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ON ANISOTROPIC DECAY RATE ESTIMATES FOR THE NAVIER-STOKES EQUATIONS

Dedicated to Professor Yoshikazu Giga of Tokyo University on the occasion of his retirement

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Abstract. We consider the nonstationary incompressible Navier-Stokes equations with nonzero external forces and zero initial data in $\mathbf{R}^n$ $(n \geq 3)$. Spatial decay rates of the solutions in the so called Serrin's class are shown, while decay rates of data are anisotropically prescribed. To consider anisotropic measurable functions, we recall the Lebesgue space equipped with the mixed norm. A weighted equation approach is applied here, that is, by multiplying anisotropically growing function to the equation, we estimate the weighted solution by the weighted and non-weighted data in the space with the mixed norm. In case the temporal decay rates of the external forces are almost zero, we obtain the spatial decay rate estimates such that the decay rates of the solutions are almost same as those of the external forces in the $n-2$ dimensional spatial directions and are grater than 2 plus those of the external forces in the remaining 2 dimensional spatial directions.

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1 Introduction

Space-time decay rate estimates We consider the nonstationary incompressible Navier-Stokes equations in $\mathbf{R}^n$ ($n \geq 3$) with the zero initial data:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = f, \quad \nabla \cdot u = 0, \quad \text{in } \mathbf{R}^n \times (0, \infty), \\ u(\cdot, 0) = 0, \quad |u(x,t)| \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{NS}$$

Here $u = (u_j(x,t))$ ($j = 1, \cdots, n$) and $\pi = \pi(x,t)$ denote unknown velocity and pressure, respectively, and $f = (f_j(x,t))$ denotes a given external force.

We are interested in anisotropic decay rates of a solution under natural assumption on external force. We begin with reviewing a few typical known results. Knightly [17] constructed an *analytic, in space-time, solution* for the Navier-Stokes equations in $\mathbf{R}^n$ with zero external force and nonzero continuous initial data $g(x)$ with $|g(x)| \leq A \min(1, |x|^{-r})$ for some $r \geq 1$ and sufficiently small A. He assumed that for some constant vector u_∞ and some $T > 0$, u, u_t, ∇u, Δu, π and $\nabla \pi$ are all continuous on $\mathbf{R}^n \times (0, T)$ and $|u(x,t) - u_\infty| = o(1)$, $|\nabla u(x,t)| = o(|x|)$ and $|\pi(x,t)| = o(|x|)$ as $|x| \to \infty$, locally uniformly in t. He represented the solution by an integral equation with the fundamental solution of the Stokes operator and obtained that

$$|u(x,t)| \leq C\min(1, |x|^{-r}, t^{-r/2}), \ r < n. \tag{1.1}$$

Here $n \geq 3$ for $\pi(x,t)$. In (1.1), we see $|u(x,t)| \leq C|x|^{-r}$ in $\{(x,t) \in \mathbf{R}^n \times (0,\infty); |x| \geq \sqrt{t} \geq 1\}$ and $|u(x,t)| \leq Ct^{-r/2}$ in $\{(x,t) \in \mathbf{R}^n \times (0,\infty); \sqrt{t} \geq |x| \geq 1\}$. Thus (1.1) are not *uniform* decay estimates and actually yields $|u(x,t)| \leq C(1 + t + |x|^2)^{-r/2}$.

As well-known, if a weak solution u of (NS) is in the class

$$L^s(0, \infty; L^q(\mathbf{R}^n)^n) \quad \text{with} \quad n/q + 2/s = 1 \tag{1.2}$$

(and if its norm there $||u||_{q,s}$ is sufficiently small when $(q, s) = (n, \infty)$), u is C^∞ regular in space after a finite time (See Serrin [23], Giga [13] and Takahashi [24]). Takahashi [25] showed uniform estimates for weak solutions of (NS) in the class (1.2) with $||u||_{q,s} \ll 1$, while the decay rates of external forces are prescribed. That is, for $f(x,t) = o(|x|^{-\mu} t^{-\lambda})$ as $|x|, t \to \infty$ for some $\mu, \lambda > 0$, Takahashi [25] showed that

$$|u(x,t)| \leq C(1 + |x|)^{-\rho}(1 + t)^{-\sigma}$$

with $C = C(n, f, \rho, \sigma)$. Here for any $c > 0$, (ρ, σ) is given as follows.
(a) For decays in space, $\{(\rho, \sigma), (\mu, \lambda)\} = \{(n - c, 0), (n, 1)\}$.
(b) For time decays,

$$\{(\rho, \sigma), (\mu, \lambda)\} = \begin{cases} \{(0, n/2 - c), (n, n/2)\} \text{ for even } n, \\ \{(0, n/2 - c), (n, \frac{n}{2} + \frac{1}{n+1})\} \text{ for odd } n \end{cases}$$

(c) For decays in space-time, $\{(\rho, \sigma), (\mu, \lambda)\} = \{(n - 2k - c, k - c), (n, k)\}$ for $k \in \mathbf{N}$ with $k < n/2$.
Hereafter we simply denote

$$L^{q,s} := L^s(0, \infty; L^q(\mathbf{R}^n)^n).$$

Thus space $L^{q,s}$ is equipped with the norm

$$||u||_{q,s} = \left\{ \int_0^\infty (\int_{\mathbf{R}^n} |u(x,t)|^q \ dx)^{s/q} dt \right\}^{1/s}.$$

Anisotropic estimates We call function $f(x)$ decays isotropically if and only if there exists a positive constant α such that

$$\frac{C_1}{|x|^\alpha} \le |f(x)| \le \frac{C_2}{|x|^\alpha} \ \text{ as } |x| \to \infty.$$

We note that Young's inequality implies

$$\frac{1}{|x|^n} = \frac{1}{\sqrt{x_1^2 + \cdots + x_n^2}^n} \le \frac{1}{n^{n/2}} \frac{1}{|x_1 \cdots x_n|}$$

and that

$$\frac{1}{(1+|x|)^{n\alpha}}, \ \frac{1}{(1+|x_1|)^\alpha \cdots (1+|x_n|)^\alpha} \in L^p(\mathbf{R}^n) \ \text{ if } \alpha p > 1,$$

where $L^p(\mathbf{R}^n)$ is an isotropic space.

As isotropic estimates of the heat eqation $v_t - \Delta v = f$ in $\mathbf{R}^n$ with the zero initial data, the anisotropic estimates can be obtained as follows,

$$\begin{aligned} E(x,t) &= (4\pi t)^{-i/2} \exp\{-\tfrac{(x_1^2+\cdots+x_i^2)}{4t}\} \cdot (4\pi t)^{-(n-i)/2} \exp\{-\tfrac{(x_{i+1}^2+\cdots+x_n^2)}{4t}\} \\ &\le \frac{C}{(x_1^2+\cdots+x_i^2)^{a/2}(x_{i+1}^2+\cdots+x_n^2)^{b/2}t^c}. \end{aligned}$$

Here $a+b+2c = n, \ a \le i, \ b \le n-i$. For external force f with

$$|f| \le \frac{1}{(x_1^2+\cdots+x_i^2)^{\alpha/2}(x_{i+1}^2+\cdots+x_n^2)^{\beta/2}t^\gamma}, \tag{1.3}$$

where $\alpha \le i$, $\beta \le n-i$ and $\gamma \le 1$, the solution v is estimated as

$$|v| \le \frac{C}{(x_1^2+\cdots+x_i^2)^{\frac{\alpha-\alpha'}{2}}(x_{i+1}^2+\cdots+x_n^2)^{\frac{\beta-\beta'}{2}}t^{\gamma-\gamma'}}, \tag{1.4}$$

where $\alpha' + \beta' + 2\gamma' = 2$. We here take $a = i - \alpha'$, $b = n - i - \beta'$ and $c = 1 - \gamma'$.

We here especially consider the case $i = 2, \ \beta' = \gamma' = 0, \ \alpha' = 2$. Hereafter we denote that

$$\mathbf{R}^n \ = \mathbf{R}^{n-2} \times \mathbf{R}^2 = \{x' = (x_1, x_2, \cdots, x_{n-2})\} \times \{\tilde{x} = (x_{n-1}, x_n)\}$$

with the norm

$$|x'| = \sqrt{x_1^2 + x_2^2 + \cdots x_{n-2}^2}, \quad |\tilde{x}| = \sqrt{x_{n-1}^2 + x_n^2}. \tag{1.5}$$

It follows that

$$|f| \le \frac{1}{|x'|^\beta |\tilde{x}|^2 t^\gamma} \Rightarrow |v| \le \frac{C}{|x'|^\beta t^\gamma}. \tag{1.6}$$

Main results In this paper we will show that

$$f(x,t) = o(|x'|^{-\mu}|\tilde{x}|^{-2}t^{-\lambda}) \;\Rightarrow\; |u(x,t)| \le C(1+|x'|)^{-\rho} \tag{1.7}$$

for so-called Serrin's class solutions u of (NS). Here for $f \in L^s(0,\infty; L^{\tilde{q}}(\mathbf{R}^2); L^{q'}(\mathbf{R}^{n-2}))$ with $q' > (n-2)/\mu,\ \tilde{q} > 1$ and $s > 1/\lambda$, decay rate ρ must satisfy

$$\rho < \mu, \quad \rho < \frac{n-2}{q'} + \frac{2}{\tilde{q}} + \frac{2}{s} - 2.$$

Here we can take $\tilde{q}$ close to 1. In case λ is sufficiently small, we can take s such that $2/s$ is close to 0. Taking q' close to $(n-2)/\mu$, we can take ρ close to μ. Thus we can say that estimate (1.7) is close to estimate (1.6).

Our arguement is parallel to that of [25]. We first consider the linearized equations, so called the Stokes equations, which turns to the heat equations in $\mathbf{R}^n$ by applying the Helmholtz decomposition. We multiply a growing function $M(x')$ to the equations and obtain the a priori estimates of the weighted solution Mu. The embedding inequality in space-time yields us the estimates in so-called mixed norm space $L^s(0,\infty; L^{\mathbf{q}}(\mathbf{R}^n)^n)$. The Gagliardo-Nirenberg inequality of parabolic type gives the uniform bound of Mu.

Since we can choose weighting functions $M(x')$ as $\bar{d_m}^{\gamma}(x') := (1 + d_m(x'))^\gamma := (1 + \sqrt{x_1^2 + \cdots + x_m^2})^\gamma$ $(0 < \gamma < 1)$ far from the origin, we obtain the decay rates of the solution by applying this operation several times.

To estimate (NS) we treat the nonlinear terms $u \cdot \nabla u$ as external forces. Under the smallness condition $||u||_{\mathbf{p},p_0} \ll 1$ in the class

$$u \in L^{\mathbf{p},p_0} \text{ for all } \mathbf{p} = (p_1, \cdots, p_n), p_0 \ge 1, \quad \text{with } \sum_{i=1}^n \frac{1}{p_i} + \frac{2}{p_0} = 1, \tag{1.8}$$

the nonlinear term is estimated by the parabolic derivatives, i.e., the time and spatial second derivatives. Hence we obtain the same estimates of Mu as those for the Stokes equations.

Literature overview for spatial decay properties Amrouche et al. [1], Miyakawa [21], Brandolese [8] and Brandolese and Vigneron [9] constructed strong solutions $u(x,t)$ for the Navier-Stokes equations in $\mathbf{R}^n$ with zero external force and nonzero initial data with some suitable conditions. In [1] strong solutions which decay in L^2 at the rate of $||u(t)||_2 \le C(1+t)^{-\mu}$ satisfiy for $0 \le k \le n/2$

$$|D^\alpha u(x,t)| \le C_{k,m}(1+t)^{-\rho}(1+|x|)^{-k/2},$$

where $\rho = (1-2k/n)(m/2+\mu+n/4), |\alpha| = m,\ \mu > n/4$ and $2 \le n \le 5$. In [21], the initial data are assumed to satisfy solenoidal, some cyclically symmetricity and sufficient decay properties, then the solutions satisfy

$$|u(x,t)| \le C(1+|x|)^{-n-3}, \quad |u(x,t)| \le C(1+t)^{-(n+3)/2}.$$

In [8] the solutions which are left invariant under the action of discrete subgroups of the orthogonal group decay much faster than in generic case, where $n = 2, 3$. In [9] for well localized initial data the solution satisfies

$$Ct|x|^{-(n+1)} \le |u(x,t)| \le C't|x|^{-(n+1)}$$

for sufficiently small t and sufficiently large $|x|$.

Bae and Brandolese [2] also constructed a strong solution $u(x,t)$ for the Navier-Stokes equations in $\mathbf{R}^n$ with external force with non-zero mean and initial data with some suitable conditions satisfying

$$Ct|x|^{-n} \leq |u(x,t)| \leq C't|x|^{-n}$$

for sufficiently small t and sufficiently large $|x|$.

There is abundant literature on the spatial decay properties of solutions of Navier-Stokes equations by many mathematicians. (See [3],[7],[11],[12],[19],[20].)

There seems no literature studying anisotropic decay rate estimates for the solutions of the Navier-Stokes equations, although the Navier-Stokes equations are considered in the spaces with the mixed norm.(See [10],[16],[22].)

2 Mixed norm spaces $L^{\mathbf{p}}$

In this section we recall some fundamental estimates in anisotropic Lebesgue spaces, so called *mixed norm spaces*, and prepare additional estimates needed later on.

Study of mixed norm spaces goes back to Benedek and Panzone [5] and many efforts for anisotropic spaces have been especially made by Russian mathematicians (see Besov, Il'in and Nikol'skii [6], Kudryavtsev and Nikol'skii [18] and references there in). A survey is recently given by [15].

For measureable function $f(x)$ defined on $\mathbf{R}^n$ we write $f \in L_i^p(\mathbf{R})$ if

$$||f||_{L_i^p} = (\int_{\mathbf{R}} |f(x)|^p dx_i)^{\frac{1}{p}} < \infty.$$

For component $\mathbf{p} = (p_1, \cdots, p_n)$ $(1 \leq p_i \leq \infty$ for $i = 1, \cdots, n)$, the mixed norm space $L^{\mathbf{p}}(\mathbf{R}^n)$ is equipped with the norm

$$||f||_{\mathbf{p}} = (\int \cdots (\int (\int |f(x)|^{p_1} dx_1)^{\frac{p_2}{p_1}} dx_2)^{\frac{p_3}{p_2}} \cdots dx_n)^{\frac{1}{p_n}}.$$

That is, $L^{\mathbf{p}}(\mathbf{R}^n) = L_n^{p_n}(\cdots (L_2^{p_2}(\mathbf{R}, L_1^{p_1}(\mathbf{R})) \cdots)$. We also consider *restricted* integration

$$L^{\mathbf{p}}_{1,\cdots,i} = L_i^{p_i}(\cdots L_2^{p_2}(\mathbf{R}, L_1^{p_1}(\mathbf{R})) \cdots).$$

Throughout the paper, the boldface small letters designate n-tuples $(n \geq 2)$ with components between 1 and ∞, e.g., $\mathbf{p} = (p_1, p_2, \cdots, p_n)$. Relations or equations among n-tuples $\psi(\mathbf{p}, \mathbf{q}, \cdots, \mathbf{r})$ mean that those among numbers $\psi(p_i, q_i, \cdots, r_i)$ hold for each component, e.g., $1 < \mathbf{p}(\mathbf{p}-1)^{-1} < \infty$ means $1 < p_i(p_i - 1)^{-1} < \infty$ for all $i = 1, 2, \cdots, n$. In case all components are same, we simply use the italic letters instead of the boldface letters, e.g., $p = (p, p, \cdots, p)$. We note that the order in which the mixed norms are taken is fundamental because in general,

$$(\int (\int (\int |f(x_1,x_2)|^{p_1} dx_1)^{\frac{p_2}{p_1}} dx_2)^{\frac{1}{p_2}} \neq (\int (\int (\int |f(x_1,x_2)|^{p_2} dx_2)^{\frac{p_1}{p_2}} dx_1)^{\frac{1}{p_1}}.$$

By succesive applications of the Minkowski, Hölder and Young inequalities, it follows that

$$||f+g||_{\mathbf{p}} \leq ||f||_{\mathbf{p}}+||g||_{\mathbf{p}}, 1 \leq \mathbf{p} \leq \infty, \tag{2.1}$$

$$||fg||_{\mathbf{p}} \leq ||f||_{\mathbf{q}}||g||_{\mathbf{r}}, \text{ for } \mathbf{p} \leq \mathbf{q}, \mathbf{r} \text{ with } \frac{1}{\mathbf{p}}=\frac{1}{\mathbf{q}}+\frac{1}{\mathbf{r}}, \tag{2.2}$$

$$||f*g||_{\mathbf{p}} \leq ||f||_{\mathbf{q}}||g||_{\mathbf{r}}, \text{ with } 1+\frac{1}{\mathbf{p}}=\frac{1}{\mathbf{q}}+\frac{1}{\mathbf{r}}. \tag{2.3}$$

Here $f*g$ is the usual convolution of f and g. For the fundamental properties of the space with the mixed norm, e.g., reflexivity, completeness and so on, see [5] and [6]. The following theorem is also given by [5].

Theorem 2.1. *For $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\cdots,\lambda_n)$ with $0<\boldsymbol{\lambda}<1$ and $\mathbf{p}=(p_1,p_2,\cdots,p_n)$ with $1<\mathbf{p}<1/\boldsymbol{\lambda}$, let $\mathbf{q}$ satisfy $1/\mathbf{p}=1/\boldsymbol{\lambda}+1/\mathbf{q}$. Then it follows that*

$$||f*\frac{1}{|x|^{n-|\lambda|}}||_{\mathbf{q}} \leq C||f||_{\mathbf{p}} \tag{2.4}$$

for all $f \in L^{\mathbf{p}}$, where $|\lambda|=\sum_{i=1}^n \lambda_i$ and $C=C(\boldsymbol{\lambda},\mathbf{p})$.

Here and hereafter we denote that for $x=(x_1,x_2,\cdots,x_n)$, $i<n$ and $i<j$,

$$d_i(x)=(x_1^2+\cdots x_i^2)^{1/2}, \quad d_i'(x)=(x_{i+1}^2+\cdots+x_n^2)^{1/2},$$
$$d_{i,j}(x)=(x_i^2+\cdots+\cdots+x_j^2)^{1/2}.$$

Lemma 2.2. *(1) For $1=i_0<i_1<i_2<\cdots<i_k<i_{k+1}=n$ and $0<\lambda_\ell<i_\ell-i_{\ell-1}$ ($\ell=1,2,\cdots,k+1$), it follows that*

$$||f*\frac{1}{d_{1,i_1}^{i_1-\lambda_1}d_{i_1+1,i_2}^{(i_2-i_1)-\lambda_2}\cdots d_{i_k+1,n}^{(n-i_k)-\lambda_{k+1}}}||_{\mathbf{p}} \leq C||f||_{\mathbf{q}} \tag{2.5}$$

with

$$\frac{1}{q_j}-\frac{1}{p_j}=\theta_j, \qquad \sum_{j=i_\ell+1}^{i_{\ell+1}}\theta_j=\lambda_\ell \ (\ell=1,2,\cdots,k+1) \tag{2.6}$$

that is,

$$||\{(-\Delta_{1,i_1})^{-\lambda_1/2}+(-\Delta_{i_1+1,i_2})^{-\lambda_2/2}+\cdots+(-\Delta_{i_{k+1},n})^{-\lambda_{k+1}/2}\}f||_{\mathbf{p}} \leq C||f||_{\mathbf{q}} \tag{2.7}$$

for $\Delta_{i,j}=\dfrac{\partial^2}{\partial x_i^2}+\cdots+\dfrac{\partial^2}{\partial x_j^2}$ $(i<j)$

(2) Let $1\leq i<n$ and $0<\delta<i$. Let $r\in(1,\infty)$ satisfy $\dfrac{n}{\delta+i}<r<\dfrac{n}{\delta}$. Then it follows that

$$||f*\frac{d_i^\delta}{|x|^n}||_{\mathbf{p}} \leq C||f||_{\mathbf{q}} \tag{2.8}$$

with

$$\sum_{j=1}^{i}(\frac{1}{q_j}-\frac{1}{p_j})=i-\frac{n}{r}+\delta, \qquad \sum_{j=i+1}^{n}(\frac{1}{q_j}-\frac{1}{p_j})=\frac{n}{r}-i \tag{2.9}$$

Proof. (1) For simplicity it is enough to estimate $f * \dfrac{1}{d_i^{i-\lambda} d_i'^{(n-i)-\lambda'}}$.

$$\begin{aligned}
& f * \frac{1}{d_i^{i-\lambda} d_i'^{(n-i)-\lambda'}} \\
&= \int f(y_1, \cdots, y_n) \frac{1}{d_i^{i-\lambda}(x-y) d_i'^{(n-i)-\lambda'}(x-y)} dy \\
&= \int \left(\int f(y_1, \cdots, y_n) \frac{1}{d_i^{i-\lambda}(x-y)} dy_1 \cdots dy_i \right) \frac{1}{d_i'^{(n-i)-\lambda'}(x-y)} dy_{i+1} \cdots dy_n
\end{aligned}$$

Denoting that $v(x_1, \cdots, x_i, y_{i+1}, \cdots, y_n) = \displaystyle\int f(y_1, \cdots, y_n) \frac{1}{d_i^{i-\lambda}(x-y)} dy_1 \cdots dy_i$ and $*_i^j$ the convolution in $\mathbf{R}_i \times \cdots \times \mathbf{R}_j$, we see

$$\begin{aligned}
& ||f * \frac{1}{d_i^{i-\lambda} d_i'^{(n-i)-\lambda'}}||_{\mathbf{p}} \\
&= ||v(x_1, \cdots, x_i, \cdot, \cdots, \cdot) *_{i+1}^n \frac{1}{d_i'^{(n-i)-\lambda'}}||_{\mathbf{p}} \\
&= || \, ||v||_{L^{p_1, \cdots, p_i}(\mathbf{R}_1 \times \cdots \times \mathbf{R}_i)} *_{i+1}^n \frac{1}{d_i'^{(n-i)-\lambda'}}||_{L^{p_{i+1}, \cdots, p_n}(\mathbf{R}_{i+1} \times \cdots \times \mathbf{R}_n)}
\end{aligned}$$

By (2.4), it follows that

$$||v(x_1, \cdots, x_i, \cdot, \cdots, \cdot) *_{i+1}^n \frac{1}{d_i'^{(n-i)-\lambda'}}||_{\mathbf{p}} \leq ||v||_{L^{p_1, \cdots, p_i, q_{i+1}, \cdots, q_n}} \tag{2.10}$$

with

$$\frac{1}{q_j} - \frac{1}{p_j} = \theta_j \ (j = i+1, \cdots, n), \quad \lambda' = \sum_{j=i+1}^{n} \theta_j \tag{2.11}$$

By (2.4), it also follows that

$$\begin{aligned}
||v(\cdot, \cdots, \cdot, x_{i+1}, \cdots, x_n)||_{L^{p_1, \cdots, p_i}} &= ||f *_1^i \frac{1}{d_i^{i-\lambda}}||_{L^{p_1, \cdots, p_i}}(x_{i+1}, \cdots, x_n) \\
&\leq C ||f||_{L^{q_1, \cdots, q_n}}(x_{i+1}, \cdots, x_n)
\end{aligned} \tag{2.12}$$

with

$$\frac{1}{q_j} - \frac{1}{p_j} = \theta_j \ (j = 1, \cdots, i), \quad \lambda = \sum_{j=1}^{i} \theta_j \tag{2.13}$$

Thus it follows that

$$||f * \frac{1}{d_i^{i-\lambda} d_i'^{(n-i)-\lambda'}}||_{\mathbf{p}} \leq C ||f||_{\mathbf{q}} \tag{2.14}$$

(2) Young's inequality yields

$$d_i^{1/r} d_i'^{1/r'} \leq \left(\frac{d_i^2}{r} + \frac{d_i'^2}{r'}\right)^{1/2} \leq C|x|$$

for $1 < r < \infty$, $\dfrac{1}{r} + \dfrac{1}{r'} = 1$. Thus we obtain

$$\frac{d_i^\delta}{|x|^n} \le C \frac{1}{d_i^{n/r-\delta} d_i'^{n/r'}} = C \frac{1}{d_i^{i-\alpha} d_i'^{(n-i)-\alpha'}} \tag{2.15}$$

with

$$\begin{aligned} &0 < \alpha < i, \quad \frac{n}{r} - \delta = i - \alpha \\ &0 < \alpha' < n - i, \quad \frac{n}{r'} = (n-i) - \alpha'. \end{aligned} \tag{2.16}$$

Here it must be satisfied that

$$\frac{n}{r} - i < \delta < \frac{n}{r}. \tag{2.17}$$

Inequlity (2.14) with (2.11) and (2.13) now yields (2.8) with (2.9). □

In the whole space $\mathbf{R}^n$, the following a priori estimates can be shown in the same manner as those of the usual isotropic spaces.

A priori estimates for the Stokes system 2.3. *Let $n \ge 2$, $1 < \mathbf{q}, s < \infty$ and let $f \in L^s(0,\infty;\ L^{\mathbf{q}}_\sigma(\mathbf{R}^n)^n)$. Then there exists a unique solution $(u, \nabla\pi)$ of the Stokes system*

$$\begin{cases} \partial_t u - \Delta u + \nabla\pi = f, \quad \nabla \cdot u = 0, & \text{in } \mathbf{R}^n \times (0,\infty), \\ u(\cdot,0) = 0, \quad |u(x,t)| \to 0 \textit{ as } |x| \to \infty, \end{cases} \tag{S}$$

such that $\partial_t u, \nabla\pi \in L^{\mathbf{q},s}$ and $\nabla^2 u \in L^s(0,T_0; L^{\mathbf{q}}(\mathbf{R}^n)^n)$ for all $T_0 < \infty$, and it follows that

$$||\partial_t u||_{\mathbf{q},s} + ||\nabla^2 u||_{\mathbf{q},s} + ||\nabla\pi||_{\mathbf{q},s} \le C||f||_{\mathbf{q},s}. \tag{2.18}$$

Here $C = C(n, \mathbf{q}, s)$.

The following lemma shows the embeddings of the Sobolev spaces with the parabolic derivatives, i.e., the time first and spatial second derivatives, into the Lebesgue spaces.

Embedding Lemma 2.4. *For $\mathbf{q}, s \in (1,\infty)$, let $\partial_t v,\ \nabla^2 v \in L^{\mathbf{q},s}$ and $v(\cdot,0) = 0$.*
(i) *For $\boldsymbol{\alpha} \in [\mathbf{q},\infty)$ and $\beta \in [s,\infty)$ with*

$$\sum_{i=1}^n \frac{1}{q_i} + \frac{2}{s} = 2 + \sum_{i=1}^n \frac{1}{\alpha_i} + \frac{2}{\beta}, \tag{2.19}$$

it follows that

$$||v||_{\boldsymbol{\alpha},\beta} \le C(||\partial_t v||_{\mathbf{q},s} + ||\nabla^2 v||_{\mathbf{q},s}). \tag{2.20}$$

(ii) *For $\boldsymbol{\alpha}' \in [\mathbf{q},\infty)$ and $\beta' \in [s,\infty)$ with*

$$\sum_{i=1}^n \frac{1}{q_i} + \frac{2}{s} = 1 + \sum_{i=1}^n \frac{1}{\alpha_i'} + \frac{2}{\beta'}, \tag{2.21}$$

it follows that

$$||\nabla v||_{\boldsymbol{\alpha}',\beta'} \le C(||\partial_t v||_{\mathbf{q},s} + ||\nabla^2 v||_{\mathbf{q},s}). \tag{2.22}$$

Here $C = C(n, \mathbf{q}, s)$.

Proof. Since $v(x,0)=0$, $v(x,t)$ is expressed as $v(x,t)=\displaystyle\int_0^t \mathrm{e}^{(t-\tau)\Delta}(\partial_t v-\Delta v)\,d\tau$. Here

$$\mathrm{e}^{t\Delta}f=E*f=\int_{\mathbf{R}^n}E(x-y,t)f(y,t)\,dy\quad\text{for}\quad E(x,t)=\frac{1}{(4\pi t)^{n/2}}\mathrm{e}^{-|x|^2/4t}. \tag{2.23}$$

Applying the $L^{\mathbf{p}}$-$L^{\mathbf{q}}$ estimates for the heat kernel and the Hardy-Littlewood inequalities on time yield the desired estimates. □

3 Main results

In this section we state the main results in this paper. We first define a *weak solution* of the Navier-Stokes equations

$$\begin{cases} \partial_t u-\Delta u+(u\cdot\nabla)u+\nabla\pi=f, \quad \nabla\cdot u=0, \quad \text{in } \mathbf{R}^n\times(0,\infty), \\ u(\cdot,0)=0, \quad |u(x,t)|\to 0 \text{ as } |x|\to\infty, \end{cases} \tag{NS}$$

with $f\in L^s(0,\infty;L^q_\sigma(\mathbf{R}^n)^n)$ for some $q,s\in(1,\infty)$. We call u a weak solution of (NS), if u is in the class $u\in L^\infty(0,\infty;L^2_\sigma(\mathbf{R}^n)^n)$ and $\nabla u\in L^2(0,\infty;L^2_\sigma(\mathbf{R}^n)^{n\times n})$ and satisfies

$$\begin{aligned} &-\int_0^\infty\int_{\mathbf{R}^n}u\cdot\partial_t\varphi\,dxdt+\int_0^\infty\int_{\mathbf{R}^n}\nabla u\cdot\nabla\varphi\,dxdt \\ &+\int_0^\infty\int_{\mathbf{R}^n}(u\cdot\nabla u)\cdot\varphi\,dxdt=\int_0^\infty\int_{\mathbf{R}^n}f\cdot\varphi\,dxdt, \end{aligned}$$

for all $\varphi\in C^1_0([0,\infty);C^\infty_{0,\sigma}(\mathbf{R}^n)^n)$. Here $L^q_\sigma(\mathbf{R}^n)^n$ is the closure of $C^\infty_{0,\sigma}(\mathbf{R}^n)^n=\{f\in C^\infty_0(\mathbf{R}^n)^n;\ \nabla\cdot f=0\}$ in $L^q(\mathbf{R}^n)^n$. There is a scalar distribution π such that $\nabla\pi=-\partial_t u+\Delta u-(u\cdot\nabla)u+f$ follows in the sense of distribution. Such π is uniquely determined by u up to additive functions of t and called the pressure associated with u. For the existence of weak solutions we need $f\in L^2(0,\infty;L^2_\sigma(\mathbf{R}^n)^n)$. Hereafter we simply denote $L^{\mathbf{q},s}:=L^s(0,\infty;L^{\mathbf{q}}(\mathbf{R}^n)^n)$. Thus space $L^{\mathbf{q},s}$ is equipped with the norm

$$||u||_{\mathbf{q},s}=\left\{\int_0^\infty ||u(\cdot,t)||^s_{L^{\mathbf{q}}(\mathbf{R}^n)}dt\right\}^{1/s}.$$

We do not distinguish the spaces of vector and scalar valued functions unless it causes confusion. We now state our main results in this paper.

Theorem 3.1. *Let $n\ge 3$. For $\mu\in(0,n-2]$ and $\lambda\in(0,1]$, assume that solenoidal function $f\in L^{\infty,\infty}_{\mathrm{loc}}$ satisfies $\lim\limits_{|x|,t\to\infty}f(x,t)|x'|^\mu|\tilde{x}|^2t^\lambda=0$ with (1.5), and for $q'>(n-2)/\mu$, $\tilde{q}>1$, $s>1/\lambda$ with $q',s\in(1,\infty)$ assume that*

$$f\in L^s(0,\infty;L^{\tilde{q}}(\mathbf{R}^2);L^{q'}(\mathbf{R}^{n-2})). \tag{3.1}$$

Then for a weak solution u of (NS) in the class $L^{\mathbf{p},p_0}$ for all $\mathbf{p}=(p_1,p_2,\cdots,p_n)$, $p_0\in[1,\infty]$ satisfying $\sum_{i=1}^n 1/p_i+2/p_0=1$, there exist positive constants $\varepsilon=\varepsilon(n,\mu,\lambda)$ and $\rho<\mu$ with

$$\rho<\frac{n-2}{q'}+\frac{2}{\tilde{q}}+\frac{2}{s}-2$$

such that $\sup ||u||_{\mathbf{p},p_0} < \varepsilon$ *implies that*

$$|u(x,t)| \le C(1+|x'|)^{-\rho}$$

with $C = C(n,f,\rho)$*.*

Remark. The estimate (1.6) is one of the estimates (1.3)-(1.4). Similarly we will be able to obtain the estimes of type (1.3)-(1.4) for the weak solution of (NS) in Serrin's class, by applying the weighted equation approach in space and time simultaneously as in [25].

4 Anisotropically weighted Stokes equations

Operating the Helmholtz decomposition in $L^{\mathbf{q}}(\mathbf{R}^n)^n$ to the Stokes system (S), we consider the heat equations with the solenoidal condition

$$\begin{cases} \partial_t u - \Delta u = f, \\ \nabla \cdot u = 0, \ \text{in } \mathbf{R}^n \times (0,\infty), \\ u(\cdot,0) = 0, \\ |u(x,t)| \to 0 \ (|x| \to \infty). \end{cases} \tag{H}$$

For $\gamma \in (0,1)$ and $m = 1,2,\cdots,n$, let $\alpha_0 \in (m,\infty]$ and $\alpha_1 \in (m/2, m]$ satisfy

$$\gamma < \gamma_0 := 1 - \frac{m}{\alpha_0} = 2 - \frac{m}{\alpha_1}. \tag{4.1}$$

Let $M(x') \in W^{2,p}_{\text{loc}}(\mathbf{R}^m)$ (for some $p>1$) satisfy

$$\begin{cases} 0 < M(x') \le C\bar{d_m}^{\gamma}, \\ \nabla M(x'), \ (M\Delta M(x'))^{1/2} \in L^{\alpha_0}(\mathbf{R}^m), \\ \Delta M(x') \in L^{\alpha_1}(\mathbf{R}^m), \end{cases} \tag{M}$$

with $\bar{d_m}(x') := 1 + d_m(x') = 1 + \sqrt{x_1^2 + \cdots + x_m^2}$. We can take $M(x') = C\bar{d_m}^{\gamma}(x')$ for $\gamma < 1$ since

$$|\nabla M(x')|, \ |(M\Delta M(x'))^{1/2}| \le C\bar{d_m}^{\gamma-1}(x')$$

and $|\Delta M(x')| \le C\bar{d_m}^{\gamma-2}$.

Let $1 < \mathbf{q} \le \mathbf{q}_1 < \infty$ and $1 < s \le s_1 < \infty$ satisfy

$$\sum_{i=1}^{n} \frac{1}{q_i} + \frac{2}{s} = (1 - \frac{m}{\alpha_0}) + \sum_{i=1}^{n} \frac{1}{q_{1_i}} + \frac{2}{s_1}. \tag{4.2}$$

We note that (4.1) implies $\alpha_1 < \alpha_0/2$ since $\alpha_1 > n/2$.

Key Lemma 4.1. *Let* $f \in L^{\mathbf{q},s}$ *and* $Mf \in L^{\mathbf{q}_1,s_1}$*. If* $\mathbf{q}_1 < \alpha_1$*, the solution of (H) satisfies*

$$||Mu_t||_{\mathbf{q}_1,s_1} + ||\nabla^2(Mu)||_{\mathbf{q}_1,s_1} \le C(||Mf||_{\mathbf{q}_1,s_1} + ||f||_{\mathbf{q},s}).$$

Here $C = C(n,\mathbf{q},s,\mathbf{q}_1,s_1,M,\alpha_0)$*.*

Proof. (i) Multiplying M to (H), we have $(Mu)_t - \Delta(Mu) = Mf - 2\nabla M \cdot \nabla u - u\Delta M$. Although Mu is not solenoidal in general, the estimates for the heat equations parallel to A priori estimates 2.3 yield

$$\begin{aligned}&||(Mu)_t||_{\mathbf{q}_1,s_1} + ||\nabla^2(Mu)||_{\mathbf{q}_1,s_1}\\ &\leq C(||Mf||_{\mathbf{q}_1,s_1} + ||\nabla M \nabla u||_{\mathbf{q}_1,s_1} + ||u\Delta M||_{\mathbf{q}_1,s_1}).\end{aligned}$$

Although $(Mu)_t$ and $\nabla^2(Mu)$ may not be in the space $L^{\mathbf{q}_1,s_1}$ here, the above inequality can be followed by localizing M and the usual approximate argument.

For $\mathbf{q}_1 < \alpha_1$, Hölder's inequality and Embedding Lemma 2.4 yield

$$\begin{aligned}||\nabla M \nabla u||_{\mathbf{q}_1,s_1} &\leq C|\nabla M|_{\alpha_0}(||u_t||_{\mathbf{q},s} + ||\nabla^2 u||_{\mathbf{q},s}),\\ ||u\Delta M||_{\mathbf{q}_1,s_1} &\leq C|\Delta M|_{\alpha_1}(||u_t||_{\mathbf{q},s} + ||\nabla^2 u||_{\mathbf{q},s}).\end{aligned}$$

Here and hereafter we denote $|f(x)|_{\mathbf{q}}$ the usual norm in $L^{\mathbf{q}}(\mathbf{R}^m)$.

The a priori estimate of (H) now implies the desired result. □

Since the derivarives ∇M and ΔM must decay, the growth order of M must be less than 1. Then, how many times can we apply this Key Lemma? It will determine the decay order.

For $k \in \mathbf{N}$ let $1 < \mathbf{q} \leq \mathbf{q}_1 \leq \cdots \leq \mathbf{q}_k < \infty$ and $1 < s \leq s_1 \leq \cdots \leq s_k < \infty$ satisfy

$$\sum_{i=1}^{n} \frac{1}{q_i} + \frac{2}{s} = j(1 - \frac{m}{\alpha_0}) + \sum_{i=1}^{n} \frac{1}{q_{j_i}} + \frac{2}{s_j}, \tag{4.3}$$

for $j = 1, 2, \cdots, k$. By (4.1), $\mathbf{q}_k < \alpha_0/2$ $(k \geq 2)$ implies $\mathbf{q}_1 < \alpha_1$. Hereafter we denote $(\mathbf{q}_0, s_0) := (\mathbf{q}, s)$.

Proposition 4.2. *Let $k \geq 2$. Assume that $\mathbf{q}_k < \alpha_0/2$. If $M^j f \in L^{\mathbf{q}_j,s_j}(j = 0, 1, \cdots, k)$, the weighted solution $M^k u$ satisfies*

$$||M^k u_t||_{\mathbf{q}_k,s_k} + ||\nabla^2(M^k u)||_{\mathbf{q}_k,s_k} \leq C \sum_{j=0}^{k} ||M^j f||_{\mathbf{q}_j,s_j}. \tag{4.4}$$

See [25, Proposition 4.2] for the proof. We take M such that $M^k(x') = C\bar{d}_m(x')^{k\gamma}$ at the infinity while $k\gamma$ is actually the decay order of solutions as we see below. By (4.1),

$$k\gamma < \sum_{i=1}^{n} \frac{1}{q_i} + \frac{2}{s} - (\sum_{i=1}^{n} \frac{1}{q_{k_i}} + \frac{2}{s_k}). \tag{4.5}$$

The decay rate of external force f determines $\mathbf{q}$ and s, with which f belongs to space $L^{\mathbf{q},s}$. For faster decaying external force, smaller $\mathbf{q}$ and s we can take. And greater q_k and s_k yield higher order.

We here recall the Gagliardo-Nirenberg inequalities of parabolic type.

The Gagliardo-Nirenberg-Yamazaki inequalities 4.3. *Let $n \geq 1$, $0 < \theta < 1$ and $1 < r < \infty$. Assume that $\dfrac{n+2}{2} < q \leq \infty$ satisfies equality*

$$0 = \theta(2 - \frac{n+2}{q}) - (1-\theta)\frac{n+2}{r}. \tag{4.6}$$

Then there exists a positive constant $C = C(n,q,r)$ such that

$$||v||_{\infty,\infty} \leq C||\partial_t v - \Delta v||_{q,q}^{\theta}||v||_{r,r}^{1-\theta}, \tag{4.7}$$

for all $v \in L^{r,r}$ with $v(\cdot,0) = 0$ and $\partial_t v - \Delta v \in L^{q,q}$.

See [25] for the proof. We do not have to prepare the anisotropic version of the Gagliardo-Nirenberg-Yamazaki inequalities.

The following lemma shows the decay rate estimates. Let $\alpha_0 > n+2$ and let $(\mathbf{a}_j, b_j)$ $(j = 0,1,\cdots,k)$ satisfy

$$\begin{aligned} &\mathbf{q}_j < \mathbf{a}_j, \quad s_j < a'_j, \\ &\sum_{i=1}^{n}\frac{1}{a_{0_i}} + \frac{2}{a'_0} = j(1-\frac{m}{\alpha_0}) + \sum_{i=1}^{n}\frac{1}{a_{j_i}} + \frac{2}{a'_j}, \\ &\frac{n+2}{2} < a_k = a'_k < \frac{\alpha_0}{2}. \end{aligned} \tag{4.8}$$

Lemma 4.4. (Uniform estimates for the weighted solutions) *Assume that (M), (4.1) and $\mathbf{q}_k < \alpha_0/2$ for $\alpha_0 \in (n+2,\infty]$. Let $M^j f \in L^{\mathbf{q}_j,s_j} \cap L^{\mathbf{a}_j,a'_j}$ $(j = 0,1,\cdots,k)$. Then there exist constants $C = C(n,\mathbf{q},s,k,\mathbf{q}_j,s_j,\mathbf{a}_j,a'_j,\alpha_0,M)$ and $\theta_k = \theta_k(n,\mathbf{q}_k,s_k,a_k)$ $\in (0,1)$ such that the solution of (H) satisfies*

$$\begin{aligned} ||M^k u||_{\infty,\infty} \leq\ & C(||M^k f||_{a_k,a'_k} + \cdots + ||f||_{\mathbf{a}_0,a'_0})^{\theta_k} \cdot \\ & \cdot (||M^k f||_{\mathbf{q}_k,s_k} + \cdots + ||f||_{\mathbf{q},s})^{1-\theta_k}, \end{aligned} \tag{4.9}$$

with

$$2 = \theta_k\frac{n+2}{a_k} + (1-\theta_k)(\sum_{i=1}^{n}\frac{1}{q_{k_i}} + \frac{2}{s_k}), \tag{4.10}$$

$$\frac{2(n+2)}{\alpha_0} < \frac{n+2}{a_k} < 2 < \sum_{i=1}^{n}\frac{1}{q_{k_i}} + \frac{2}{s_k}. \tag{4.11}$$

Proof. The Gagliardo-Nirenberg-Yamazaki inequalities yield

$$||M^k u||_{\infty,\infty} \leq C||(M^k u)_t - \Delta(M^k u)||_{a_k,a_k}^{\theta_k}||M^k u||_{r_k,r_k}^{1-\theta_k} \tag{4.12}$$

with $(q,r,\theta) = (a_k,r_k,\theta_k)$ in (4.6). Embedding lemma yields

$$||M^k u||_{r_k,r_k} \leq C(||M^k u_t||_{\mathbf{q}_k,s_k} + ||\nabla^2(M^k u)||_{\mathbf{q}_k,s_k})$$

with

$$\sum_{i=1}^{n} \frac{1}{q_{k_i}} + \frac{2}{s_k} = 2 + \frac{n+2}{r_k}. \tag{4.13}$$

Proposition 4.2 yields Lemma 4.4. $\square$

Lemma 4.4 now yields uniform decays $|u(x,t)| \leq CM^{-k}$ for the Stokes equations with weakly solenoidal external forces.

5 Estimates for the weighted Navier-Stokes system

Next we estimate the nonlinear terms and obtain the same estimates as in Proposition 4.2 for the weighted Navier-Stokes equations.

A regularity class for the weak solutions Applying the Helmholtz decomposition projector P of $L^{\mathbf{q}}(\mathbf{R}^n)^n$ to (NS), it follows that

$$\partial_t u - \Delta u = f - P(u \cdot \nabla u) \quad \text{and} \quad \nabla \cdot u = 0, \tag{5.1}$$

for $f(\cdot,t) \in L^{\mathbf{q}}_{\sigma}(\mathbf{R}^n)^n$. Since the boundedness of the Helmholtz decomposition projector P in $L^{\mathbf{q}}$, i.e., $|Pf|_{\mathbf{q}} \leq C|f|_{\mathbf{q}}$ is given by [5], in parallel to [25, Proposition 5.1], it follows that

Proposition 5.1. *Let $\mathbf{q}, s \in (1,\infty)$ satisfy $\sum_{i=1}^{n} 1/q_i + 2/s > 1$. Assume that $f \in L^s(0,\infty; L^{\mathbf{q}}_{\sigma}(\mathbf{R}^n)^n)$. Let $\mathbf{p}_0 \in [1,\infty]$ and $p_0' \in [2,\infty]$ satisfy $\mathbf{p}_0 > \mathbf{q}$ and $p_0' > s$ and $\sum_{i=1}^{n} 1/p_{0_i} + 2/p_0' = 1$. Then there exists a positive constant $\varepsilon = \varepsilon(n,\mathbf{q},s)$ such that , if the weak solution u of (NS) is in $L^{\mathbf{p}_0,p_0'}$ and $||u||_{\mathbf{p}_0,p_0'} < \varepsilon$, then it follows that*

$$||P(u \cdot \nabla u)||_{\mathbf{q},s} \leq ||f||_{\mathbf{q},s} \tag{5.2}$$

and

$$||u||_{\mathbf{p}_1,p_1'} \leq C||f||_{\mathbf{q},s}, \tag{5.3}$$

for all $\mathbf{p}_1 \in [\mathbf{q},\infty)$ and $p_1' \in [s,\infty)$ with

$$\sum_{i=1}^{n} \frac{1}{q_i} + \frac{2}{s} = 2 + \sum_{i=1}^{n} \frac{1}{p_{1_i}} + \frac{2}{p_1'}.$$

Here $C = C(n,\mathbf{q},s)$.

Hereafter we consider the weak solutions u in the class (1.8).

Commuting weights and the Helmholtz projector We will obtain the same estimates in Proposition 4.2 for the weak solutions in the class (1.8) under the assumptions in Lemma 4.4. Multipling M to (5.1) k-times, Proposition 4.2 implies

$$\begin{aligned} &||(M^k u)_t||_{\mathbf{q}_k,s_k} + ||\nabla^2(M^k u)||_{\mathbf{q}_k,s_k} \\ &\leq C\{\sum_{j=1}^{k}(||M^j P(u \cdot \nabla u)||_{\mathbf{q}_j,s_j} + ||M^j f||_{\mathbf{q}_j,s_j}) \\ &\quad +||P(u \cdot \nabla u)||_{\mathbf{q},s} + ||f||_{\mathbf{q},s}\}. \end{aligned} \tag{5.4}$$

To estimate $M^jP(u\cdot\nabla u)$ for $j\geq 1$ in (5.4), we commute M^j and P and consider the commutator. That is,

$$M^jP(u\cdot\nabla u)=PM^j(u\cdot\nabla u)-(PM^j-M^jP)(u\cdot\nabla u). \tag{5.5}$$

The Leibnitz rule implies

$$PM^j(u\cdot\nabla u)=P(uM^j\cdot\nabla u)=P(u\cdot\{\nabla(M^ju)-j(\nabla M)M^{j-1}u\}). \tag{5.6}$$

Hereafter we denote

$$|||u|||_{\mathbf{q},s}:=||u_t||_{\mathbf{q},s}+||\nabla^2u||_{\mathbf{q},s}.$$

For $(\mathbf{q}_k,s_k)$ with $\mathbf{q}_k<\alpha_0/2$ and (4.1), let $(\mathbf{p},p_0)$ satisfy

$$\frac{1}{q_{k_i}}>\frac{1}{\alpha_0}+\frac{1}{p_i},\ \ s_k<p_0,\ \ \sum_{i=1}^n\frac{1}{p_i}+\frac{2}{p_0}=1. \tag{5.7}$$

Applying Hölder's inequality and Embedding Lemma to (5.6) yields

$$\begin{aligned}&||PM^j(u\cdot\nabla u)||_{\mathbf{q}_j,s_j}\\&\leq C||u||_{\mathbf{p},p_0}\{|||M^ju|||_{\mathbf{q}_j,s_j}+|\nabla M|_{\alpha_0}|||M^{j-1}u|||_{\mathbf{q}_{j-1},s_{j-1}}\}.\end{aligned} \tag{5.8}$$

Here $C=C(n,\mathbf{q},s,j,\alpha_0)$.

Finally we show the estimates of the commutator in (5.5). We recall embedding

$$[M]_{1-m/r}\leq C(m,r)|\nabla M|_r$$

for $r>m$ with the Hölder norm

$$[M]_\alpha=\sup_{x',y'}\frac{|M(x')-M(y')|}{|x'-y'|^\alpha}.$$

Since we have assumed $\nabla M\in L^{\alpha_0}(\mathbf{R}^m)$ in (M), applying the above embedding with $r=\alpha_0$, we have

$$|M(x')-M(y')|\leq C|\nabla M|_{\alpha_0}|x'-y'|^{1-m/\alpha_0}. \tag{5.9}$$

Lemma 5.2. (Estimates of the commutator) *Let $j=1,2,\cdots,k$ satisfy $0<j\gamma_0=j(1-\frac{m}{\alpha_0})<n$. Assume that $\nabla M(x')\in L^{\alpha_0}(\mathbf{R}^m)$. Let $\mathbf{q}_j\in(1,\infty)$ and $\mathbf{b}_\ell\in(1,\mathbf{q}_j)$ for $\ell=0,1,\cdots,j-1$ satisfy*

$$\sum_{i=1}^n\left(\frac{1}{b_{\ell_i}}-\frac{1}{q_{j_i}}\right)=\gamma_0(j-\ell) \tag{5.10}$$

and

$$0<\sum_{i=1}^n\frac{1}{q_{j_i}}<n-j\gamma_0. \tag{5.11}$$

Then there exists a constant $C=C(n,\mathbf{q}_j,j)$ such that

$$|(PM^j-M^jP)v|_{\mathbf{q}_j}\leq C\sum_{\ell=0}^{j-1}|M^\ell v|_{\mathbf{b}_\ell}. \tag{5.12}$$

Proof. Since projector P is written of the form $P = I + \nabla(-\Delta)^{-1}\mathrm{div}$, the commutator is expressed of the form

$$(PM^j - M^jP)v = C_0(n)\int (M^j(y') - M^j(x'))\frac{(x-y)\otimes(x-y)}{|x-y|^{n+2}}v(y)\ dy.$$

By the polynomial expansion formula, we see

$$\begin{aligned} M^j(x') \quad &= \{(M(x') - M(y')) + M(y')\}^j \\ &= \sum_{\ell=0}^{j-1} C(j,\ell)(M(x') - M(y'))^{j-\ell}M^\ell(y') + M^j(y'). \end{aligned}$$

By the Hölder continuity and Lemma 2.2, it follows that

$$\begin{aligned} |(PM^j - M^jP)v|_{\mathbf{q}_j} \quad &\le C_0\sum_{\ell=0}^{j-1} C(j,\ell)|\int \frac{|M(y') - M(x')|^{j-\ell}}{|x-y|^n}|M^\ell v(y)|\ dy|_{\mathbf{q}_j} \\ &\le C\sum_{\ell=0}^{j-1}|\int \frac{|x'-y'|^{\gamma_0(j-\ell)}}{|x-y|^n}|M^\ell v(y)|\ dy|_{\mathbf{q}_j} \\ &\le C\sum_{\ell=0}^{j-1}|M^\ell v|_{\mathbf{b}_\ell}. \end{aligned} \tag{5.13}$$

□

We apply the above lemma to $v = (u\cdot\nabla)u$. In parallel to [25, Proposition 5.3], we have

Proposition 5.3. *Assume the same as in Lemma 5.2. Let* $(\mathbf{p}, p_0)$ *satisfy (5.7).and let*

$$\frac{2}{s} - \frac{2}{s_{k-1}} \le 1 - \frac{2}{p_0} = \sum_{i=1}^n \frac{1}{p_i}. \tag{5.14}$$

Then it follows that

$$||(PM^j - M^jP)(u\cdot\nabla u)||_{\mathbf{q}_j,s_j} \le C||u||_{\mathbf{p},p_0}\sum_{\ell=0}^{j-1}|||M^\ell u|||_{\mathbf{q}_\ell,s_\ell}. \tag{5.15}$$

Proof of Theorem 3.1 Substituting (5.2), (5.8) and (5.15) to (5.4), we obtain

$$\begin{aligned} &||(M^ku)_t||_{\mathbf{q}_k,s_k} + ||\nabla^2(M^ku)||_{\mathbf{q}_k,s_k} \\ &\le C\{||u||_{\mathbf{p},p_0}(||M^ku_t||_{\mathbf{q}_k,s_k} + ||\nabla^2(M^ku)||_{\mathbf{q}_k,s_k}) \\ &\qquad +||u||_{\mathbf{p},p_0}\sum_{j=0}^{k-1}(||M^ju_t||_{\mathbf{q}_j,s_j} + ||\nabla^2(M^ju)||_{\mathbf{q}_j,s_j}) + \sum_{j=0}^{k}||M^jf||_{\mathbf{q}_j,s_j}\}. \end{aligned}$$

Since the left hand side is finite by the approximate argument, the first term in the right hand side is absorbed to the left hand side provided that

$$C||u||_{\mathbf{p},p_0} < 1/2 \quad \text{with} \quad C = C(n, q, s, k, \alpha_0). \tag{5.16}$$

The second terms are inductively estimated by the weighted external forces. We thus obtain the same estimates as in Proposition 4.2 and Lemma 4.4.

In parallel to [25, Proof of Theorem 2.1], for the desired decay rate $\rho = k\gamma$, letting

$$k\gamma = \rho < \frac{n-2}{q'} + \frac{2}{\tilde{q}} + \frac{2}{s} - 2, \quad \sum_{i=1}^{n} \frac{1}{q_{k_i}} + \frac{2}{s_k} \searrow 2, \tag{5.17}$$

we conclude Theorem 3.1.

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References

[1] C.Amrouche,V.Girault,M.Schonbek and T.Schonbek Pointwise decay of solutions and higher derivatives to Navier-Stokes equations, *SIAM J.Math. Anal.* **31** (2000), 740-753.

[2] H.O.Bae and L.Brandolese, On the effect of external forces on incompressible fluid motions at large distances, *Ann.Univ.Ferrara VII Sci.* **55**(2) (2009), 225-238.

[3] H.O.Bae and B.J.Jin, Asymptotic behavior for the Navier-Stokes equations in 2D exterior domains *J.Functional Analysis* **240** (2006), 508-529.

[4] A.Benedek, A.P.Calderón and R.Panzone, Convolution operators on Banach space valued functions, *Proc. Nat. Acad. Sci. USA* **48** (1962), 356-365.

[5] A. Benedek and R. Panzone, The spaces L^P, with mixed norm, *Duke Math. J.* **28** (1961), 301-324.

[6] O.V. Besov, V.P. Il'in and S.M. Nikol'skii, Integral Representations of Functions and Imbedding Theorems vol. I, V.H. Winston & Sons, Washington, D.C., 1978.

[7] L.Brandolese, On the localization of symmetric and asymmetric solutions of the Navier- Stokes equations in $\mathbf{R}^n$, *C.R.AcadSci. Paris, Se'r. I Math.* **332**(2) (2001), 125-130.

[8] L.Brandolese, Space-time decay of Navier-Stokes flows invariant under rotations, *Math. Ann.* **329** (2004), 685-706.

[9] L.Brandolese and F.Vigneron, *New asymptotic profiles of nonstationary solutions of the Navier-Stokes system,* J. Math. Pures Appl. **88** (2007), 64-86.

[10] T.Chang and K.Kang, Estimates of anisotropic Sobolev spaces with mixed norms for the Stokes system in a half-space, *Ann.Univ.Ferrara Sez.VII Sci.Mat.* **64**(2018), 47-82.

[11] H.J.Choe and B.J.Jin, Decay rate of solutions of Navier-Stokes equations in 3-dimensional half space *J.Math.Anal.Appl.* **339** (2008), 1485-1496.

[12] Y. Fujigaki and T. Miyakawa, *Asymptotic profiles of nonstationary incompressible Navier-Stokes flows in the whole space,* SIAM J. Math. Anal. **33**(3) (2001), 523-544.

[13] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, *J. Differential Equations* **62** (1986), 186-212.

[14] Y.Giga and H.Sohr, Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains, *J. Functional Analysis* **102** (1991), 72-94.

[15] L.Huang and D.Yang, On Function Spaces with Mixed Norms-A Survey, *J.Math. Study* **54** (2021), 262-336.

[16] D.Q.Khai and N.M.Tri, Solutions in mixed-norm Sobolev-Lorentz spaces to the initial value problem for the Navier-Stokes equations, *J. Math. Anal. Appl.* **417** (2014), 819-833.

[17] G.H.Knightly, A Cauchy problem for the Navier-Stokes equations in $\mathbf{R}^n$, *SIAM J. Anal.* **3** (1972), 506-511.

[18] L.D. Kudryavtsev and S.M. Nikol'skii, Spaces of differentiable functions of several variables and imbedding theorems, Analysis III (Ed. S. M. Nikol'skii), Encyclopaedia of Mathematical Sciences vol. 26, Springer-Verlag, Berlin Heidelberg, 1991.

[19] I. Kukavica, Space-time Decay for Solutions of the Navier-Stokes Equations, *Indiana Univ. Math. J.* **50** (2001), 205-222.

[20] T.Miyakawa, On upper and lower bounds of rates of decay for nonstationary Navier-Stokes flows in the whole space *Hiroshima Math. J.* **32** (2002), 431-462.

[21] T.Miyakawa, Notes on space-time decay properties of nonstationnary incompressible Navier-Stokes flows in $\mathbf{R}^n$ *Funkcial. Ekvac.* **45** (2002), 271-289.

[22] T.Phan, Well-posedness for the Navier-Stokes equations in critical mixed-norm Lebesgue spaces, *J. Evol. Equ.* **20** (2020), 553-576.

[23] J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations, *Arch. Rational Mech. Anal.* **9** (1962), 187-195.

[24] S. Takahashi, On interior regularity criteria for weak solutions of the Navier-Stokes equations, *Manuscripta Math.* **69** (1990), 237-254.

[25] S. Takahashi, A weighted equation approach to decay rate estimates for the Navier-Stokes equations, *Nonlinear Analysis* **37** (1999), 751-789.