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CONSTRUCTION OF WEAK SOLUTIONS OF A WEIGHTED INVERSE MEAN CURVATURE FLOW

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Abstract. The inverse mean curvature flow is a geometric evolution equation related to general relativity. In this paper, we consider a weighted version of the inverse mean curvature flow and prove the existence of its weak solution formulated by a level-set method. We construct the solution by approximation by weighted p-harmonic functions analogue to the method introduced by R.Moser.

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1 Introduction

Let $n \geq 2$, and $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary such that its complement is bounded. We consider an equation

$$\operatorname{div}\left(a\frac{\nabla u}{|\nabla u|}\right) = a|\nabla u| \quad \text{in } \Omega$$

with Dirichlet condition where u is an unkown and a is a given function. This equation is a level-set formulation(explained later) of the inverse men curvature flow(IMCF) in a manifold $(\mathbb{R}^n; a^{\frac{2}{n-1}}\delta_{ij})$ where δ_{ij} means the Euclidean metric. The IMCF is a family of hypersurfaces $(E_t)_{t\geq 0} \subset M^n$(Riemannian manifold) satisfying

$$V = -\frac{1}{H},$$

where V denotes the normal velocity and $H(<0)$ denotes the mean curvature. The IMCF is studied by many mathematicians. For example, see [3], [13]. The IMCF has a lot of properties. In particular, it is closely related to general relativity. For example, a smooth solution of the IMCF is nondecreasing for the functional for 2-dimensional surfaces defined by

$$m_H(N) := \frac{|N|^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}}\left(16\pi - \int_N H^2\right),$$

where $|N|$ is a surface area. The functional is called the Hawking mass and the monotonicity property is called the Geroch monotonicity. If there is a smooth global-in-time solution for some initial surface, important inequality in general relativity called the Riemannian Penrose inequality can be proved by using this monotonicity. However, in general, this equation does not admit a smooth global-in-time solution, so Huisken and Ilmanen introduced a weak solution for the IMCF in [5]. This weak solution is based on a level-set method and a variational method. They proved the unigue existence of a weak solution, the Geroch monotonicity for this weak solution and also proved the Riemannian Penrose inequality in special cases such that ∂M has only one component.

In this paper, we consider the existence of a weak solution introduced by Huisken and Ilmanen for the IMCF in a manifold $(\mathbb{R}^n; a^{\frac{2}{n-1}}\delta_{ij})$. In this context, the normal velocity (in Euclidean sense) V is given by

$$V = \frac{1}{-H + \nabla \log a \cdot \nu}, \tag{1.1}$$

where a is some given function in $\mathbb{R}^n$ and ν is the outer unit normal. We take a different approach from previous researches considering the IMCF in a general manifold. Let us explain the weak formulation for solutions of (1.1).

Assuming that a family of surfaces satisfying (1.1) is given by a level set of a scalar function u, that is, $E_t = [u = t]$, and that $|\nabla u| \neq 0$, then

$$V = \frac{1}{|\nabla u|},$$

$$-H + \nabla \log a \cdot \nu = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \nabla \log a \cdot \frac{\nabla u}{|\nabla u|} = \frac{1}{a}\mathrm{div}\left(a\frac{\nabla u}{|\nabla u|}\right).$$

So the equaion (1.1) becomes

$$\mathrm{div}\left(a\frac{\nabla u}{|\nabla u|}\right) = a|\nabla u|. \tag{1.2}$$

On the other hand, for a compact set $K \subset \mathbb{R}^n$ and $v, w \in C^{0,1}_{\mathrm{loc}}(\Omega)$, define the functional

$$J_v(w; K) := \int_K a\left(|\nabla w| + w|\nabla v|\right). \tag{1.3}$$

We cannot regard (1.2) as an Euler-Lagrange equation of some functionals because of right hand side of (1.2). However, if a function u is a critical point of the functional $J_u(*; K)$ (note that the subscript of J is fixed to be u), we may regard u as a solution of (1.2). So we define the weak solution of (1.1) as follows.

Definition1.1. A function $u \in C^{0.1}_{\mathrm{loc}}(\Omega)$ is called *a weak solution* of (1.1) if for every precompact set $K \subset \Omega$ and every $w \in C^{0,1}_{\mathrm{loc}}(\Omega)$ with $w = u$ in Ω^c, the inequality

$$J_u(u; K) \leq J_u(w; K), \tag{1.4}$$

and

$$u = 0 \quad \text{on} \quad \partial\Omega$$

holds.

Moreover, a flow satisfying (1.1) is expected to be compact, so we define the following property ensuring E_t is bounded.

Definition1.2. A weak solution of (1.1) is called *proper* if

$$\lim_{|x|\to\infty} u(x) = \infty \tag{1.5}$$

holds.

To prove the existence of a proper weak solution of (1.1), we adapt the method introduced by R. Moser [10] for the case $a = 1$. Consider a weighted p-harmonic equation

$$\mathrm{div}\left(a|\nabla v_p|^{p-2}\nabla v_p\right) = 0. \tag{1.6}$$

For a solution v_p of (1.6), we use a kind of the Hopf-Cole transformation and set $u_p := (1-p)\log v_p$. It is easily computed that u_p satisfies

$$\mathrm{div}\left(a|\nabla u_p|^{p-2}\nabla u_p\right) = a|\nabla u_p|^p. \tag{1.7}$$

It can be expected that (1.7) approximate (1.2) as $p \to 1$, so we want to see that u_p converges to a proper weak solution of (1.1). It turns out that the strategy is successful under suitable assumptions on a.

Main Theorem Suppose $\Omega \subset \mathbb{R}^n$ is a domain with smooth boundary, such that Ω^c is bounded. Moreover, we assume that a positive function $a \in C^\infty(\mathbb{R}^n)$ satisfies following properties.

1. There exists a positive constant c_1 such that for all ball $B \subset \mathbb{R}^n$
$$c_1 \fint_B a \leq \inf_B a. \tag{1.8}$$
2. For any $x \in \mathbb{R}^n$,
$$|\nabla \log a(x)| \leq c_2, \tag{1.9}$$
$$\inf_{x \in \mathbb{R}^n} (\nabla \log a(x) \cdot x) > -n+1 \tag{1.10}$$
holds where c_2 is a positive constant.
3. There exists a positive constant c_3 such that for all $x, \xi \in \mathbb{R}^n$
$$\mathrm{Hess}(\log a(x))(\xi, \xi) \leq c_3 |\xi|^2. \tag{1.11}$$

Then there exists a unique proper weak solution u of (1.1). Moreover it satisfies the estimates
$$u(x) \leq \left(c_2 + \frac{n-1}{r}\right)(|x - x_0| - r) \ \text{ in } \Omega,$$
$$u(x) \geq (\beta - 1) \log \frac{|x|}{s} \ \text{ in } \Omega,$$
$$|\nabla u(x)| \leq \{2(c_2 + (n-1)r^{-1})c_3(|x - x_0| - r) + (c_2 + (n-1)R^{-1})^2\}^{\frac{1}{2}} \ \text{ in } \Omega,$$
provided that $B_r(x) \subset \Omega^c$, $\Omega \subset B_s(0)$ and $R := \sup\{r > 0 | \forall x_0 \in \partial\Omega, \exists x_1 \in \Omega^c \ s.t. \ x_0 \in \partial B_r(x_1)\}$, where $B_r(z)$ is a closed ball of radius r centered at z.

Remark1.3. A family of functions which satisfies inequality (1.8) is called Muckenhoupt class. This property ensures the existence of p-harmonic functions. Property 2 is needed in order to get a proper solution because, if $\nabla \log a$ does not have any restriction, the left hand side of (1.2) may become negative. See Remark on proposition 3.1. The inequality (1.11) can be interpreted as a restriction on curvature for the manifold $(\mathbb{R}^n; a^{\frac{2}{n-1}}\delta_{ij})$.

Remark1.4. Previous researches treating weak solutions of the inverse mean curvature flow in a general riemannian manifold include [5], [7], and [12]. The authors of [5] use a method which approximate the term $|\nabla u|$ by $\sqrt{|\nabla u|^2 + \epsilon^2}$, where ϵ is very small. However, the situation which they treat cannot be compared with our situation. The authors of [7] take the approach using p-harmonic functions, but the way of treating p-harmonic functions is different. Also, a metric treated in [7] has to satisfy $\lim_{r \to \infty} \nu(r) = 0$, where
$$\nu(r) := \sup_{2r \leq t < \infty} \frac{t}{V(\Omega \cap B(0, t))}.$$

However, we do not necessarily require above property. For example, If $\Omega^c = B_1(0)$ and $a = |x|^\alpha$ in Ω, then $\lim_{r\to\infty} \nu(r) = 0$ implies $\alpha < \frac{(n-1)^2}{n}$. However, our method can treat the case $0 < \alpha < n-1$. The author of [12] also uses p-harmonic functions but it does not treat the situation in which a solution is proper.

Remark1.5. Let $\Omega^c = B_1(0)$ and $a = |x|^\alpha$ in Ω, then by easy computation, we can check that $(n-1+\alpha)\log|x|$ is a solution for (1.2). If $\alpha = -n+1$, the solution is 0. The reason is that if $\alpha = -n+1$, the left hand side of (1.1) become ∞.

Remark1.6. Proposition 3.1 and Proposition 3.2 (appeared later) which state the uniform boundedness of u_p and ∇u_p are key estimates to prove the Main Theorem. Similar estimates are needed in the case that a is constant, but there are some difficulties in our case. Firstly, to get Proposition 3.1, we have to make subsolution and supersolution of (1.6) and in order to make them, we assume that a satisfies (1.9) and (1.10). Secondly, to prove Proposition 3.2, we assume that a satisfies (1.11) to use a kind of Bernstein method. Also see Remark 3.5.1.

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2 Preliminary

In this section, we arrange some properties of weighted p-harmonic functions needed to prove Main Theorem without proof. The theory of weighted p-harmonic functions is well developed. The reader is referred to [4] for example.

Here we define the weighted Sobolev spaces to describe some properties of weighted p-harmonic functions.
For a function $\varphi \in C^\infty(\Omega)$ we let

$$||\varphi||_{1,p} = (\int_\Omega a|\varphi|^p)^{\frac{1}{p}} + (\int_\Omega a|\nabla\varphi|^p)^{\frac{1}{p}}.$$

The Sobolev spase $W^{1,p}(\Omega; a)$ is defined to be the completion of

$$\{\varphi \in C^\infty(\Omega); ||\varphi||_{1,p} < \infty\}$$

with respect to the norm $||*||_{1,p}$.
The space $W_0^{1,p}(\Omega; a)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega; a)$.
The local version of $W^{1,p}(\Omega; a)$ denoted to $W_{\mathrm{loc}}^{1,p}(\Omega; a)$ is a set of function u such that for all relatively compact open set Ω' in Ω, $u \in W^{1,p}(\Omega'; a)$.
To prove the existence of a weighted p-harmonic function needed to prove our main theorem, the following version of the Sobolev space is needed.

$$L^{1,p}(\Omega; a) = \{u \in W_{\mathrm{loc}}^{1,p}(\Omega; a); \nabla u \in L^p(\Omega; a)\}$$

and $L_0^{1,p}(\Omega;a)$ is the closure of $C_0^\infty(\Omega)$ with respect to the seminorm

$$(\int_\Omega a|\nabla u|^p)^{\frac{1}{p}}.$$

Proposition 2.1 (Existence of a weighted p-harmonic function) If a is a locally integrable, nonnegative function in $\mathbb{R}^n$ satisfying (1.8) for all ball $B \subset \mathbb{R}^n$, there exists a function $g \in L_0^{1,p}(\Omega;a)$ satisfying (1.6) and $g \le 1$ in Ω with $g = 1$ in Ω^c.

The proof of this proposition is a minor modification of the proof of [4](Theorem 5.26 and 5.27).

Proposition 2.2 (Regularity of a weighted p-harmonic function) Assume that $a \in C^2(\mathbb{R}^n)$ and $\partial\Omega$ is C^2. If v_p satisfies (1.6) in Ω with boundary value 1, then v_p is in $C^{1,\alpha}(\overline{\Omega})$.

Proposition 2.3 If v_p satisfies (1.6) in Ω, there exists a constant c_4 depending on n, p such that for all ball B with $2B \subset \Omega$

$$\sup_B |\nabla v_p|^p \le c_4 \fint_{2B} |\nabla v_p|^p$$

holds.

For the proof of Proposition 2.2.2 and 2.2.3, see for example [2], [9].

Proposition 2.4 (the Harnack inequality) If v_p satisfies (1.6) in Ω, there exists a constant c_5 depending on n, p such that for all ball B with $2B \subset \Omega$

$$\sup_B v_p \le c_5 \inf_B v_p$$

holds.

For the proof, see [4](Theorem6.2).

Proposition 2.5 (Comparison principle) Let $u \in L^{1,p}(\Omega;a)$ be a continuous supersolution, $v \in L^{1,p}(\Omega;a)$ be a continuous subsolution of (1.6) in Ω and Ω be connected. If $v \le u$ on $\partial\Omega$, then $v \le u$ in Ω.

The proof is a minor modification of the proof of [4](Lemma 3.18).

3 Proof of Main Theorem

By proposition 2.1, there exists a weighted p-harmonic function in Ω with boundary value 1 (denoted by v_p), and set $u_p := (1-p)\log v_p$. To prove the Main Theorem, it is sufficient to ensure the local uniform boundedness of u_p and ∇u_p due to the Ascoli-Arzela theorem. In fact, following estimates are valid.

Proposition 3.1. (Uniform boundedness of u_p) If $B_r(x_0) \subset \Omega^c$ and $\Omega \subset B_s(0)$, u_p satisfies the following two estimates.

$$u_p(x) \leq \left(c_2 + \frac{n-1}{r}\right)(|x - x_0| - r) \text{ in } \Omega,$$

$$u_p(x) \geq (\beta - p) \log \frac{|x|}{s} \text{ in } \Omega,$$

where the positive constant c_2 is independent of p.

Proposition 3.2. (Uniform boundedness of ∇u_p) If $B_r(x_0) \subset \Omega^c$, ∇u_p satisfies

$$|\nabla u_p(x)| \leq \{2(c_2 + (n-1)r^{-1})c_3(|x - x_0| - r) + (c_2 + (n-1)r^{-1})^2\}^{\frac{1}{2}} \text{ in } \Omega,$$

where the positive constant c_3 is independent of p.

To prove Proposition 3.1, we construct an appropriate supersolution and subsolution for a weighted p-harmonic equation and use Proposition 2.5 (comparison principle). To prove Proposition 3.2, we want to get a gradient estimate of u_p on boundary and get a interior estimate by a kind of Bernstein method.

Proof of proposition 3.1. Setting $w = \mathrm{e}^{\alpha|x-x_0|}$ $(\alpha < 0)$ and computing the p-laplacian of w, we get

$$\begin{aligned}&\operatorname{div}\left(a|\nabla w|^{p-2}\nabla w\right)\\ =&a\alpha|\alpha|^{p-2}\mathrm{e}^{\alpha|x-x_0|(p-1)}\{\nabla \log a \cdot \frac{(x-x_0)}{|x-x_0|} + (n-1)|x-x_0|^{-1} + \alpha(p-1)\}.\end{aligned}$$

Therefore, if w is a subsolution in $x \neq x_0$,

$$\nabla \log a \cdot \frac{(x-x_0)}{|x-x_0|} + (n-1)|x-x_0|^{-1} + \alpha(p-1) \leq 0$$

holds for every $x \neq x_0$.
By (1.9) we see that

$$|\nabla \log a \cdot \frac{x-x_0}{|x-x_0|}| \leq c_2$$

As a result, it is sufficient to assume

$$\alpha \leq -(p-1)^{-1}\left(c_2 + \frac{n-1}{r}\right)$$

so that w is a subsolution in Ω. In particular, if $B_r(x_0) \subset \Omega^c$,

$$w_1 := \mathrm{e}^{\frac{c_2+(n-1)r^{-1}}{1-p}|x-x_0| - r\frac{c_2+(n-1)r^{-1}}{1-p}}$$

is a subsolution of (1.6) in Ω.
On the contrary, setting $w = |x|^\alpha$ and computing the p-laplacian of w, we get

$$\begin{aligned}&\operatorname{div}\left(a|\nabla w|^{p-2}\nabla w\right)\\ =&a\alpha|\alpha|^{p-2}|x|^{(\alpha-1)(p-1)-1}(\nabla\log a\cdot x+\alpha(p-1)+n-p).\end{aligned}$$

So if w is a supersolution in $x \neq 0$,

$$\alpha(p-1) \geq -n+p-\nabla\log a\cdot x$$

holds for every $x \neq 0$.
By (1.10) we see that

$$-\nabla\log a\cdot x-n+p<p-1.$$

Therefore, to conclude that w is a supersolution, it is sufficient to assume

$$\alpha \geq \frac{\beta-p}{1-p},$$

where β is constant satisfying $\beta > 1$. In particular, if $\Omega^c \subset B_s(0)$,

$$w_2 := |\frac{x}{s}|^{\frac{\beta-p}{1-p}}$$

is a supersolution of (1.6) in $\Omega\backslash\{0\}$. Moreover, $w_1 \leq 1$ on $\partial\Omega$ and $w_2 \geq 1$ on $\partial\Omega$, so by comparison princile (Proposition 2.5), we observe that

$$\begin{aligned}v_p &\geq w_1 \qquad \text{in } \Omega,\\ v_p &\leq w_2 \qquad \text{in } \Omega.\end{aligned}$$

Recall that $u_p = (1-p)\log v_p$. By above inequalities, we see

$$u_p \leq \left(c_2+\frac{n-1}{r}\right)(|x-x_0|-r) \qquad \text{in } \Omega,$$

$$u_p \geq (\beta-p)\log\frac{|x|}{s} \qquad \text{in } \Omega.$$

□

Remark 3.1.1 Let $\Omega^c = B_r(0)$ for some $r > 0$ and $a = |x|^{-(n-1)}$ in Ω. Then

$$\nabla\log a\cdot x=-n+1.$$

So such a does not satisfy (1.10). However, in this case, we can check that

$$\exp\left(\frac{\epsilon}{1-p}(|x|+1)\right)$$

is a subsolution of equation (1.6) and $v_p \le 1$, where ϵ is any positive constant. Therefore, we can obtain

$$0 \le u_p(x) \le \epsilon(|x|+1).$$

Taking $p \to 1$ and $\epsilon \to 0$, we obtain $u = 0$. This is not a proper solution. Thus the lower bound of (1.10) is important in our method. Also in this case, the left hand side of (1.1) become ∞, so it seems that there are no proper solution if we take a as above.

Remark 3.1.2 If $a = 1$, the function

$$|x - x_0|^{\frac{n-p}{1-p}}$$

satisfies the p-harmonic equation. So the estimate of v_p above is easier than the situation in which a is not constant.

To prove Proposition 3.2, we prepare some lemmas.

Lemma 3.3. If $B_r(x_0) \subset \Omega$, there exists a positive constant c_9 depending only on n and p such that

$$|\nabla u_p(x_0)| \le \frac{c_9}{r}$$

holds.

Proof : Let $\eta \in C_0^\infty(\Omega)$ be a cut-off function. Taking $\eta^p v_p$ as a test function of (1.6), we get

$$\begin{aligned}\int_\Omega a\eta^p |\nabla v_p|^p &= -p\int_\Omega a\eta^{p-1} v_p |\nabla v_p|^{p-2}\nabla v_p \cdot \nabla\eta \\ &\le p\left(\int_\Omega a\eta^p|\nabla v_P|^p\right)^{\frac{p-1}{p}}\left(\int_\Omega a(v_p)^p|\nabla\eta|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Thus

$$\int_\Omega a\eta^p|\nabla v_p|^p \le p^p\int_\Omega a(v_p)^p|\nabla\eta|^p$$

holds. Taking η with supp $\eta \subset B_{\frac{r}{2}}(x_0)$ and $\eta = 1$ on $B_{\frac{r}{4}}(x_0)$ in this inequality, we get

$$\int_{B_{\frac{r}{4}}(x_0)} a|\nabla v_p|^p \le p^p\int_{B_{\frac{r}{2}}(x_0)} a(v_p)^p|\nabla\eta|^p.$$

Using the property (1.8) and Proposition 2.4, we obtain

$$p^p\int_{B_{\frac{r}{2}}(x_0)} a(v_p)^p|\nabla\eta|^p \le c_6 r^{n-p}\inf_{B_{\frac{r}{2}}(x_0)} a \inf_{B_{\frac{r}{2}}(x_0)} v_p.$$

Also using Proposition 2.3, we obtain

$$\int_{B_{\frac{r}{4}}(x_0)} a|\nabla v_p|^p \ge c_7 r^n \inf_{B_{\frac{r}{4}}(x_0)} a \sup_{B_{\frac{r}{4}}(x_0)} |\nabla v_p|^p.$$

Therefore we can get

$$\sup_{B_{\frac{r}{4}}(x_0)} |\nabla v_p|^p \leq c_8 r^{-p} \inf_{B_{\frac{r}{4}}(x_0)} v_p^p.$$

In particular,

$$|\nabla u_p(x_0)| \leq \frac{c_9}{r}$$

holds.

□

Lemma 3.4. Set $R := \sup\{r > 0 | \forall x_0 \in \partial\Omega, \exists x_1 \in \Omega^c \ s.t. \ x_0 \in \partial B_r(x_1)\}$. For all $x \in \partial\Omega$

$$|\nabla u_p(x)| \leq c_2 + \frac{n-1}{R}$$

holds.

Proof : Take any $x \in \partial\Omega$. If $\nabla u_p(x) = 0$, the inequality is clearly satisfied, so we consider the case $\nabla u_p \neq 0$. By Proposition 3.1, $u_p \geq 0$ in Ω, and $u_p = 0$ on $\partial\Omega$,

$$\begin{aligned} & u_p\left(x + \epsilon\frac{\nabla u_p(x)}{|\nabla u_p(x)|}\right) - u_p(x) \\ \leq & (c_2 + (n-1)R^{-1})\left(|x + \epsilon\frac{\nabla u_p(x)}{|\nabla u_p(x)|} - x_0| - R\right) - (c_2 + (n-1)R^{-1})(|x - x_0| - R) \end{aligned}$$

holds, here $x \in \partial B(x_0, R)$ (such x_0 exists by the definition of R) and $\epsilon > 0$ sufficiently small. Dividing both sides by ϵ and letting $\epsilon \to 0$, we get

$$\begin{aligned} |\nabla u_p(x)| &\leq \left(c_2 + \frac{n-1}{R}\right)\frac{x - x_0}{|x - x_0|} \cdot \frac{\nabla u_p(x)}{|\nabla u_p(x)|} \\ &\leq \left(c_2 + \frac{n-1}{R}\right). \end{aligned}$$

This is the desired inequality.

□

Lemma 3.5. Assume that u_p is smooth and set $f := |\nabla u_p|^2$, $g := \frac{f}{2} - c_3 u_p$. Then

$$\operatorname{div}(f^{\frac{p}{2}-1} A\nabla g) + h \cdot \nabla g \geq \frac{p-1}{4} f^{\frac{p}{2}-2} |\nabla f|^2$$

holds, where

$$A := I + (p-2)\frac{\nabla u_p \otimes \nabla u_p}{f},$$

$$h := -pf^{\frac{p}{2}-1}\nabla u_p + |\nabla u_p|^{p-2}\nabla \log a + (p-2)f^{\frac{p}{2}-2}(\nabla u_p \cdot \nabla \log a)\nabla u_p.$$

Proof : Denote u_p by u. If u is smooth, the equation (1.7) is transformed into

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |\nabla u|^{p-2}\nabla u\cdot\nabla\log a = |\nabla u|^p.$$

Differentiating this equation, we obtain

$$\begin{aligned}
&\operatorname{div}\left(|\nabla u|^{p-2}\nabla u_{x_i} + (p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla u_{x_i})\nabla u\right)\\
&+(p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla u_{x_i})\nabla u\cdot\nabla\log a\\
&+|\nabla u|^{p-2}\nabla u\cdot\nabla(\log a)_{x_i} + |\nabla u|^{p-2}\nabla u_{x_i}\cdot\nabla\log a\\
&= p|\nabla u|^{p-2}\nabla u\cdot\nabla u_{x_i}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\operatorname{div}\left(|\nabla u|^{p-2}u_{x_i}\nabla u_{x_i} + (p-2)|\nabla u|^{p-4}u_{x_i}(\nabla u\cdot\nabla u_{x_i})\nabla u\right)\\
&= |\nabla u|^{p-2}|\nabla u_{x_i}|^2 + (p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla u_{x_i})^2\\
&+ u_{x_i}\operatorname{div}\left(|\nabla u|^{p-2}\nabla u_{x_i} + (p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla u_{x_i})\nabla u\right)\\
&= |\nabla u|^{p-2}|\nabla u_{x_i}|^2 + (p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla u_{x_i})^2 + pu_{x_i}|\nabla u|^{p-2}\nabla u\cdot\nabla u_{x_i}\\
&- u_{x_i}|\nabla u|^{p-2}\nabla u\cdot\nabla(\log a)_{x_i} - u_{x_i}|\nabla u|^{p-2}\nabla u_{x_i}\cdot\nabla\log a\\
&- (p-2)u_{x_i}|\nabla u|^{p-4}(\nabla u\cdot\nabla u_{x_i})(\nabla u\cdot\log a)
\end{aligned}$$

holds. Summing this over 1 to n, we obtain

$$\begin{aligned}
&\frac{1}{2}\operatorname{div}(f^{\frac{p}{2}-1}A\nabla f)\\
&= |\nabla u|^{p-2}\sum_{i=1}^{n}|\nabla u_{x_i}|^2 + \frac{p-2}{4}|\nabla u|^{p-4}|\nabla f|^2 + \frac{p}{2}|\nabla u|^{p-2}\nabla u\cdot\nabla f\\
&- |\nabla u|^{p-2}\mathrm{Hess}(\log a)(\nabla u,\nabla u) - \frac{1}{2}|\nabla u|^{p-2}\nabla\log a\cdot\nabla f\\
&- \frac{p-2}{2}f^{\frac{p}{2}-2}(\nabla u\cdot\nabla f)(\nabla u\cdot\nabla\log a).
\end{aligned}$$

By the Schwarz inequality

$$|\nabla u|^2|\nabla u_{x_i}|^2 \geq \frac{1}{4}|\nabla f|^2.$$

So

$$\begin{aligned}
&\frac{1}{2}\operatorname{div}(f^{\frac{p}{2}-1}A\nabla f)\\
&\geq \frac{p-1}{4}|\nabla u|^{p-4}|\nabla f|^2 + \frac{p}{2}|\nabla u|^{p-2}\nabla u\cdot\nabla f - |\nabla u|^{p-2}\mathrm{Hess}(\log a)(\nabla u,\nabla u)\\
&- \frac{1}{2}|\nabla u|^{p-2}\nabla\log a\cdot\nabla f - \frac{p-2}{2}f^{\frac{p}{2}-2}(\nabla u\cdot\nabla f)(\nabla u\cdot\nabla\log a).
\end{aligned}$$

On the other hand

$$\begin{aligned}&\mathrm{div}(f^{\frac{p}{2}-1}A\nabla u)\\=&(p-1)\mathrm{div}(|\nabla u|^{p-2}\nabla u)\\=&(p-1)(|\nabla u|^p-|\nabla u|^{p-2}\nabla u\cdot\nabla\log a).\end{aligned}$$

Thus we observe that

$$\begin{aligned}&\mathrm{div}(f^{\frac{p}{2}-1}A\nabla g)\\=&\frac{1}{2}\mathrm{div}(f^{\frac{p}{2}-1}A\nabla f)-c_3\mathrm{div}(f^{\frac{p}{2}-1}A\nabla u)\\\geq&\frac{p-1}{4}|\nabla u|^{p-4}|\nabla f|^2+\frac{p}{2}|\nabla u|^{p-2}\nabla u\cdot\nabla f-|\nabla u|^{p-2}\mathrm{Hess}(\log a)(\nabla u,\nabla u)\\&-\frac{1}{2}|\nabla u|^{p-2}\nabla\log a\cdot\nabla f-\frac{p-2}{2}f^{\frac{p}{2}-2}(\nabla u\cdot\nabla f)(\nabla u\cdot\nabla\log a)\\&-c_3(p-1)(|\nabla u|^p-|\nabla u|^{p-2}\nabla u\cdot\nabla\log a).\end{aligned}$$

We conclude that

$$\begin{aligned}&\mathrm{div}(f^{\frac{p}{2}-1}A\nabla g)+h\cdot\nabla g\\\geq&\frac{p-1}{4}|\nabla u|^{p-4}|\nabla f|^2-|\nabla u|^{p-2}\mathrm{Hess}(\log a)(\nabla u,\nabla u)\\&-c_3(p-1)(|\nabla u|^p-|\nabla u|^{p-2}\nabla u\cdot\nabla\log a)+pc_3f^{\frac{p}{2}-1}|\nabla u|^2\\&-c_3|\nabla u|^{p-2}\nabla u\cdot\nabla\log a-(p-2)c_3(\nabla u\cdot\nabla\log a)f^{\frac{p}{2}-1}\\\geq&\frac{p-1}{4}|\nabla u|^{p-4}|\nabla f|^2-c_3|\nabla u|^p-c_3(p-1)|\nabla u|^p+pc_3|\nabla u|^p\\=&\frac{p-1}{4}|\nabla u|^{p-4}|\nabla f|^2.\end{aligned}$$

□

Remark 3.5.1. If a is constant, we can obtain

$$\begin{aligned}&\frac{1}{2}\mathrm{div}(f^{\frac{p}{2}-1}A\nabla f)-\frac{p}{2}|\nabla u|^{p-2}\nabla u\cdot\nabla f\\\geq&\ \frac{p-1}{4}|\nabla u|^{p-4}|\nabla f|^2.\end{aligned}$$

So we do not need to consider the function g. In order to squeeze the second-order differentiation of a, we need to introduce g.

Proof of Proposition 3.2. Fix β with $\beta > c_2+(n-1)R^{-1}$ and set $\Omega_\beta := \{x\in\Omega|\ g>\frac{\beta^2}{2}\}$. Assume that $\Omega_\beta\neq\emptyset$. By Proposition 3.1 and Lemma 3.3, $g\to-\infty$ as $|x|\to\infty$. So Ω_β is bounded open set and by Lemma 3.4, $g=\frac{\beta^2}{2}$ on $\partial\Omega_\beta$. Moreover, $|\nabla u_p|$ is bounded from below by a positive constant in Ω_β. So u_p is smooth in Ω_β. Therefore, by Lemma 3.5, we can apply maximum principle for divergence form to g and get

$$\sup_{\Omega_\beta} g\leq\sup_{\partial\Omega_\beta} g.$$

This contracts to $\Omega_\beta \neq \emptyset$. So

$$g < \frac{\beta^2}{2}$$

holds in Ω. Thus we get

$$g \leq \frac{1}{2}\left(c_2 + \frac{n-1}{R}\right)^2$$

in Ω. By Proposition 3.1 again, we obtain

$$|\nabla u_p(x)| \leq \{2(c_2 + (n-1)r^{-1}c_3(|x-x_0| - r) + (c_2 + (n-1)R^{-1})^2\}^{\frac{1}{2}} \quad \text{in } \Omega.$$

□

By Proposition 3.1 and Proposition 3.2, it turns out that there exists a subsequence $\{u_{p_k}\}_{k\in N} \subset \{u_p\}_{p>1}$ which converges uniformly to some $u \in C^{0,1}_{\text{loc}}(\Omega)$ as $p_k \to 1$. We have to prove that u is a proper weak solution of (1.1). It is proceeded by the same way as Moser [10].

To prove that u is a solution, we invoke the following lemma.

Lemma 3.6. For a compact set $K \subset \Omega$ and $v, w \in W^{1,p}_{\text{loc}}(\Omega; a)$, define the functional

$$J^p_v(w; K) := \int_K \frac{1}{p} a|\nabla w|^p + aw|\nabla v|^p.$$

Then for every precompact set $K \subset \Omega$ and every $w \in W^{1,p}_{\text{loc}}(\Omega; a)$ with $w = u_p$ in Ω^c, the inequality

$$J^p_{u_p}(u_p; K) \leq J^p_{u_p}(w; K)$$

holds.

Proof : Taking $u_p - w$ as test function in (1.7), we obtain

$$\begin{aligned}
\int_K a(u_p - w)|\nabla u_p|^p &= \int_\Omega a|\nabla u_p|^{p-2}\nabla u \cdot \nabla(w - u_p) \\
&\leq (\int_\Omega a|\nabla u_p|^p)^{\frac{p-1}{p}}(\int_\Omega a|\nabla w|^p)^{\frac{1}{p}} - \int_\Omega a|\nabla u_p|^p \\
&\leq \frac{1}{p}\int_K a(|\nabla w|^p - |\nabla u_p|).
\end{aligned}$$

□

Proof of the Main Theorem. By Proposition 3.1 and 3.2, it is easily checked that u is proper and satisfies the estimates in Main theorem. Also, the proof of uniqueness is almost same as the proof of [5](Therem 2.2). So it remains to show that u minimizes J_u in the sense of Definition 1.1. Take any precompact set $K \subset \Omega$ and $v \in C^{0,1}_{\text{loc}}(\Omega)$ with

$v = u$ in $\Omega\backslash K$. Choose $\eta \in C_0^\infty(\Omega)$ with $0 \le \eta \le 1$ and $\eta = 1$ in K. Applying Lemma 3.6 for the function $w = \eta v + (1-\eta)u_p$ and compact set supp η, we obtain

$$\begin{aligned}
&\int_{\text{supp }\eta} a\{\frac{1}{p}|\nabla u_p|^p + u_p|\nabla u_p|^p\} \\
\le &\int_{\text{supp }\eta} a\{\frac{1}{p}|\nabla\{\eta v + (1-\eta)u_p\}|^p + \{\eta v + (1-\eta)u_p\}|\nabla u_p|^p\} \\
\le &\int_{\text{supp }\eta} a\{\frac{1}{p}|\eta\nabla v + (1-\eta)\nabla u_p + (v-u_p)\nabla\eta|^p + \{\eta v + (1-\eta)u_p\}|\nabla u_p|^p\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{\text{supp }\eta} a\{\frac{1}{p}|\nabla u_p|^p + \eta(u_p - v)|\nabla u_p|^p\} \\
\le &\frac{1}{p}\int_{\text{supp }\eta} a|\eta\nabla v + (1-\eta)\nabla u_p + (v-u_p)\nabla\eta|^p \\
\le &\frac{3^{p-1}}{p}\int_{\text{supp }\eta} a\{\eta^p|\nabla v|^p + (1-\eta)^p|\nabla u_p|^p + |v-u_p|^p|\nabla\eta|^p\}. \qquad (*)
\end{aligned}$$

Setting $v = u$ and taking $p_k \to 1$, we get

$$\int_{\text{supp }\eta} a\eta|\nabla u| \ge \limsup_{k\to\infty}\int_{\text{supp }\eta} a\eta|\nabla u_{p_k}|^{p_k}.$$

The lower semicontinuity implies

$$\liminf_{k\to\infty}\int_{\text{supp }\eta} a|\nabla u_{p_k}|^{p_k} \ge \int_{\text{supp }\eta} a|\nabla u|.$$

Therefore taking supp $\eta\backslash K$ small enough, we obtain

$$\lim_{k\to\infty}\int_{\text{supp }\eta} a|\nabla u_{p_k}|^{p_k} = \int_{\text{supp }\eta} a|\nabla u|.$$

By taking $p_k \to 1$ in $(*)$, the left hand side becomes

$$\begin{aligned}
&\int_{\text{supp }\eta} a\{|\nabla u| + \eta(u-v)|\nabla u|\} \\
= &\int_K a|\nabla u| + (u-v)|\nabla u| + \int_{\text{supp }\eta\backslash K} a|\nabla u|,
\end{aligned}$$

and right hand side of $(*)$ becomes

$$\begin{aligned}
&\int_{\text{supp }\eta} a\{\eta|\nabla v| + (1-\eta)|\nabla u| + |v-u||\nabla\eta|\} \\
= &\int_K a|\nabla v| + \int_{\text{supp }\eta\backslash K} a\{\eta|\nabla u| + (1-\eta)|\nabla u| + |v-u||\nabla\eta|\} \\
\le &\int_K a|\nabla v| + \int_{\text{supp }\eta\backslash K} a|\nabla u|.
\end{aligned}$$

This implies that u is a weak solution.

□

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