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A NEW STRUCTURE-PRESERVING SCHEME WITH THE STAGGERED SPACE MESH FOR THE CAHN–HILLIARD EQUATION UNDER A DYNAMIC BOUNDARY CONDITION

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Abstract. We propose a structure-preserving scheme with the staggered space mesh for the Cahn–Hilliard equation under a dynamic boundary condition. Although we use the discrete variational derivative method (DVDM) [13] for designing the scheme, there are some improvements in the conventional way of taking the space mesh of the standard DVDM for the problem with the Neumann boundary condition. In this study, we improve the scheme by modifying the conventional manner, and by applying the modified method to the problems with dynamic boundary conditions, we propose a new structure-preserving scheme. Moreover, we give the stability, the existence, and the uniqueness of the solution for our proposed scheme as the main result.

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1 Introduction

Let $L > 0$ be the length of the one-dimensional material. In this paper, we study the following Cahn–Hilliard equation [2]:

$$\begin{cases} \partial_t u = \partial_x^2 p & \text{in } (0, L) \times (0, \infty), \quad (1) \\ p = -\gamma \partial_x^2 u + F'(u) & \text{in } (0, L) \times (0, \infty), \quad (2) \end{cases}$$

under the dynamic boundary condition and the homogeneous Neumann boundary condition:

$$\begin{cases} \partial_t u(0, t) = \partial_x u(x, t)|_{x=0} & \text{in } (0, \infty), \quad (3) \\ \partial_t u(L, t) = - \partial_x u(x, t)|_{x=L} & \text{in } (0, \infty), \quad (4) \\ \partial_x p(x, t)|_{x=0} = \partial_x p(x, t)|_{x=L} = 0 & \text{in } (0, \infty). \quad (5) \end{cases}$$

The unknown functions u: $[0, L] \times [0, \infty) \to \mathbb{R}$ and p: $[0, L] \times [0, \infty) \to \mathbb{R}$ are corresponding to the order parameter and the chemical potential, respectively. Moreover, γ is a positive constant. Furthermore, F: $\mathbb{R} \to \mathbb{R}$ is a given potential, and F' is its derivative. For example, F can be a double-well potential, i.e.,

$$F(s) = \frac{q}{4} s^4 - \frac{r}{2} s^2 \quad \text{for all } s \in \mathbb{R},$$

where q and r are positive constants. Throughout this article, we assume that the potential F is bounded from below. Let us define the "local energy" G and the "global energy" J, which characterize the equations (1)–(2), as follows:

$$G(u, \partial_x u) := \frac{\gamma}{2} |\partial_x u|^2 + F(u),$$
$$J(u) := \int_0^L G(u, \partial_x u) dx.$$

We remark that the above words "local energy" and "global energy" are ones for space, not for time, and that this "global" is different from the one of the words "global boundedness" and "global existence," which appear later. Also, let us define the "mass" M as follows:

$$M(u) := \int_0^L u dx.$$

Clearly, the solution of the system (1)–(5) has the following properties:

$$\frac{d}{dt} M(u(t)) = 0, \quad (6)$$

$$\frac{d}{dt} J(u(t)) = -\gamma |\partial_t u(0, t)|^2 - \gamma |\partial_t u(L, t)|^2 - \int_0^L |\partial_x p(x, t)|^2 dx \le 0. \quad (7)$$

From a mathematical point of view, the problem (1)–(5) with initial conditions are studied in [5–10, 14–16, 20, 21, 26–28]. First, in the case of $F(s) = (q/4)s^4 - (r/2)s^2$, Racke and Zheng [27] proved the global existence and uniqueness of the solution to the problem, and Prüss et al. [26] obtained the result on the maximal L^p-regularity and proved the existence of a global attractor. Also, Wu and Zheng [28] and Chill et al. [6] proved the convergence of the solution of the problem to an equilibrium as time goes to infinity. Moreover, various results of the existence, uniqueness, and regularity of the solution, the existence of a global attractor, the convergence to steady states, and the optimal control are obtained under more general assumptions on the nonlinearities in [5,7–10,15,16,20,21]. Gal [14] compared the problem under other dynamic boundary conditions with that under our target dynamic boundary conditions. We remark that in these papers, the problem is considered in the multi-dimensional case, where the boundary conditions (3) and (4) include the Laplace–Beltrami operator.

From a numerical point of view, there are some interesting studies of the Cahn–Hilliard equation with dynamic boundary conditions (see, for example, [1, 3–5, 11, 17–19, 22, 23]). In [3, 18], Cherfils et al. and Israel et al. have considered the finite element space semi-discretizations of the problem in the two-dimensional or three-dimensional cases and shown the error estimate. Moreover, the unconditional stability of fully discrete schemes based on the backward Euler scheme for the time discretization, and the convergence of the solution to a steady-state have been obtained. See also [5] for other numerical results. In addition, Nabet has performed an interesting analysis for the problem in the two-dimensional case by using the finite-volume method and proved the convergence of the numerical solution in [22], and the first-order convergence with respect to the H^1-norms in [23]. Besides, the results of the problem with other dynamic boundary conditions have been obtained [1, 4, 17, 19]. As we have just seen, although the numerical analysis of the Cahn–Hilliard equation with various types of dynamic boundary conditions has recently attracted much attention, there are only a few results by the finite difference method. Recently, however, finite difference structure-preserving schemes for (1)–(5) based on the discrete variational derivative method (DVDM) proposed by Furihata and Matsuo [13] have been designed and analyzed. Throughout this article, "structure-preserving" means that the scheme inherits the conservative property such as (6) or the dissipative property such as (7). In the standard DVDM for problems under the Neumann boundary condition, and in the method of designing the structure-preserving schemes of the previous studies [24, 25] based on DVDM for problems under dynamic boundary conditions, an approximation of the equations in the interior of the domain is imposed even on the fixed boundary (further details are described later). We consider this to be artificial and an area for improvement. In the previous study [11], a structure-preserving scheme has been proposed, improving the problem. However, in this scheme, all outward normal derivatives in the boundary conditions have been approximated only by a forward difference, and the time difference of the discrete boundary condition has been considered at a point off the boundary. Namely, the scheme has room for improvement in the discretization of boundary conditions. Motivated by this, we design a new structure-preserving scheme that improves on the above problems.

We explain the problem of the structure-preserving schemes based on DVDM [13] in the case of the Neumann boundary condition. For the sake of simplicity, we consider the following heat equation with the Neumann boundary condition:

$$\begin{cases} \partial_t u = \partial_x^2 u & \text{in } (0,L)\times(0,\infty), \quad (8)\\ \partial_x u(x,t)|_{x=0} = \partial_x u(x,t)|_{x=L} = 0 & \text{in } (0,\infty). \quad (9) \end{cases}$$

For the solution of (8)–(9), it holds that

$$\frac{d}{dt}M(u(t)) = 0, \tag{10}$$

$$\frac{d}{dt}E_1(u(t)) = -\int_0^L |\partial_x u(x,t)|^2 dx \le 0, \tag{11}$$

$$\frac{d}{dt}E_2(u(t)) = -\int_0^L |\partial_x^2 u(x,t)|^2 dx \le 0, \tag{12}$$

where energy E_1 and E_2 are defined by

$$E_1(u) := \int_0^L \frac{|\partial_x u|^2}{2} dx, \quad E_2(u) := \int_0^L \frac{u^2}{2} dx.$$

We introduce a structure-preserving scheme for (8)–(9) based on the usual DVDM, which retains the above properties in a discrete sense. We note some notation in preparation for this purpose. Let K be a positive integer. We define $U_k^{(n)}$ $(k = -1, 0, \ldots, K, K+1, n = 0, 1, \ldots)$ to be the approximation to $u(x,t)$ at location $x = k\Delta x$ and time $t = n\Delta t$, where Δx is a space mesh size, i.e., $\Delta x := L/K$, and Δt is a time mesh size, and $U_{-1}^{(n)}$ and $U_{K+1}^{(n)}$ are artificial quantities and determined by the imposed discrete boundary condition. To describe the scheme, we define basic discrete operators as follows: we denote average operators μ_k^+, μ_k^- concerning subscript k by

$$\mu_k^+ f_k := \frac{f_k + f_{k+1}}{2}, \quad \mu_k^- f_k := \frac{f_k + f_{k-1}}{2},$$

and the difference operators δ_k^+, δ_k^-, $\delta_k^{\langle 1\rangle}$, and $\delta_k^{\langle 2\rangle}$ concerning subscript k by

$$\delta_k^+ f_k := \frac{f_{k+1} - f_k}{\Delta x}, \qquad \delta_k^- f_k := \frac{f_k - f_{k-1}}{\Delta x},$$
$$\delta_k^{\langle 1\rangle} f_k := \frac{f_{k+1} - f_{k-1}}{2\Delta x}, \qquad \delta_k^{\langle 2\rangle} f_k := \frac{f_{k+1} - 2f_k + f_{k-1}}{(\Delta x)^2}.$$

Similarly, we define the difference operator δ_n^+ corresponding superscript (n) by

$$\delta_n^+ f^{(n)} := \frac{f^{(n+1)} - f^{(n)}}{\Delta t}.$$

By applying the standard DVDM to the above problem, we obtain the following structure-preserving scheme: for $n = 0, 1, \ldots,$

$$\begin{cases} \delta_n^+ U_k^{(n)} = \delta_k^{\langle 2\rangle}\left(\dfrac{U_k^{(n+1)} + U_k^{(n)}}{2}\right) & (k = 0, \ldots, K), \quad (13)\\ \delta_k^{\langle 1\rangle} U_k^{(n+1)} = 0 & (k = 0, K), \quad (14) \end{cases}$$

where the initial value $\boldsymbol{U}^{(0)}$ also satisfies (14). In the scheme (13)–(14), $\boldsymbol{U}^{(n+1)}$ is unknown, and $\boldsymbol{U}^{(n)}$ is known. Besides, we consider the values U_{-1}, U_{K+1} at artificial lattice points in the above scheme. These U_{-1} and U_{K+1} are artificial quantities to define the above energy naturally, and their values are determined by the imposed discrete boundary conditions. The scheme (13)–(14) guarantees that discrete versions of (10)–(12) hold (see [12, 13] for details). As can be seen in (13), in the structure-preserving scheme by the standard DVDM for problems with boundary conditions including the flux, such as the Neumann boundary condition, the approximation of the equation in the interior of the domain are often considered even on the boundary, i.e., $k = 0$ and $k = K$. It is artificial and should be improved. Thus, by adopting the staggered space mesh and modifying the discretization of energy and the summation-by-parts formula, we design a structure-preserving scheme in which the discrete equations corresponding to the equation in the interior of the domain are not imposed at the points $k = 0, K$.

Here, we describe our idea of how to design a structure-preserving scheme in more detail. We derive the desired new structure-preserving scheme by adopting the staggered space mesh $(k - 1/2)\Delta x$ $(k = 0, 1, \ldots, K, K + 1)$ used in the previous work [11] and improving the discretization of energy and the summation-by-parts formula. In other words, from now on, we define $U_k^{(n)}$ $(k = 0, 1, \ldots, K, K + 1, n = 0, 1, \ldots)$ to be the approximation to $u(x, t)$ at location $x = (k - 1/2)\Delta x$ and time $t = n\Delta t$. They are also written in vector as $\boldsymbol{U}^{(n)} := (U_1^{(n)}, \ldots, U_K^{(n)})^\top$ or $\boldsymbol{U}^{(n)} := (U_0^{(n)}, U_1^{(n)}, \ldots, U_K^{(n)}, U_{K+1}^{(n)})^\top$. Guess the meaning of $\boldsymbol{U}^{(n)}$ from the context. We remark that $U_0^{(n)}$ and $U_{K+1}^{(n)}$ are artificial quantities and determined by the imposed discrete boundary condition.

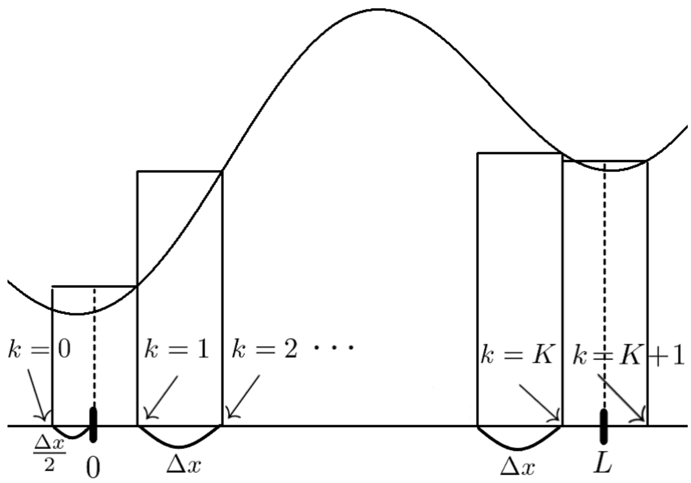

Figure 1: Staggered space mesh and relationship between x and k

In this paper, we define the discrete energy E_{1d}: $\mathbb{R}^{K+2} \to \mathbb{R}$ as follows:

$$E_{\mathrm{1d}}(\boldsymbol{U}) := \sum_{k=1}^{K} \frac{\left|\delta_k^+ U_k\right|^2 + \left|\delta_k^- U_k\right|^2}{2} \Delta x \quad \text{for all } \boldsymbol{U} \in \mathbb{R}^{K+2}.$$

Remark 1.1. In the previous result [11], the discrete energy $\sum_{k=1}^{K}(|\delta_k^- U_k|^2/2)\Delta x$ consisting only of a backward difference is adopted as a discrete version of the energy E_1.

Besides, we use the following new formula as a summation by parts formula, which is essential to derive a structure-preserving scheme in DVDM. All the proofs can be obtained by direct calculation and here omitted.

Lemma 1.1 (Second-order summation by parts formula)**.** *The following second-order summation by parts formula holds:*

$$\sum_{k=1}^{K}\frac{\left(\delta_k^+ f_k\right)\left(\delta_k^+ g_k\right)+\left(\delta_k^- f_k\right)\left(\delta_k^- g_k\right)}{2}\Delta x=-\sum_{k=1}^{K}\left(\delta_k^{\langle 2\rangle} f_k\right) g_k\Delta x+\left(\delta_k^- f_k\big|_{k=K+1}\right)\left(\mu_k^- g_k\big|_{k=K+1}\right)$$
$$-\left(\delta_k^+ f_k\big|_{k=0}\right)\left(\mu_k^+ g_k\big|_{k=0}\right)$$

for all $\{f_k\}_{k=0}^{K+1}, \{g_k\}_{k=0}^{K+1}\in\mathbb{R}^{K+2}$. *In particular, when it holds that* $f_k = g_k$ $(k=0,\ldots,K+1)$*, we have*

$$\sum_{k=1}^{K}\frac{\left|\delta_k^+ f_k\right|^2+\left|\delta_k^- f_k\right|^2}{2}\Delta x=-\sum_{k=1}^{K}\left(\delta_k^{\langle 2\rangle} f_k\right) f_k\Delta x+\left(\delta_k^- f_k\big|_{k=K+1}\right)\left(\mu_k^- f_k\big|_{k=K+1}\right)$$
$$-\left(\delta_k^+ f_k\big|_{k=0}\right)\left(\mu_k^+ f_k\big|_{k=0}\right).$$

Lemma 1.2 (Summation of a difference)**.** *The following fundamental formula holds:*

$$\sum_{k=1}^{K}\left(\delta_k^{\langle 2\rangle} f_k\right)\Delta x=\delta_k^- f_k\big|_{k=K+1}-\delta_k^+ f_k\big|_{k=0}\quad \textit{for all } \{f_k\}_{k=0}^{K+1}\in\mathbb{R}^{K+2}.$$

Based on DVDM, we calculate the discrete variation $E_{1\mathrm{d}}(\boldsymbol{U}) - E_{1\mathrm{d}}(\boldsymbol{V})$ for all $\boldsymbol{U},\boldsymbol{V}\in\mathbb{R}^{K+2}$ by using the summation by parts formula and obtain the following new structure-preserving scheme for (8)–(9): for $n=0,1,\ldots,$

$$\begin{cases}\delta_n^+ U_k^{(n)}=\delta_k^{\langle 2\rangle}\left(\dfrac{U_k^{(n+1)}+U_k^{(n)}}{2}\right) & (k=1,\ldots,K), \qquad (15)\\[2ex] \delta_k^+ U_k^{(n+1)}\Big|_{k=0}=\delta_k^- U_k^{(n+1)}\Big|_{k=K+1}=0, & \qquad (16)\end{cases}$$

where the initial value $\boldsymbol{U}^{(0)}$ also satisfies (16). In the scheme (15)–(16), $\boldsymbol{U}^{(n+1)}$ is unknown, and $\boldsymbol{U}^{(n)}$ is known. The solution to the scheme (15)–(16) satisfies the following properties:

$$\delta_n^+ E_{1\mathrm{d}}(\boldsymbol{U}^{(n)})=-\sum_{k=1}^{K}\left|\delta_k^{\langle 2\rangle}\left(\frac{U_k^{(n+1)}+U_k^{(n)}}{2}\right)\right|^2\Delta x\le 0\quad (n=0,1,\ldots),$$

$$\delta_n^+ E_{2\mathrm{d}}(\boldsymbol{U}^{(n)})=-\sum_{k=1}^{K}\frac{1}{2}\left\{\left|\delta_k^+\left(\frac{U_k^{(n+1)}+U_k^{(n)}}{2}\right)\right|^2+\left|\delta_k^-\left(\frac{U_k^{(n+1)}+U_k^{(n)}}{2}\right)\right|^2\right\}\Delta x\le 0\quad (n=0,1,\ldots),$$

$$\delta_n^+ M_{\mathrm{d}}(\boldsymbol{U}^{(n)})=0\quad (n=0,1,\ldots),$$

where the discrete energy $E_{2\mathrm{d}}\colon \mathbb{R}^{K+2} \to \mathbb{R}$ and the discrete mass $M_{\mathrm{d}}\colon \mathbb{R}^{K+2s} \to \mathbb{R}$ are defined by

$$E_{2\mathrm{d}}(\boldsymbol{U}) := \sum_{k=1}^{K} \frac{U_k^2}{2}\Delta x \quad \text{for all } \boldsymbol{U} \in \mathbb{R}^{K+2},$$
$$M_{\mathrm{d}}(\boldsymbol{U}) := \sum_{k=1}^{K} U_k \Delta x \quad \text{for all } \boldsymbol{U} \in \mathbb{R}^{K+2s}, \text{ where } s = 0, 1.$$

Remark 1.2. The scheme (15)–(16), designed by adopting the staggered space mesh and modifying the discretization of energy and the summation-by-parts formula, has the following advantages:

- On the fixed boundary, we do not have to consider the equations in the interior of the domain.
- The number of elements of unknown $\boldsymbol{U}^{(n+1)}$ is one less than that of schemes (13)–(14), accordingly, the number of equations is also one less, and thus the computational cost is lower.

Because of the above advantages and the fact that the dynamic boundary condition we consider in this paper also involves the flux, we apply the method described above of designing the scheme to the problem with the dynamic boundary condition and design a new structure-preserving scheme. Furthermore, in this paper, we give the stability and the solvability of the scheme as the main results.

Here, we show the solvability of the above scheme (15)–(16). It holds from (16) and that

$$U_0^{(n)} = U_1^{(n)},\ U_{K+1}^{(n)} = U_K^{(n)} \quad (n = 0, 1, \ldots). \tag{17}$$

By using (17), we eliminate artificial quantities U_0 and U_{K+1} in (15) and transform (15) as follows:

$$\begin{cases} U_1^{(n+1)} - \dfrac{\Delta t}{2(\Delta x)^2}\left(U_2^{(n+1)} - U_1^{(n+1)}\right) = U_1^{(n)} + \dfrac{\Delta t}{2(\Delta x)^2}\left(U_2^{(n)} - U_1^{(n)}\right) \quad (n = 0, 1, \ldots), \\ U_k^{(n+1)} - \dfrac{\Delta t}{2(\Delta x)^2}\left(U_{k+1}^{(n+1)} - 2U_k^{(n+1)} + U_{k-1}^{(n+1)}\right) = U_k^{(n)} + \dfrac{\Delta t}{2(\Delta x)^2}\left(U_{k+1}^{(n)} - 2U_k^{(n)} + U_{k-1}^{(n)}\right) \\ \qquad\qquad (k = 2, \ldots, K-1,\ n = 0, 1, \ldots), \\ U_K^{(n+1)} - \dfrac{\Delta t}{2(\Delta x)^2}\left(U_{K-1}^{(n+1)} - U_K^{(n+1)}\right) = U_K^{(n)} + \dfrac{\Delta t}{2(\Delta x)^2}\left(U_{K-1}^{(n)} - U_K^{(n)}\right) \quad (n = 0, 1, \ldots). \end{cases}$$

Next, we express the above simultaneous linear equations as follows by using matrices:

$$\left(I - \frac{\Delta t}{2} D_2\right) \boldsymbol{U}^{(n+1)} = \left(I + \frac{\Delta t}{2} D_2\right) \boldsymbol{U}^{(n)} \quad (n = 0, 1, \ldots), \tag{18}$$

where I is the K-dimensional identity matrix. Besides, D_2 is the K-dimensional matrix corresponding to the difference operator $\delta_k^{\langle 2 \rangle}$ with the homogeneous Neumann boundary condition (16) and is defined by

$$D_2 := \frac{1}{(\Delta x)^2}\begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix}.$$

If the coefficient matrix $I-(\Delta t/2)D_2$ of $\boldsymbol{U}^{(n+1)}$ in (18) is nonsingular, then the scheme (15)–(16) has a unique solution. In fact, the scheme (15)–(16) has a unique solution since the coefficient matrix $I-(\Delta t/2)D_2$ is a strictly diagonally dominant matrix. The solvability of the standard scheme (13)–(14) can also be proved from the same argument.

The rest of this paper proceeds as follows. In Section 2, we propose a structure-preserving scheme for (1)–(5), whose solution satisfies the discrete version of the conservative property (6) and the dissipative property (7). In Section 3, we show the boundedness for the solution of the scheme globally in space. In Section 4, we provide a sufficient condition for the existence of the unique solution to the scheme.

2 Proposed scheme

In this section, we design a structure-preserving scheme for (1)–(5) by using the discretization of energy and the summation by parts formula given in Section 1.

2.1 Preparation

For later use, we define the difference quotient. Let Ω be a domain in $\mathbb{R}$. For a function $F \in C^1(\Omega)$ and $\xi, \eta \in \Omega$, the difference quotient $dF/d(\xi,\eta)$ of F at (ξ,η) is defined as follows:

$$\frac{dF}{d(\xi,\eta)} := \begin{cases} \dfrac{F(\xi)-F(\eta)}{\xi-\eta} & (\xi \neq \eta), \\ F'(\eta) & (\xi = \eta). \end{cases}$$

Here, let us define two discrete local energies $G_{\mathrm{d}}\colon \mathbb{R}^{K+2} \to \mathbb{R}^K$ with the elements given by

$$G_{\mathrm{d},k}(\boldsymbol{U}) := \frac{\gamma}{2}\frac{(\delta_k^+ U_k)^2 + (\delta_k^- U_k)^2}{2} + F(U_k) \quad (k=1,\dots,K),$$

for all $\boldsymbol{U} \in \mathbb{R}^{K+2}$. Namely, $G_{\mathrm{d},k}(\boldsymbol{U}) = \{G_{\mathrm{d}}(\boldsymbol{U})\}_k$. Furthermore, we define the discrete global energy $J_{\mathrm{d}}\colon \mathbb{R}^{K+2} \to \mathbb{R}$ as follows:

$$J_{\mathrm{d}}(\boldsymbol{U}) := \sum_{k=1}^{K} G_{\mathrm{d},k}(\boldsymbol{U})\Delta x = \frac{\gamma}{2}\sum_{k=1}^{K}\frac{\left(\delta_k^+ U_k\right)^2 + \left(\delta_k^- U_k\right)^2}{2}\Delta x + \sum_{k=1}^{K} F(U_k)\,\Delta x.$$

From the idea on DVDM in [13], we take a discrete variation to derive a structure-preserving scheme for (1)–(5). That is, we need the difference $J_{\mathrm{d}}(\boldsymbol{U}) - J_{\mathrm{d}}(\boldsymbol{V})$ for all $\boldsymbol{U}, \boldsymbol{V} \in \mathbb{R}^{K+2}$. By using Lemma 1.1, we have the following lemma:

Lemma 2.1. *It holds:*

$$\begin{aligned} J_{\mathrm{d}}(\boldsymbol{U}) - J_{\mathrm{d}}(\boldsymbol{V}) = &\sum_{k=1}^{K} \left\{ -\gamma \delta_k^{\langle 2 \rangle} \left(\frac{U_k + V_k}{2} \right) + \frac{dF}{d(U_k, V_k)} \right\} (U_k - V_k)\, \Delta x \\ &+ \gamma \left\{ \delta_k^{-} \left(\frac{U_k + V_k}{2} \right) \bigg|_{k=K+1} \right\} \left(\mu_k^{-} U_k \big|_{k=K+1} - \mu_k^{-} V_k \big|_{k=K+1} \right) \\ &- \gamma \left\{ \delta_k^{+} \left(\frac{U_k + V_k}{2} \right) \bigg|_{k=0} \right\} \left(\mu_k^{+} U_k \big|_{k=0} - \mu_k^{+} V_k \big|_{k=0} \right) \end{aligned}$$

for all $\boldsymbol{U}, \boldsymbol{V} \in \mathbb{R}^{K+2}$.

The proof can be obtained from direct calculation and is omitted here. The discrete energy dissipation in Proposition 2.1 will be proved by applying Lemma 2.1.

2.2 Proposed scheme

The concrete form of our scheme for (1)–(5) is

$$\delta_n^{+} U_k^{(n)} = \delta_k^{\langle 2 \rangle} P_k^{(n)} \quad (k = 1, \ldots, K,\ n = 0, 1, \ldots), \tag{19}$$

$$P_k^{(n)} = -\gamma \delta_k^{\langle 2 \rangle} \left(\frac{U_k^{(n+1)} + U_k^{(n)}}{2} \right) + \frac{dF}{d(U_k^{(n+1)}, U_k^{(n)})} \quad (k = 1, \ldots, K,\ n = 0, 1, \ldots), \tag{20}$$

$$\delta_n^{+} \left(\mu_k^{+} U_k^{(n)} \Big|_{k=0} \right) = \delta_k^{+} \left(\frac{U_k^{(n+1)} + U_k^{(n)}}{2} \right) \Bigg|_{k=0} \quad (n = 0, 1, \ldots), \tag{21}$$

$$\delta_n^{+} \left(\mu_k^{-} U_k^{(n)} \Big|_{k=K+1} \right) = -\, \delta_k^{-} \left(\frac{U_k^{(n+1)} + U_k^{(n)}}{2} \right) \Bigg|_{k=K+1} \quad (n = 0, 1, \ldots), \tag{22}$$

$$\delta_k^{+} P_k^{(n)} \Big|_{k=0} = \delta_k^{-} P_k^{(n)} \Big|_{k=K+1} = 0 \quad (n = 0, 1, \ldots). \tag{23}$$

Remark 2.1. Note that in our scheme, boundaries $x = 0, 1$ are at the middle of points $k = 0, 1$ and the middle of points $k = K, K+1$, respectively.

Clearly, the following properties corresponding to (6) and (7) are obtained.

Proposition 2.1. *The solution* $\boldsymbol{U}^{(n)}$ *to the scheme* (19)–(23) *satisfies the following inequality:*

$$\begin{aligned} \delta_n^{+} J_{\mathrm{d}}(\boldsymbol{U}^{(n)}) = &-\gamma \left| \delta_n^{+} \left(\mu_k^{+} U_k^{(n)} \Big|_{k=0} \right) \right|^2 - \gamma \left| \delta_n^{+} \left(\mu_k^{-} U_k^{(n)} \Big|_{k=K+1} \right) \right|^2 \\ &- \sum_{k=1}^{K} \frac{\left| \delta_k^{+} P_k^{(n)} \right|^2 + \left| \delta_k^{-} P_k^{(n)} \right|^2}{2} \Delta x \leq 0 \quad (n = 0, 1, \ldots). \end{aligned}$$

Proposition 2.2. *The solution* $\boldsymbol{U}^{(n)}$ *of the scheme* (19)–(23) *satisfies*

$$\delta_n^{+} M_{\mathrm{d}}(\boldsymbol{U}^{(n)}) = 0 \quad (n = 0, 1, \ldots).$$

Since Proposition 2.1 is a direct application of Lemmas 1.1, 2.1 and (19)–(23), and Proposition 2.2 can be obtained from (19), (23), and Lemma 1.2, we omit these proofs.

3 Global boundedness of the solution to the proposed scheme

In this section, we show that, if the proposed scheme has a solution, then it satisfies the global boundedness. Firstly, let us give definitions of the discrete L^∞-norm and the discrete Dirichlet semi-norm.

Definition 3.1. We define the discrete L^∞-norm $\|\cdot\|_{L^\infty_{\mathrm{d}}}$ by

$$\|\boldsymbol{f}\|_{L^\infty_{\mathrm{d}}} := \max_{k=0,1,\ldots,K+1} |f_k| \quad \text{for all } \boldsymbol{f} = \{f_k\}_{k=0}^{K+1} \in \mathbb{R}^{K+2}.$$

For all $\boldsymbol{f} = \{f_k\}_{k=0}^{K+1} \in \mathbb{R}^{K+2}$, we define the discrete Dirichlet semi-norm $\|D\boldsymbol{f}\|$ of $\boldsymbol{f}$ by

$$\|D\boldsymbol{f}\| := \sqrt{\sum_{k=1}^{K} \frac{|\delta_k^+ f_k|^2 + |\delta_k^- f_k|^2}{2}\Delta x}.$$

For the proof of the global boundedness of the numerical solution, we use the following lemmas.

Lemma 3.1. *The solution to the scheme* (19)–(23) *satisfies the following inequality:*

$$\left\|D\boldsymbol{U}^{(n)}\right\| \le \left\{\frac{2}{\gamma}\left(J_{\mathrm{d}}(\boldsymbol{U}^{(0)}) + L\left|\min\left\{\inf_{\xi\in\mathbb{R}} F(\xi), 0\right\}\right|\right)\right\}^{\frac{1}{2}} \quad (n = 0, 1, \ldots). \tag{24}$$

This lemma follows from the same argument as in Lemma 3.2 of [25].

Lemma 3.2 (Discrete Poincaré–Wirtinger inequality)**.** *For all $\boldsymbol{f} = \{f_k\}_{k=0}^{K+1} \in \mathbb{R}^{K+2}$, the following inequality holds:*

$$\left|f_m - \frac{1}{L}\sum_{k=1}^{K} f_k \Delta x\right| \le 2L^{\frac{1}{2}} \|D\boldsymbol{f}\| \quad (m = 0, 1, \ldots, K+1). \tag{25}$$

Proof. For any fixed $m = 0, 1, \ldots, K+1$, we see that

$$Lf_m - \sum_{k=1}^{K} f_k \Delta x = f_m \sum_{k=1}^{K} \Delta x - \sum_{k=1}^{K} f_k \Delta x = \sum_{k=1}^{K} (f_m - f_k)\Delta x. \tag{26}$$

From the direct calculation, it is seen that

$$f_m - f_k = \begin{cases} \displaystyle\sum_{l=k+1}^{m} \delta_l^- f_l \Delta x & (m > k), \\ 0 & (m = k), \\ \displaystyle\sum_{l=m+1}^{k} \delta_l^- f_l \Delta x & (m < k) \end{cases}$$

for $k = 1, \ldots, K$. Hence, we have

$$
\begin{aligned}
|f_m - f_k| \leq \sum_{l=1}^{K+1} |\delta_l^- f_l| \Delta x &= \sum_{l=1}^{K} |\delta_l^- f_l| \Delta x + \left| \delta_l^- f_l \big|_{l=K+1} \right| \Delta x \\
&= \sum_{l=1}^{K} |\delta_l^- f_l| \Delta x + \left| \delta_l^+ f_l \big|_{l=K} \right| \Delta x \\
&\leq \sum_{l=1}^{K} \left(|\delta_l^+ f_l| + |\delta_l^- f_l| \right) \Delta x \quad (k = 1, \ldots, K).
\end{aligned} \tag{27}
$$

Using (26), (27), and the Hölder inequality, we obtain

$$
\begin{aligned}
\left| L f_m - \sum_{k=1}^{K} f_k \Delta x \right| &\leq \sum_{k=1}^{K} |f_m - f_k| \Delta x \\
&\leq \sum_{l=1}^{K} \left(|\delta_l^+ f_l| + |\delta_l^- f_l| \right) \Delta x \sum_{k=1}^{K} \Delta x \\
&\leq L \left(\sum_{l=1}^{K} 1^2 \Delta x \right)^{\frac{1}{2}} \left\{ \sum_{l=1}^{K} \left(|\delta_l^+ f_l| + |\delta_l^- f_l| \right)^2 \Delta x \right\}^{\frac{1}{2}} \\
&\leq L^{\frac{3}{2}} \left\{ 4 \sum_{l=1}^{K} \frac{|\delta_l^+ f_l|^2 + |\delta_l^- f_l|^2}{2} \Delta x \right\}^{\frac{1}{2}} \\
&= 2 L^{\frac{3}{2}} \left\| D \boldsymbol{f} \right\|.
\end{aligned}
$$

Dividing both sides of the above inequality by L, we have (25). □

Applying Lemma 3.2 to (24) and using the conservative property in Proposition 2.2, we obtain the following global boundedness.

Theorem 3.1 (Global boundedness)**.** *The solution to the scheme* (19)–(23) *satisfies the following inequality:*

$$
\left\| \boldsymbol{U}^{(n)} \right\|_{L_{\mathrm{d}}^{\infty}} \leq \frac{1}{L} \left| M_{\mathrm{d}}(\boldsymbol{U}^{(0)}) \right| + \left\{ \frac{8L}{\gamma} \left(J_{\mathrm{d}}(\boldsymbol{U}^{(0)}) + L \left| \min\left\{ \inf_{\xi \in \mathbb{R}} F(\xi), 0 \right\} \right| \right) \right\}^{\frac{1}{2}} \quad (n = 0, 1, \ldots).
$$

4 Existence and uniqueness of the solution for the proposed scheme

In this section, using the energy method in [11,24,25,29–32], we give a sufficient condition such that the proposed scheme (19)–(23) has a unique solution, as Theorem 4.1.

4.1 Preparation

Let Ω be a domain in $\mathbb{R}$. We define $\bar{F}''$ for $F \in C^2(\Omega)$ and give several lemmas necessary for the proof of the existence and uniqueness of the solution.

Definition 4.1. For a function $F \in C^2(\Omega)$, we define $\bar{F}''\colon \Omega^4 \to \mathbb{R}$ by

$$\begin{aligned}\bar{F}''(\xi,\tilde{\xi};\eta,\tilde{\eta}) &:= \frac{d}{d(\xi,\tilde{\xi})}\left(\frac{dF}{d(\cdot,\eta)}+\frac{dF}{d(\cdot,\tilde{\eta})}\right)\\ &= \begin{cases} \dfrac{1}{\xi-\tilde{\xi}}\left\{\left(\dfrac{dF}{d(\xi,\eta)}+\dfrac{dF}{d(\xi,\tilde{\eta})}\right)-\left(\dfrac{dF}{d(\tilde{\xi},\eta)}+\dfrac{dF}{d(\tilde{\xi},\tilde{\eta})}\right)\right\} & (\xi\neq\tilde{\xi}),\\ \partial_\xi\left(\dfrac{dF}{d(\xi,\eta)}+\dfrac{dF}{d(\xi,\tilde{\eta})}\right)\Bigg|_{\xi=\tilde{\xi}} & (\xi=\tilde{\xi})\end{cases}\end{aligned}$$

for all $(\xi,\tilde{\xi},\eta,\tilde{\eta}) \in \Omega^4$.

Since proofs of the following Lemmas 4.1, 4.2 can be found in [31], we omit them.

Lemma 4.1 ([31, Lemma 2.4])**.** *If $F \in C^2(\Omega)$, then $\bar{F}'' \in C(\Omega^4)$. Moreover, we have*

$$\left|\bar{F}''(\xi,\tilde{\xi};\eta,\tilde{\eta})\right| \le \sup_{\eta,\tilde{\eta}\in\Omega}\sup_{\xi\in\Omega}\left|\frac{\partial}{\partial\xi}\left(\frac{dF}{d(\xi,\eta)}+\frac{dF}{d(\xi,\tilde{\eta})}\right)\right| \le \sup_{\xi\in\Omega}|F''(\xi)|$$

for all $(\xi,\tilde{\xi},\eta,\tilde{\eta}) \in \Omega^4$.

Lemma 4.2 ([31, Proposition 2.5])**.** *Assume that $F \in C^2(\Omega)$. For any ξ, $\tilde{\xi}$, η, $\tilde{\eta} \in \Omega$, we have*

$$\frac{dF}{d(\xi,\eta)}-\frac{dF}{d(\tilde{\xi},\tilde{\eta})} = \frac{1}{2}\bar{F}''(\xi,\tilde{\xi};\eta,\tilde{\eta})(\xi-\tilde{\xi})+\frac{1}{2}\bar{F}''(\eta,\tilde{\eta};\xi,\tilde{\xi})(\eta-\tilde{\eta}).$$

Lemma 4.3. *For all $\boldsymbol{f}=\{f_k\}_{k=0}^{K+1}, \boldsymbol{g}=\{g_k\}_{k=0}^{K+1} \in \mathbb{R}^{K+2}$, the following inequality holds:*

$$\|D(\boldsymbol{fg})\| \le \|\boldsymbol{f}\|_{L^\infty_{\mathrm{d}}}\|D\boldsymbol{g}\| + \|\boldsymbol{g}\|_{L^\infty_{\mathrm{d}}}\|D\boldsymbol{f}\|, \tag{28}$$

where $\boldsymbol{fg}=\{f_kg_k\}_{k=0}^{K+1} \in \mathbb{R}^{K+2}$.

Proof. By using the following decomposition:

$$ab-cd = \frac{a+c}{2}(b-d)+(a-c)\frac{b+d}{2} \quad \text{for all } a,b,c,d \in \mathbb{R},$$

we see that

$$
\begin{aligned}
&\|D(\boldsymbol{f}\boldsymbol{g})\|^2 \\
&= \sum_{k=1}^{K} \frac{|\delta_k^+(f_k g_k)|^2 + |\delta_k^-(f_k g_k)|^2}{2} \Delta x \\
&= \frac{1}{2}\sum_{k=1}^{K} \left\{ \left| (\delta_k^+ f_k)\frac{g_{k+1}+g_k}{2} + \frac{f_{k+1}+f_k}{2}(\delta_k^+ g_k) \right|^2 + \left| (\delta_k^- f_k)\frac{g_k+g_{k-1}}{2} + \frac{f_k+f_{k-1}}{2}(\delta_k^- g_k) \right|^2 \right\} \Delta x \\
&= \frac{1}{8}\sum_{k=1}^{K} \left\{ (\delta_k^+ f_k)^2 (g_{k+1}+g_k)^2 + (\delta_k^- f_k)^2 (g_k + g_{k-1})^2 \right\} \Delta x \\
&\quad + \frac{1}{4}\sum_{k=1}^{K} \left\{ (\delta_k^+ f_k)(g_{k+1}+g_k)(f_{k+1}+f_k)(\delta_k^+ g_k) + (\delta_k^- f_k)(g_k+g_{k-1})(f_k+f_{k-1})(\delta_k^- g_k) \right\} \Delta x \\
&\quad + \frac{1}{8}\sum_{k=1}^{K} \left\{ (f_{k+1}+f_k)^2 (\delta_k^+ g_k)^2 + (f_k + f_{k-1})^2 (\delta_k^- g_k)^2 \right\} \Delta x.
\end{aligned} \tag{29}
$$

By using the inequality: $(a+b)^2 \le 2(a^2+b^2)$ for all $a, b \in \mathbb{R}$, we have

$$
\frac{1}{8}\sum_{k=1}^{K} \left\{ (\delta_k^+ f_k)^2 (g_{k+1}+g_k)^2 + (\delta_k^- f_k)^2 (g_k + g_{k-1})^2 \right\} \Delta x \le \|\boldsymbol{g}\|_{L^\infty_{\mathrm{d}}}^2 \|D\boldsymbol{f}\|^2, \tag{30}
$$

$$
\frac{1}{8}\sum_{k=1}^{K} \left\{ (f_{k+1}+f_k)^2 (\delta_k^+ g_k)^2 + (f_k + f_{k-1})^2 (\delta_k^- g_k)^2 \right\} \Delta x \le \|\boldsymbol{f}\|_{L^\infty_{\mathrm{d}}}^2 \|D\boldsymbol{g}\|^2. \tag{31}
$$

Furthermore, by the fundamental inequality:

$$
\sum_{k=1}^{K} (a_k b_k + c_k d_k) \le \sqrt{\sum_{k=1}^{K} (a_k^2 + c_k^2)} \sqrt{\sum_{k=1}^{K} (b_k^2 + d_k^2)} \tag{32}
$$

for all $\{a_k\}_{k=1}^K, \{b_k\}_{k=1}^K, \{c_k\}_{k=1}^K, \{d_k\}_{k=1}^K \in \mathbb{R}^K$, we are led to

$$
\begin{aligned}
&\frac{1}{4}\sum_{k=1}^{K} \left\{ (\delta_k^+ f_k)(g_{k+1}+g_k)(f_{k+1}+f_k)(\delta_k^+ g_k) + (\delta_k^- f_k)(g_k+g_{k-1})(f_k+f_{k-1})(\delta_k^- g_k) \right\} \Delta x \\
&\le \frac{1}{4}\sqrt{\sum_{k=1}^{K} \left\{ (\delta_k^+ f_k)^2 (g_{k+1}+g_k)^2 + (\delta_k^- f_k)^2 (g_k + g_{k-1})^2 \right\} \Delta x} \\
&\quad \times \sqrt{\sum_{k=1}^{K} \left\{ (f_{k+1}+f_k)^2 (\delta_k^+ g_k)^2 + (f_k + f_{k-1})^2 (\delta_k^- g_k)^2 \right\} \Delta x} \\
&\le \|\boldsymbol{g}\|_{L^\infty_{\mathrm{d}}} \|\boldsymbol{f}\|_{L^\infty_{\mathrm{d}}} \sqrt{\sum_{k=1}^{K} \left\{ (\delta_k^+ f_k)^2 + (\delta_k^- f_k)^2 \right\} \Delta x} \sqrt{\sum_{k=1}^{K} \left\{ (\delta_k^+ g_k)^2 + (\delta_k^- g_k)^2 \right\} \Delta x} \\
&= 2\|\boldsymbol{g}\|_{L^\infty_{\mathrm{d}}} \|D\boldsymbol{f}\| \, \|\boldsymbol{f}\|_{L^\infty_{\mathrm{d}}} \|D\boldsymbol{g}\|.
\end{aligned} \tag{33}
$$

From (29)–(33), we obtain

$$\begin{aligned}\|D(\boldsymbol{f}\boldsymbol{g})\|^2 &\le \|\boldsymbol{g}\|^2_{L^\infty_{\rm d}} \|D\boldsymbol{f}\|^2 + 2\|\boldsymbol{g}\|_{L^\infty_{\rm d}} \|D\boldsymbol{f}\| \, \|\boldsymbol{f}\|_{L^\infty_{\rm d}} \|D\boldsymbol{g}\| + \|\boldsymbol{f}\|^2_{L^\infty_{\rm d}} \|D\boldsymbol{g}\|^2 \\ &= (\|\boldsymbol{f}\|_{L^\infty_{\rm d}} \|D\boldsymbol{g}\| + \|\boldsymbol{g}\|_{L^\infty_{\rm d}} \|D\boldsymbol{f}\|)^2.\end{aligned}$$

Namely, we have (28). □

Lemma 4.4. *Assume that $F \in C^3(\Omega)$. For any $\boldsymbol{f}_1 = \{f_{1,k}\}_{k=0}^{K+1}$, $\boldsymbol{f}_2 = \{f_{2,k}\}_{k=0}^{K+1}$, $\boldsymbol{f}_3 = \{f_{3,k}\}_{k=0}^{K+1}$, $\boldsymbol{f}_4 = \{f_{4,k}\}_{k=0}^{K+1} \in \mathbb{R}^{K+2}$, all the elements of which are in Ω, we have*

$$\left\| D\bar{F}''(\boldsymbol{f}_1, \boldsymbol{f}_2; \boldsymbol{f}_3, \boldsymbol{f}_4) \right\| \le \frac{1}{6} \sup_{\xi\in\Omega} |F'''(\xi)| \left(2\|D\boldsymbol{f}_1\| + 2\|D\boldsymbol{f}_2\| + \|D\boldsymbol{f}_3\| + \|D\boldsymbol{f}_4\|\right).$$

Proof. Let us define

$$\begin{aligned} H(\xi,\eta,\zeta) &:= \frac{d}{d(\xi,\eta)}\left(\frac{dF}{d(\cdot,\zeta)}\right) \\ &= \begin{cases} \dfrac{1}{\xi-\eta}\left(\dfrac{dF}{d(\xi,\zeta)} - \dfrac{dF}{d(\eta,\zeta)}\right) & (\xi\ne\eta), \\ \partial_\xi \left(\dfrac{dF}{d(\xi,\zeta)}\right)\Bigg|_{\xi=\eta} & (\xi=\eta) \end{cases} \end{aligned}$$

for all $\xi, \eta, \zeta \in \Omega$. Namely, for any fixed $\boldsymbol{f}_1 = \{f_{1,k}\}_{k=0}^{K+1}$, $\boldsymbol{f}_2 = \{f_{2,k}\}_{k=0}^{K+1}$, $\boldsymbol{f}_3 = \{f_{3,k}\}_{k=0}^{K+1}$, $\boldsymbol{f}_4 = \{f_{4,k}\}_{k=0}^{K+1} \in \mathbb{R}^{K+2}$, all the elements of which are in Ω, we see that

$$\begin{aligned} \bar{F}''(f_{1,k}, f_{2,k}; f_{3,k}, f_{4,k}) &= \frac{d}{d(f_{1,k}, f_{2,k})}\left(\frac{dF}{d(\cdot, f_{3,k})} + \frac{dF}{d(\cdot, f_{4,k})}\right) \\ &= H(f_{1,k}, f_{2,k}, f_{3,k}) + H(f_{1,k}, f_{2,k}, f_{4,k}) \quad (k = 0, 1, \ldots, K, K+1). \end{aligned} \tag{34}$$

Moreover, it holds that

$$\begin{aligned} \delta_k^+ H(f_{1,k}, f_{2,k}, f_{3,k}) &= (\delta_k^+ f_{1,k}) \frac{d}{d(f_{1,k+1}, f_{1,k})} H(\cdot, f_{2,k+1}, f_{3,k+1}) \\ &\quad + (\delta_k^+ f_{2,k}) \frac{d}{d(f_{2,k+1}, f_{2,k})} H(f_{1,k}, \cdot, f_{3,k+1}) \\ &\quad + (\delta_k^+ f_{3,k}) \frac{d}{d(f_{3,k+1}, f_{3,k})} H(f_{1,k}, f_{2,k}, \cdot) \quad (k = 1, \ldots, K), \\ \delta_k^- H(f_{1,k}, f_{2,k}, f_{3,k}) &= (\delta_k^- f_{1,k}) \frac{d}{d(f_{1,k}, f_{1,k-1})} H(\cdot, f_{2,k}, f_{3,k}) \\ &\quad + (\delta_k^- f_{2,k}) \frac{d}{d(f_{2,k}, f_{2,k-1})} H(f_{1,k-1}, \cdot, f_{3,k}) \\ &\quad + (\delta_k^- f_{3,k}) \frac{d}{d(f_{3,k}, f_{3,k-1})} H(f_{1,k-1}, f_{2,k-1}, \cdot) \quad (k = 1, \ldots, K). \end{aligned}$$

From now on, to avoid complications, we use the following expressions:

$$
\begin{aligned}
H_{1,k} &:= \frac{d}{d(f_{1,k}, f_{1,k-1})} H(\cdot, f_{2,k}, f_{3,k}) \quad (k = 1, \dots, K), \\
H_{2,k} &:= \frac{d}{d(f_{2,k}, f_{2,k-1})} H(f_{1,k-1}, \cdot, f_{3,k}) \quad (k = 1, \dots, K), \\
H_{3,k} &:= \frac{d}{d(f_{3,k}, f_{3,k-1})} H(f_{1,k-1}, f_{2,k-1}, \cdot) \quad (k = 1, \dots, K).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\frac{|\delta_k^+ H(f_{1,k}, f_{2,k}, f_{3,k})|^2 + |\delta_k^- H(f_{1,k}, f_{2,k}, f_{3,k})|^2}{2} \\
&= \frac{1}{2} \left\{ |\delta_k^+ f_{1,k}|^2 \, |H_{1,k+1}|^2 + |\delta_k^+ f_{2,k}|^2 \, |H_{2,k+1}|^2 + |\delta_k^+ f_{3,k}|^2 \, |H_{3,k+1}|^2 \right\} \\
&\quad + \frac{1}{2} \left\{ |\delta_k^- f_{1,k}|^2 \, |H_{1,k}|^2 + |\delta_k^- f_{2,k}|^2 |H_{2,k}|^2 + |\delta_k^- f_{3,k}|^2 \, |H_{3,k}|^2 \right\} \\
&\quad + (\delta_k^+ f_{1,k}) H_{1,k+1} (\delta_k^+ f_{2,k}) H_{2,k+1} + (\delta_k^- f_{1,k}) H_{1,k} (\delta_k^- f_{2,k}) H_{2,k} \\
&\quad + (\delta_k^+ f_{2,k}) H_{2,k+1} (\delta_k^+ f_{3,k}) H_{3,k+1} + (\delta_k^- f_{2,k}) H_{2,k} (\delta_k^- f_{3,k}) H_{3,k} \\
&\quad + (\delta_k^+ f_{3,k}) H_{3,k+1} (\delta_k^+ f_{1,k}) H_{1,k+1} + (\delta_k^- f_{3,k}) H_{3,k} (\delta_k^- f_{1,k}) H_{1,k}.
\end{aligned}
$$

By the fundamental inequality (32), we obtain

$$
\begin{aligned}
&\sum_{k=1}^{K} \left\{ (\delta_k^+ f_{1,k}) H_{1,k+1} (\delta_k^+ f_{2,k}) H_{2,k+1} + (\delta_k^- f_{1,k}) H_{1,k} (\delta_k^- f_{2,k}) H_{2,k} \right\} \Delta x \\
&\leq \left\{ \sum_{k=1}^{K} \left(|\delta_k^+ f_{1,k}|^2 \, |H_{1,k+1}|^2 + |\delta_k^- f_{1,k}|^2 \, |H_{1,k}|^2 \right) \Delta x \right\}^{\frac{1}{2}} \\
&\quad \times \left\{ \sum_{k=1}^{K} \left(|\delta_k^+ f_{2,k}|^2 \, |H_{2,k+1}|^2 + |\delta_k^- f_{2,k}|^2 \, |H_{2,k}|^2 \right) \Delta x \right\}^{\frac{1}{2}}.
\end{aligned}
$$

Using the mean value theorem, we get

$$
H_{1,k+1} \leq \sup_{\eta,\zeta \in \Omega} \sup_{\xi \in \Omega} |\partial_\xi H(\xi, \eta, \zeta)|, \;\; H_{1,k} \leq \sup_{\eta,\zeta \in \Omega} \sup_{\xi \in \Omega} |\partial_\xi H(\xi, \eta, \zeta)| \quad (k = 1, \dots, K).
$$

Thus, we have

$$
\left\{ \sum_{k=1}^{K} \left(|\delta_k^+ f_{1,k}|^2 \, |H_{1,k+1}|^2 + |\delta_k^- f_{1,k}|^2 \, |H_{1,k}|^2 \right) \Delta x \right\}^{\frac{1}{2}} \leq \sqrt{2} \left(\sup_{\eta,\zeta \in \Omega} \sup_{\xi \in \Omega} |\partial_\xi H(\xi, \eta, \zeta)| \right) \|D \boldsymbol{f}_1\|.
$$

Similarly, we obtain

$$
\left\{ \sum_{k=1}^{K} \left(|\delta_k^+ f_{2,k}|^2 \, |H_{2,k+1}|^2 + |\delta_k^- f_{2,k}|^2 \, |H_{2,k}|^2 \right) \Delta x \right\}^{\frac{1}{2}} \leq \sqrt{2} \left(\sup_{\xi,\zeta \in \Omega} \sup_{\eta \in \Omega} |\partial_\eta H(\xi, \eta, \zeta)| \right) \|D \boldsymbol{f}_2\|.
$$

Hence, we are led to

$$\begin{aligned}
&\sum_{k=1}^{K}\left\{(\delta_k^+ f_{1,k})H_{1,k+1}(\delta_k^+ f_{2,k})H_{2,k+1}+(\delta_k^- f_{1,k})H_{1,k}(\delta_k^- f_{2,k})H_{2,k}\right\}\Delta x\\
&\le 2\left(\sup_{\eta,\zeta\in\Omega}\sup_{\xi\in\Omega}|\partial_\xi H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_1\|\left(\sup_{\xi,\zeta\in\Omega}\sup_{\eta\in\Omega}|\partial_\eta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_2\|.
\end{aligned}$$

In the same manner, we obtain

$$\begin{aligned}
&\sum_{k=1}^{K}\left\{(\delta_k^+ f_{2,k})H_{2,k+1}(\delta_k^+ f_{3,k})H_{3,k+1}+(\delta_k^- f_{2,k})H_{2,k}(\delta_k^- f_{3,k})H_{3,k}\right\}\Delta x\\
&\le 2\left(\sup_{\xi,\zeta\in\Omega}\sup_{\eta\in\Omega}|\partial_\eta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_2\|\left(\sup_{\xi,\eta\in\Omega}\sup_{\zeta\in\Omega}|\partial_\zeta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_3\|,\\
&\sum_{k=1}^{K}\left\{(\delta_k^+ f_{3,k})H_{3,k}(\delta_k^+ f_{1,k})H_{1,k}+(\delta_k^- f_{3,k})H_{3,k}(\delta_k^- f_{1,k})H_{1,k}\right\}\Delta x\\
&\le 2\left(\sup_{\xi,\eta\in\Omega}\sup_{\zeta\in\Omega}|\partial_\zeta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_3\|\left(\sup_{\eta,\zeta\in\Omega}\sup_{\xi\in\Omega}|\partial_\xi H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_1\|.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\sum_{k=1}^{K}\frac{|\delta_k^+ H(f_{1,k},f_{2,k},f_{3,k})|^2+|\delta_k^- H(f_{1,k},f_{2,k},f_{3,k})|^2}{2}\Delta x\\
&\le\left(\sup_{\eta,\zeta\in\Omega}\sup_{\xi\in\Omega}|\partial_\xi H(\xi,\eta,\zeta)|\right)^2\|D\boldsymbol{f}_1\|^2+\left(\sup_{\xi,\zeta\in\Omega}\sup_{\eta\in\Omega}|\partial_\eta H(\xi,\eta,\zeta)|\right)^2\|D\boldsymbol{f}_2\|^2\\
&\quad+\left(\sup_{\xi,\eta\in\Omega}\sup_{\zeta\in\Omega}|\partial_\zeta H(\xi,\eta,\zeta)|\right)^2\|D\boldsymbol{f}_3\|^2+2\left\{\left(\sup_{\eta,\zeta\in\Omega}\sup_{\xi\in\Omega}|\partial_\xi H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_1\|\right.\\
&\quad\times\left(\sup_{\xi,\zeta\in\Omega}\sup_{\eta\in\Omega}|\partial_\eta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_2\|+\left(\sup_{\xi,\zeta\in\Omega}\sup_{\eta\in\Omega}|\partial_\eta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_2\|\\
&\quad\times\left(\sup_{\xi,\eta\in\Omega}\sup_{\zeta\in\Omega}|\partial_\zeta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_3\|+\left(\sup_{\xi,\eta\in\Omega}\sup_{\zeta\in\Omega}|\partial_\zeta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_3\|\\
&\quad\left.\times\left(\sup_{\eta,\zeta\in\Omega}\sup_{\xi\in\Omega}|\partial_\xi H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_1\|\right\}\\
&\quad=\left\{\left(\sup_{\eta,\zeta\in\Omega}\sup_{\xi\in\Omega}|\partial_\xi H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_1\|+\left(\sup_{\xi,\zeta\in\Omega}\sup_{\eta\in\Omega}|\partial_\eta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_2\|\right.\\
&\qquad\left.+\left(\sup_{\xi,\eta\in\Omega}\sup_{\zeta\in\Omega}|\partial_\zeta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_3\|\right\}^2.
\end{aligned}$$

Namely, it holds that

$$\begin{aligned}
\|DH(\boldsymbol{f}_1,\boldsymbol{f}_2,\boldsymbol{f}_3)\|\le&\left(\sup_{\eta,\zeta\in\Omega}\sup_{\xi\in\Omega}|\partial_\xi H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_1\|+\left(\sup_{\xi,\zeta\in\Omega}\sup_{\eta\in\Omega}|\partial_\eta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_2\|\\
&+\left(\sup_{\xi,\eta\in\Omega}\sup_{\zeta\in\Omega}|\partial_\zeta H(\xi,\eta,\zeta)|\right)\|D\boldsymbol{f}_3\|.
\end{aligned}$$

From the same argument as the proof of Lemma 2.6 in [31], we have

$$\begin{aligned}
&\sup_{\xi\in\Omega}|\partial_\xi H(\xi,\eta,\zeta)| \le \frac{1}{6}\sup_{s\in\Omega}|F'''(s)| \quad \text{for all } \eta,\zeta\in\Omega,\\
&\sup_{\eta\in\Omega}|\partial_\eta H(\xi,\eta,\zeta)| \le \frac{1}{6}\sup_{s\in\Omega}|F'''(s)| \quad \text{for all } \xi,\zeta\in\Omega,\\
&\sup_{\zeta\in\Omega}|\partial_\zeta H(\xi,\eta,\zeta)| \le \frac{1}{6}\sup_{s\in\Omega}|F'''(s)| \quad \text{for all } \xi,\eta\in\Omega.
\end{aligned}$$

Therefore, we obtain

$$\|DH(\boldsymbol{f}_1,\boldsymbol{f}_2,\boldsymbol{f}_3)\| \le \frac{1}{6}\sup_{\xi\in\Omega}|F'''(\xi)|(\|D\boldsymbol{f}_1\|+\|D\boldsymbol{f}_2\|+\|D\boldsymbol{f}_3\|). \tag{35}$$

Similarly, we get

$$\|DH(\boldsymbol{f}_1,\boldsymbol{f}_2,\boldsymbol{f}_4)\| \le \frac{1}{6}\sup_{\xi\in\Omega}|F'''(\xi)|(\|D\boldsymbol{f}_1\|+\|D\boldsymbol{f}_2\|+\|D\boldsymbol{f}_4\|). \tag{36}$$

Hence, from (34), (35), and (36), we conclude that

$$\begin{aligned}
\left\|D\bar{F}''(\boldsymbol{f}_1,\boldsymbol{f}_2;\boldsymbol{f}_3,\boldsymbol{f}_4)\right\| &\le \|DH(\boldsymbol{f}_1,\boldsymbol{f}_2,\boldsymbol{f}_3)\| + \|DH(\boldsymbol{f}_1,\boldsymbol{f}_2,\boldsymbol{f}_4)\|\\
&\le \frac{1}{6}\sup_{\xi\in\Omega}|F'''(\xi)|(2\|D\boldsymbol{f}_1\|+2\|D\boldsymbol{f}_2\|+\|D\boldsymbol{f}_3\|+\|D\boldsymbol{f}_4\|).
\end{aligned}$$

This completes the proof. □

At the end of this subsection, we mention about the elementary lemma from the matrix theory. This lemma will be applied in the proof of Theorem 4.1.

Lemma 4.5. *Let A be an arbitrary $n\times n$ Hermitian positive semi-definite matrix and let B be an arbitrary $n\times n$ Hermitian positive definite matrix. Then, the eigenvalues of AB are all real and nonnegative.*

4.2 Existence and uniqueness of the solution

Theorem 4.1. *Assume that the potential function F is in C^3. For any given $\boldsymbol{U}^{(0)} = \{U_k^{(0)}\}_{k=0}^{K+1} \in \mathbb{R}^{K+2}$, let*

$$B_0 := \left\{\frac{2}{\gamma}\left(J_{\mathrm{d}}(\boldsymbol{U}^{(0)}) + L\left|\min\left\{\inf_{\xi\in\mathbb{R}}F(\xi),0\right\}\right|\right)\right\}^{\frac{1}{2}}, \quad \tilde{B}_0 := \frac{1}{L}\left|M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\right| + 2L^{\frac{1}{2}}B_0.$$

If Δt satisfies

$$\max\left\{\frac{3}{2}\max_{|\xi|\le 2\tilde{B}_0}|F''(\xi)|,\ \frac{1}{2}\max_{|\xi|\le 2\tilde{B}_0}|F''(\xi)| + \frac{5L^{\frac{1}{2}}B_0}{3}\max_{|\xi|\le 2\tilde{B}_0}|F'''(\xi)|\right\}\sqrt{\frac{\Delta t}{2\gamma}} < 1, \tag{37}$$

then there exists a unique solution $\{U_k^{(n)}\}_{k=0}^{K+1}\in\mathbb{R}^{K+2}$ $(n=1,2,\dots)$ of (19)–(23).

Remark 4.1. The condition (37) is independent of the space mesh size Δx. It is the first advantage of our numerical method. Additionally, (37) can be derived explicitly with respect to Δt. It is the second advantage.

Proof. We show the existence of $\boldsymbol{U}^{(n+1)} = \{U_k^{(n+1)}\}_{k=0}^{K+1} \in \mathbb{R}^{K+2}$ for any given $\boldsymbol{U}^{(n)} = \{U_k^{(n)}\}_{k=0}^{K+1} \in \mathbb{R}^{K+2}$ satisfying (19)–(23). For the purpose, we define the nonlinear mapping Ψ: $\{U_k\}_{k=0}^{K+1} \mapsto \{\tilde{U}_k\}_{k=0}^{K+1}$ by

$$\frac{\tilde{U}_k - U_k^{(n)}}{\Delta t} = \delta_k^{\langle 2\rangle} \tilde{P}_k^{(n)} \quad (k = 1, \ldots, K), \tag{38}$$

$$\tilde{P}_k^{(n)} = -\gamma \delta_k^{\langle 2\rangle} \left(\frac{\tilde{U}_k + U_k^{(n)}}{2} \right) + \frac{dF}{d(U_k, U_k^{(n)})} \quad (k = 1, \ldots, K), \tag{39}$$

$$\frac{\mu_k^+ \tilde{U}_k \Big|_{k=0} - \mu_k^+ U_k^{(n)} \Big|_{k=0}}{\Delta t} = \delta_k^+ \left(\frac{\tilde{U}_k + U_k^{(n)}}{2} \right) \Bigg|_{k=0}, \tag{40}$$

$$\frac{\mu_k^- \tilde{U}_k \Big|_{k=K+1} - \mu_k^- U_k^{(n)} \Big|_{k=K+1}}{\Delta t} = -\,\delta_k^{\langle 1\rangle} \left(\frac{\tilde{U}_k + U_k^{(n)}}{2} \right) \Bigg|_{k=K}, \tag{41}$$

$$\delta_k^+ \tilde{P}_k^{(n)} \Big|_{k=0} = \delta_k^- \tilde{P}_k^{(n)} \Big|_{k=K+1} = 0. \tag{42}$$

First, we show that the mapping Ψ is well-defined. For any fixed $\boldsymbol{U} = \{U_k\}_{k=0}^{K+1} \in \mathbb{R}^{K+2}$, from (40) and (41), $\tilde{U}_0$ and $\tilde{U}_{K+1}$ can be explicitly written as

$$\tilde{U}_0 = \alpha \tilde{U}_1 - \alpha U_0^{(n)} + U_1^{(n)}, \tag{43}$$

$$\tilde{U}_{K+1} = \alpha \tilde{U}_K + U_K^{(n)} - \alpha U_{K+1}^{(n)}, \tag{44}$$

where α is defined by

$$\alpha := \frac{\dfrac{1}{\Delta x} - \dfrac{1}{\Delta t}}{\dfrac{1}{\Delta x} + \dfrac{1}{\Delta t}}.$$

Thus, it is sufficient to show that $\tilde{U}_k$ $(k = 1, \ldots, K)$ can be explicitly written by given $\boldsymbol{U}$ and $\boldsymbol{U}^{(n)}$. Using (42)–(44), we eliminate $\tilde{U}_0$, $\tilde{U}_{K+1}$, $\tilde{P}_0^{(n)}$, and $\tilde{P}_{K+1}^{(n)}$ in (38) and (39). Thus, we have

$$\frac{\tilde{U}_1 - U_1^{(n)}}{\Delta t} = \frac{1}{(\Delta x)^2} \left(\tilde{P}_2^{(n)} - \tilde{P}_1^{(n)} \right), \tag{45}$$

$$\frac{\tilde{U}_k - U_k^{(n)}}{\Delta t} = \delta_k^{\langle 2\rangle} \tilde{P}_k^{(n)} \quad (k = 2, \ldots, K-1), \tag{46}$$

$$\frac{\tilde{U}_K - U_K^{(n)}}{\Delta t} = \frac{1}{(\Delta x)^2} \left(\tilde{P}_{K-1}^{(n)} - \tilde{P}_K^{(n)} \right), \tag{47}$$

$$\tilde{P}_0^{(n)} = -\frac{\gamma}{2(\Delta x)^2} \left\{ \tilde{U}_2 - (2-\alpha)\tilde{U}_1 \right\} - \frac{\gamma}{2(\Delta x)^2} (U_1^{(n)} - \alpha U_0^{(n)}) - \frac{\gamma}{2}\, \delta_k^{\langle 2\rangle} U_k^{(n)} \Big|_{k=1} + \frac{dF}{d(U_1, U_1^{(n)})}, \tag{48}$$

$$\tilde{P}_k^{(n)} = -\gamma \delta_k^{\langle 2\rangle} \left(\frac{\tilde{U}_k + U_k^{(n)}}{2} \right) + \frac{dF}{d(U_k, U_k^{(n)})} \quad (k = 2, \ldots, K-1), \tag{49}$$

$$\tilde{P}_K^{(n)} = -\frac{\gamma}{2(\Delta x)^2} \left\{ \tilde{U}_{K-1} - (2-\alpha)\tilde{U}_K \right\} - \frac{\gamma}{2(\Delta x)^2} (U_K^{(n)} - \alpha U_{K+1}^{(n)}) - \frac{\gamma}{2}\, \delta_k^{\langle 2\rangle} U_k^{(n)} \Big|_{k=K} + \frac{dF}{d(U_K, U_K^{(n)})}. \tag{50}$$

Here, we give the following matrix expression of (45)–(50):

$$A\tilde{\boldsymbol{U}} = f(\boldsymbol{U}, \boldsymbol{U}^{(n)}).$$

By using (48)–(50), we eliminate $\tilde{P}_k^{(n)}$ in (45)–(47). Then, the $K \times K$ matrix A is defined by

$$A := I + \beta \begin{pmatrix} 3-\alpha & -3 & 1 & & & & & & \\ -(4-\alpha) & 6 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & -4 & 6 & -4 & 1 & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -4 & 6 & -(4-\alpha) \\ & & & & & & 1 & -3 & 3-\alpha \end{pmatrix},$$

where β is defined by $\beta := (\gamma\Delta t)/(2(\Delta x)^4)$. Also, f is a K-dimensional vector consisting only of $\boldsymbol{U}$ and $\boldsymbol{U}^{(n)}$. The concrete definition of f is omitted here since it does not concern the essence of the following proof. If the matrix A is nonsingular, then the mapping Ψ is well-defined. In fact, A is nonsingular. For the purpose, we show that the determinant of A is positive. Firstly, let us define the $K \times K$ matrix $\tilde{D}_2$ by

$$\tilde{D}_2 := \begin{pmatrix} -2+\alpha & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2+\alpha \end{pmatrix}.$$

Then, we obtain

$$D_2\tilde{D}_2 = \begin{pmatrix} 3-\alpha & -3 & 1 & & & & & & \\ -(4-\alpha) & 6 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & -4 & 6 & -4 & 1 & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -4 & 6 & -(4-\alpha) \\ & & & & & & 1 & -3 & 3-\alpha \end{pmatrix}.$$

Namely, $A = I + \beta D_2\tilde{D}_2$. Moreover, $-D_2$ is a symmetric positive semi-definite matrix, and $-\tilde{D}_2$ is a symmetric positive definite matrix. Actually, for any non-zero vector $\boldsymbol{x} =$

$(x_1, x_2, \dots, x_K)^\top \in \mathbb{R}^K$, it holds that

$$(-\tilde{D}_2)\boldsymbol{x} = \begin{pmatrix} (2-\alpha)x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 - x_4 \\ \vdots \\ -x_{K-3} + 2x_{K-2} - x_{K-1} \\ -x_{K-2} + 2x_{K-1} - x_K \\ -x_{K-1} + (2-\alpha)x_K \end{pmatrix}.$$

Furthermore, we have $\alpha < 1$ from $\Delta t > 0$. Thus, we obtain

$$\begin{aligned} &\boldsymbol{x}^\top(-\tilde{D}_2)\boldsymbol{x} \\ &= (2-\alpha)x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2 - x_2x_3 - x_2x_3 + 2x_3^2 - x_3x_4 - \cdots \\ &\quad \cdots - x_{K-3}x_{K-1} + 2x_{K-2}^2 - x_{K-2}x_{K-1} - x_{K-2}x_{K-1} + 2x_{K-1}^2 - x_{K-1}x_K - x_{K-1}x_K + (2-\alpha)x_K^2 \\ &= (1-\alpha)x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots \\ &\quad \cdots + (x_{K-2} - x_{K-1})^2 + (x_{K-1} - x_K)^2 + (1-\alpha)x_K^2 \geq 0. \end{aligned}$$

Suppose that $\boldsymbol{x}^\top(-\tilde{D}_2)\boldsymbol{x} = 0$. Then, we get $x_1 = \cdots = x_K = 0$. This is contradictory to $\boldsymbol{x} \neq \boldsymbol{0}$. Namely, $-\tilde{D}_2$ is positive definite. Similarly, by the direct calculation, we can see that $-D_2$ is positive semi-definite. Therefore, from Lemma 4.5, eigenvalues of $D_2\tilde{D}_2 = (-D_2)(-\tilde{D}_2)$ are all real and nonnegative. Hence, eigenvalues of A are all positive from two facts that eigenvalues of $D_2\tilde{D}_2$ are all real and nonnegative and that β is positive. Thus, it is clear that A is nonsingular.

Next, we prove the existence and uniqueness of the solution to the proposed scheme by the fixed-point theorem for a contraction mapping. For the purpose, we define the mapping $\Theta : \mathbb{R}^{K+2} \to \mathbb{R}^{K+2}$ by

$$\Theta(\boldsymbol{V}) := \boldsymbol{V} + \frac{1}{L} M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\boldsymbol{1} \quad \text{for all } \boldsymbol{V} \in \mathbb{R}^{K+2}, \tag{51}$$

where $\boldsymbol{1} := (1, 1, \dots, 1)^\top \in \mathbb{R}^{K+2}$. Then, its inverse mapping Θ^{-1} is written as

$$\Theta^{-1}(\boldsymbol{V}) = \boldsymbol{V} - \frac{1}{L} M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\boldsymbol{1} \quad \text{for all } \boldsymbol{V} \in \mathbb{R}^{K+2}. \tag{52}$$

Let us define the mapping $\Phi : \mathbb{R}^{K+2} \to \mathbb{R}^{K+2}$ by

$$\Phi(\boldsymbol{V}) := (\Theta^{-1} \circ \Psi \circ \Theta)(\boldsymbol{V}) = \Theta^{-1}(\Psi(\Theta(\boldsymbol{V}))) \quad \text{for all } \boldsymbol{V} \in \mathbb{R}^{K+2}. \tag{53}$$

Moreover, let

$$X_0 := \left\{ \boldsymbol{f} \in \mathbb{R}^{K+2}; \|D\boldsymbol{f}\| \leq 2B_0,\ M_{\mathrm{d}}(\boldsymbol{f}) = 0 \right\}.$$

We show that Φ is a contraction mapping on X_0 under the assumption (37) on Δt. If Φ is a contraction mapping, Φ has a unique fixed-point $\boldsymbol{V}^*$ in the closed ball X_0 from the Banach fixed point theorem. That is, $\boldsymbol{V}^*$ satisfies $\Phi(\boldsymbol{V}^*) = \boldsymbol{V}^*$. From (52) and (53), we have

$$\Phi(\boldsymbol{V}^*) = \Theta^{-1}(\Psi(\Theta(\boldsymbol{V}^*))) = \Psi(\Theta(\boldsymbol{V}^*)) - \frac{1}{L} M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\boldsymbol{1}. \tag{54}$$

Furthermore, from (51), we obtain

$$\boldsymbol{V}^* = \Theta\left(\boldsymbol{V}^*\right) - \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\mathbf{1}. \tag{55}$$

Hence, from (54) and (55), it holds that $\Psi\left(\Theta\left(\boldsymbol{V}^*\right)\right) = \Theta\left(\boldsymbol{V}^*\right)$. Namely, $\Theta\left(\boldsymbol{V}^*\right)$ is the solution $\boldsymbol{U}^{(n+1)}$ to the scheme (19)–(23). Firstly, we show that $\Phi(X_0) \subset X_0$. For the purpose, we check $\|D(\Phi(\boldsymbol{V}))\| \leq 2B_0$ and $M_{\mathrm{d}}(\Phi(\boldsymbol{V})) = 0$ for any fixed $\boldsymbol{V} \in X_0$. Let $\boldsymbol{U} := \Theta(\boldsymbol{V})$. Then, from (51), we have

$$U_k = V_k + \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)}) \quad (k = 0, 1, \ldots, K, K+1). \tag{56}$$

Hence, it holds that

$$\delta_k^+ U_k = \delta_k^+\left(V_k + \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\right) = \delta_k^+ V_k \quad (k = 1, \ldots, K), \tag{57}$$

$$\delta_k^- U_k = \delta_k^-\left(V_k + \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\right) = \delta_k^- V_k \quad (k = 1, \ldots, K). \tag{58}$$

Let us define $\tilde{\boldsymbol{U}} := \Psi(\boldsymbol{U})$ and $\tilde{\boldsymbol{V}} := \Phi(\boldsymbol{V}) = \Theta^{-1}(\tilde{\boldsymbol{U}})$. Then, from (52), we obtain

$$\tilde{V}_k = \tilde{U}_k - \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)}) \quad (k = 0, 1, \ldots, K, K+1). \tag{59}$$

Hence, we have

$$\delta_k^+ \tilde{V}_k = \delta_k^+\left(\tilde{U}_k - \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\right) = \delta_k^+ \tilde{U}_k \quad (k = 1, \ldots, K), \tag{60}$$

$$\delta_k^- \tilde{V}_k = \delta_k^-\left(\tilde{U}_k - \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\right) = \delta_k^- \tilde{U}_k \quad (k = 1, \ldots, K). \tag{61}$$

In addition, it follows from (38), (42), Lemma 1.2, and Proposition 2.2 that

$$\begin{aligned} M_{\mathrm{d}}(\tilde{\boldsymbol{U}}) = \sum_{k=1}^{K} \tilde{U}_k \Delta x &= \sum_{k=1}^{K} U_k^{(n)} \Delta x + \Delta t \sum_{k=1}^{K} \delta_k^{\langle 2\rangle} \tilde{P}_k^{(n)} \Delta x \\ &= M_{\mathrm{d}}(\boldsymbol{U}^{(n)}) + \Delta t\left(\delta_k^- \tilde{P}_k^{(n)}\Big|_{k=K+1} - \delta_k^+ \tilde{P}_k^{(n)}\Big|_{k=0}\right) \\ &= M_{\mathrm{d}}(\boldsymbol{U}^{(0)}). \end{aligned}$$

Thus, it holds from (59) and the above inequality that

$$M_{\mathrm{d}}(\tilde{\boldsymbol{V}}) = \sum_{k=1}^{K} \tilde{V}_k \Delta x = \sum_{k=1}^{K}\left(\tilde{U}_k - \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\right)\Delta x = M_{\mathrm{d}}(\tilde{\boldsymbol{U}}) - M_{\mathrm{d}}(\boldsymbol{U}^{(0)}) = 0.$$

Therefore, all that is left is to show $\|D\tilde{\boldsymbol{V}}\| \leq 2B_0$. By using Lemma 2.1 and the Young inequality: $ab \leq (\varepsilon/2)a^2 + (1/(2\varepsilon))b^2$ for all $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we obtain from (38)–(42)

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left(\left\|D\tilde{\boldsymbol{U}}\right\|^2 - \|D\boldsymbol{U}^{(n)}\|^2\right)\\
&= \sum_{k=1}^{K} \frac{\left\{\delta_k^+\left(\dfrac{\tilde{U}_k + U_k^{(n)}}{2}\right)\right\}\left\{\delta_k^+\left(\dfrac{\tilde{U}_k - U_k^{(n)}}{\Delta t}\right)\right\} + \left\{\delta_k^-\left(\dfrac{\tilde{U}_k + U_k^{(n)}}{2}\right)\right\}\left\{\delta_k^-\left(\dfrac{\tilde{U}_k - U_k^{(n)}}{\Delta t}\right)\right\}}{2}\Delta x\\
&= -\sum_{k=1}^{K}\left\{\delta_k^{\langle 2\rangle}\left(\frac{\tilde{U}_k+U_k^{(n)}}{2}\right)\right\}\frac{\tilde{U}_k-U_k^{(n)}}{\Delta t}\Delta x + \left\{\delta_k^-\left(\frac{\tilde{U}_k+U_k^{(n)}}{2}\right)\bigg|_{k=K+1}\right\}\left\{\mu_k^-\left(\frac{\tilde{U}_k-U_k^{(n)}}{\Delta t}\right)\bigg|_{k=K+1}\right\}\\
&\quad - \left\{\delta_k^+\left(\frac{\tilde{U}_k+U_k^{(n)}}{2}\right)\bigg|_{k=0}\right\}\left\{\mu_k^+\left(\frac{\tilde{U}_k-U_k^{(n)}}{\Delta t}\right)\bigg|_{k=0}\right\}\\
&= -\frac{1}{\gamma}\sum_{k=1}^{K}\left(-\tilde{P}_k^{(n)} + \frac{dF}{d(U_k, U_k^{(n)})}\right)\left(\delta_k^{\langle 2\rangle}\tilde{P}_k^{(n)}\right)\Delta x - \left|\delta_k^-\left(\frac{\tilde{U}_k+U_k^{(n)}}{2}\right)\bigg|_{k=K+1}\right|^2 - \left|\delta_k^+\left(\frac{\tilde{U}_k+U_k^{(n)}}{2}\right)\bigg|_{k=0}\right|^2\\
&\le -\frac{1}{\gamma}\sum_{k=1}^{K}\left(\delta_k^{\langle 2\rangle}\tilde{P}_k^{(n)}\right)\left(-\tilde{P}_k^{(n)} + \frac{dF}{d(U_k, U_k^{(n)})}\right)\Delta x\\
&= \frac{1}{\gamma}\sum_{k=1}^{K}\frac{\left(\delta_k^+\tilde{P}_k^{(n)}\right)\left\{\delta_k^+\left(-\tilde{P}_k^{(n)} + \dfrac{dF}{d(U_k, U_k^{(n)})}\right)\right\} + \left(\delta_k^-\tilde{P}_k^{(n)}\right)\left\{\delta_k^-\left(-\tilde{P}_k^{(n)} + \dfrac{dF}{d(U_k, U_k^{(n)})}\right)\right\}}{2}\Delta x\\
&\quad - \frac{1}{\gamma}\left(\delta_k^-\tilde{P}_k^{(n)}\Big|_{k=K+1}\right)\left\{\mu_k^-\left(-\tilde{P}_k^{(n)} + \frac{dF}{d(U_k, U_k^{(n)})}\right)\bigg|_{k=K+1}\right\}\\
&\quad - \frac{1}{\gamma}\left(\delta_k^+\tilde{P}_k^{(n)}\Big|_{k=0}\right)\left\{\mu_k^+\left(-\tilde{P}_k^{(n)} + \frac{dF}{d(U_k, U_k^{(n)})}\right)\bigg|_{k=0}\right\}\\
&= -\frac{1}{\gamma}\left\|D\tilde{\boldsymbol{P}}^{(n)}\right\|^2 + \frac{1}{\gamma}\sum_{k=1}^{K}\frac{\left(\delta_k^+\tilde{P}_k^{(n)}\right)\left\{\delta_k^+\left(\dfrac{dF}{d(U_k, U_k^{(n)})}\right)\right\} + \left(\delta_k^-\tilde{P}_k^{(n)}\right)\left\{\delta_k^-\left(\dfrac{dF}{d(U_k, U_k^{(n)})}\right)\right\}}{2}\Delta x\\
&\le -\frac{1}{\gamma}\left\|D\tilde{\boldsymbol{P}}^{(n)}\right\|^2\\
&\quad + \frac{1}{2\gamma}\sum_{k=1}^{K}\left\{\left|\delta_k^+\tilde{P}_k^{(n)}\right|^2 + \frac{1}{4}\left|\delta_k^+\left(\frac{dF}{d(U_k, U_k^{(n)})}\right)\right|^2 + \left|\delta_k^-\tilde{P}_k^{(n)}\right|^2 + \frac{1}{4}\left|\delta_k^-\left(\frac{dF}{d(U_k, U_k^{(n)})}\right)\right|^2\right\}\Delta x\\
&= \frac{1}{4\gamma}\left\|D\left(\frac{dF}{d(\boldsymbol{U}, \boldsymbol{U}^{(n)})}\right)\right\|^2.
\end{aligned}
$$

Consequently, using the triangle inequality: $\sqrt{a^2+b^2} \le |a| + |b|$ for all $a, b \in \mathbb{R}$, we get

$$
\left\|D\tilde{\boldsymbol{U}}\right\| \le \|D\boldsymbol{U}^{(n)}\| + \sqrt{\frac{\Delta t}{2\gamma}}\left\|D\left(\frac{dF}{d\left(\boldsymbol{U}, \boldsymbol{U}^{(n)}\right)}\right)\right\|. \tag{62}
$$

Thus, from (60), (61), and (62), it is sufficient to show that the right-hand side of (62) is not greater than $2B_0$. For $k = 1, \dots, K$, using Lemma 4.2, we have

$$\delta_k^+\left(\frac{dF}{d(U_k, U_k^{(n)})}\right) = \frac{1}{2}\bar{F}''(U_{k+1}, U_k; U_{k+1}^{(n)}, U_k^{(n)})\delta_k^+ U_k + \frac{1}{2}\bar{F}''(U_{k+1}^{(n)}, U_k^{(n)}; U_{k+1}, U_k)\delta_k^+ U_k^{(n)}, \tag{63}$$

$$\delta_k^-\left(\frac{dF}{d(U_k, U_k^{(n)})}\right) = \frac{1}{2}\bar{F}''(U_k, U_{k-1}; U_k^{(n)}, U_{k-1}^{(n)})\delta_k^- U_k + \frac{1}{2}\bar{F}''(U_k^{(n)}, U_{k-1}^{(n)}; U_k, U_{k-1})\delta_k^- U_k^{(n)}. \tag{64}$$

From the same argument as in the proof of Lemma 4.3, by using (63), (64), and the fundamental inequality (32), we have

$$\begin{aligned}
&\left\| D\left(\frac{dF}{d\left(\boldsymbol{U}, \boldsymbol{U}^{(n)}\right)}\right)\right\|^2 \\
&= \frac{1}{2}\sum_{k=1}^{K}\left(\left|\frac{1}{2}\bar{F}''(U_{k+1}, U_k; U_{k+1}^{(n)}, U_k^{(n)})\delta_k^+ U_k + \frac{1}{2}\bar{F}''(U_{k+1}^{(n)}, U_k^{(n)}; U_{k+1}, U_k)\delta_k^+ U_k^{(n)}\right|^2\right. \\
&\quad \left. + \left|\frac{1}{2}\bar{F}''(U_k, U_{k-1}; U_k^{(n)}, U_{k-1}^{(n)})\delta_k^- U_k + \frac{1}{2}\bar{F}''(U_k^{(n)}, U_{k-1}^{(n)}; U_k, U_{k-1})\delta_k^- U_k^{(n)}\right|^2\right)\Delta x \\
&\le \frac{1}{4}\left\{\left(\max_{k=1,\dots,K+1}\left|\bar{F}''(U_k, U_{k-1}; U_k^{(n)}, U_{k-1}^{(n)})\right|\right)\|D\boldsymbol{U}\|\right. \\
&\quad \left. + \left(\max_{k=1,\dots,K+1}\left|\bar{F}''(U_k^{(n)}, U_{k-1}^{(n)}; U_k, U_{k-1})\right|\right)\left\|D\boldsymbol{U}^{(n)}\right\|\right\}^2.
\end{aligned}$$

Namely, we have

$$\begin{aligned}
\left\| D\left(\frac{dF}{d\left(\boldsymbol{U}, \boldsymbol{U}^{(n)}\right)}\right)\right\| \le{}& \frac{1}{2}\left(\max_{k=1,\dots,K+1}\left|\bar{F}''(U_k, U_{k-1}; U_k^{(n)}, U_{k-1}^{(n)})\right|\right)\|D\boldsymbol{U}\| \\
&+ \frac{1}{2}\left(\max_{k=1,\dots,K+1}\left|\bar{F}''(U_k^{(n)}, U_{k-1}^{(n)}; U_k, U_{k-1})\right|\right)\left\|D\boldsymbol{U}^{(n)}\right\|.
\end{aligned}$$

Next, we consider $|\bar{F}''(U_k, U_{k-1}; U_k^{(n)}, U_{k-1}^{(n)})|$ and $|\bar{F}''(U_k^{(n)}, U_{k-1}^{(n)}; U_k, U_{k-1})|$. It follows from Lemma 3.2 that

$$\|\boldsymbol{U}\|_{L^\infty_{\mathrm{d}}} \le \frac{1}{L}\left|M_{\mathrm{d}}(\boldsymbol{U})\right| + 2L^{\frac{1}{2}}\|D\boldsymbol{U}\|.$$

Since it holds from $\boldsymbol{V} \in X_0$ that $M_{\mathrm{d}}(\boldsymbol{V}) = 0$, we see from (56) that

$$M_{\mathrm{d}}(\boldsymbol{U}) = \sum_{k=1}^{K} U_k \Delta x = M_{\mathrm{d}}(\boldsymbol{V}) + M_{\mathrm{d}}(\boldsymbol{U}^{(0)}) = M_{\mathrm{d}}(\boldsymbol{U}^{(0)}).$$

Because it holds from $\boldsymbol{V} \in X_0$, (57), and (58) that $\|D\boldsymbol{U}\| = \|D\boldsymbol{V}\| \le 2B_0$, we obtain

$$\|\boldsymbol{U}\|_{L^\infty_{\mathrm{d}}} \le \frac{1}{L}\left|M_{\mathrm{d}}(\boldsymbol{U})\right| + 4L^{\frac{1}{2}}B_0 \le \frac{1}{L}\left|M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\right| + 4L^{\frac{1}{2}}B_0 \le 2\tilde{B}_0.$$

Also, using Theorem 3.1, we get $\|\boldsymbol{U}^{(n)}\|_{L^\infty_{\mathrm{d}}} \le \tilde{B}_0$. Therefore, from Lemma 4.1, we obtain

$$\left|\bar{F}''(U_k, U_{k-1}; U_k^{(n)}, U_{k-1}^{(n)})\right| \le \max_{|\xi|\le 2\tilde{B}_0} |F''(\xi)| \quad (k = 1, \dots, K+1),$$
$$\left|\bar{F}''(U_k^{(n)}, U_{k-1}^{(n)}; U_k, U_{k-1})\right| \le \max_{|\xi|\le 2\tilde{B}_0} |F''(\xi)| \quad (k = 1, \dots, K+1).$$

Hence, it holds that

$$\left\| D\left(\frac{dF}{d(\boldsymbol{U}, \boldsymbol{U}^{(n)})}\right)\right\| \le \frac{1}{2}\max_{|\xi|\le 2\tilde{B}_0} |F''(\xi)| \left(\|D\boldsymbol{U}\| + \|D\boldsymbol{U}^{(n)}\|\right). \tag{65}$$

Consequently, using (62), (65), and Lemma 3.1, the following estimate holds:

$$\left\|D\tilde{\boldsymbol{U}}\right\| \le B_0 + \frac{3\max_{|\xi|\le 2\tilde{B}_0}|F''(\xi)|}{2}\sqrt{\frac{\Delta t}{2\gamma}}B_0.$$

Now, from (60), (61), and the assumption (37), we have $\|D\tilde{\boldsymbol{V}}\| = \|D\tilde{\boldsymbol{U}}\| \le 2B_0$. From the above, it holds that $\Phi(\boldsymbol{V}) = \tilde{\boldsymbol{V}} \in X_0$, i.e., $\Phi(X_0) \subset X_0$. Next, we prove that Φ is contractive. For any $\boldsymbol{V}_1, \boldsymbol{V}_2 \in X_0$, let $\boldsymbol{U}_1 := \Theta(\boldsymbol{V}_1)$ and $\boldsymbol{U}_2 := \Theta(\boldsymbol{V}_2)$. From (51), it holds that

$$U_{i,k} = V_{i,k} + \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)}) \quad (k = 0, 1, \dots, K, K+1,\ i = 1, 2). \tag{66}$$

It follows from (66) that

$$\delta_k^+ U_{i,k} = \delta_k^+\left(V_{i,k} + \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\right) = \delta_k^+ V_{i,k} \quad (k = 1, \dots, K,\ i = 1, 2), \tag{67}$$
$$\delta_k^- U_{i,k} = \delta_k^-\left(V_{i,k} + \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\right) = \delta_k^- V_{i,k} \quad (k = 1, \dots, K,\ i = 1, 2). \tag{68}$$

Furthermore, from (66) and $\boldsymbol{V}_i \in X_0$ $(i = 1, 2)$, we have

$$M_{\mathrm{d}}(\boldsymbol{U}_i) = \sum_{k=1}^{K} U_{i,k}\Delta x = M_{\mathrm{d}}(\boldsymbol{V}_i) + M_{\mathrm{d}}(\boldsymbol{U}^{(0)}) = M_{\mathrm{d}}(\boldsymbol{U}^{(0)}) \quad (i = 1, 2). \tag{69}$$

Moreover, it follows from $U_{1,k} - U_{2,k} = V_{1,k} - V_{2,k}$ $(k = 0, 1, \dots, K, K+1)$ that $\|D(\boldsymbol{U}_1 - \boldsymbol{U}_2)\| = \|D(\boldsymbol{V}_1 - \boldsymbol{V}_2)\|$. Now, let us define $\tilde{\boldsymbol{U}}_i := \Psi(\boldsymbol{U}_i)$ and $\tilde{\boldsymbol{V}}_i := \Phi(\boldsymbol{V}_i) = \Theta^{-1}(\tilde{\boldsymbol{U}}_i)$ $(i = 1, 2)$. Then, from (52), we have

$$\tilde{V}_{i,k} = \tilde{U}_{i,k} - \frac{1}{L}M_{\mathrm{d}}(\boldsymbol{U}^{(0)}) \quad (k = 0, 1, \dots, K, K+1,\ i = 1, 2).$$

Hence, it holds that $\tilde{V}_{1,k} - \tilde{V}_{2,k} = \tilde{U}_{1,k} - \tilde{U}_{2,k}$ $(k = 0, 1 \dots, K, K+1)$. Namely, $\|D(\tilde{\boldsymbol{V}}_1 - \tilde{\boldsymbol{V}}_2)\| = \|D(\tilde{\boldsymbol{U}}_1 - \tilde{\boldsymbol{U}}_2)\|$. Now, from the definition of Ψ, the vector $\tilde{\boldsymbol{U}}_i = \Psi(\boldsymbol{U}_i)$ satisfies (38)–(42) $(i = 1, 2)$. Subtracting these relations, we obtain

$$\frac{\tilde{U}_{1,k}-\tilde{U}_{2,k}}{\Delta t} = \delta_k^{\langle 2\rangle}\left(\tilde{P}_{1,k}^{(n)} - \tilde{P}_{2,k}^{(n)}\right) \quad (k=1,\ldots,K), \tag{70}$$

$$\delta_k^{\langle 2\rangle}\left(\tilde{U}_{1,k}-\tilde{U}_{2,k}\right) = \frac{2}{\gamma}\left\{-\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)+\left(\frac{dF}{d(U_{1,k},U_k^{(n)})}-\frac{dF}{d(U_{2,k},U_k^{(n)})}\right)\right\} \quad (k=1,\ldots,K), \tag{71}$$

$$\frac{\mu_k^+\tilde{U}_{1,k}\Big|_{k=0} - \mu_k^+\tilde{U}_{2,k}\Big|_{k=0}}{\Delta t} = \delta_k^+\left(\frac{\tilde{U}_{1,k}-\tilde{U}_{2,k}}{2}\right)\Bigg|_{k=0}, \tag{72}$$

$$\frac{\mu_k^-\tilde{U}_{1,k}\Big|_{k=K+1} - \mu_k^-\tilde{U}_{2,k}\Big|_{k=K+1}}{\Delta t} = -\,\delta_k^-\left(\frac{\tilde{U}_{1,k}-\tilde{U}_{2,k}}{2}\right)\Bigg|_{k=K+1}, \tag{73}$$

$$\delta_k^+\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\Big|_{k=0} = \delta_k^-\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\Big|_{k=K+1} = 0. \tag{74}$$

From Lemma 1.1, (70)–(74), and the Young inequality, we have

$$\begin{aligned}
&\left\|D(\tilde{\boldsymbol{U}}_1-\tilde{\boldsymbol{U}}_2)\right\|^2 \\
&= -\sum_{k=1}^{K}\left\{\delta_k^{\langle 2\rangle}\left(\tilde{U}_{1,k}-\tilde{U}_{2,k}\right)\right\}\left(\tilde{U}_{1,k}-\tilde{U}_{2,k}\right)\Delta x + \left\{\delta_k^-\left(\tilde{U}_{1,k}-\tilde{U}_{2,k}\right)\Big|_{k=K+1}\right\}\left\{\mu_k^-\left(\tilde{U}_{1,k}-\tilde{U}_{2,k}\right)\Big|_{k=K+1}\right\} \\
&\quad - \left\{\delta_k^+\left(\tilde{U}_{1,k}-\tilde{U}_{2,k}\right)\Big|_{k=0}\right\}\left\{\mu_k^+\left(\tilde{U}_{1,k}-\tilde{U}_{2,k}\right)\Big|_{k=0}\right\} \\
&= \frac{2\Delta t}{\gamma}\sum_{k=1}^{K}\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\left\{\delta_k^{\langle 2\rangle}\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\right\}\Delta x \\
&\quad - \frac{2\Delta t}{\gamma}\sum_{k=1}^{K}\left(\frac{dF}{d(U_{1,k},U_{1,k}^{(n)})}-\frac{dF}{d(U_{2,k},U_{2,k}^{(n)})}\right)\left\{\delta_k^{\langle 2\rangle}\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\right\}\Delta x \\
&\quad - \frac{2}{\Delta t}\left|\mu_k^-\left(\tilde{U}_{1,k}-\tilde{U}_{2,k}\right)\Big|_{k=K+1}\right|^2 - \frac{2}{\Delta t}\left|\mu_k^+\left(\tilde{U}_{1,k}-\tilde{U}_{2,k}\right)\Big|_{k=0}\right|^2 \\
&\le -\frac{2\Delta t}{\gamma}\left\|D\left(\tilde{\boldsymbol{P}}_1^{(n)}-\tilde{\boldsymbol{P}}_2^{(n)}\right)\right\|^2 + \frac{2\Delta t}{\gamma}\left\{\delta_k^-\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\Big|_{k=K+1}\right\}\left\{\mu_k^-\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\Big|_{k=K+1}\right\} \\
&\quad - \frac{2\Delta t}{\gamma}\left\{\delta_k^+\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\Big|_{k=0}\right\}\left\{\mu_k^+\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\Big|_{k=0}\right\} \\
&\quad + \frac{\Delta t}{\gamma}\sum_{k=1}^{K}\left\{\delta_k^+\left(\frac{dF}{d(U_{1,k},U_{1,k}^{(n)})}-\frac{dF}{d(U_{2,k},U_{2,k}^{(n)})}\right)\right\}\left\{\delta_k^+\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\right\}\Delta x \\
&\quad + \frac{\Delta t}{\gamma}\sum_{k=1}^{K}\left\{\delta_k^-\left(\frac{dF}{d(U_{1,k},U_{1,k}^{(n)})}-\frac{dF}{d(U_{2,k},U_{2,k}^{(n)})}\right)\right\}\left\{\delta_k^-\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\right\}\Delta x \\
&\quad - \frac{2\Delta t}{\gamma}\left\{\delta_k^-\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\Big|_{k=K+1}\right\}\left\{\mu_k^-\left(\frac{dF}{d(U_{1,k},U_{1,k}^{(n)})}-\frac{dF}{d(U_{2,k},U_{2,k}^{(n)})}\right)\Bigg|_{k=K+1}\right\} \\
&\quad - \frac{2\Delta t}{\gamma}\left\{\delta_k^+\left(\tilde{P}_{1,k}^{(n)}-\tilde{P}_{2,k}^{(n)}\right)\Big|_{k=0}\right\}\left\{\mu_k^+\left(\frac{dF}{d(U_{1,k},U_{1,k}^{(n)})}-\frac{dF}{d(U_{2,k},U_{2,k}^{(n)})}\right)\Bigg|_{k=0}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le -\frac{2\Delta t}{\gamma}\left\| D\left(\tilde{\boldsymbol{P}}_1^{(n)} - \tilde{\boldsymbol{P}}_2^{(n)}\right)\right\|^2 \\
&\quad + \frac{\Delta t}{\gamma}\sum_{k=1}^{K}\left\{\frac{1}{4}\left|\delta_k^+\left(\frac{dF}{d(U_{1,k},U_{1,k}^{(n)})} - \frac{dF}{d(U_{2,k},U_{2,k}^{(n)})}\right)\right|^2 + \left|\delta_k^+\left(\tilde{P}_{1,k}^{(n)} - \tilde{P}_{2,k}^{(n)}\right)\right|^2\right\}\Delta x \\
&\quad + \frac{\Delta t}{\gamma}\sum_{k=1}^{K}\left\{\frac{1}{4}\left|\delta_k^-\left(\frac{dF}{d(U_{1,k},U_{1,k}^{(n)})} - \frac{dF}{d(U_{2,k},U_{2,k}^{(n)})}\right)\right|^2 + \left|\delta_k^-\left(\tilde{P}_{1,k}^{(n)} - \tilde{P}_{2,k}^{(n)}\right)\right|^2\right\}\Delta x \\
&= -\frac{2\Delta t}{\gamma}\left\| D\left(\tilde{\boldsymbol{P}}_1^{(n)} - \tilde{\boldsymbol{P}}_2^{(n)}\right)\right\|^2 + \frac{2\Delta t}{\gamma}\left\| D\left(\tilde{\boldsymbol{P}}_1^{(n)} - \tilde{\boldsymbol{P}}_2^{(n)}\right)\right\|^2 \\
&\quad + \frac{\Delta t}{2\gamma}\left\| D\left(\frac{dF}{d(\boldsymbol{U}_1,\boldsymbol{U}_1^{(n)})} - \frac{dF}{d(\boldsymbol{U}_2,\boldsymbol{U}_2^{(n)})}\right)\right\|^2 \\
&= \frac{\Delta t}{2\gamma}\left\| D\left(\frac{dF}{d(\boldsymbol{U}_1,\boldsymbol{U}_1^{(n)})} - \frac{dF}{d(\boldsymbol{U}_2,\boldsymbol{U}_2^{(n)})}\right)\right\|^2.
\end{aligned}
$$

Namely,

$$
\left\| D(\tilde{\boldsymbol{U}}_1 - \tilde{\boldsymbol{U}}_2)\right\| \le \sqrt{\frac{\Delta t}{2\gamma}}\left\| D\left(\frac{dF}{d(\boldsymbol{U}_1,\boldsymbol{U}_1^{(n)})} - \frac{dF}{d(\boldsymbol{U}_2,\boldsymbol{U}_2^{(n)})}\right)\right\|. \tag{75}
$$

Using Lemma 4.2, we get

$$
\frac{dF}{d(U_{1,k},U_k^{(n)})} - \frac{dF}{d(U_{2,k},U_k^{(n)})} = \frac{1}{2}\bar{F}''(U_{1,k},U_{2,k};U_k^{(n)},U_k^{(n)})(U_{1,k}-U_{2,k})
$$

for $k = 0,1,\ldots,K,K+1$. Hence, it follows from Lemma 4.3 that

$$
\begin{aligned}
\left\| D\left(\frac{dF}{d(\boldsymbol{U}_1,\boldsymbol{U}^{(n)})} - \frac{dF}{d(\boldsymbol{U}_2,\boldsymbol{U}^{(n)})}\right)\right\| &= \frac{1}{2}\left\| D\left\{\bar{F}''(\boldsymbol{U}_1,\boldsymbol{U}_2;\boldsymbol{U}^{(n)},\boldsymbol{U}^{(n)})(\boldsymbol{U}_1-\boldsymbol{U}_2)\right\}\right\| \\
&\le \frac{1}{2}\left\|\bar{F}''(\boldsymbol{U}_1,\boldsymbol{U}_2;\boldsymbol{U}^{(n)},\boldsymbol{U}^{(n)})\right\|_{L_{\mathrm{d}}^\infty}\|D(\boldsymbol{U}_1-\boldsymbol{U}_2)\| \\
&\quad + \frac{1}{2}\|\boldsymbol{U}_1-\boldsymbol{U}_2\|_{L_{\mathrm{d}}^\infty}\left\| D\bar{F}''(\boldsymbol{U}_1,\boldsymbol{U}_2;\boldsymbol{U}^{(n)},\boldsymbol{U}^{(n)})\right\|.
\end{aligned} \tag{76}
$$

We give estimates for $\|\bar{F}''(\boldsymbol{U}_1,\boldsymbol{U}_2;\boldsymbol{U}^{(n)},\boldsymbol{U}^{(n)})\|_{L_{\mathrm{d}}^\infty}$ and $\|D\bar{F}''(\boldsymbol{U}_1,\boldsymbol{U}_2;\boldsymbol{U}^{(n)},\boldsymbol{U}^{(n)})\|$. From $\boldsymbol{V}_i \in X_0$ $(i=1,2)$, (67), and (68), we have $\|D\boldsymbol{U}_i\| = \|D\boldsymbol{V}_i\| \le 2B_0$ $(i=1,2)$. Therefore, using Lemma 3.2 and (69), we obtain

$$
\|\boldsymbol{U}_i\|_{L_{\mathrm{d}}^\infty} \le \frac{1}{L}\left|M_{\mathrm{d}}(\boldsymbol{U}_i)\right| + 2L^{\frac{1}{2}}\|D\boldsymbol{U}_i\| \le \frac{1}{L}\left|M_{\mathrm{d}}(\boldsymbol{U}^{(0)})\right| + 4L^{\frac{1}{2}}B_0 \le 2\tilde{B}_0 \quad (i=1,2).
$$

Hence, using Theorem 3.1 and Lemma 4.1, we get

$$
\left\|\bar{F}''(\boldsymbol{U}_1,\boldsymbol{U}_2;\boldsymbol{U}^{(n)},\boldsymbol{U}^{(n)})\right\|_{L_{\mathrm{d}}^\infty} \le \max_{|\xi|\le 2\tilde{B}_0}\left|F''(\xi)\right|. \tag{77}
$$

Furthermore, from Lemma 3.1 and Lemma 4.4, the following estimate holds:

$$
\begin{aligned}
\left\| D\bar{F}''(\boldsymbol{U}_1,\boldsymbol{U}_2;\boldsymbol{U}^{(n)},\boldsymbol{U}^{(n)})\right\| &\le \frac{1}{3}\max_{|\xi|\le 2\tilde{B}_0}|F'''(\xi)|\left(\|D\boldsymbol{U}_1\| + \|D\boldsymbol{U}_2\| + \left\|D\boldsymbol{U}^{(n)}\right\|\right) \\
&\le \frac{5}{3}\max_{|\xi|\le 2\tilde{B}_0}|F'''(\xi)|B_0.
\end{aligned} \tag{78}
$$

Now, it follows from $U_{1,k} - U_{2,k} = V_{1,k} - V_{2,k}$ $(k = 0, \dots, K)$ and $\boldsymbol{V}_i \in X_0$ $(i = 1, 2)$ that

$$M_{\mathrm{d}}(\boldsymbol{U}_1 - \boldsymbol{U}_2) = M_{\mathrm{d}}(\boldsymbol{V}_1 - \boldsymbol{V}_2) = M_{\mathrm{d}}(\boldsymbol{V}_1) - M_{\mathrm{d}}(\boldsymbol{V}_2) = 0.$$

Hence, from Lemma 3.2, we have

$$\|\boldsymbol{U}_1 - \boldsymbol{U}_2\|_{L^\infty_{\mathrm{d}}} \leq \frac{1}{L} |M_{\mathrm{d}}(\boldsymbol{U}_1 - \boldsymbol{U}_2)| + 2L^{\frac{1}{2}} \|D(\boldsymbol{U}_1 - \boldsymbol{U}_2)\| = 2L^{\frac{1}{2}} \|D(\boldsymbol{U}_1 - \boldsymbol{U}_2)\|. \tag{79}$$

Thus, using (76)–(79), we get the following estimate:

$$\left\| D \left(\frac{dF}{d(\boldsymbol{U}_1, \boldsymbol{U}^{(n)})} - \frac{dF}{d(\boldsymbol{U}_2, \boldsymbol{U}^{(n)})} \right) \right\| \leq \left(\frac{\max\limits_{|\xi| \leq 2\tilde{B}_0} |F''(\xi)|}{2} + \frac{5L^{\frac{1}{2}} B_0 \max\limits_{|\xi| \leq 2\tilde{B}_0} |F'''(\xi)|}{3} \right) \times \|D(\boldsymbol{U}_1 - \boldsymbol{U}_2)\|. \tag{80}$$

Consequently, from (75) and (80), we obtain

$$\begin{aligned} \left\|D(\tilde{\boldsymbol{V}}_1 - \tilde{\boldsymbol{V}}_2)\right\| &= \left\|D(\tilde{\boldsymbol{U}}_1 - \tilde{\boldsymbol{U}}_2)\right\| \\ &\leq \left(\frac{\max\limits_{|\xi| \leq 2\tilde{B}_0} |F''(\xi)|}{2} + \frac{5L^{\frac{1}{2}} B_0 \max\limits_{|\xi| \leq 2\tilde{B}_0} |F'''(\xi)|}{3} \right) \sqrt{\frac{\Delta t}{2\gamma}} \|D(\boldsymbol{U}_1 - \boldsymbol{U}_2)\| \\ &= \left(\frac{\max\limits_{|\xi| \leq 2\tilde{B}_0} |F''(\xi)|}{2} + \frac{5L^{\frac{1}{2}} B_0 \max\limits_{|\xi| \leq 2\tilde{B}_0} |F'''(\xi)|}{3} \right) \sqrt{\frac{\Delta t}{2\gamma}} \|D(\boldsymbol{V}_1 - \boldsymbol{V}_2)\|. \end{aligned}$$

Since it holds from the assumption (37) on Δt that

$$\left(\frac{\max\limits_{|\xi| \leq 2\tilde{B}_0} |F''(\xi)|}{2} + \frac{5L^{\frac{1}{2}} B_0 \max\limits_{|\xi| \leq 2\tilde{B}_0} |F'''(\xi)|}{3} \right) \sqrt{\frac{\Delta t}{2\gamma}} < 1,$$

the mapping Φ is contraction into X_0. This completes the proof. $\square$

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