Advances in Mathematical Sciences and Applications
Vol. 30, No. 1 (2021), pp. 209–216

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A REMARK ON ANISOTROPIC DECAY RATE ESTIMATES FOR THE NAVIER-STOKES EQUATIONS

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Abstract. For the nonstationary incompressible Navier-Stokes equations with anisotropically decaying external forces and zero initial data in $\mathbf{R}^n$ $(n \geq 3)$, we continue to consider anisotropic spatial decay rate estimates of the solutions in the so called Serrin's class. We utilize the weighted equation approach in the Lebesgue space equipped with the mixed norm in this paper again. But we here multiply two anisotropically growing functions to the equations at the same time. The double-weighted solutions for the linearized equations are estimated with the weighted and non-weighted external forces. The estimates for the nonlinear terms are more complicated.

In case the temporal decay rates of the external forces are grater than 1, we show the spatial decay rates of the solutions are almost same as those of the external forces in all directions as well as the isotropic decay rate estimates.

Communicated by Yoshikazu, Giga; Received April 19, 2021
AMS Subject Classification: 76D05, 35Q30.
Keywords: mixed norm, regularity class, uniform bound of weighted solutions, the Gagliardo-Nirenberg-Yamazaki inequality.

1 Introduction

This paper gives a remark on an extension of the work [1]. We consider the nonstationary incompressible Navier-Stokes equations in $\mathbf{R}^n$ $(n \geq 3)$ with the zero initial data:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = f, \quad \nabla \cdot u = 0, \quad \text{in } \mathbf{R}^n \times (0, \infty), \\ u(\cdot, 0) = 0, \quad |u(x,t)| \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{NS}$$

Here $u = (u_j(x,t))$ $(j = 1, \cdots, n)$ and $\pi = \pi(x,t)$ denote unknown velocity and pressure, respectively, and $f = (f_j(x,t))$ denotes a given external force.

In [1] it has been shown that

$$\begin{aligned} &f(x,t) = o\Big(\frac{1}{\sqrt{x_1^2 + x_2^2 + \cdots x_{n-2}^2}^{\mu} \sqrt{x_{n-1}^2 + x_n^2}^{2} t^{\lambda}}\Big) \\ &\Rightarrow \ |u(x,t)| \leq C(1 + \sqrt{x_1^2 + x_2^2 + \cdots x_{n-2}^2})^{-\rho} \end{aligned} \tag{1.1}$$

for so-called Serrin's class solutions u of (NS). Here for $f \in L^s(0, \infty; L^{\tilde{q}}(\mathbf{R}^2); L^{q'}(\mathbf{R}^{n-2}))$ with $q' > (n-2)/\mu$, $\tilde{q} > 1$ and $s > 1/\lambda$, decay rate ρ must satisfy

$$\rho < \mu, \quad \rho < \frac{n-2}{q'} + \frac{2}{\tilde{q}} + \frac{2}{s} - 2.$$

For $x = (x_1, x_2, \cdots, x_n)$, we denote that

$$\begin{aligned} &\tilde{x} = (x_1, x_2, \cdots, x_i), \quad x' = (x_{i+1}, \cdots, x_n) \\ &d_i(\tilde{x}) = (x_1^2 + \cdots + x_i^2)^{1/2}, \quad d_i'(x') = (x_{i+1}^2 + \cdots + x_n^2)^{1/2}. \end{aligned} \tag{1.2}$$

In this paper we will show that for $\rho < i, \rho' < n - i$,

$$f(x,t) = o(d_i^{-i} d_i'^{-(n-i)} t) \ \Rightarrow \ |u(x,t)| \leq C(1 + d_i)^{-\rho}(1 + d_i')^{-\rho'} \tag{1.3}$$

for so-called Serrin's class solutions u of (NS).

Our arguement is parallel to that of [1]. We first consider the linearized equations, so called the Stokes equations, which turns to the heat equations in $\mathbf{R}^n$ by applying the Helmholtz decomposition. We multiply a growing function $M_1(x_1, \cdots, x_i)$ and $M_2(x_{i+1}, \cdots, x_n)$ to the equations and obtain the a priori estimates of the weighted solution $M_1 M_2 u$. The embedding inequality in space-time yields us the estimates in $L^s(0, \infty; L^{q'}(\mathbf{R}^{n-i})^{n-i}; L^q(\mathbf{R}^i)^i)$. The Gagliardo-Nirenberg inequality of parabolic type gives the uniform bound of $M_1 M_2 u$.

Since we can choose weighting functions M_1 and M_2 respectively as

$$\begin{aligned} &\bar{d_i}^{\gamma} := (1 + d_i)^{\gamma} = (1 + \sqrt{x_1^2 + \cdots + x_i^2})^{\gamma}, \quad 0 < \gamma < 1 \\ &\bar{d_i'}^{\gamma'} := (1 + d_i')^{\gamma'} = (1 + \sqrt{x_{n+1}^2 + \cdots + x_n^2})^{\gamma'}, \quad 0 < \gamma' < 1 \end{aligned} \tag{1.4}$$

far from the origin, we obtain the decay rates of the solution by applying this operation several times.

To estimate (NS) we treat the nonlinear terms $u \cdot \nabla u$ as external forces. Under the smallness condition $||u||_{p,p',p_0} \ll 1$ in the class

$$u \in L^{p,p',p_0} \text{ for all } p, p', p_0 \geq 1, \quad \text{with } \frac{i}{p} + \frac{n-i}{p'} + \frac{2}{p_0} = 1, \tag{1.5}$$

the nonlinear term is estimated by the parabolic derivatives. The estimates for the commutator between the Helmholtz decomposition projector and the double weights are more complicated than those in [1]. Hence we obtain the same estimates of $M_1 M_2 u$ as those for the Stokes equations.

2 Main results

Theorem 2.1. *Let $n \geq 3$ and $i = 1, \cdots, n-1$. Let $\mu \in (0, i]$, $\mu' \in (0, n-i]$, $q > i/\mu$, $q' > (n-i)/\mu'$ and $s > 1$. Assume that solenoidal function $f \in L^{\infty,\infty}_{\mathrm{loc}}$ satisfies*

$$\lim_{|x|,t \to \infty} f(x,t) \sqrt{x_1^2 + \cdots x_i^2}^{\,\mu} \sqrt{x_{i+1}^2 + \cdots + x_n^2}^{\,\mu'} t = 0$$

and $f \in L^s(0,\infty; L^q(\mathbf{R}^i); L^{q'}(\mathbf{R}^{n-i}))$. Then for a weak solution u of (NS) in the class L^{p,p',p_0} for all p, p', $p_0 \in [1,\infty]$ satisfying $i/p + (n-i)/p' + 2/p_0 = 1$, there exist positive constants $\varepsilon = \varepsilon(n, \mu, \mu')$ and $\rho < \mu$, $\rho' < \mu'$ with

$$\rho + \rho' < \frac{i}{q} + \frac{n-i}{q'} + \frac{2}{s} - 2$$

such that $\sup ||u||_{p,p',p_0} < \varepsilon$ *implies that*

$$|u(x,t)| \leq C \left(1 + \sqrt{x_1^2 + \cdots x_i^2}\right)^{-\rho} \left(1 + \sqrt{x_{i+1}^2 + \cdots + x_n^2}\right)^{-\rho'}$$

with $C = C(n, f, \rho, \rho')$.

3 Anisotropically double weighted Stokes equations

Operating the Helmholtz decomposition in $L^{q,q'}(\mathbf{R}^n)^n$ to the Stokes system, we consider the heat equations with the solenoidal condition

$$\begin{cases} \partial_t u - \Delta u = f, \\ \nabla \cdot u = 0, \ \text{in } \mathbf{R}^n \times (0,\infty), \\ u(\cdot, 0) = 0, \\ |u(x,t)| \to 0 \ (|x| \to \infty). \end{cases} \tag{H}$$

Let $i = 1, 2, \cdots, n-1$. For $\gamma, \gamma' \in (0,1)$, let $\alpha_0 \in (i, \infty]$, $\alpha_1 \in (i/2, i]$, $\alpha_0' \in (n-i, \infty]$, $\alpha_1' \in ((n-i)/2, n-i]$ satisfy

$$\gamma < \gamma_0 := 1 - \frac{i}{\alpha_0} = 2 - \frac{i}{\alpha_1}, \quad \gamma' < \gamma_0' := 1 - \frac{n-i}{\alpha_0'} = 2 - \frac{n-i}{\alpha_1'}. \tag{3.1}$$

Let $M_1(\tilde{x}) \in W^{2,p}_{\rm loc}(\mathbf{R}^i),\ M_2(x') \in W^{2,p'}_{\rm loc}(\mathbf{R}^{n-i})$ (for some $p, p' > 1$) satisfy

$$\begin{cases} 0 < M_1 \le C\bar{d}_i^{\,\gamma}, \\ \nabla M_1 \in L^{\alpha_0}(\mathbf{R}^i), \\ \Delta M_1 \in L^{\alpha_1}(\mathbf{R}^i), \end{cases} \tag{M$_1$}$$

$$\begin{cases} 0 < M_2 \le C\bar{d}_i'^{\,\gamma'}, \\ \nabla M_2 \in L^{\alpha_0'}(\mathbf{R}^{n-i}), \\ \Delta M_2 \in L^{\alpha_1'}(\mathbf{R}^{n-i}). \end{cases} \tag{M$_2$}$$

We can take $M_1 = C\bar{d}_i^{\,\gamma},\ M_2 = C\bar{d}_i'^{\,\gamma'}$. We denote that

$$L^{q,q',s} = L^s(0,\infty; L^{q'}(\mathbf{R}^{n-i}); L^q(\mathbf{R}^i))$$

with norm $||\cdot||_{q,q',s}$ and

$$|||u|||_{q,q',s} := ||\partial_t u||_{q,q',s} + ||\nabla^2 u||_{q,q',s}.$$

We also denote that $M := M_1 M_2$.

Proposition 3.1. *For $k \ge 2$, let $1 < q \le q_1 \le \cdots \le q_k < \infty, 1 < q' \le q_1' \le \cdots \le q_k' < \infty$ and $1 < s \le s_1 \le \cdots \le s_k < \infty$ satisfy*

$$\begin{aligned} &1 \le j, m \le k, \\ &\frac{i}{q} + \frac{n-i}{q'} + \frac{2}{s} \\ &= (1 - \frac{i}{\alpha_0})j + (1 - \frac{n-i}{\alpha_0'})m + \frac{i}{q_{j,m}} + \frac{n-i}{q_{j,m}'} + \frac{2}{s_{j,m}}, \\ &(q_j, q_j', s_j) := (q_{j,j}, q_{j,j}', s_{j,j}),\ (q_0, q_0', s_0) := (q, q', s). \end{aligned} \tag{3.2}$$

Assume that $q_k < \alpha_0/2, q_k' < \alpha_0'/2$. If $M_1^j M_2^m f \in L^{q_{j,m}, q_{j,m}', s_{j,m}} (j, m = 0, 1, \cdots, k)$, the solution of (H) satisfies

$$|||M^k u|||_{q_k, q_k' s_k} \le C \sum_{j,m=0}^{k} ||M_1^j M_2^m f||_{q_{j,m}, q_{j,m}', s_{j,m}}. \tag{3.3}$$

Proof. Multiplying $M = M_1 M_2$ to (H), we have

$$\begin{aligned} &(Mu)_t - \Delta(Mu) \\ &= Mf - 2\nabla M \cdot \nabla u - u\Delta M \\ &= Mf + 2\nabla M_1 \nabla M_2 \cdot u - 2(\nabla M_1 \cdot \nabla(M_2 u) + \nabla M_2 \cdot \nabla(M_1 u)) \\ &\quad -((\Delta M_2) M_1 u + (\Delta M_1) M_2 u). \end{aligned} \tag{3.4}$$

The a priori estimates, Hölder's inequality and Embedding Lemma in [1] yield

$$
\begin{aligned}
&|||Mu|||_{q_1,q_1',s_1} \\
&\leq C(||Mf||_{q_1,q_1',s_1} + ||M_1 f||_{q_{1,0},q_{1,0}',s_{1,0}} + ||M_2 f||_{q_{0,1},q_{0,1}',s_{0,1}} + ||f||_{q,q',s})
\end{aligned}
\tag{3.5}
$$

with

$$
\begin{aligned}
\frac{i}{q} + \frac{n-i}{q'} + \frac{2}{s} &= (1 - \frac{i}{\alpha_0}) + \frac{i}{q_{1,0}} + \frac{n-i}{q_{1,0}'} + \frac{2}{s_{1,0}} \\
&= (1 - \frac{n-i}{\alpha_0'}) + \frac{i}{q_{0,1}} + \frac{n-i}{q_{0,1}'} + \frac{2}{s_{0,1}} \\
&= (1 - \frac{i}{\alpha_0}) + (1 - \frac{n-i}{\alpha_0'}) + \frac{i}{q_1} + \frac{n-i}{q_1'} + \frac{2}{s_1}.
\end{aligned}
\tag{3.6}
$$

The desired results are given by

$$
\begin{aligned}
&(M^k u)_t - \Delta(M^k u) \\
&= M^k f - 2k\nabla M \cdot \nabla(M^{k-1}u) + k(k-1)(\nabla M)^2 M^{k-2}u - kM^{k-1}u\Delta M \\
&= M^k f - 2k\{\nabla M_1 \cdot \nabla(M_1^{k-1}M_2^k u) + \nabla M_2 \cdot \nabla(M_1^k M_2^{k-1}u)\} \\
&\quad +2k(k-3)\nabla M_1 \nabla M_2 \cdot \nabla(M^{k-1}u) \\
&\quad +k(k-1)\{(\nabla M_1)^2 M_1^{k-2}M_2^k u + (\nabla M_2)^2 M_1^k M_2^{k-2}u\} \\
&\quad -k\{(\Delta M_2)(M_1^k M_2^{k-1}u) + (\Delta M_1)(M_1^{k-1}M_2^k u)\}.
\end{aligned}
\tag{3.7}
$$

□

Since we take M_1, M_2 such that $M_1^k = C\bar{d}_i^{k\gamma}, M_2^k = C\bar{d}_i'^{k\gamma'}$ at the infinity while $k\gamma$ and $k\gamma'$ are actually the decay order of solutions in $\mathbf{R}^i$ and $\mathbf{R}^{n-i}$ respectively, (3.1) implies

$$
k(\gamma + \gamma') < \frac{i}{q} + \frac{n-i}{q'} + \frac{2}{s} - (\frac{i}{q_k} + \frac{n-i}{q_k'} + \frac{2}{s_k}).
\tag{3.8}
$$

By the Gagliardo-Nirenberg-Yamazaki inequalities in [1], we obtain uniform decays

$$
|u(x,t)| \leq CM^{-k}
$$

for the Stokes equations with weakly solenoidal external forces.

4 Estimates for the weighted Navier-Stokes system

Hereafter we consider the weak solutions u of (NS) in the class (1.5). Applying the Helmholtz decomposition projector to (NS), Proposition 3.1 implies for $M = M_1 M_2$,

$$
\begin{aligned}
&|||M^k u|||_{q_k,q_k',s_k} \\
&\leq C \sum_{j,m=0}^{k} (||M_1^j M_2^m P(u\cdot\nabla u)||_{q_{j,m},q_{j,m}',s_{j,m}} + ||M_1^j M_2^m f||_{q_{j,m},q_{j,m}',s_{j,m}}).
\end{aligned}
\tag{4.1}
$$

For (q_k, q_k', s_k) with $q_k < \alpha_0/2, q_k' < \alpha_0'/2$ and (3.1), let (p, p', p_0) satisfy

$$\frac{1}{q_k} > \frac{1}{\alpha_0} + \frac{1}{p},\ \frac{1}{q_k'} > \frac{1}{\alpha_0'} + \frac{1}{p'},\ s_k < p_0,\ \frac{i}{p} + \frac{n-i}{p'} + \frac{2}{p_0} = 1. \tag{4.2}$$

To estimate $M_1^j M_2^m P(u \cdot \nabla u)$ in (4.1), we have

$$M_1^j M_2^m P(u \cdot \nabla u) = P M_1^j M_2^m (u \cdot \nabla u) - (P M_1^j M_2^m - M_1^j M_2^m P)(u \cdot \nabla u), \tag{4.3}$$

$$\begin{aligned} &M_1^j M_2^m (u \cdot \nabla u) \\ &= u \cdot \{\nabla (M_1^j M_2^m u) - (m(\nabla M_2) M_1^j M_2^{m-1} + j(\nabla M_1) M_1^{j-1} M_2^m) u\}. \end{aligned} \tag{4.4}$$

Applying Hölder's inequality and Embedding Lemma in [1] to the first term in the right hand side of (4.3) with (4.4) yield

$$\begin{aligned} &||P M_1^j M_2^m (u \cdot \nabla u)||_{q_{j,m}, q'_{j,m}, s_{j,m}} \\ &\leq C||u||_{p,p',p_0} \{|||M_1^j M_2^m u|||_{q_{j,m}, q'_{j,m}, s_{j,m}} + |||M_1^j M_2^{m-1} u|||_{q_{j,m-1}, q'_{j,m-1}, s_{j,m-1}} \\ &\qquad + |||M_1^{j-1} M_2^m u|||_{q_{j-1,m}, q'_{j-1,m}, s_{j-1,m}}\}. \end{aligned} \tag{4.5}$$

Next we show the estimates of the commutator in (4.3).

Lemma 4.1. *Let $1 \leq i < n$, $0 < \delta < i$ and $0 < \delta' < n - i$. Let $r \in (1, \infty)$ satisfy*

$$\frac{\delta}{n} < \frac{1}{r} < \frac{\delta + i}{n}, \quad \frac{\delta'}{n} < 1 - \frac{1}{r} < \frac{\delta' + (n-i)}{n}. \tag{4.6}$$

Then it follows that

$$||f * \frac{d_i^{\delta} d_i'^{\delta'}}{|x|^n}||_{\mathbf{p}} \leq C||f||_{\mathbf{q}} \tag{4.7}$$

with

$$\sum_{j=1}^{i} (\frac{1}{q_j} - \frac{1}{p_j}) = i - \frac{n}{r} + \delta, \quad \sum_{j=i+1}^{n} (\frac{1}{q_j} - \frac{1}{p_j}) = \frac{n}{r} - i + \delta'. \tag{4.8}$$

Proof.
Young's inequality yields

$$\frac{d_i^{\delta} d_i'^{\delta'}}{|x|^n} \leq C \frac{1}{d_i^{n/r - \delta} d_i'^{n/r' - \delta'}} = C \frac{1}{d_i^{i - \alpha} d_i'^{(n-i) - \alpha'}} \tag{4.9}$$

with

$$\begin{aligned} &1 < r < \infty,\ \frac{1}{r} + \frac{1}{r'} = 1, \\ &0 < \alpha < i, \quad \frac{n}{r} - \delta = i - \alpha, \\ &0 < \alpha' < n - i, \quad \frac{n}{r'} - \delta' = (n-i) - \alpha'. \end{aligned} \tag{4.10}$$

Here it must be satisfied that (4.6). We obtain the desired results in parallel to Lemma 2.2 (2) in [1]. □

Lemma 4.2. (Estimates of the commutator) *Assume that* $\nabla M_1 \in L^{\alpha_0}(\mathbf{R}^i)$, $\nabla M_2 \in L^{\alpha_0'}(\mathbf{R}^{n-i})$. *Let* $j, m = 1, 2, \cdots, k$ *and* $0 < j\gamma_0 + m\gamma_0' < n$. *For* $q_{j,m}, q_{j,m}' \in (1, \infty)$, *let* $c_{\ell,\ell'} \in (1, q_{j,m})$, $c_{\ell,\ell'}' \in (1, q_{j,m}')$ *for* $\ell = 0, 1, \cdots, j$ *and* $\ell' = 0, 1, \cdots, m$ *satisfy*

$$i\Big(\frac{1}{c_{\ell,\ell'}} - \frac{1}{q_{j,m}}\Big) + (n-i)\Big(\frac{1}{c_{\ell,\ell'}'} - \frac{1}{q_{j,m}'}\Big) = \gamma_0(j-\ell) + \gamma_0'(m-\ell'), \tag{4.11}$$

and

$$0 < \frac{i}{q_{j,m}} + \frac{n-i}{q_{j,m}'} < n - (j\gamma_0 + m\gamma_0'). \tag{4.12}$$

Then it follows that

$$||(PM_1^j M_2^m - M_1^j M_2^m P)v||_{q_{j,m},q_{j,m}'} \leq C \sum_{(\ell,\ell')\neq(j,m)} ||M_1^\ell M_2^{\ell'} v||_{c_{\ell,\ell'},c_{\ell,\ell'}'}. \tag{4.13}$$

Proof. The polynomial expansion formula implies

$$\begin{aligned} &M_1^j M_2^m(y) - M_1^j M_2^m(x) \\ &= -\sum_{\ell=0}^{j-1} C(j,\ell) A_{j,m,\ell} - \sum_{\ell'=0}^{m-1} C(m,\ell') A_{j,m,\ell'}' - \sum_{\ell=0}^{j-1}\sum_{\ell'=0}^{m-1} C(j,m,\ell,\ell') B_{j,m,\ell,\ell'} \end{aligned} \tag{4.14}$$

with

$$\begin{aligned} A_{j,m,\ell} &= (M_1(\tilde{x}) - M_1(\tilde{y}))^{j-\ell} M_1^\ell(\tilde{y}) M_2^m(y'), \\ A_{j,m,\ell'}' &= (M_2(x') - M_2(y'))^{m-\ell'} M_2^{\ell'}(y') M_1^j(\tilde{y}), \\ B_{j,m,\ell,\ell'} &= (M_1(\tilde{x}) - M_1(\tilde{y}))^{j-\ell}(M_2(x') - M_2(y'))^{m-\ell'} M_1^\ell(\tilde{y}) M_2^{\ell'}(y'). \end{aligned} \tag{4.15}$$

By Lemma 4.1 and

$$\begin{aligned} |M_1(\tilde{x}) - M_1(\tilde{y})| &\leq C|\nabla M_1|_{\alpha_0} |\tilde{x} - \tilde{y}|^{1-i/\alpha_0}, \\ |M_2(x') - M_2(y')| &\leq C|\nabla M_2|_{\alpha_0'} |x' - y'|^{1-(n-i)/\alpha_0'}, \end{aligned} \tag{4.16}$$

it follows that

$$\begin{aligned} &||(PM_1^j M_2^m - M_1^j M_2^m P)v||_{q_{j,m},q_{j,m}'} \\ &\leq C\{\sum_{\ell=0}^{j-1} || \int \frac{|\tilde{x} - \tilde{y}|^{\gamma_0(j-\ell)}}{|x-y|^n} |M_1^\ell(\tilde{y}) M_2^m(y') v(y)|\, dy||_{q_{j,m},q_{j,m}'} \\ &\quad + \sum_{\ell'=0}^{m-1} || \int \frac{|x' - y'|^{\gamma_0'(m-\ell')}}{|x-y|^n} |M_1^j(\tilde{y}) M_2^{\ell'}(y') v(y)|\, dy||_{q_{j,m},q_{j,m}'} \\ &\quad + \sum_{\ell=0}^{j-1}\sum_{\ell'=0}^{m-1} || \int \frac{|x'-y'|^{\gamma_0(j-\ell)} |\tilde{x} - \tilde{y}|^{\gamma_0'(m-\ell')}}{|x-y|^n} |M_1^\ell(\tilde{y}) M_2^{\ell'}(y') v(y)|\, dy||_{q_{j,m},q_{j,m}'}\} \\ &\leq C\left\{ \sum_{\ell'=0}^{m-1} ||M_1^j M_2^{\ell'} v||_{c_{j,\ell'},c_{j,\ell'}'} + \sum_{\ell=0}^{j-1} ||M_1^\ell M_2^m v||_{c_{\ell,m},c_{\ell,m}'} + \sum_{\ell=0}^{j-1}\sum_{\ell'=0}^{m-1} ||M_1^\ell M_2^{\ell'} v||_{c_{\ell,\ell'},c_{\ell,\ell'}'} \right\} \\ &= C \sum_{(\ell,\ell')\neq(j,m)} ||M_1^\ell M_2^{\ell'} v||_{c_{\ell,\ell'},c_{\ell,\ell'}'}. \end{aligned}$$

□

We apply the above lemma to $v = (u \cdot \nabla)u$.

Proposition 4.3. *Assume the same as in Lemma 4.2. Let (p, p', p_0) satisfy (4.2) and let*

$$\frac{2}{s} - \frac{2}{s_{k-1}} \leq 1 - \frac{2}{p_0} = \frac{i}{p} + \frac{n-i}{p'}. \tag{4.17}$$

Then it follows that

$$\begin{aligned} &||(PM_1^j M_2^m - M_1^j M_2^m P)(u \cdot \nabla u)||_{q_{jm}, q'_{j,m}, s_{j,m}} \\ &\leq C||u||_{p,p',p_0} \sum_{(\ell,\ell') \neq (j,m)} |||M_1^\ell M_2^{\ell'} u|||_{q_{\ell,\ell'}, q'_{\ell,\ell'}, s_{\ell,\ell'}}. \end{aligned} \tag{4.18}$$

Proof.
Lemma 4.2 implies

$$\begin{aligned} &||(PM_1^j M_2^m - M_1^j M_2^m P)(u \cdot \nabla u)||_{q_{jm}, q'_{j,m}, s_{j,m}} \\ &\leq C \sum_{(\ell,\ell') \neq (j,m)} ||M_1^\ell M_2^{\ell'} (u \cdot \nabla u)||_{c_{\ell,\ell'}, c'_{\ell,\ell'}, s_{j,m}}. \end{aligned} \tag{4.19}$$

In parallel to (4.5), the Leibnitz rule (4.4) implies

$$\begin{aligned} &||(PM^j - M^j P)(u \cdot \nabla u)||_{q_j, q'_j, s_j} \\ &\leq C||u||_{p,p',p_0} \sum_{(\ell,\ell') \neq (j,m)} \{|||M_1^\ell M_2^{\ell'} u|||_{q_{\ell,\ell'}, q'_{\ell,\ell'}, s_{\ell,\ell'}} \\ &\quad + |||M_1^{\ell-1} M_2^{\ell'} u|||_{q_{\ell-1,\ell'}, q'_{\ell-1,\ell'}, s_{\ell-1,\ell'}} + |||M_1^\ell M_2^{\ell'-1} u|||_{q_{\ell,\ell'-1}, q'_{\ell,\ell'-1}, s_{\ell,\ell'-1}}\}. \end{aligned} \tag{4.20}$$

Thus we obtain (4.18). □

Proof of Theorem 2.1 We finally obtain

$$\begin{aligned} &|||M^k u|||_{q_k, q'_k, s_k} \\ &\leq C\{||u||_{p,p',p_0} |||M^k u|||_{q_k, q'_k, s_k} + \sum_{(\ell,\ell') \neq (k,k)} |||M_1^\ell M_2^{\ell'} u|||_{q_{\ell,\ell'}, q'_{\ell,\ell'}, s_{\ell,\ell'}} \\ &\quad + \sum_{j,m=0}^{k} ||M_1^j M_2^m f||_{q_{j,m}, q'_{j,m}, s_{j,m}}\}. \end{aligned} \tag{4.21}$$

In parallel to [1], for the desired decay rate $\rho = k(\gamma + \gamma')$, letting

$$k(\gamma + \gamma') = \rho < \frac{n-2}{q'} + \frac{2}{\tilde{q}} + \frac{2}{s} - 2, \quad \frac{i}{q_k} + \frac{n-i}{q'_k} + \frac{2}{s_k} \searrow 2, \tag{4.22}$$

we conclude Theorem 2.1.

References

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