Advances in Mathematical
Sciences and Applications
Vol. 30, No. 2 (2021), pp. 249–265

GAKKOTOSHO
TOKYO JAPAN

REMARKS ON EXTENSION OF CONVEX FUNCTIONS AND APPLICATION TO EVOLUTION INCLUSIONS GENERATED BY $-\Delta\beta$

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Abstract. This paper is concerned with a generalization of the extension result of proper lower semicontinuous convex functions on reflexive Banach spaces onto their dual spaces obtained in [6]. In the analysis of variational inequalities it is often necessary to extend convex functions onto a larger space than the original one in connection with the weak variational formulation.

In this paper, letting φ be a proper, lower semicontinuous convex function on a Hilbert space H and X be a Banach space which dense and continuously embedded in H, we show that φ can be extended onto the dual space X^* of X as a proper, lower semicontinuous and convex function, and discuss the solvability of an evolution inclusion generated by $-\Delta\beta$ as an application.

Communicated by Editors; Received 28 June, 2021
AMS Subject Classification: 80A22, 34G25, 35K51, 35K57, 35K59
Keywords: Stefan problem, convex function, subdifferential, conjugate function

1. Introduction

Let Ω be a smooth bounded domain in $\mathbf{R}^N,\ 1 \le N < \infty$, and

$$Q := \Omega \times (0,T),\ 0 < T < \infty,\quad \Gamma := \partial\Omega,\ \Sigma := \Gamma \times (0,T).$$

For simplicity, we put

$$V := H_0^1(\Omega),\quad V^* := H^{-1}(\Omega),\quad H := L^2(\Omega),$$

employing $|v|_V := (\int_\Omega |\nabla v|^2 dx)^{\frac{1}{2}}$ as the norm of $v \in V$. We denote by $\langle \cdot,\cdot \rangle$ the duality between V^* and V with the duality mapping $J : V \to V^*$ given by

$$\langle Jv, w \rangle = \int_\Omega \nabla v \cdot \nabla w dx,\quad \forall v,\ w \in V,$$

and by $(\cdot,\cdot)_H$ the usual inner product in H. Moreover, we introduce an inner product $(\cdot,\cdot)_*$ in V^* as

$$(v,w)_* := \langle v, J^{-1}w \rangle,\quad \forall v,\ w \in V^*.$$

Clearly, V^* is a Hilbert space equipped with this inner product $(\cdot,\cdot)_*$.

In this paper, given a maximal monotone graph in β in $\mathbf{R}^2$. we discuss an evolution inclusion of the form:

$$u_t - \Delta\beta(u) \ni f \text{ in Q},\ \beta(u) \ni h \text{ on } \Sigma,\ u(\cdot,0) = u_0 \text{ on } \Omega, \tag{1.1}$$

where $f \in L^2(Q),\ h \in W^{1,2}(0,T; H^{\frac{1}{2}}(\Gamma))$ and $u_0 \in L^2(\Omega)$ are prescribed as data. For simplicity we suppose that $0 \in \beta(0)$, and fix a proper, lower semicontinuous (l.s.c.) convex function $\hat\beta$ on $\mathbf{R}$ such that

$$\hat\beta(0) = 0,\quad \partial_{\mathbf{R}}\hat\beta = \beta,$$

where $\partial_{\mathbf{R}}\hat\beta$ is the subdifferential of $\hat\beta$ in $\mathbf{R}$. In case $\hat\beta$ satisfies some growth condition, for instance,

$$\hat\beta(r) \ge c_\beta |r|^{1+\gamma} - c'_\beta,\quad \forall r \in \mathbf{R},\ \text{with some constant } \gamma > 0, \tag{1.2}$$

the initial-boundary value problem (1.1) was earlier treated by A. Damlamian [5] as the enthalpy formulation for multi-phase Stefan problems in $H^{-1}(\Omega)$. However, without growth condition, the solvabilty of (1.1) is more delicate, even if β is Lipschitz continuous on $\mathbf{R}$. The first step of the analysis for (1.1) is to consider the proper, l.s.c. convex function $\varphi_{h,0}^t$ on H given by

$$\varphi_{h,0}^t(z) := \begin{cases} \displaystyle\int_\Omega (\hat\beta(z(x)) - u_h(x,t)z(x))dx, & \text{if } z \in H,\ \hat\beta(z) \in L^1(\Omega), \\ \infty, & \text{otherwise}, \end{cases}$$

where $u_h(t)$ is a function in $H^1(\Omega)$ with $u_h(t) = h(t)$ a.e. on Γ, and the second step is to extend it as a proper, l.s.c. and convex function onto V^*. To do so, under the growth condition with $\gamma = 1$, it is enough to take its trivial extension φ_h^t onto V^*, namely

$$\varphi_h^t(z) := \begin{cases} \varphi_{h,0}^t(z), & \text{if } z \in H, \\ \infty, & \text{otherwise}. \end{cases}$$

However, without growth condision of $\hat{\beta}$, the above extension is not necessaily lower semi-continuous on V^*, in general. One of important questions is how to get the subdifferential expression of $-\Delta\beta$ in V^*. In this paper we are ready to discuss (1.1) without any growth condition of $\hat{\beta}$, but under some restrictions on the data h and u_0.

We shall prove an extension result of proper, l.s.c. and convex functions in Section 2 and discuss the evolution inclusions generated by $\partial\varphi_h^t$ in V^* in Section 3. Finally, we shall treat (1.1) in a weak variational sense in V^* as an application in Section 4.

2. Extension of proper, l.s.c. convex functions

2.1. General result on extension

Let X be a reflexive Banach space which is dense and continuously embedded in a Hilbert space H. Hence,

$$X \subset H \subset X^* \text{ with dense and continuous embeddings.}$$

Let φ_0 be a proper, l.s.c. and convex function on H and denote by φ_0^* the conjugate function of φ_0 (see [7, 10] for the definition and fundamental properties of conjugate functions). Suppose that

$$\begin{cases} \text{for any } z \in D(\varphi_0^*) \text{ there is a sequence } \{z_n\} \subset X \text{ such that} \\ \qquad z_n \to z \text{ in } H, \quad \varphi_0^*(z_n) \to \varphi_0^*(z) \text{ (as } n \to \infty); \end{cases} \tag{2.1}$$

hence $D(\varphi^*) \cap X$ is dense in $D(\varphi^*)$ with respect to the topology of H.

Given a proper l.s.c. convex function on H or X or X^*, we denote by $\partial_H\psi$ the subdifferential of ψ in H, by $\partial_{X,X^*}\psi$ that of ψ from X into X^* and by $\partial_{X^*,X}\psi$ that of ψ from X^* into $X := X^{**}$.

The following theorem asserts that φ_0 can be extended onto X^* as a proper, l.s.c. and convex function.

Theorem 2.1. *Let φ_0 and φ_0^* be as above. Suppose that (2.1) holds. Then there is a proper, l.s.c. and convex function φ on X^* such that*

(i) $\varphi(z) = \varphi_0(z)$ for all $z \in H$. Namely, $\varphi : X^* \to \mathbf{R} \cup \{\infty\}$ is a proper l.s.c. convex extension of φ_0 onto V^*.

(ii) The following (a) and (b) are equivalent:

(a) $v \in D(\partial_{X^*,X}\varphi) \cap H$,

(b) $v \in D(\partial_H\varphi_0)$ and $\partial_H\varphi_0(v) \cap X \neq \emptyset$.

(iii) Moreover,

$$\partial_H\varphi_0(v) \cap X = \partial_{X^*,X}\varphi(v), \quad \forall v \in D(\partial_{X^*,X}\varphi) \cap H.$$

Proof. The main idea is found in the paper [6]. We consider the conjugate function φ_0^* of φ_0 which is a proper, l.s.c. and convex function on H given by

$$\varphi_0^*(v) := \sup_{w\in H}\{(v,w)_H - \varphi_0(w)\}, \quad \forall v \in H.$$

Now, let φ_X^* be the restriction of φ_0^* onto X, and denote its conjugate function by φ, namely

$$\varphi(v) := \sup_{w\in X}\{\langle v,w\rangle - \varphi_X^*(w)\}, \quad \forall v \in X^*,$$

where $\langle\cdot,\cdot\rangle$ stands for the duality between X^* and X. In the case of $v \in H$, by assumtion (2.1), $D(\varphi_X^*)$ is a dense subset of $D(\varphi_0^*)$ in H, so that

$$\varphi(v) = \sup_{w\in X}\{\langle v,w\rangle - \varphi_X^*(w)\} = \sup_{w\in H}\{(v,w)_H - \varphi_0^*(w)\} = \varphi_0(v).$$

Hence (i) is obtained. For the proof of (ii) we use the general results on conjugate functions (cf. [7, 10]) and (i) to see that

$$\begin{aligned} \tilde v \in \partial_{X^*,X}\varphi(v),\ v\in H \quad &\Longleftrightarrow \quad \tilde v \in X,\ v\in H,\ \varphi(v)+\varphi^*(\tilde v) = \langle v,\tilde v\rangle \\ &\Longleftrightarrow \quad \tilde v \in X,\ v\in H,\ \varphi_0(v)+\varphi^*(\tilde v) = (v,\tilde v)_H, \end{aligned}$$

where φ^* is the conjugate function of φ. Hence, if $\tilde v \in \partial_{X^*,X}\varphi(v)$ and $v \in H$, then

$$\varphi_0(v) + \varphi_0^*(\tilde v) \ge (v,\tilde v)_H = \varphi_0(v) + \varphi^*(\tilde v),$$

which implies that $\varphi_0^*(\tilde v) \ge \varphi^*(\tilde v)$. On the other hand, by the definition of conjugate functions we have

$$\varphi_0^*(\tilde v) = \sup_{w\in H}\{(w,\tilde v)_H - \varphi_0(w)\} \le \sup_{w\in X^*}\{\langle w,\tilde v\rangle - \varphi(w)\} = \varphi^*(\tilde v).$$

Hence, $\varphi_0^*(\tilde v) = \varphi^*(\tilde v)$ and $\varphi_0(v) + \varphi_0^*(\tilde v) = (v,\tilde v)_H$. As a consequence it follows that

$$\tilde v \in \partial_{X^*,X}\varphi(v),\ v \in H \implies \tilde v \in \partial_H\varphi_0(v)\cap X.$$

Now, assume (a) and let $\tilde v \in \partial_{X^*,X}\varphi(v)$ with $v \in H$. Then,

$$\langle w-v,\tilde v\rangle = (w-v,\tilde v)_H \le \varphi_0(w) - \varphi_0(v), \quad \forall w \in H,$$

which implies $\tilde v \in \partial_H\varphi_0(v)\cap X$ and (b) holds. Conversely, assume (b) and $\tilde v \in \partial_H\varphi_0(v)\cap X$. Then, since $\varphi_X^* = \varphi^*$ on X and $\varphi = \varphi_0$ on H, we have

$$\varphi(v) + \varphi^*(\tilde v) = \varphi_0(v) + \varphi_X^*(\tilde v) = \varphi_0(v) + \varphi_0^*(\tilde v) = (v,\tilde v)_H,$$

whence $\tilde v \in \partial_{X^*,X}\varphi(v) \in X$. Thus (a) is obtained and (iii) holds. □

Remark 2.1. Asummption (2.1) is quite technical for the proof of Theorem 2.1. This assumption is checked in all the concrete examples which will be treated in this paper. Also, for some related works in $X := W_0^{p,s}(\Omega)$, see H. Brezis [4]).

2.2. Examples of conjugate functions and their extension

In this subsection, let β be a maximal monotone graph in $\mathbf{R}^2$ such that $0 \in \beta(0)$, and $\hat{\beta}$ be a proper, l.s.c. and convex function on $\mathbf{R}$ such that

$$\hat{\beta} \geq 0 \text{ on } \mathbf{R},\ \hat{\beta}(0) = 0,\ \partial_{\mathbf{R}}\hat{\beta} = \beta \text{ in } \mathbf{R}. \tag{2.2}$$

Also, let $V,\ V^*$ and H be as in the introduction.

Example 2.1. We define a proper l.s.c. convex function φ_0 on H by

$$\varphi_0(z) := \begin{cases} \displaystyle\int_\Omega \hat{\beta}(z(x))dx, & \text{if } z \in H,\ \hat{\beta}(z) \in L^1(\Omega), \\ \infty, & \text{otherwise.} \end{cases} \tag{2.3}$$

Then we have:

Lemma 2.1. *Let $\hat{\beta}^*$ be the conjugate function of $\hat{\beta}$ on $\mathbf{R}$. Then:*

(i) The conjugate function φ_0^* of φ_0 is given by

$$\varphi_0^*(z) := \begin{cases} \displaystyle\int_\Omega \hat{\beta}^*(z(x))dx, & \text{if } z \in H,\ \hat{\beta}^*(z) \in L^1(\Omega), \\ \infty, & \text{otherwise.} \end{cases} \tag{2.4}$$

(ii) φ_0^* satisfies condition (2.1) for $X = W^{1,p}(\Omega), 2 \leq p < \infty$.

Proof. This lemma is well known. But for the completeness we recall here the proof. First, assume that $\mathrm{int}D(\hat{\beta}^*) = \emptyset$. Since $0 \in \beta(0)$ by assumption, it follows that $0 \in \beta^{-1}(0) \subset D(\hat{\beta}^*)$ which shows $D(\hat{\beta}^*) = \{0\}$ and $\hat{\beta}^*(r) = \infty$ for all $r \neq 0$. Here we observe that $\hat{\beta}$ is constant on $\mathbf{R}$; in fact, if not, there would exists $r_0 \neq 0$ such that $\beta(r_0) \ni \tilde{r}_0$ for some $\tilde{r}_0 \neq 0$ and hence $r_0 \in \beta^{-1}(\tilde{r}_0)$, which contradicts $D(\hat{\beta}^*) = \{0\}$. As a consequence, $\hat{\beta} \equiv \hat{\beta}(0) = 0$ on $\mathbf{R}$. Besides, $\hat{\beta}^*(0) = 0$ by $\hat{\beta}(0) + \hat{\beta}^*(0) = 0$. Consequently,

$$\varphi_0^*(0) := \sup_{v \in H}\{(0,v)_H - \varphi_0(v)\} = \sup_{v \in H}\left\{(0,v)_H - \int_\Omega \hat{\beta}(0)dx\right\} = 0 = \int_\Omega \hat{\beta}^*(0)dx,$$

namely (2.4) holds.

Next, assume that $\mathrm{int}D(\hat{\beta}^*) \neq \emptyset$. In this case, $\overline{D(\hat{\beta}^*)} = [a_1, a_2]$ for some constants $a_1,\ a_2$ with $-\infty \leq a_1 < a_2 \leq \infty$; we may assume without loss of generality that $a_1 < 0 < a_2$. Now, choose sequences $\{r_n\}$ and $\{R_n\}$ in $\mathbf{R}$ such that

$$a_1 < r_n < 0,\quad 0 < R_n < a_2,\quad r_n \downarrow a_1,\ R_n \uparrow a_2 \text{ (as } n \to \infty).$$

Given $z \in H$ with $\int_\Omega \hat{\beta}^*(z(x))dx < \infty$, define

$$w_n(x) := (z(x) \vee r_n) \wedge R_n,\ x \in \Omega,\ n = 1, 2, \cdots. \tag{2.5}$$

Then, $w_n \to z$ in H and there is a bounded sequence $\{\tilde{w}_n\}$ in $L^\infty(\Omega)$ such that $w_n \in \beta(\tilde{w}_n)$ a.e. on Ω for all n. Hence

$$
\begin{aligned}
&\int_\Omega \hat{\beta}^*(z)dx = \lim_{n\to\infty}\int_\Omega \hat{\beta}^*(w_n)dx = \lim_{n\to\infty}\int_\Omega \{w_n\tilde{w}_n - \hat{\beta}(\tilde{w}_n)\}dx \\
= &\lim_{n\to\infty}\left\{\int_\Omega (z\tilde{w}_n - \hat{\beta}(\tilde{w}_n))dx + (w_n - z, \tilde{w}_n)_H\right\} \\
\le &\sup_{v\in H}\int_\Omega (zv - \hat{\beta}(v))dx + \lim_{n\to\infty}(w_n - z, \tilde{w}_n)_H = \sup_{v\in H}\{(z,v)_H - \varphi_0(v)\} = \varphi_0^*(z).
\end{aligned}
$$

Thus $\int_\Omega \hat{\beta}^*(z)dx \le \varphi_0^*(z)$. On the other hand, since $\hat{\beta}^*(z(x)) + \hat{\beta}(v(x)) \ge z(x)v(x)$ a.e. $x \in \Omega$ for every $v \in H$ with $\hat{\beta}(v) \in L^1(\Omega)$, it follows that

$$
\int_\Omega \hat{\beta}^*(z)dx \ge \int_\Omega (zv - \hat{\beta}(v))dx = (z,v)_H - \varphi_0(v), \quad \forall v \in D(\varphi_0),
$$

which implies that

$$
\int_\Omega \hat{\beta}^*(z)dx \ge \sup_{v\in H}\{(z,v)_H - \varphi_0(v)\} = \varphi_0^*(z).
$$

As a consequent, we have (2.4).

Next we show (ii). Let $z \in H$ with $\varphi_0^*(z) < 0$. When $\mathrm{int}D(\hat{\beta}^*) = \emptyset$, by (2.2) we have $D(\varphi_0^*) = \{0\}$, so that (2.1) trivially holds. In the case of $\mathrm{int}D(\hat{\beta}^*) \ne \emptyset$, we take a sequence $\{w_n\}$ given by (2.5) and furthermore approximate each w_n by a smooth function $\tilde{w}_n$ so that

$$
|\tilde{w}_n| \le |w_n| \text{ a.e. on } \Omega, \ |\tilde{w}_n - w_n|_H \le \frac{1}{n}, \quad |\varphi_0^*(w_n) - \varphi_0^*(\tilde{w}_n)| \le \frac{1}{n}.
$$

The sequence $\{z_n := \tilde{w}_n\}$ fulfills condition (2.1) for $X := W^{1,p}(\Omega)$. □

By virtue of Theorem 2.1 we have:

Proposition 2.1. *Let $X := W^{1,p}(\Omega)$, $2 \le p < \infty$. Then φ_0 can be extended onto the dual space X^* of X as a proper, l.s.c. and convex function, denoted by φ, so that the statements (ii) and (iii) of Theorem 2.1 hold.*

Example 2.2. Let us take as X the space $V := H_0^1(\Omega)$. Given $h \in H^{\frac{1}{2}}(\Gamma)$, we denote by $u_h \in H^1(\Omega)$ the solution to the variational problem

$$
u_h \in H^1(\Omega), \ u_h = h \text{ a.e. on } \Gamma, \int_\Omega \nabla u_h \cdot \nabla \zeta dx = 0, \quad \forall \zeta \in V. \tag{2.6}
$$

and define a proper, l.s.c. and convex function $\varphi_{h,0}$ on H by

$$
\varphi_{h,0}(z) := \begin{cases} \displaystyle\int_\Omega \{\hat{\beta}(z) - u_h z\}dx, & \text{if } z \in H, \ \hat{\beta}(z) \in L^1(\Omega), \\ \infty, & \text{otherwise.} \end{cases}
$$

Then, just as Proposition 2.1, we have:

Proposition 2.2. *Let $h \in H^{\frac{1}{2}}(\Gamma)$. Then $\varphi_{h,0}$ can be extended onto $V^* := H^{-1}(\Omega)$ as a proper, l.s.c. and convex function, denoted by φ_h, so that the statements (ii) and (iii) of Theorem 2.1 hold for $\varphi = \varphi_h$.*

In this case, the conjugate function $\varphi^*_{h,0}$ of $\varphi_{h,0}$ on H is given by

$$\varphi^*_{h,0}(z) = \int_\Omega \hat{\beta}^*(z + u_h)dx, \quad \forall z \in H. \tag{2.7}$$

In fact, (2.7) holds, since

$$\sup_{r' \in \mathbf{R}} \{rr' - \hat{\beta}(r') + u_h(x)r'\} = \sup_{r' \in \mathbf{R}} \{(r + u_h(x))r' - \hat{\beta}(r')\} = \hat{\beta}^*(r + u_h(x))$$

for a.e. $x \in \Omega$. Therefore, by a slight modification of (i) of Lemma 2.1 we get (2.7) and condition (2.1) is similarly checked. Hence by Theorem 2.1 we have Proposition 2.2.

Example 2.3. Let β and $\hat{\beta}$ be as above and φ_0 be as defined by (2.3). Now, consider the Moreau-Yosida approximation $\varphi_{0,\lambda}$ of φ_0,

$$\varphi_{0,\lambda}(z) := \inf_{v \in H} \left\{ \frac{1}{2\lambda} |z - v|^2_H + \varphi_0(v) \right\}, \quad \forall z \in H, \ \forall \lambda > 0.$$

Then its conjugate function $\varphi^*_{0,\lambda}$ on H is given by

$$\varphi^*_{0,\lambda}(z) = \varphi^*_0(z) + \frac{\lambda}{2} |z|^2_H, \quad \forall z \in H,$$

which follows from the following lemma.

Lemma 2.2. *Let $\hat{\beta}^*_\lambda := (\hat{\beta}_\lambda)^*$, $\lambda > 0$, be the conjugate function of the Moreau-Yosida approximation $\hat{\beta}_\lambda$ of $\hat{\beta}$. Then,*

$$\hat{\beta}^*_\lambda(r) = \hat{\beta}^*(r) + \frac{\lambda}{2} r^2, \quad \forall r \in \mathbf{R}. \tag{2.8}$$

Proof. By definition,

$$\begin{aligned} \hat{\beta}^*_\lambda(r) &= \sup_{r' \in \mathbf{R}} \{rr' - \hat{\beta}_\lambda(r')\} = \sup_{r' \in \mathbf{R}} \left\{ rr' - \inf_{r'' \in \mathbf{R}} \left\{ \frac{1}{2\lambda} |r' - r''|^2 + \hat{\beta}(r'') \right\} \right\} \\ &= \sup_{r' \in \mathbf{R}} \left\{ \sup_{r'' \in \mathbf{R}} \left\{ rr' - \frac{1}{2\lambda} |r' - r''|^2 - \hat{\beta}(r'') \right\} \right\} \\ &= \sup_{r' \in \mathbf{R}} \left\{ \sup_{r'' \in \mathbf{R}} \left\{ rr'' - \hat{\beta}(r'') + r(r' - r'') - \frac{1}{2\lambda} |r' - r''|^2 \right\} \right\} \end{aligned} \tag{2.9}$$

From the last equality of (2.9) it follows that

$$\hat{\beta}^*_\lambda(r) \le \hat{\beta}^*(r) + \sup_{r', r'' \in \mathbf{R}} \left\{ r(r' - r'') - \frac{1}{2\lambda} |r' - r''|^2 \right\} \le \hat{\beta}^*(r) + \frac{\lambda}{2} |r|^2. \tag{2.10}$$

On the other hand, for any $\varepsilon > 0$, choose r''_ε so that

$$\hat{\beta}^*(r) \le r r''_\varepsilon - \hat{\beta}(r''_\varepsilon) + \varepsilon.$$

Then, from the last equality of (2.9) again we see that

$$\begin{aligned}\hat{\beta}^*_\lambda(r) &\ge r r''_\varepsilon - \hat{\beta}(r''_\varepsilon) + r(r' - r''_\varepsilon) - \frac{1}{2\lambda}|r' - r''_\varepsilon|^2 \\ &\ge \hat{\beta}^*(r) - \varepsilon + r(r' - r''_\varepsilon) - \frac{1}{2\lambda}|r' - r''_\varepsilon|^2, \quad \forall r' \in \mathbf{R}.\end{aligned}$$

Here, taking $r' \in \mathbf{R}$ so that $r = \frac{1}{\lambda}(r' - r''_\varepsilon)$, we have $r(r' - r''_\varepsilon) - \frac{1}{2\lambda}|r' - r''_\varepsilon|^2 = \frac{\lambda}{2}r^2$, whence

$$\hat{\beta}^*_\lambda(r) \ge \hat{\beta}^*(r) + \frac{\lambda}{2}r^2 - \varepsilon. \tag{2.11}$$

Since $\varepsilon > 0$ is arbitrary, (2.8) is obtained from (2.10) and (2.11). $\square$

3. Evolution inclusions generated by $-\Delta\beta$

In this section, we suppose that β is a maximal monotone graph in $\mathbf{R}^2$ and $\hat{\beta}$ is a proper, l.s.c. convex function on $\mathbf{R}$ such that

$$0 \in \beta(0), \quad \hat{\beta}(0) = 0, \ \partial_{\mathbf{R}}\hat{\beta} = \beta. \tag{3.1}$$

3.1. Lemmas

We denote by S the operator which assigns to each $h \in H^{\frac{1}{2}}(\Gamma)$ the solution u_h to the variational problem (2.6). It is easy to see that S is a bounded and linear from $H^{\frac{1}{2}}(\Gamma)$ into $H^1(\Omega)$; in fact, for some positive constant $C_S > 0$,

$$|Sh|_{H^1(\Omega)} \le C_S |h|_{H^{\frac{1}{2}}(\Gamma)}.$$

For each $h \in H^{\frac{1}{2}}(\Gamma)$, we define a proper, l.s.c. convex function $\varphi_{h,0}(\cdot)$ on H by

$$\varphi_{h,0}(z) := \begin{cases} \displaystyle\int_\Omega \{\hat{\beta}(z(x)) - Sh(x)z(x)\}dx, & \text{if } \hat{\beta}(z) \in L^1(\Omega) \\ \infty, & \text{otherwise.}\end{cases}$$

Just as in Example 2.2 of the previous section, for each $h \in H^{\frac{1}{2}}(\Gamma)$, consider the conjugate function $\varphi^*_{h,0}$ of $\varphi_{h,0}$, which is given by

$$\varphi^*_{h,0}(w) := \sup_{z\in H}\{(w,z)_H - \varphi_{h,0}(z)\} = \int_\Omega \hat{\beta}^*(w + Sh)dx, \quad \forall w \in H.$$

Furthermore, for the restriction $\varphi^*_{h,V}$ of $\varphi^*_{h,0}$ onto V, let φ_h be its conjugate function, namely

$$\varphi_h(v) := \sup_{w\in V}\{\langle v, w\rangle - \varphi^*_{h,V}(w)\}, \quad \forall v \in V^*.$$

Now, for $h \in W^{1,2}(0,T;H^{\frac{1}{2}}(\Gamma))$ and for each $t \in [0,T]$ we consider the family of time-dependent proper l.s.c. and convex functions $\{\varphi_h^t\}$ on V^* by putting $\varphi_h^t := \varphi_{h(t)}$.

In order to investigate the time-dependence of φ_h^t, we approximate β by

$$\alpha_n(r) := \beta_{\lambda_n}(r) + \lambda_n r, \quad \forall r \in \mathbf{R}, \tag{3.2}$$

where $\{\lambda_n\}$ is a sequence in $(0,1]$ such that $\lambda_n \downarrow 0$ (as $n \to \infty$) and β_{λ_n} is the Yosida approximation of β, i.e.

$$\beta_{\lambda_n}(r) := \frac{1}{\lambda_n}\{r - (I + \lambda_n \beta)^{-1} r\}, \quad \forall r \in \mathrm{R}.$$

It is easy to see that

$$\lambda_n \le \alpha_n'(r) := \frac{d}{dr}\alpha_n(r) \le \frac{1}{\lambda_n} + \lambda_n, \quad \frac{\lambda_n r^2}{2} \le \hat{\alpha}_n(r) \le \frac{1}{2}\left(\frac{1}{\lambda_n} + \lambda_n\right) r^2, \ \forall r \in \mathbf{R}, \tag{3.3}$$

where $\hat{\alpha}_n$ is the primitive of α_n with $\hat{\alpha}_n(0) = 0$; note that the inverse α_n^{-1} is singlevalued and α_n is bi-Lipschitz continuous on $\mathbf{R}$ and α_n converges to β in the graph sense, or equivalently

$$(I + \lambda \alpha_n)^{-1} r \to (I + \lambda \beta)^{-1} r, \quad \forall r \in \mathbf{R}, \ \forall \lambda > 0.$$

Now, for each $t \in [0,T]$, we define a proper, l.s.c. and convex function $\varphi_{nh,0}^t$ on H by

$$\varphi_{nh,0}^t(z) := \int_\Omega (\hat{\alpha}_n(z) - Sh(t)z)dx, \quad \text{if } z \in H.$$

As was seen in Example 2.2, its conjugate function $\varphi_{nh,0}^{t*} := (\varphi_{nh,0}^t)^*$ is given by $D(\varphi_{nh,0}^{t*}) = H$ and

$$\varphi_{nh,0}^{t*}(z) := \int_\Omega \hat{\alpha}_n^*(z + Sh(t))dx, \quad \forall z \in H,$$

where $\hat{\alpha}_n^*$ is the conjugate function of $\hat{\alpha}_n$. Denoting the restriction of $\varphi_{nh,0}^{t*}$ on V by $\varphi_{nh,V}^{t*}$, we infer from Proposition 2.2 that the conjugate function of $\varphi_{nh,V}^{t*}$ gives a proper, l.s.c. convex extension of $\varphi_{nh,0}^t$ onto V^*, denoted by φ_{nh}^t. We should notice that

$$\varphi_{nh}^t(z) := \begin{cases} \varphi_{nh,0}^t(z) < \infty, & \text{if } z \in H, \\ \infty, & \text{if } z \in V^* - H, \end{cases} \tag{3.4}$$

since $\varphi_{nh,0}^t$ satisfies a growth condition of the from

$$\varphi_{nh,0}^t(z) \ge \frac{\lambda_n}{4}|z|_H^2 - c_0, \quad \forall z \in H, \quad \text{for some constant } c_0 > 0. \tag{3.5}$$

Lemma 3.1. *Let φ_{nh}^t be the proper, l.s.c. and convex function on V^* given by (3.4) for each $t \in [0,T]$. Then, for every $t \in [0,T]$, φ_{nh}^t converges to φ_h^t on V^* in the sense of Mosco (as $n \to \infty$). Actually, it holds (cf. [11]) that*

(M1) If $z_n \to z$ weakly in V^*, then $\liminf_{n\to\infty} \varphi_{nh}^t(z_n) \ge \varphi_h^t(z)$.

(M2) For every $z \in D(\varphi_h^t)$, there is a sequence $\{z_n\} \subset H$ such that

$$z_n \to z \quad in\ V^*, \quad \varphi_{nh}^t(z_n) \to \varphi_h^t(z).$$

Proof. By duality of the convex analysis, it is enough to show that $\varphi_{nh,V}^{t*} \to \varphi_{h,V}^{t*}$ on V in the sense of Mosco (cf. [12]).

First, assuming $z_n \to z$ weakly in V, we check $\liminf_{n\to\infty} \varphi_{nh,V}^{t*}(z_n) \geq \varphi_{h,V}^{t*}(z)$. In fact,

$$\liminf_{n\to\infty} \varphi_{nh,V}^{t*}(z_n) = \liminf_{n\to\infty} \varphi_{nh,0}^{t*}(z_n) \geq \varphi_{h,0}^{t*}(z) = \varphi_{h,V}^{t*}(z), \tag{3.6}$$

since $\varphi_{nh,0}^{t*} \to \varphi_{h,0}^{t*}$ on H in the sense of Mosco. Next we check another condition of Mosco convergence, which is that for each $z \in D(\varphi_{h,V}^{t*})$ there is a sequence $\{z_n\} \subset V$ such that

$$z_n \to z \text{ in } V, \quad \varphi_{nh,V}^{t*}(z_n) \to \varphi_{h,V}^{t*}(z). \tag{3.7}$$

We claim that it is enough to choose $z_n = z$. Indeed, by (3.6), $\liminf_{n\to\infty} \varphi_{nh,V}^{t}(z) \geq \varphi_{h,V}^{t}(z)$. On the other hand, noting from Lemma 2.2 that

$$\hat{\alpha}_n^*(r) \leq \hat{\beta}_{\lambda_n}^*(r) = \hat{\beta}^*(r) + \frac{\lambda_n}{2} r^2, \quad \forall r \in \mathbf{R},$$

we have

$$\varphi_{nh,V}^{t*}(z) \leq \varphi_{h,0}^{t*}(z) + \frac{\lambda_n}{2}|z + Sh(t)|_H^2.$$

Consequently,

$$\limsup_{n\to\infty} \varphi_{nh,V}^{t*}(z) \leq \varphi_{h,0}^{t*}(z) = \varphi_{h,V}^{t*}(z),$$

so that the required convergence (3.7) with $z_n = z$ follows. Hence $\varphi_{nh}^t \to \varphi_h^t$ on V^* in the sense of Mosco. $\square$

Lemma 3.2. *For every $s,\ t \in [0, T]$, we have*

$$D(\varphi_h^t) = D(\varphi_h^s) =: D_0 \tag{3.8}$$

and

$$|\varphi_h^t(v) - \varphi_h^s(v)| \leq C_0 |h(t) - h(s)|_{H^{\frac{1}{2}}(\Gamma)}, \quad \forall v \in D_0, \tag{3.9}$$

where C_0 is a positive constant.

Proof. Assume that $v \in D(\varphi_h^t)$. Then, by Lemma 3.1, there is $\{v_n\}$ in H such that $v_n \to v$ in V^* and $\varphi_{nh}^t(v_n) \to \varphi_h^t(v)$, and

$$\begin{aligned}
\varphi_{nh}^s(v_n) &= \int_\Omega (\hat{\alpha}_n(v_n) - Sh(s)v_n)dx \\
&= \int_\Omega (\hat{\alpha}_n(v_n) - Sh(t)v_n)dx + \int_\Omega (S(h(t) - h(s))v_n)dx \\
&= \varphi_{nh}^t(v_n) + \int_\Omega S(h(t) - h(s))v_n dx.
\end{aligned}$$

Now, let $n \to \infty$ to see from the above relations that

$$\begin{aligned}\varphi_h^s(v) &\le \liminf_{n\to\infty} \varphi_{nh}^s(v_n) = \lim_{n\to\infty} \left\{ \varphi_{nh}^t(v_n) - \int_\Omega S(h(s)-h(t))v_n dx \right\} \\ &= \lim_{n\to\infty} \{\varphi_{nh}^t(v_n) - \langle v_n, S(h(s)-h(t))\rangle\} = \varphi_h^t(v) - \langle v, S(h(s)-h(t))\rangle \\ &< \infty.\end{aligned}$$

Hence $v \in D(\varphi_h^s)$. Similarly, if $v \in D(\varphi_h^s)$, then $v \in D(\varphi_h^t)$ and

$$\varphi_h^t(v) \le \varphi_h^s(v) - \langle v, S(h(t)-h(s))\rangle.$$

Thus we obtain (3.8) and (3.9) with $C_0 = |S|_{L(H^{\frac{1}{2}}(\Gamma), H^1(\Omega))}$. □

We denote by $\partial_*\varphi_h^t$ the subdifferential of φ_h^t in V^* with respect to the inner product $(\cdot,\cdot)_*$; by definition $z^* \in \partial_*\varphi_h^t(z)$ if and only if $z^* \in V^*$, $z \in D(\varphi_h^t)$ and

$$(z^*, v-z)_* \le \varphi_h^t(v) - \varphi_h^t(z), \quad \forall v \in V^*.$$

Lemma 3.3. *If $v^* \in \partial_*\varphi_h^t(v)$ and $v \in H$, then there is $\tilde{v} \in H^1(\Omega)$ such that*

$$\tilde{v} \in \beta(v), \text{ a.e. on } \Omega, \quad \tilde{v} - Sh(t) \in V, \tag{3.10}$$

and $v^* = J(\tilde{v} - Sh(t))$, namely

$$\langle v^*, \zeta\rangle = \int_\Omega \nabla(\tilde{v} - Sh(t)) \cdot \nabla\zeta dx, \quad \forall \zeta \in V. \tag{3.11}$$

Proof. On account of Proposition 2.2, $v^* \in J\partial_{V^*,V}\varphi_h^t(v)$. Therefore there is $\tilde{v}_1 \in \partial_{V^*,V}\varphi_h^t(v) = \partial_H\varphi_{h,0}^t(v) \cap V$ such that $v^* = J\tilde{v}_1$ and

$$\tilde{v}_1 = \tilde{v} - Sh(t) \in V, \quad \tilde{v} \in \beta(v) \text{ a.e. on } \Omega,$$

since $\partial_H\varphi_{h,0}(v) = \beta(v) - Sh(t)$. Now, we observe that $J^{-1}v^* = \tilde{v} - Sh(t) \in V$, namely $v^* = J(\tilde{v} - Sh(t))$, and (3.10) and (3.11) hold. □

3.2 Evolution inclusions generated by $\partial_*\varphi_h^t$ and $\partial_*\varphi_{nh}^t$

As a consequence of Lemma 3.2, by the general theory on evolution inclusions (cf. [2, 8]) we have:

Proposition 3.1. *For every $h \in W^{1,2}(0,T; H^{\frac{1}{2}}(\Gamma))$, $u_0 \in \overline{D_0}^{V^*}$ and $\ell \in L^2(0,T;V^*)$, the evolution problem*

$$u'(t) + \partial_*\varphi_h^t(u(t)) \ni \ell(t) \text{ in } V^*, \text{ a.e. } t \in (0,T), \quad u(0) = u_0. \tag{3.12}$$

possesses a unique solution u in $C([0,T];V^*)$ such that

$$\sqrt{t}u' \in L^2(0,T;V^*), \quad \sup_{t\in(0,T]} t\varphi_h^t(u(t)) < \infty, \ \varphi_h^t(u) \in L^1(0,T),$$

and $t \to \varphi_h^t(u(t))$ *is absolutely continuous on* $[\delta, T]$ *for every* $\delta \in (0,T)$.

In particular, if $u_0 \in D_0$*, then the solution* u *to (3.12) satisfies that* $u \in W^{1,2}(0,T;V^*)$ *and* $t \to \varphi_h^t(u(t))$ *is absolutely continuous on* $[0,T]$.

Also, we consider a sequence of approximate evolution problems to (3.12) as follows. Let φ_{nh}^t be the proper, l.s.c. and convex function on V^* given by (3.4), and apply Proposition 3.1 to

$$u_n'(t) + \partial_* \varphi_{nh}^t(u_n(t)) \ni \ell_n(t) \ \textit{in} \ V^*, \ \textit{a.e.} \ \ t \in (0,T), \quad u_n(0) = u_{n0}, \tag{3.13}$$

where $\ell_n \in L^2(0,T;V^*)$ and $u_{n0} \in H$. Then (3.13) admits a solution $u_n \in W^{1,2}(0,T;V^*)$ such that $t \to \varphi_{nh}^t(u_n(t))$ is absolutely continuous on $[0,T]$. Therefore, $u_n \in C_w([0,T];H)$ for each $n = 1,2,\cdots$, by the growth condition (3.3). Moreover, since $\varphi_{nh}^t \to \varphi_h^t$ on V^* for each $t \in [0,T]$ in the sense of Mosco (as $n \to \infty$), it follows from the general convergence result on evolution inclusions (cf. [8; Theorem 2.7.1]) that if $\ell_n \to \ell$ in $L^2(0,T;V^*)$ and $u_{n0} \to u_0$ in V^*, then u_n converges to the solution u of (3.12) in the sense that

$$u_n \to u \text{ in } C([0,T];V^*), \quad \int_0^T \varphi_{nh}^t(u_n(t))dt \to \int_0^T \varphi_h^t(u(t))dt.$$

4. Weak solvability of (1.1)

The weak formulation of problem (1.1) is given in the following definition.

Definition 4.1. Let $h \in W^{1,2}(0,T;H^{\frac{1}{2}}(\Gamma))$, $u_0 \in H$ and $f \in L^2(0,T;H)$. Then u is said to be a (weak) solution of (1.1), if $u \in C([0,T];V^*)$, $u(0) = u_0$, u is weakly continuous from $(0,T]$ into H, $u' \in L^2_{loc}((0,T];V^*)$ and there exists a function $\tilde{u} \in L^2_{loc}((0,T];H^1(\Omega))$ such that

$$\tilde{u} \in \beta(u) \text{ a.e. on } Q, \ \ \tilde{u}(t) - S_0(h(t)) \in V, \ \text{ a.e. } t \in (0,T), \tag{4.1}$$

and

$$\begin{gathered}\langle u'(t), \zeta \rangle + \int_\Omega \nabla(\tilde{u}(t) - S_0(h(t))) \cdot \nabla \zeta dx = (f(t), \zeta)_H, \\ \forall \zeta \in V, \ \text{ a.e. } t \in (0,T).\end{gathered} \tag{4.2}$$

This weak formulation is denoted by $CP_D(\beta; f, h, u_0)$.

According to Lemma 3.3, the solution u of

$$u'(t) + \partial_* \varphi_h^t(u(t)) \ni f(t) \text{ in } V^*, \text{ a.e. } t \in (0,T), \quad u(0) = u_0,$$

is that of $CP_D(\beta; f, h, u_0)$, if and only if $u(t) \in H$ for a.e. $t \in (0,T)$. The following theorem is concerned with the construction of such a solution.

Theorem 4.1. *Assume that (3.1) holds, and let* d_*, d^* *be any finite constants such that* $d_* \le 0 \le d^*$ *and* $[d_*, d^*] \subset \mathrm{int}R(\beta)$*, where* $R(\beta)$ *is the range of* β*. Given data* $f \in L^\infty(Q)$,

$h \in W^{1,2}(0,T;H^{\frac{1}{2}}(\Gamma)) \cap L^\infty(\Sigma)$ and $u_0 \in L^\infty(\Omega)$ with $\int_\Omega \hat\beta(u_0)dx < \infty$, suppose that

$$d_* \le h \le d^* \text{ a.e. on } \Sigma, \quad u_0 \in L^\infty(\Omega), \quad \int_\Omega \hat\beta(u_0)dx < \infty. \tag{4.3}$$

Then there exists a unique solution u with $\tilde u$ of $CP_D(\beta; f, h, u_0)$ such that

$$\begin{aligned} &|u|^2_{W^{1,2}(0,T;V_0^*)} + |u|^2_{L^\infty(Q)} + \sup_{t\in[0,T]} \int_\Omega \hat\beta(u(t))dx + |\tilde u - Sh|^2_{L^2(0,T;V)} \\ &\le C_1 \left(1 + \int_\Omega \hat\beta(u_0)dx + |u_0|^2_{L^\infty(\Omega)} + |f|^2_{L^\infty(Q))} + |h|^2_{L^\infty(\Sigma)} + |h|^2_{W^{1,2}(0,T;H^{\frac{1}{2}}(\Gamma))}\right), \end{aligned} \tag{4.4}$$

where C_1 is a positive constant independent of all the data β, f, h, u_0.

Proof. (Step 1: Construction of approximate solutions)
Let α_n be the same approximation of β as given by (3.2) of subsection 3.1, $\alpha_n(r) := \beta_{\lambda_n}(r) + \lambda_n r$ for all $r \in \mathbf{R}$ and $\{f_n\}$ be a sequence in $W^{1,2}(0,T;H)$ such that

$$|f_n| \le |f| \text{ a.e. on } Q, \quad f_n \to f \text{ in } L^2(Q).$$

Also, let $\{u_{0n}\}$ be a sequence of approximate initial data in $H^1(\Omega)$ such that

$$|u_{0n}| \le |u_0| \text{ a.e. on } \Omega, \ u_{0n} \to u_0 \text{ in } H, \ \int_\Omega \hat\alpha_n(u_{0n})dx \to \int_\Omega \hat\beta(u_0)dx.$$

It is clear that the convex function $\hat\alpha_n$ satisfies a growth condition

$$\hat\alpha_n(r) \ge \frac{\lambda_n}{2} r^2, \quad \forall r \in \mathbf{R}, \quad \forall n.$$

Therefore, by virtue of Lemma 3.3 for φ^t_{nh}, for each n the problem (3.13) is written as

$$\alpha_n(u_n) - Sh \in L^2(0,T;V), \quad u_n(0) = u_{n0},$$

$$\langle u_n'(t), \zeta\rangle + \int_\Omega \nabla(\alpha_n(u_n(t)) - Sh(t)) \cdot \nabla\zeta dx = (f_n(t), \zeta)_H, \quad \forall \zeta \in V. \text{ a.e. } t \in (0,T). \tag{4.5}$$

Now, in connection with (4.5), let us consider a doubly nonlinear evolution equation of the form in H:

$$\bar u_n'(t) + \partial_H \psi^t(\alpha_n(\bar u_n(t))) = f_n(t) \text{ in } H, \text{ a.e. } t \in (0,T), \quad \bar u_n(0) = u_{n0}, \tag{4.6}$$

where ψ^t is a proper, l.s.c. and convex function on H given by

$$\psi^t(z) := \begin{cases} \dfrac{1}{2}|\nabla(z - Sh(t))|^2_H, & \text{if } z - Sh(t) \in V, \\ \infty, & \text{otherwise,} \end{cases}$$

According to an abstract result in Theorem 2.8.1 in [8], problem (4.6) admits a unique solution $\bar u_n$ in $W^{1,2}(0,T;H)$ such that $\bar u_n - Sh \in L^\infty(0,T;V)$. It is easy to see from (4.6) that $\bar u_n - Sh \in L^2(0,T;V)$, $\bar u_n(0) = u_{n0}$ and

$$(\bar u_n'(t), \zeta)_H + \int_\Omega \nabla(\alpha_n(\bar u_n(t)) - Sh(t)) \cdot \nabla\zeta dx = (f_n(t), \zeta)_H, \quad \forall \zeta \in V. \text{ a.e. } t \in (0,T), \tag{4.7}$$

so that $\bar{u}_n$ satisfies (4.5), too. By the uniqueness of solutions to (4.5), it results that $u_n \equiv \bar{u}_n \in W^{1,2}(0,T;H)$.

(Step 2: Uniform estimates for approximate solutions u_n)

Next we show a uniform estimate of u_n from above. To do so we choose constants $M_0,\ M_1,\ M_2$ so that

$$|f| \le M_0 \text{ a.e. on } Q, \quad d^* \in \beta(M_1), \quad M_2 := |u_0^+|_{L^\infty(\Omega)} \text{ a.e. on } \Omega.$$

Here we note by assumption (4.3) that M_1 is finite and positive, and $d^* + \delta_0 \in \mathrm{int}R(\beta)$ for a small $\delta_0 > 0$ and $M_2 \le \sup D(\beta)$. Hence there is a positive number $\delta_1 > 0$ such that

$$d^* + \delta_0 \le \beta^o(M_1 + \delta_1) < \infty, \tag{4.8}$$

where β^o stands for the minimal section of β. With these $M_0,\ M_1,\ \delta_1$ and M_2, put

$$v_h(t) := M_0 t + M_1 + \delta_1 + M_2, \quad \forall t \in [0,T].$$

First we show that u_n is uniformly bounded from above. Now, we compare u_n with v_h as follows. It follows from (4.7) that

$$(u_n'(\tau) - v_h'(\tau), \zeta)_H + \int_\Omega \nabla \alpha_n(u_n(\tau)) \cdot \nabla \zeta dx = (f_n(\tau) - M_0, \zeta)_H, \ \forall \zeta \in V_0, \tag{4.9}$$

for a.e. $\tau \in (0,T)$. We take as ζ the function $\sigma_\varepsilon([\alpha_n(u_n(\tau)) - \alpha_n(v_h(\tau))]^+)$ for all small $\varepsilon > 0$, where σ_ε is a smooth non-decreasing function on $\mathbf{R}$ such that

$$\sigma_\varepsilon(r) := [\rho_\varepsilon * \sigma_0](r) \ \text{ with } \sigma_0(r) := \begin{cases} 1, & \text{if } r > 0, \\ 0, & \text{if } r = 0, \\ -1, & \text{if } r < 0. \end{cases}$$

and ρ_ε is the usual one-dimensional mollifier. It is possible to take such a function as a test function ζ. In fact, since $\beta_\lambda(r) \uparrow \beta^o(r)$ for all $r > 0$ as $\lambda \downarrow 0$, given any $\nu > 0$ with $\nu < \delta_0$ there is a positive integer n_ν such that

$$\beta_{\lambda_n}(M_1 + \delta_1) \ge \beta^o(M_1 + \delta_1) - \nu, \quad \forall n \ge n_\nu,$$

whence by (4.8)

$$\begin{aligned} \alpha_n(v_h) &= \beta_{\lambda_n}(v_h) + \lambda_n v_h \ge \beta_{\lambda_n}(v_h) \ge \beta_{\lambda_n}(M_1 + \delta_1) \\ &\ge \beta^o(M_1 + \delta_1) - \nu \ge d^* + \delta_0 - \nu \\ &\ge d^* \ge h, \quad \forall n \ge n_\nu. \end{aligned}$$

This shows that $\alpha_n(u_n) - \alpha_n(v_h) = h - \alpha_n(v_h) \le 0$ a.e. on Σ, which implies that

$$[\alpha_n(u_n) - \alpha_n(v_h)]^+ \in V, \quad \sigma_\varepsilon([\alpha_n(u_n(\tau)) - \alpha_n(v_h(\tau))]^+) \in V$$

for a.e. $\tau \in (0,T)$. Therefore, by (4.8) and (4.9),

$$(u_n'(\tau) - v_h'(\tau), \sigma_\varepsilon([\alpha_n(u_n(\tau)) - \alpha_n(v_h(\tau))]^+)_H$$

$$+\int_\Omega \nabla\alpha_n(u_n(\tau))\cdot\nabla\sigma_\varepsilon([\alpha_n(u_n)(\tau)-\alpha_n(v_h(\tau))]^+)dx \tag{4.10}$$
$$=(f_n(\tau)-M_0,\sigma_\varepsilon([\alpha_n(u_n)(\tau)-\alpha_n(v_h(\tau))]^+)_H\le 0,\ \text{a.e. } \tau\in(0,T)$$

and

$$\int_\Omega \nabla\alpha_n(u_n(\tau))\cdot\nabla\sigma_\varepsilon([\alpha_n(u_n)(\tau)-\alpha_n(v_h(\tau))]^+)dx$$
$$=\int_{\{u_n(x,\tau)>v_h(x,\tau)\}} |\nabla\alpha_n(u_n(\tau))|^2\sigma'_\varepsilon([\alpha_n(u_n)(\tau)-\alpha_n(v_h(\tau))]^+)dx\ge 0,$$

for a.e. $\tau\in(0,T)$. Consequently it follows from (4.10) that

$$(u'_n(\tau)-v'_h(\tau),\sigma_\varepsilon([\alpha_n(u_n)(\tau)-\alpha_n(v_h(\tau))]^+)_H\le 0,\quad \text{a.e. } \tau\in(0,T).$$

First, letting $\varepsilon\to 0$ we get $\frac{d}{d\tau}|[u_n(\tau)-v_h(\tau)]^+|_{L^1(\Omega)}\le 0$ for a.e. $\tau\in(0,T)$, since $\sigma_0([\alpha_n(u_n)-\alpha_n(v_h)]^+)=\sigma_0([u_n-v_h]^+)$. Secondly, integrating it in time over $[0,t]$, we obtain by (4.8) that

$$|[u_n(t)-v_h(t)]^+|_{L^1(\Omega)}\le|[u_{0n}-(M_1+\delta_1+M_2)]^+|_{L^1(\Omega)}=0,\quad \forall t\in[0,T],$$

namely $u_n(x,t)\le M_0T+M_1+\delta_1+M_2=:M^*=M^*(f,h,u_0)$ a.e. on Q. Similarly, $u_n(x,t)\ge m^*$ a.e. on Q for some constant $m^*=m^*(f,h,u_0)$. Thus we have

$$m^*\le u_n(x,t)\le M^*\ \text{ a.e. on } Q,\ \forall\text{large } n. \tag{4.11}$$

By taking $\zeta=\alpha_n(u_n(\tau))-Sh(\tau)$ in (4.9) we have that

$$\frac{d}{d\tau}\int_\Omega\{\hat\alpha_n(u_n(\tau))-u'_n(\tau)Sh(\tau)\}dx+|\alpha_n(u_n(\tau))-Sh(\tau)|_V^2$$
$$=(f_n(\tau),\alpha_n(u_n(\tau))-Sh(\tau))_H\ \text{ a.e. } \tau\in(0,T).$$

Besides integrate it in τ over $[0,t]$, $0\le t\le T$, to get

$$\int_\Omega\hat\alpha_n(u_n(t))dx-(u_n(t),Sh(t))_H+\int_0^t|\alpha_n(u_n)-Sh|_V^2d\tau$$
$$=\int_\Omega\hat\alpha_n(u_{n0})dx-(u_{n0},Sh(0))_H+\int_0^t(u_n,(Sh)')_Hd\tau+\int_0^t(f_n,\alpha_n(u_n)-Sh)_Hd\tau$$

It is easy to see the following type of estimate from this equality with (4.11) and assumptions on the data $f_n,\ h,\ u_{0n}$:

$$\int_\Omega\hat\alpha_n(u_n(t))dx+|\alpha_n(u_n)-Sh|^2_{L^2(0,t;V)}$$
$$\le C'_1\left(1+\int_\Omega\hat\alpha_n(u_{0n})dx+|u_{0n}|^2_{L^\infty(\Omega)}+|f_n|^2_{L^\infty(Q)}+|h|^2_{W^{1,2}(0,T;H^{\frac12}(\Gamma))}+|h|^2_{L^\infty(\Sigma)}\right) \tag{4.12}$$
$$\le C''_1\left(1+\int_\Omega\hat\beta(u_0)dx+|u_0|^2_{L^\infty(\Omega)}+|f|^2_{L^\infty(Q)}+|h|^2_{W^{1,2}(0,T;H^{\frac12}(\Gamma))}+|h|^2_{L^\infty(\Sigma)}\right),$$

for all $t \in [0,T]$, where C_1', C_1'' are positive constants independent of all the data.

(Step 3: Convergence of approximate solutions)

On account of uniform estimates (4.11) and (4.12), by choosing a subsequence of $\{u_n\}$ if necessary, we may assume that

$$u_n \to u \text{ weakly}^* \text{ in } L^\infty(0,T;H), \quad \alpha_n(u_n) \to \tilde{u} \text{ weakly in } L^2(0,T;H^1(\Omega)) \tag{4.13}$$

for some functions $u \in L^\infty(0,T;H)$ and $\tilde{u} \in L^2(0,T;H^1(\Omega))$ with $\tilde{u} - Sh \in L^2(0,T;V)$. Moreover, we see from (4.5) that

$$u_n' \to u' \text{ weakly in } L^2(0,T;V^*), \quad \text{and } u_n \to u \text{ in } C([0,T];V^*). \tag{4.14}$$

Hence $u_n(t) \to u(t)$ weakly in H and uniformly on $[0,T]$.

In order to complete the proof of Theorem 4.1 we make use of proper, l.s.c. convex functionals Φ_{nh} and Φ_h on $L^2(0,T;V^*)$ defined by

$$\Phi_{nh}(w) := \begin{cases} \displaystyle\int_0^T \varphi_{nh}^t(w(t))dt \ \left(= \int_Q \{\hat{\alpha}_n(w) - (Sh)w\} dxdt\right), & \forall w \in L^2(0,T;H), \\ \infty, & \text{otherwise,} \end{cases}$$

and

$$\Phi_h(w) := \int_0^T \varphi_h^t(w(t))dt, \quad \forall w \in L^2(0,T;V^*).$$

We observe that their subdifferentials in $L^2(0,T;V^*)$, denoted by $\partial_*\Phi_{nh}$ and $\partial_*\Phi_h$, are given by

$$\partial_*\Phi_{nh}(w) = \{\tilde{w} \in L^2(0,T;V^*) \mid \tilde{w}(t) \in \partial_*\varphi_{nh}^t(w(t)), \text{ a.e. } t \in (0,T)\}$$

and

$$\partial_*\Phi_h(w) = \{\tilde{w} \in L^2(0,T;V^*) \mid \tilde{w}(t) \in \partial_*\varphi_h^t(w(t)), \text{ a.e. } t \in (0,T)\}.$$

Moreover, Φ_{nh} converges to Φ_h on $L^2(0,T;V^*)$ in the sense of Mosco (as $n \to \infty$), which immediately follows from the fact that φ_{nh}^t converges to φ_h^t on H in the sense of Mosco for every $t \in [0,T]$ (see Lemma 3.1). Hence $\partial_*\Phi_{nh}$ converges to $\partial_*\Phi_h$ in $L^2(0,T;V^*$ with respect to the graph sense. For the precise proofs of the above results are found, for instance, see [1; Chapter 3].

(Step 4: Completion of the proof of Theorem 2.1)

By using the weak continuity of J from V into V^* and relations

$$(v,w)_H = \langle Jv, J^{-1}w \rangle = (Jv, w)_*, \quad \forall v \in V, \ \forall w \in H,$$

we see from (4.11)$\sim$(4.14) that

$$\begin{aligned} &u_n \to u \ \text{ in } C([0,T];V^*) \text{ and } C_w([0,T];H), \\ &J(\alpha_n(u_n) - Sh) \to J(\tilde{u} - Sh) \text{ weakly in } L^2(0,T;V^*), \\ &J(\alpha_n(u_n) - Sh) = f_n - u_n' \in \partial_*\Phi_{nh}(u_n). \end{aligned}$$

Therefore, by the maximal monotonicity principle (cf. Proposition 2.2 in [9]), we conclude that $J(\tilde{u} - Sh) = \ell - u' \in \partial_* \Phi_h(u)$, namely

$$J(\tilde{u}(t) - Sh(t)) = f(t) - u'(t) \in \partial_* \varphi_h^t(u(t)), \text{ a.e. } t \in (0,T), \quad u(0) = u_0,$$

and $\tilde{u} \in \beta(u)$ a.e. on Q (cf. Lemma 3.3). This shows that u together with $\tilde{u}$ gives a unique solution of $CP_D(\beta; f, h, u_0)$ which satisfies the required bound (4.4) by (4.12). □

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