligned}
\mathrm{J}_{1N}^{(1)} \;=\; & \frac{1}{l_{N*}} \left| f(\varepsilon_N(t,x)) \left\{ \frac{u(t,x+l_{N*}) - u(t,x)}{l_N(t,x)} - \frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|} \right\} \right. \\
& \left. - \left\{ f(\varepsilon_N(t,x-l_{N*})) \left\{ \frac{u(t,x) - u(t,x-l_{N*})}{l_N(t,x-l_{N*})} + \frac{u_x(t,x)}{|u_x(t,x)|} \right\} \right\} \right|, \\
\mathrm{J}_{1N}^{(2)} \;=\; & \frac{1}{l_{N*}} \left| \{ f(\varepsilon_N(t,x)) - f(\varepsilon(t,x+l_{N*})) \} \frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|} \right. \\
& \left. - \{ f(\varepsilon_N(t,x-l_{N*})) - f(\varepsilon(t,x)) \} \frac{u_x(t,x)}{|u_x(t,x)|} \right| \quad \text{for } N \in \mathbb{Z}_{>0}.
\end{aligned}$$

2nd step. $J_{1N}^{(1)}(t,x) \to 0$ as $N \to \infty$.
Proof of 2nd step. Immediately, we see that

$$\mathrm{J}_{1N}^{(1)} \;\leq\; \frac{|f(\varepsilon_N(t,x))|}{l_{N*}} \left| \mathrm{A}_{1N}^{(1)} - \mathrm{B}_{1N}^{(1)} \right| + \frac{1}{l_{N*}} \left| \{ f(\varepsilon_N(t,x)) - f(\varepsilon_N(t,x-l_{N*})) \} \mathrm{B}_{1N}^{(1)} \right|, \quad (3.5)$$

where

$$\begin{aligned}
\mathrm{A}_{1N}^{(1)} &= \frac{u(t,x+l_{N*}) - u(t,x)}{l_N(t,x)} - \frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|}, \\
\mathrm{B}_{1N}^{(1)} &= \frac{u(t,x) - u(t,x-l_{N*})}{l_N(t,x-l_{N*})} + \frac{u_x(t,x)}{|u_x(t,x)|}.
\end{aligned}$$

It is clear that

$$\mathrm{A}_{1N}^{(1)} \;=\; \frac{\frac{u(t,x+l_{N*})-u(t,x)}{l_{N*}}}{\frac{l_N(t,x)}{l_{N*}}} - \frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|}$$

$$= \frac{\frac{u(t,x+l_{N*})-u(t,x)}{l_{N*}}\left\{|u_x(t,x+l_{N*})| - \left|\frac{u(t,x+l_{N*})-u(t,x)}{l_{N*}}\right|\right\}}{\left|\frac{u(t,x+l_{N*})-u(t,x)}{l_{N*}}\right| |u_x(t,x+l_{N*})|} + \frac{\frac{u(t,x+l_{N*})-u(t,x)}{l_{N*}} - u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|}.$$

By putting

$$\begin{aligned}
\mathrm{A}_{1N}^{(1,1)} &= \frac{u(t,x+l_{N*})-u(t,x)}{l_{N*}}, \\
\mathrm{A}_{1N}^{(1,2)} &= \left|\frac{u(t,x+l_{N*})-u(t,x)}{l_{N*}}\right| |u_x(t,x+l_{N*})|, \\
\mathrm{A}_{1N}^{(1,3)} &= |u_x(t,x+l_{N*})| - \left|\frac{u(t,x+l_{N*})-u(t,x)}{l_{N*}}\right|, \\
\mathrm{A}_{1N}^{(1,4)} &= \frac{u(t,x+l_{N*})-u(t,x)}{l_{N*}} - u_x(t,x+l_{N*}), \\
\mathrm{A}_{1N}^{(1,5)} &= |u_x(t,x+l_{N*})|,
\end{aligned}$$

we have

$$\mathrm{A}_{1N}^{(1)} = \frac{\mathrm{A}_{1N}^{(1,1)}\mathrm{A}_{1N}^{(1,3)}}{\mathrm{A}_{1N}^{(1,2)}} + \frac{\mathrm{A}_{1N}^{(1,4)}}{\mathrm{A}_{1N}^{(1,5)}}.$$

Similarly, we have

$$\mathrm{B}_{1N}^{(1)} = \frac{\frac{u(t,x)-u(t,x-l_{N*})}{l_{N*}}\left\{|u_x(t,x)| - \left|\frac{u(t,x)-u(t,x-l_{N*})}{l_{N*}}\right|\right\}}{\left|\frac{u(t,x)-u(t,x-l_{N*})}{l_{N*}}\right| |u_x(t,x)|} + \frac{\frac{u(t,x)-u(t,x-l_{N*})}{l_{N*}} - u_x(t,x)}{|u_x(t,x)|}.$$

By putting

$$\begin{aligned}
\mathrm{B}_{1N}^{(1,1)} &= \frac{u(t,x)-u(t,x-l_{N*})}{l_{N*}}, \\
\mathrm{B}_{1N}^{(1,2)} &= \left|\frac{u(t,x)-u(t,x-l_{N*})}{l_{N*}}\right| |u_x(t,x)|, \\
\mathrm{B}_{1N}^{(1,3)} &= |u_x(t,x)| - \left|\frac{u(t,x)-u(t,x-l_{N*})}{l_{N*}}\right|, \\
\mathrm{B}_{1N}^{(1,4)} &= \frac{u(t,x)-u(t,x-l_{N*})}{l_{N*}} - u_x(t,x), \\
\mathrm{B}_{1N}^{(1,5)} &= |u_x(t,x)|,
\end{aligned}$$

we have

$$\mathrm{B}_{1N}^{(1)} = \frac{\mathrm{B}_{1N}^{(1,1)}\mathrm{B}_{1N}^{(1,3)}}{\mathrm{B}_{1N}^{(1,2)}} + \frac{\mathrm{B}_{1N}^{(1,4)}}{\mathrm{B}_{1N}^{(1,5)}}. \tag{3.6}$$

Thanks to 1st step, there exist positive constants C_1 and C_2 such that

$$|\varepsilon_N(t,x)| \leq C_1, \quad |f(\varepsilon_N(t,x))| \leq C_2 \quad \text{for } N = 1, 2, \ldots. \tag{3.7}$$

By using (3.7), we can estimate the first term in the right hand side of (3.5) as follows

$$\begin{aligned}
& \frac{f(\varepsilon_N(t,x))}{l_{N*}} \left| \mathrm{A}_{1N}^{(1)} - \mathrm{B}_{1N}^{(1)} \right| \\
\leq \ & \frac{C_2}{l_{N*}} \left| \mathrm{A}_{1N}^{(1)} - \mathrm{B}_{1N}^{(1)} \right| \\
\leq \ & C_2 \left| \frac{1}{l_{N*}} \left\{ \frac{\mathrm{A}_{1N}^{(1,1)} \mathrm{A}_{1N}^{(1,3)}}{\mathrm{A}_{1N}^{(1,2)}} + \frac{\mathrm{A}_{1N}^{(1,4)}}{\mathrm{A}_{1N}^{(1,5)}} \right\} - \frac{1}{l_{N*}} \left\{ \frac{\mathrm{B}_{1N}^{(1,1)} \mathrm{B}_{1N}^{(1,3)}}{\mathrm{B}_{1N}^{(1,2)}} + \frac{\mathrm{B}_{1N}^{(1,4)}}{\mathrm{B}_{1N}^{(1,5)}} \right\} \right|.
\end{aligned} \tag{3.8}$$

By using Taylor's series expansion, we can take $\theta_{1N} \in [0,1]$ such that

$$u(t,x) = u(t, x + l_{N*}) - l_{N*} u_x(t, x + l_{N*}) + \frac{l_{N*}^2}{2} u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*}). \tag{3.9}$$

We note that

$$\mathrm{A}_{1N}^{(1,3)} = |u_x(t, x + l_{N*})| - \left| u_x(t, x + l_{N*}) - \frac{l_{N*}}{2} u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*}) \right|,$$

and by putting

$$\begin{aligned}
\mathrm{A}_{1N}^{(1,3,1)} &= \sqrt{|u_x(t, x + l_{N*})|^2}, \\
\mathrm{A}_{1N}^{(1,3,2,1)} &= \left(u_x(t, x + l_{N*}), u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*}) \right), \\
\mathrm{A}_{1N}^{(1,3,2)} &= \sqrt{|u_x(t, x + l_{N*})|^2 - l_{N*} \mathrm{A}_{1N}^{(1,3,2,1)} + \frac{l_{N*}^2}{4} |u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})|^2},
\end{aligned}$$

we have

$$\begin{aligned}
\mathrm{A}_{1N}^{(1,3)} &= \frac{\left(\mathrm{A}_{1N}^{(1,3,1)} - \mathrm{A}_{1N}^{(1,3,2)} \right) \left(\mathrm{A}_{1N}^{(1,3,1)} + \mathrm{A}_{1N}^{(1,3,2)} \right)}{\mathrm{A}_{1N}^{(1,3,1)} + \mathrm{A}_{1N}^{(1,3,2)}} \\
&= -\frac{1}{\mathrm{A}_{1N}^{(1,3,1)} + \mathrm{A}_{1N}^{(1,3,2)}} \left\{ l_{N*} \mathrm{A}_{1N}^{(1,3,2,1)} + \frac{l_{N*}^2}{4} |u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})|^2 \right\}.
\end{aligned}$$

By using this equation, we can deal with the first term in the right hand side of (3.8) as follows:

$$\begin{aligned}
& \frac{1}{l_{N*}} \frac{\mathrm{A}_{1N}^{(1,1)} \mathrm{A}_{1N}^{(1,3)}}{\mathrm{A}_{1N}^{(1,2)}} \\
&= -\frac{\mathrm{A}_{1N}^{(1,1)}}{\mathrm{A}_{1N}^{(1,2)}} \frac{1}{\mathrm{A}_{1N}^{(1,3,1)} + \mathrm{A}_{1N}^{(1,3,2)}} \\
& \quad \times \left\{ \left(u_x(t, x + l_{N*}), u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*}) \right) + \frac{l_{N*}}{4} |u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*})|^2 \right\}.
\end{aligned}$$

It is obvious that

$$\frac{1}{l_{N*}}\frac{\mathrm{A}_{1N}^{(1,1)}\mathrm{A}_{1N}^{(1,3)}}{\mathrm{A}_{1N}^{(1,2)}} \to -\frac{u_x(t,x)}{2\left|u_x(t,x)\right|^3}\left(u_x(t,x),u_{xx}(t,x)\right) \quad \text{as } N\to\infty. \tag{3.10}$$

Similarly, by (3.9), we have

$$\mathrm{A}_{1N}^{(1,4)} = -\frac{l_{N*}}{2}u_{xx}(t,x+l_{N*}-\theta_{1N}l_{N*}),$$

and

$$\begin{aligned}\frac{1}{l_{N*}}\frac{\mathrm{A}_{1N}^{(1,4)}}{\mathrm{A}_{1N}^{(1,5)}} &= -\frac{u_{xx}(t,x+l_{N*}-\theta_{1N}l_{N*})}{2\left|u_x(t,x+l_{N*})\right|}\\ &\to -\frac{u_{xx}(t,x)}{2\left|u_x(t,x)\right|} \quad \text{as } N\to\infty.\end{aligned} \tag{3.11}$$

Moreover, by a similar argument as above, we obtain the following convergences.

$$\frac{1}{l_{N*}}\frac{\mathrm{B}_{1N}^{(1,1)}\mathrm{B}_{1N}^{(1,3)}}{\mathrm{B}_{1N}^{(1,2)}} \to -\frac{u_x(t,x)}{2\left|u_x(t,x)\right|^3}\left(u_x(t,x),u_{xx}(t,x)\right) \quad \text{as } N\to\infty, \tag{3.12}$$

$$\frac{1}{l_{N*}}\frac{\mathrm{B}_{1N}^{(1,4)}}{\mathrm{B}_{1N}^{(1,5)}} \to -\frac{u_{xx}(t,x)}{2\left|u_x(t,x)\right|} \quad \text{as } N\to\infty. \tag{3.13}$$

By using (3.10)–(3.13), we obtain

$$\begin{aligned}&\frac{|f(\varepsilon_N(t,x))|}{l_{N*}}\left|\mathrm{A}_{1N}^{(1)}-\mathrm{B}_{1N}^{(1)}\right|\\ &\le C_2\left|\frac{1}{l_{N*}}\left\{\frac{\mathrm{A}_{1N}^{(1,1)}\mathrm{A}_{1N}^{(1,3)}}{\mathrm{A}_{1N}^{(1,2)}}+\frac{\mathrm{A}_{1N}^{(1,4)}}{\mathrm{A}_{1N}^{(1,5)}}\right\}-\frac{1}{l_{N*}}\left\{\frac{\mathrm{B}_{1N}^{(1,1)}\mathrm{B}_{1N}^{(1,3)}}{\mathrm{B}_{1N}^{(1,2)}}+\frac{\mathrm{B}_{1N}^{(1,4)}}{\mathrm{B}_{1N}^{(1,5)}}\right\}\right| \to 0 \text{ as } N\to\infty.\end{aligned} \tag{3.14}$$

Here, we deal with in the right hand side of (3.5) as follows:

$$\begin{aligned}&\frac{1}{l_{N*}}\left|f(\varepsilon_N(t,x))-f(\varepsilon_N(t,x-l_{N*}))\right|\left|\mathrm{B}_{1N}^{(1)}\right|\\ &=\left|f(\varepsilon_N(t,x))-f(\varepsilon(t,x))+f(\varepsilon(t,x))-f(\varepsilon_N(t,x-l_{N*}))\right|\left|\frac{\mathrm{B}_{1N}^{(1)}}{l_{N*}}\right|.\end{aligned}$$

By using (3.6), (3.12) and (3.13), we have

$$\left|\frac{\mathrm{B}_{1N}^{(1)}}{l_{N*}}\right| \to \frac{1}{2\left|u_x(t,x)\right|}\left|\frac{u_x(t,x)}{\left|u_x(t,x)\right|^2}\left(u_x(t,x),u_{xx}(t,x)\right)+u_{xx}(t,x)\right| \quad \text{as } N\to\infty.$$

Since $f(\varepsilon_N)\to f(\varepsilon)$ as $N\to\infty$, it holds that

$$\frac{1}{l_{N*}}\left|f(\varepsilon_N(t,x))-f(\varepsilon_N(t,x-l_{N*}))\right|\left|\mathrm{B}_{1N}^{(1)}\right| \to 0 \quad \text{as } N\to\infty. \tag{3.15}$$

Hence, by (3.14) and (3.15), we obtain

$$\mathrm{J}_{1N}^{(1)} \to 0 \quad \text{as } N \to \infty.$$

□

3rd step. $J_{1N}^{(2)} \to 0$ as $N \to \infty$.
Proof of 3rd step. First, we see that

$$\begin{aligned}\mathrm{J}_{1N}^{(2)} \;\leq\; & \frac{1}{l_{N*}} \left| f(\varepsilon_N(t,x)) - f(\varepsilon(t,x+l_{N*})) \right| \left| \frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|} - \frac{u_x(t,x)}{|u_x(t,x)|} \right| \\ & + \frac{1}{l_{N*}} \left| \{ f(\varepsilon_N(t,x)) - f(\varepsilon(t,x+l_{N*})) - f(\varepsilon_N(t,x-l_{N*}) + f(\varepsilon(t,x)) \} \frac{u_x(t,x)}{|u_x(t,x)|} \right| \\ & =: \mathrm{J}_{1N}^{(2,1)} + \mathrm{J}_{1N}^{(2,2)}.\end{aligned}$$

Since, f is locally Lipschitz continuous on $(-1,\infty)$, we can take a positive constant C_f such that

$$|f(\varepsilon_N(t,x)) - f(\varepsilon(t,x+l_{N*}))| \leq C_f \left| \varepsilon_N(t,x) - \varepsilon(t,x+l_{N*}) \right| \text{ for } N \in \mathbb{Z}_{>0}.$$

By using this inequality and (3.3), we obtain

$$\begin{aligned}\mathrm{J}_{1N}^{(2,1)} \;\leq\; & \frac{C_f}{l_{N*}} \left| \varepsilon_N(t,x) - \varepsilon(t,x+l_{N*}) \right| \left| \frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|} - \frac{u_x(t,x)}{|u_x(t,x)|} \right| \\ \leq\; & \frac{C_f}{l_{N*}} \left| \frac{1}{l_{N*}} \{ u(t,x+l_{N*}) - u(t,x) \} - u_x(t,x+l_{N*}) \right| \left| \frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|} - \frac{u_x(t,x)}{|u_x(t,x)|} \right|.\end{aligned}$$

Here, by using (3.9), we have

$$\mathrm{J}_{1N}^{(2,1)} \leq \frac{C_f}{2} \left| u_{xx}(t,x+l_{N*} - \theta_{1N} l_{N*}) \right| \left| \frac{u_x(t,x+l_{N*})}{|u_x(t,x+l_{N*})|} - \frac{u_x(t,x)}{|u_x(t,x)|} \right|.$$

Moreover, since we assume that u_{xx} is continuous, there exists a positive constant C such that

$$|u_{xx}(t,x+l_{N*} - \theta_{1N} l_{N*})| \leq C \quad \text{for all } N = 1,2,\ldots. \tag{3.16}$$

By $u_x \in C(\overline{Q(T)})$ and $u_x \neq 0$ on $\overline{Q(T)}$, we obtain

$$\mathrm{J}_{1N}^{(2,1)} \to 0 \quad \text{as } N \to \infty.$$

Next, we shall show the convergence of $\mathrm{J}_{1N}^{(2,2)}$. By using Taylor's series expansion for $f(\varepsilon_N)$ we can choose $\theta_{2N}, \theta_{3N} \in [0,1]$ such that

$$\begin{aligned}& f(\varepsilon_N(t,x)) = f(\varepsilon(t,x+l_{N*})) + \\ & \quad \{ \varepsilon_N(t,x) - \varepsilon(t,x+l_{N*}) \} f' \left(\varepsilon(t,x+l_{N*}) + \theta_{2N} \{ \varepsilon_N(t,x-l_{N*}) - \varepsilon(t,x+l_{N*}) \} \right),\end{aligned}$$

$$
\begin{aligned}
f(\varepsilon_N(t, x - l_{N*})) &= f(\varepsilon(t,x)) \\
&\quad + \{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t,x)\} f'\left(\varepsilon(t,x) + \theta_{3N}\{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t,x)\}\right).
\end{aligned}
$$

Here, by using (3.3) and (3.9), we have

$$
\varepsilon_N(t,x) - \varepsilon(t, x + l_{N*}) = \left| u_x(t, x + l_{N*}) - \frac{l_{N*}}{2} u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*}) \right| - |u_x(t, x + l_{N*})|.
$$

Also, by using Taylor's series expansion for $u(t, x - l_{N*})$, there exists $\theta_{4N} \in [0,1]$ such that

$$
\varepsilon_N(t, x - l_{N*}) - \varepsilon(t,x) = \left| u_x(t,x) - \frac{l_{N*}}{2} u_{xx}(t, x - \theta_{4N} l_{N*}) \right| - |u_x(t,x)|.
$$

By putting

$$
\begin{aligned}
\mathrm{C}_N &= \left| u_x(t, x + l_{N*}) - \frac{l_{N*}}{2} u_{xx}(t, x + l_{N*} - \theta_{1N} l_{N*}) \right| - |u_x(t, x + l_{N*})|, \\
\mathrm{D}_N &= f'\left(\varepsilon(t, x + l_{N*}) + \theta_{2N}\{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x + l_{N*})\}\right), \\
\mathrm{E}_N &= \left| u_x(t,x) - \frac{l_{N*}}{2} u_{xx}(t, x - \theta_{4N} l_{N*}) \right| - |u_x(t,x)|, \\
\mathrm{F}_N &= f'\left(\varepsilon(t,x) + \theta_{3N}\{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t,x)\}\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathrm{J}_{1N}^{(2,2)} &= \frac{1}{l_{N*}} |\mathrm{C}_N \mathrm{D}_N - \mathrm{E}_N \mathrm{F}_N| \\
&\le \frac{1}{l_{N*}} |\mathrm{C}_N| \, |\mathrm{D}_N - \mathrm{F}_N| + \frac{1}{l_{N*}} |\mathrm{C}_N - \mathrm{E}_N| \, |\mathrm{F}_N|.
\end{aligned}
$$

Since f' is locally Lipschitz continuous on $(-1, \infty)$ and $\varepsilon > 0$ on $\overline{Q(T)}$, there exists a positive constant C'_f such that

$$
\begin{aligned}
|\mathrm{D}_N - \mathrm{F}_N| \le C'_f \, &|\varepsilon(t, x + l_{N*}) + \theta_{2N}\{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x + l_{N*})\} \\
&- \{\varepsilon(t,x) + \theta_{3N}\{\varepsilon_N(t, x - l_{N*}) - \varepsilon(t,x)\}\}| \qquad \text{for all } N = 1, 2, \ldots.
\end{aligned}
$$

By using this inequality and (3.16), we have

$$
\begin{aligned}
&\frac{1}{l_{N*}} |\mathrm{C}_N| \, |\mathrm{D}_N - \mathrm{F}_N| \\
\le\ & \frac{C'_f C}{2} \{|\varepsilon(t, x + l_{N*}) - \varepsilon(t,x)| + \theta_{2N} |\varepsilon_N(t, x - l_{N*}) - \varepsilon(t, x + l_{N*})| \\
&\qquad + \theta_{3N} |\varepsilon_N(t, x - l_{N*}) - \varepsilon(t,x)|\} \qquad \text{for all } N = 1, 2, \ldots.
\end{aligned}
$$

Thanks to 1st step, we obtain the following convergence:

$$
\frac{1}{l_{N*}} |\mathrm{C}_N| \, |\mathrm{D}_N - \mathrm{F}_N| \to 0 \quad \text{as } N \to \infty.
$$

Since f' is continuous on $(-1,\infty)$, the 1st step guarantees existence of a positive constant C' such that

$$|f'\left(\varepsilon(t,x)+\theta_{3N}\left\{\varepsilon_N(t,x-l_{N*})-\varepsilon(t,x)\right\}\right)|\le C' \quad \text{for all } N=1,2,\dots.$$

On account of this boundedness, we have

$$\begin{aligned}
&\frac{1}{l_{N*}}\left|\mathrm{C}_N-\mathrm{E}_N\right|\left|\mathrm{F}_N\right| \\
&\le \frac{C'}{l_{N*}}\left|\mathrm{C}_N^{(1)}-\mathrm{C}_N^{(2)}-\left(\mathrm{E}_N^{(1)}-\mathrm{E}_N^{(2)}\right)\right| \\
&= C'\left|\frac{\mathrm{C}_N^{(1,2)}+\frac{l_{N*}}{4}\left|u_{xx}(t,x+l_{N*}-\theta_{1N}l_{N*})\right|^2}{\mathrm{C}_N^{(1)}+\mathrm{C}_N^{(2)}}-\frac{\mathrm{E}_N^{(1,2)}+\frac{l_{N*}}{4}\left|u_{xx}(t,x+l_{N*}-\theta_{N4}l_{N*})\right|^2}{\mathrm{E}_N^{(1)}+E_N^{(2)}}\right|,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{C}_N^{(1)} &= \sqrt{\left|u_x(t,x+l_{N*})\right|^2+l_{N*}\mathrm{C}_N^{(1,2)}+\frac{l_{N*}^2}{4}\left|u_{xx}(t,x+l_{N*}-\theta_{1N}l_{N*})\right|^2}, \\
\mathrm{C}_N^{(1,2)} &= \left(u_x(t,x+l_{N*}),u_{xx}(t,x+l_{N*}-\theta_{1N}l_{N*})\right), \\
\mathrm{C}_N^{(2)} &= \sqrt{\left|u_x(t,x+l_{N*})\right|^2}, \\
\mathrm{E}_N^{(1)} &= \sqrt{\left|u_x(t,x)\right|^2+l_{N*}\mathrm{E}_N^{(1,2)}+\frac{l_{N*}^2}{4}\left|u_{xx}(t,x+l_{N*}-\theta_{4N}l_{N*}))\right|^2}, \\
\mathrm{E}_N^{(1,2)} &= \left(u_x(t,x),u_{xx}(t,x-\theta_{4N}l_{N*})\right), \\
\mathrm{E}_N^{(2)} &= \sqrt{\left|u_x(t,x+l_{N*})\right|^2}.
\end{aligned}$$

Now, the following convergences hold:

$$\begin{aligned}
\frac{\mathrm{C}_N^{(1,2)}+\frac{l_{N*}}{4}\left|u_{xx}(t,x+l_{N*}-\theta_{1N}l_{N*})\right|^2}{\mathrm{C}_N^{(1)}+\mathrm{C}_N^{(2)}} &\to \frac{(u_x(t,x),u_{xx}(t,x))}{2\left|u_x(t,x)\right|}, \\
\frac{\mathrm{E}_N^{(1,2)}+\frac{l_{N*}}{4}\left|u_{xx}(t,x+l_{N*}-\theta_{N4}l_{N*})\right|^2}{\mathrm{E}_N^{(1)}+\mathrm{E}_N^{(2)}} &\to \frac{(u_x(t,x),u_{xx}(t,x))}{2\left|u_x(t,x)\right|} \quad \text{as } N\to\infty.
\end{aligned}$$

Thus, by using these convergences, we obtain

$$\frac{1}{l_{N*}}\left|\mathrm{C}_N-\mathrm{E}_N\right|\left|\mathrm{F}_N\right|\to 0 \quad \text{as } N\to\infty.$$

Namely, it holds that $\mathrm{J}_{1N}^{(2,2)}\to 0$ as $N\to\infty$, and $\mathrm{J}_{1N}^{(2)}\to 0$ as $N\to\infty$. □

Hence, we have proved (3.4). Therefore, we obtain (3.1) as the kinetic equation.

It is not easy to solve (3.1), since the coefficient of u_x is nonlinear and singular. Accordingly, in order to deal with the equation, mathematically, we add the fourth derivative term to (3.1) as follows:

$$\rho\frac{\partial^2 u}{\partial t^2}+\gamma\frac{\partial^4 u}{\partial x^4}-\frac{\partial}{\partial x}\left(f(\varepsilon)\frac{\frac{\partial u}{\partial x}}{\left|\frac{\partial u}{\partial x}\right|}\right)=0, \varepsilon=\left|\frac{\partial u}{\partial x}\right|-1 \text{ on } Q(T), \tag{3.17}$$

The equation (3.17) is called a beam equation, and appears when we approximate the motion of a 3-dimensional elastic material by the 1-dimensional model. Also, in [2] the fourth derivative term is regarded as a description of non-local effects as induced by interfacial energy. For simplicity, we put $f_0(\varepsilon) = f(\varepsilon)/(1+\varepsilon)$ and write f_0 by f. Moreover, we impose the periodic boundary condition (3.19) representing the material that is smoothly connected at $x = 0, 1$, and the initial condition (3.20). Thus, we obtain the following initial and boundary value problem (PP) for partial differential equation.

Problem (PP):

$$\rho\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial^4 u}{\partial x^4} - \frac{\partial}{\partial x}\left(f(\varepsilon)\frac{\partial u}{\partial x}\right) = 0, \varepsilon = \left|\frac{\partial u}{\partial x}\right| - 1 \text{ on } Q(T), \tag{3.18}$$

$$\frac{\partial^i u}{\partial x^i}(t,0) = \frac{\partial^i u}{\partial x^i}(t,1) \text{ for all } t \in [0,T] \text{ and } i = 0,1,2,3, \tag{3.19}$$

$$u(0,x) = u_0(x), \frac{\partial u}{\partial x}(0,x) = v_0(x) \text{ for all } x \in [0,1]. \tag{3.20}$$

3.2 Numerical results

In this section, we show numerical results for (PP) on the domain $(0,T) \times (0, l_*)$. The purpose of our numerical simulation is to clarify the role of the term γu_{xxxx} in (3.18). For this purpose, we provide graphs of approximate solutions in varying the parameter γ.

We choose a grid $x_j = (j-1)\Delta x$, $j = 1, 2, \ldots, N$, with $\Delta x = l_*/N$ and we denote the approximation of $u(t, x_j)$ by $u_j(t)$. The fourth derivative in (3.18) is approximated using

$$\begin{aligned}\frac{\partial^4 u}{\partial x^4}(t,x_j) &\approx \frac{u_{xx}(t,x_{j+1}) - 2u_{xx}(t,x_j) + u_{xx}(t,x_{j-1})}{(\Delta x)^2} \\ &\approx \frac{\frac{u_{j+2}-2u_{j+1}+u_j}{(\Delta x)^2} - 2\frac{u_{j+1}-2u_j+u_{j-1}}{(\Delta x)^2} + \frac{u_j-2u_{j-1}+u_{j-2}}{(\Delta x)^2}}{(\Delta x)^2} \\ &= \frac{u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}}{(\Delta x)^4}.\end{aligned} \tag{3.21}$$

The other spatial derivative in (3.18) is approximated using central differences, so

$$\frac{\partial}{\partial x}\left(f(\varepsilon)\frac{\partial u}{\partial x}\right)(t,x_j) \approx \frac{\left(f(\varepsilon)\frac{\partial u}{\partial x}\right)(t,x_{j+1}) - \left(f(\varepsilon)\frac{\partial u}{\partial x}\right)(t,x_{j-1})}{2\Delta x}, \tag{3.22}$$

$$\frac{\partial u}{\partial x}(t,x_j) \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x}. \tag{3.23}$$

The approximation (3.23) is also used for the computation of ε. We have now discretized the PDE (3.18) in space.

We introduce

$$v(t,x) = \frac{\partial u}{\partial t}(t,x),$$

and denote the approximation of $v(t,x_j)$ by $v_j(t)$. Substituting the approximations (3.21)–(3.23) into (3.18), we find the first order ODE system ($j = 1, 2, \ldots, N$)

$$u_j' = v_j, \tag{3.24}$$

$$v_j' = \frac{1}{\rho}\left(\frac{-\gamma}{(\Delta x)^4}\left(u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}\right)\right.$$
$$\left. + \frac{f(\varepsilon_{j+1})(u_{j+2} - u_j) - f(\varepsilon_{j-1})(u_j - u_{j-2})}{4(\Delta x)^2}\right). \tag{3.25}$$

The periodic boundary computations (3.19) are implemented as $u_{-1} = u_{N-1}$, $u_0 = u_N$, $u_{N+1} = u_1$, $u_{N+2} = u_2$.

The ODE system can be solved using standard methods for time integration. In the numerical experiments below we have used Matlab's `ode45` routine which uses the explicit Runge-Kutta (4,5) pair.

Here, we list the values of parameters: $N = 12$, $l_* = 2\pi$, $\rho = 100$,

$$f(\varepsilon) = \frac{\kappa}{2}\left(\varepsilon + \frac{1}{2} - \frac{1}{2(1+\varepsilon)^2}\right),$$

$\kappa = 0.1$, $u(0,x) = 0.6(\cos x, \sin x)$, $v(0,x) = (0,0)$ for $x \in (0, 2\pi)$.

In the following figures, we give graphs of R_i where

$$R_i(t) = \left|u\left(t, \frac{2\pi}{N}i\right)\right| \qquad \text{for } i = 0, 1, \ldots, N-1.$$

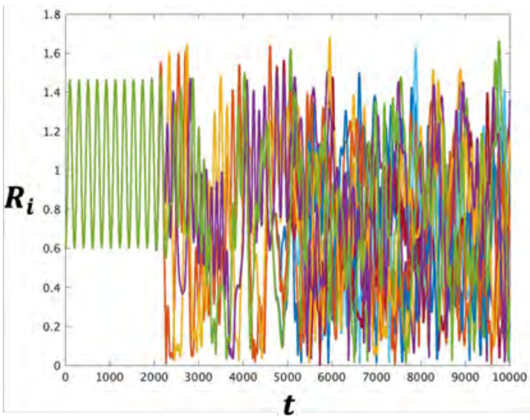

Figure 3.2: $\gamma = 0.0001$

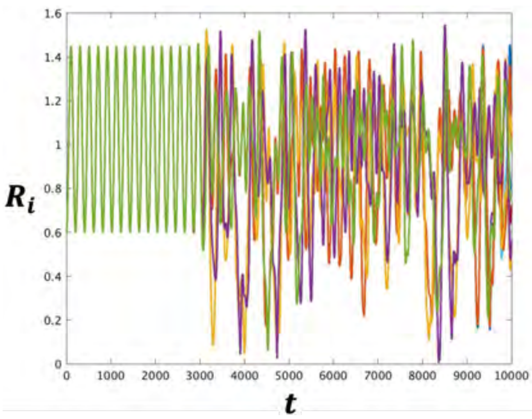

Figure 3.3: $\gamma = 0.001$

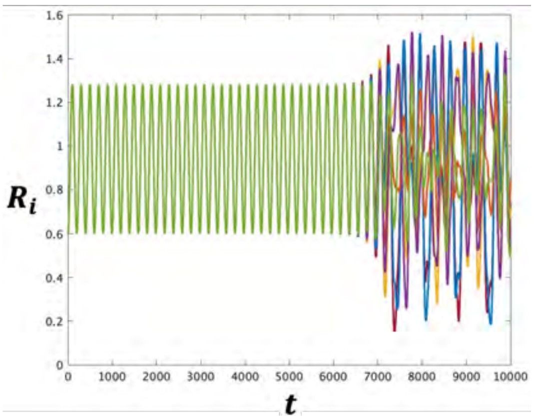

Figure 3.4: $\gamma = 0.01$

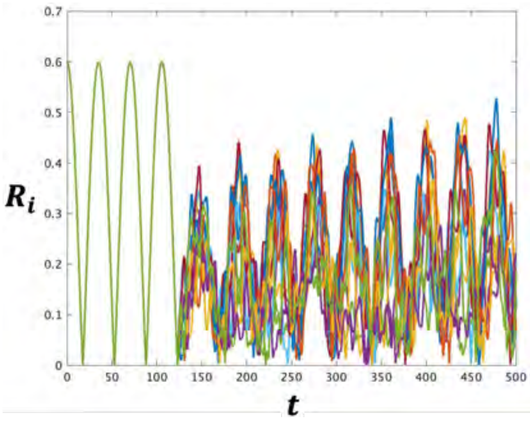

Figure 3.5: $\gamma = 1.0$

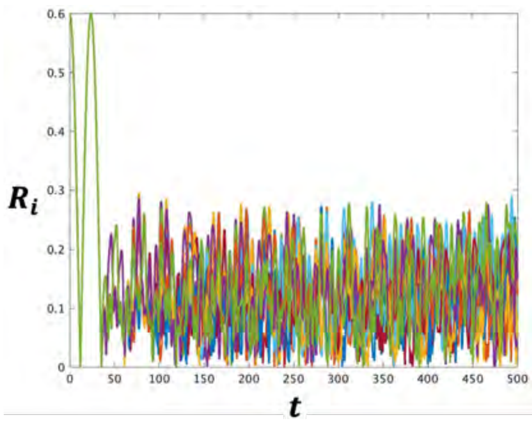

Figure 3.6: $\gamma = 2.0$

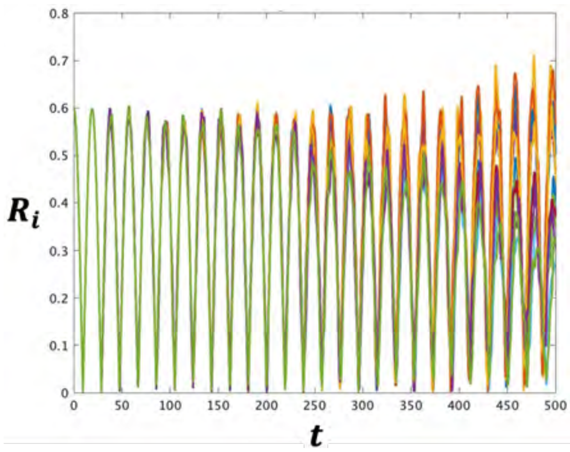

Figure 3.7: $\gamma = 3.0$

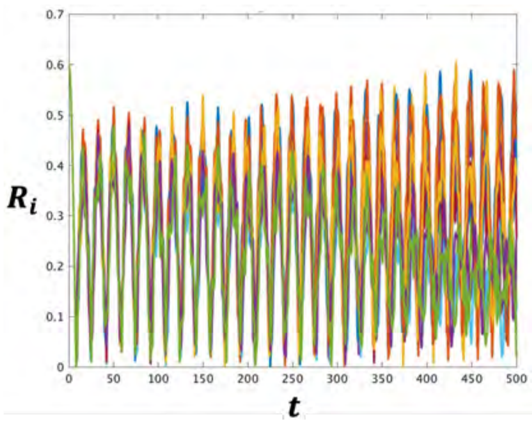

Figure 3.8: $\gamma = 4.0$

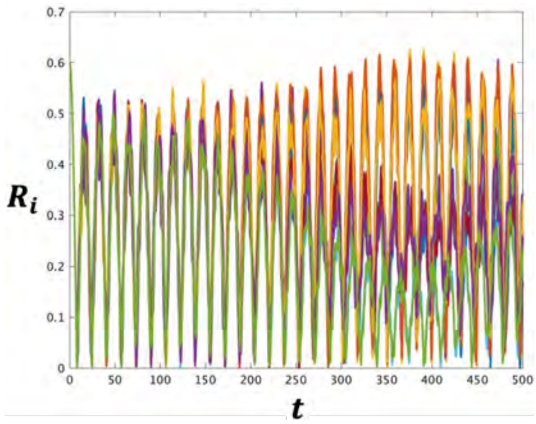

Figure 3.9: $\gamma = 4.1$

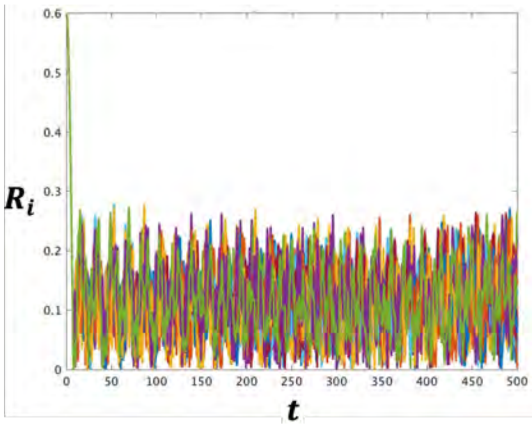

Figure 3.10: $\gamma = 4.3$

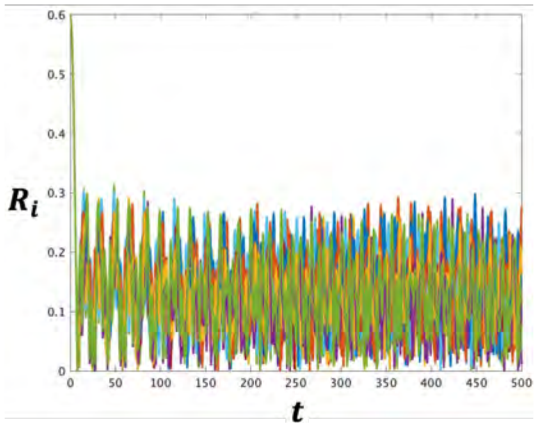

Figure 3.11: $\gamma = 4.33$

In the simulations above, we impose symmetry for the initial condition. In this case we consider that the solution must be periodic in time. However, we do not prove it, yet. For these results obtained by numerical calculations, we give some remarks from two points of view.

The 1st remark is concerned with the relationship between the value of γ and accuracy of our numerical method. From Figures 3.2–3.11 we observe that the solution for $\gamma = 0.01$ exhibits periodicity in longer time than other cases. In particular, for $\gamma > 4$ little periodicity is observed. From these observations we conjecture that an optimal value of γ exists for numerical stability.

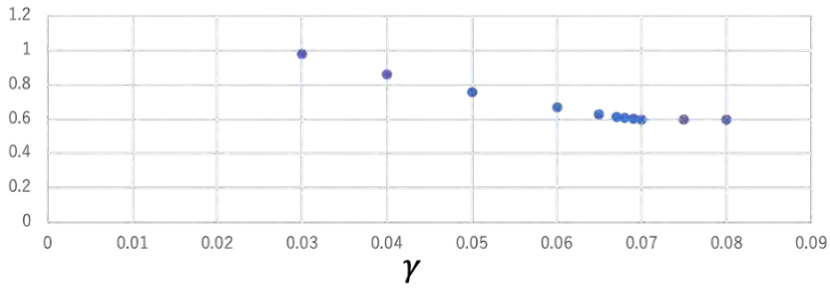

Figure 3.12: Maximum radius vs. γ

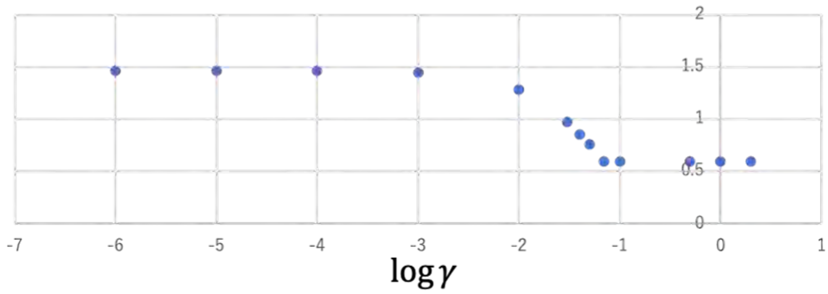

Figure 3.13: Maximum radius vs. $\log\gamma$

The second remark is on the effect of γ on the behavior of the solution. As easily seen, the following energy is conserved:

$$\frac{\rho}{2}\int_0^1 |u_t(t,x)|^2\,dx + \frac{\gamma}{2}\int_0^1 |u_{xx}(t,x)|^2\,dx + \frac{1}{2}\int_0^1 \hat{f}\left(|u_x(t,x)|^2\right)dx \qquad \text{for all } t \in [0,T],$$

where

$$\hat{f}(r) = \frac{\kappa}{4}\left(\frac{2}{3}r^{\frac{3}{2}} - \frac{1}{2}r - \frac{1}{2}\log r\right) \text{ for } r > 0.$$

Accordingly, we suppose that adding the fourth derivative term make the curve closer to straight. However, from the results plotting γ and $\max\limits_{i,t\in[0,T]} R_i(t)$, and $\log\gamma$ and $\max\limits_{i,t\in[0,T]} R_i(t)$ in Figures 3.12 and 3.13, respectively, our conjecture does not seem to be true. Actually, for the larger γ, the radius becomes smaller, namely, the curve goes farther away from the straight line.

4 Conclusion

We give some remarks for the numerical results to the ODE and PDE models. From these observations we have the following issues on the models.

1) Accuracy of the numerical methods
As mentioned above, the periodicity of the solution does not keep for long time in several numerical methods. In order to overcome the numerical failure, we propose the following two approaches.

   - The first approach is to develop a proper method to our ODE and PDE model. As we point out in Section 3.2, by adding the term γu_{xxxx} with $\gamma = 0.01$ the periodicity keeps well. From the analysis to find an optimal value of γ, we have a chance to develop a new proper numerical method.
   - It may be impossible to construct a numerical method for long time. In other words, we conjecture that periodic solutions of our model are unstable with respect to the initial values. If this conjecture is true, then accumulation of errors by truncation and rounding causes dissipation of the periodicity.

2) Role of the term γu_{xxxx}
The conjecture concerned with γu_{xxxx} seems not be true, as shown in Section 3.2. Hence, we will deal with steady solutions for (PP), and clarify the relationship between the behavior of the solution and the value of γ.

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