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MATHEMATICAL ANALYSIS OF CYCLE LENGTH-AGE STRUCTURED CELL POPULATION WITH AGGREGATE TRANSITION RULE: ASYNCHRONOUS GROWTH PROPERTY

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Abstract. This work is a continuation of [1] in which we have analyzed a mathematical model of a structured cell population. Each cell is distinguished by its cycle length and by its age. The daughter cells are correlated to the whole cell population thanks to the *Aggregate Transition Rule.* We investigate then the asymptotic behavior of the generated semigroup which allows us to get the *Asynchronous Growth Property* of the whole cell population.

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1 Introduction

This work is a continuation of [1] in which we have analyzed a mathematical model of a structured cell population. In this model, each cell is distinguished by its cell cycle length ℓ $(0 \leqslant \ell_1 < \ell < \ell_2 \leqslant \infty)$ and by its age a $(0 \leqslant a \leqslant \ell)$. The age a of each cell is null $(a = 0)$ at birth and equals to its cell cycle length ℓ $(a = \ell)$ at division.

If $f = f(t, a, \ell)$ denotes, at time t, the cell density with respect to the age a and the cell cycle length ℓ, then the cell population is governed by the following equation

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial a} - \mu f + \int_{\ell_1}^{\ell}\int_{0}^{\ell'} \eta(a, \ell, a', \ell') f(t, a', \ell') \mathrm{d}a' \mathrm{d}\ell', \tag{1.1}$$

where $\mu = \mu(a, \ell)$ stands for the *cell mortality rate* and $\eta(a, \ell, a', \ell')$ denotes the *transition rate* in which cells change their cell cycle length from ℓ' to ℓ and their age from a' to a.

We endow the master equation (1.1) with a boundary condition describing, at each mitosis, the correlation between the daughter cells and the whole cell population (for more explanation, see [1]). We mathematically describe this correlation (which we call : *Aggregate Transition Rule*) by

$$f(t, 0, \ell) = \alpha \int_{\ell_1}^{\ell_2}\int_{0}^{\ell'} \mathrm{k}(\ell, a', \ell') f(t, a', \ell') \mathrm{d}a' \mathrm{d}\ell', \qquad \ell \in (\ell_1, \ell_2), \tag{1.2}$$

where $\mathrm{k} = \mathrm{k}(\ell, a', \ell')$ denotes the correlation kernel and $\alpha \geqslant 0$ is the *fractional number* of daughter cells viable per mitosis.

To the best of our knowledge, the boundary conditions (1.2) have never been considered in connection with the master equation (1.1). This obviously shows the novelty of the full model (1.1)–(1.2) and motivates this work.

In [1], we have proved that the full model (1.1)–(1.2) is governed by a C_0-semigroup. We also have studied the positivity and the irreducibility of the generated semigroup. In particular, we have constructed the explicit form of the C_0-semigroup governing the unperturbed model (1.1)–(1.2)–$(\mu = \eta = 0)$. This explicit form will be extensively used in this work.

Our challenge, in this work, is to prove that the whole cell population possesses the *Asynchronous Growth Property.* This can be done by describing the asymptotic behavior of the generated semigroup, governing the full model (1.1)–(1.2), in the uniform topology as follows

Lemma 1.1 ([6, Theorems 9.10 and 9.11])**.** *Let $\mathbb{U} = (\mathbb{U}(t))_{t\geqslant 0}$ be a positive and irreducible C_0-semigroup, on the Banach lattice space $\mathscr{X}$, satisfying the inequality $\omega_{\text{ess}}(\mathbb{U}) < \omega_0(\mathbb{U})$. Then there exist a rank one projector $\mathbb{P}$ into $\mathscr{X}$ and an $\varepsilon > 0$ such that : for any $\eta \in (0, \varepsilon)$, there exists $M_\eta \geqslant 1$ satisfying*

$$\left\| e^{-\omega_0(\mathbb{U})t}\, \mathbb{U}(t) - \mathbb{P} \right\|_{\mathscr{L}(X)} \leqslant M_\eta e^{-\eta t} \qquad t \geqslant 0.$$

Let $\mathbb{U} = (\mathbb{U}(t))_{t\geqslant 0}$ be a C_0-semigroup whose generator is U. The *type*, $\omega_0(\mathbb{U})$, and the *essential type*, $\omega_{\text{ess}}(\mathbb{U})$ are defined by

$$\omega_0(\mathbb{U}) := \lim_{t\to\infty} t^{-1}\ln\|\mathbb{U}(t)\|_{\mathscr{L}(\mathscr{X})} \quad \text{and} \quad \omega_{\text{ess}}(\mathbb{U}) := \lim_{t\to\infty} t^{-1}\ln\|\mathbb{U}(t)\|_{\text{ess}} \tag{1.3}$$

where $\|B\|_{\text{ess}} = \inf\{\|B-K\| : K \in \mathcal{K}\}$ ($\mathcal{K} \subset \mathscr{X}$ is the subspace of all compact operators). The spectral bound $s(\mathrm{U})$ of the generator U is defined by

$$s(\mathrm{U}) := \begin{cases} \sup\{\mathrm{Re}(\lambda),\ \lambda \in \sigma(\mathrm{U})\}, & \text{if} \quad \sigma(\mathrm{U}) \neq \emptyset, \\ -\infty, & \text{if} \quad \sigma(\mathrm{U}) = \emptyset. \end{cases}$$

All these quantities are related by

$$\omega_0(\mathbb{U}) = \max\Big\{s(\mathrm{U})\ ,\ \omega_{\text{ess}}(\mathbb{U})\Big\}. \tag{1.4}$$

We say that the C_0-semigroup $\mathbb{U} = (\mathbb{U}(t))_{t\geqslant 0}$ possesses the *Asynchronous Growth Property* whenever all the required conditions of Lemma 1.1 are fulfilled. In this case $s(\mathrm{U})$ is called the *Intrinsic Growth Constant*.

Lemma 1.1 describes the bacterial profile of the whole population whose privileged direction is driven by the projector $\mathbb{P}$ and that is what the biologist could see in his laboratory.

2 Mathematical tools

This section deals with some theoretical results which we will use throughout this work. Let $\mathscr{X} := L^1(\Omega)$.

Lemma 2.1. *Let* S *and* T *be two bounded linear operators from* $\mathscr{X}$ *into itself. Then the following assertions hold.*

(1) *If* T *and* S *are weakly compact, then* ST *is compact.*
(2) *Suppose that* $0 \leqslant \mathrm{S} \leqslant \mathrm{T}$. *If* T *is weakly compact, then so is* S.
(3) *Suppose that* $0 \leqslant |\mathrm{S}| \leqslant \mathrm{T}$. *If* T *is weakly compact, then so is* S.

Proof. (1) and (2) follow from Dunford-Pettis Property (see for instance [2]).
(3). If $0 \leqslant |\mathrm{S}| \leqslant \mathrm{T}$ then $-\mathrm{T} \leqslant \mathrm{S} \leqslant \mathrm{T}$ which leads to $0 \leqslant \mathrm{S} + T \leqslant 2\mathrm{T}$. Hence $S + \mathrm{T}$ is weakly compact because of (2) and so is $\mathrm{S} = (\mathrm{S}+\mathrm{T}) - \mathrm{T}$. ∎

Lemma 2.2. *Let* $\mathbb{V} = (\mathbb{V}(t))_{t\geqslant 0}$ *and* $\mathbb{W} = (\mathbb{W}(t))_{t\geqslant 0}$ *be two* C_0-*semigroups on* $\mathscr{X}$. *If* $\mathbb{W}(\mathrm{t}_0) - \mathbb{V}(\mathrm{t}_0)$ *is a weakly compact operator for some* $\mathrm{t}_0 > 0$, *then* $\omega_{\text{ess}}(\mathbb{W}) = \omega_{\text{ess}}(\mathbb{V})$.

Proof. Let $\lambda \in \rho(\mathbb{V}(\mathrm{t}_0))$. Then $((\mathbb{W}(\mathrm{t}_0) - \mathbb{V}(\mathrm{t}_0))(\lambda - \mathbb{V}(\mathrm{t}_0))^{-1})^2$ is a compact operator because of Lemma 2.1(1). Now [4] ends the proof. ∎

Lemma 2.3. *Let* $\mathbb{V} = (\mathbb{V}(t))_{t\geqslant 0}$ *and* $\mathbb{W} = (\mathbb{W}(t))_{t\geqslant 0}$ *be two* C_0-*semigroups, on* $\mathscr{X}$, *whose generators are* V *and* $\mathrm{W} := \mathrm{V} + \mathrm{P}$ *where* P *is a bounded linear operator from* $\mathscr{X}$ *into itself. If* P *is a weakly compact operator, then* $\omega_{\text{ess}}(\mathbb{W}) = \omega_{\text{ess}}(\mathbb{V})$.

Proof. Let $\mathrm{t}_0 > 0$. Then $\mathrm{P}\mathbb{V}(\mathrm{t}_0)\mathrm{P}$ is a compact operator on $\mathscr{X}$ because of Lemma 2.1(1). Now [5] ends the proof. ∎

3 The full model (1.1)–(1.2)

The aim of this section is to complete all the results already proved in [1]. Let then ℓ_1 and ℓ_2 be such that $0 \leqslant \ell_1 < \ell_2 \leqslant \infty$ and let

$$\Omega := \Big\{(a,\ell) \quad : \quad 0 < a < \ell \qquad \text{and} \qquad \ell_1 < \ell < \ell_2\Big\}.$$

Let $\mathscr{X}$ and $\mathscr{Y}$ be the following Banach spaces

$$\mathscr{X} := L^1(\Omega) \qquad \text{whose norm is} \qquad \|\varphi\|_{\mathscr{X}} := \int_{\ell_1}^{\ell_2}\int_0^{\ell} |\varphi(a,\ell)|\,\mathrm{d}a\mathrm{d}\ell,$$

and $$\mathscr{Y} := L^1(\ell_1,\ell_2) \qquad \text{whose norm is} \qquad \|\psi\|_{\mathscr{Y}} := \int_{\ell_1}^{\ell_2} |\psi(\ell)|\,\mathrm{d}\ell,$$

and let K_α be the linear operator from $\mathscr{X}$ into $\mathscr{Y}$ defined by

$$\mathrm{K}_\alpha\varphi(\ell) := \alpha\int_{\ell_1}^{\ell_2}\int_0^{\ell'} \mathrm{k}(\ell,a',\ell')\varphi(a',\ell')\mathrm{d}a'\mathrm{d}\ell', \qquad \ell \in (\ell_1,\ell_2), \tag{3.1}$$

where $\mathrm{k} = \mathrm{k}(\ell,a',\ell')$ denotes the correlation between daughter cells and the whole cell population and $\alpha \geqslant 0$ is the fractional number of daughter cells viable per mitosis. The kernel k is subject to the following assumptions

$\big(\mathbf{A}_\mathrm{k}^1\big)$ $$\overline{\kappa} < \infty,$$

$\big(\mathbf{A}_\mathrm{k}^2\big)$ $$\mathrm{k} \text{ is positive},$$

$\big(\mathbf{A}_\mathrm{k}^2\big)'$ $$\mathrm{k}(\ell,a',\ell') > 0 \quad \text{for a.e.} \quad (\ell,a',\ell') \in (\ell_1,\ell_2)\times\Omega,$$

where $$\overline{\kappa} := \operatorname*{ess\,sup}_{(a',\ell')\in\Omega}\int_{\ell_1}^{\ell_2} |\mathrm{k}(\ell,a',\ell')|\,\mathrm{d}\ell. \tag{3.2}$$

If $\big(\mathbf{A}_\mathrm{k}^1\big)$ holds true then the linear operator K_α is bounded from $\mathscr{X}$ into $\mathscr{Y}$ satisfying (see [1, Lemma 3.1])

$$\|\mathrm{K}_\alpha\varphi\|_{\mathscr{Y}} \leqslant \alpha\overline{\kappa}\|\varphi\|_{\mathscr{X}} \qquad \text{for all} \qquad \varphi \in \mathscr{X}. \tag{3.3}$$

Let $\mathscr{W}$ be the following Banach space

$$\mathscr{W} := \Big\{\varphi \in \mathscr{X} \quad : \quad \frac{\partial\varphi}{\partial a} \in \mathscr{X} \quad \text{and} \quad \frac{1}{\ell}\varphi \in \mathscr{X}\Big\}$$

normed by $$\|\varphi\|_{\mathscr{W}} := \|\varphi\|_{\mathscr{X}} + \left\|\frac{\partial\varphi}{\partial a}\right\|_{\mathscr{X}} + \left\|\frac{1}{\ell}\varphi\right\|_{\mathscr{X}}$$

and let T_α, $\alpha \geqslant 0$, be the following unbounded linear operator

$$\begin{aligned} \mathrm{T}_\alpha\varphi &:= -\frac{\partial\varphi}{\partial a} \qquad \text{on the domain} \\ \mathrm{D}(\mathrm{T}_\alpha) &:= \Big\{\varphi \in \mathscr{W} \quad : \quad \varphi(0,\cdot) = \mathrm{K}_\alpha\varphi\Big\}. \end{aligned} \tag{3.4}$$

The definition of $\mathrm{D}(\mathrm{T}_\alpha)$ makes sense because of [1, Lemma 4.1].

Lemma 3.1. *Let $\alpha \geqslant 0$. If $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)$ holds, then T_{α} generates on $\mathscr{X}$ a C_0-semigroup $\mathbb{T}_{\alpha} = (\mathbb{T}_{\alpha}(t))_{t\geqslant 0}$ given by*

$$\mathbb{T}_{\alpha}(t) = \mathbb{T}_{0}(t) + \mathbb{M}_{\alpha}(t) + \mathbb{N}_{\alpha}(t), \qquad t \geqslant 0, \tag{3.5}$$

where $\mathbb{T}_{0} = (\mathbb{T}_{0}(t))_{t\geqslant 0}$ is a positive C_0-semigroup on $\mathscr{X}$ defined by

$$\mathbb{T}_{0}(t)\varphi(a,\ell) := \chi\,(t,a,\ell)\,\varphi(a-t,\ell), \qquad (a,\ell) \in \Omega, \tag{3.6}$$

with
$$\chi(t,a,\ell) := \begin{cases} 1 & \text{if} \quad a \geqslant t; \\ 0 & \text{if} \quad a < t, \end{cases} \tag{3.7}$$

and, $\mathbb{M}_{\alpha}(t)$ and $\mathbb{N}_{\alpha}(t)$ are two bounded linear operators from $\mathscr{X}$ into itself defined by

$$\mathbb{M}_{\alpha}(t)\varphi(a,\ell) := \alpha\overline{\chi}(t,a,\ell)\int_{\ell_1}^{\ell_2}\int_{0}^{\ell'} \mathrm{k}(\ell,a',\ell')\mathbb{T}_{0}(t-a)\varphi(a',\ell')\mathrm{d}a'\mathrm{d}\ell', \tag{3.8}$$

$$\begin{aligned} \mathbb{N}_{\alpha}(t)\varphi(a,\ell) := \alpha^2\overline{\chi}(t,a,\ell)&\int_{\ell_1}^{\ell_2}\int_{0}^{\ell'}\int_{\ell_1}^{\ell_2}\int_{0}^{\ell''} \mathrm{k}(\ell,a',\ell')\mathrm{k}(\ell',a'',\ell'') \\ &\overline{\chi}(t-a,a',\ell')\mathbb{T}_{\alpha}(t-a-a')\varphi(a'',\ell'')\mathrm{d}a''\mathrm{d}\ell''\mathrm{d}a'\mathrm{d}\ell', \end{aligned} \tag{3.9}$$

with
$$\overline{\chi}(t,a,\ell) := \begin{cases} 0 & \text{if} \quad a \geqslant t; \\ 1 & \text{if} \quad a < t, \end{cases} \qquad (a,\ell) \in \Omega. \tag{3.10}$$

Proof. Firstly, the existence of the C_0-semigroup $\mathbb{T}_{\alpha} = (\mathbb{T}_{\alpha}(t))_{t\geqslant 0}$ has already been proved (see [1, Theorem 4.4]). Furthermore [1, Theorem 5.4] yields, for all $\varphi \in \mathscr{X}$, that

$$\mathbb{T}_{\alpha}(t)\varphi := \mathbb{T}_{0}(t)\varphi + \mathbb{A}_{\alpha}(t)\varphi, \qquad t \geqslant 0, \tag{3.11}$$

where $\mathbb{A}_{\alpha}(t)$ is a bounded linear operator from $\mathscr{X}$ into itself ([1, Lemma 5.3]) satisfying, for almost all $(a,\ell) \in \Omega$,

$$\mathbb{A}_{\alpha}(t)\varphi(a,\ell) = \overline{\chi}(t,a,\ell)\mathrm{K}_{\alpha}\Big(\mathbb{T}_{0}(t-a)\varphi + \mathbb{A}_{\alpha}(t-a)\varphi\Big)(\ell). \tag{3.12}$$

Next, let $t \geqslant 0$ and let $\varphi \in \mathscr{X}$. Combining (3.11) and (3.12) and then using the explicit form (3.1) we get, for almost all $(a,\ell) \in \Omega$, that

$$\mathbb{A}_{\alpha}(t)\varphi(a,\ell) = \alpha\overline{\chi}(t,a,\ell)\int_{\ell_1}^{\ell_2}\int_{0}^{\ell'} \mathrm{k}(\ell,a',\ell')\mathbb{T}_{\alpha}(t-a)(a',\ell')\mathrm{d}a'\mathrm{d}\ell'. \tag{3.13}$$

Putting (3.11) in (3.13), we obtain that

$$\begin{aligned} \mathbb{A}_{\alpha}(t)\varphi(a,\ell) &= \mathbb{M}_{\alpha}(t)\varphi(a,\ell) \\ &\quad + \alpha\overline{\chi}(t,a,\ell)\int_{\ell_1}^{\ell_2}\int_{0}^{\ell'} \mathrm{k}(\ell,a',\ell')\mathbb{A}_{\alpha}(t-a)\varphi(a',\ell')\mathrm{d}a'\mathrm{d}\ell' \end{aligned}$$

in which we put again (3.13) to finally get, $\mathbb{A}_\alpha(t)\varphi = \mathbb{M}_\alpha(t)\varphi + \mathbb{N}_\alpha(t)\varphi$. This together with (3.11) lead to the desired (3.5).

The boundedness of $\mathbb{T}_0(t)$, from $\mathscr{X}$ into itself, follows from [1, Lemma 4.2(3)]. Those of both operators $\mathbb{M}_\alpha(t)$ and $\mathbb{N}_\alpha(t)$ follow as composition of bounded linear operators from $\mathscr{X}$ into itself. ■

Now, let Q_μ and P_η be the following linear operators

$$\mathrm{Q}_\mu\varphi := -\mu\varphi \quad \text{and} \quad \mathrm{P}_\eta\varphi(a,l) := \int_{\ell_1}^{\ell}\int_0^{\ell'} \eta(a,l,a',l')\varphi(a',l')\mathrm{d}a'\mathrm{d}\ell', \tag{3.14}$$

where μ and η are subject to the following relevant biological assumptions

$(\mathbf{A}_\mu^1)$
$$\overline{\mu} := \operatorname*{ess\,sup}_{(a,\ell)\in\Omega} |\mu(a,\ell)| < \infty, \tag{3.15}$$

$(\mathbf{A}_\mu^2)$
$$\mu \text{ is positive,}$$

$(\mathbf{A}_\eta^1)$
$$\overline{\eta} := \sup_{(a',\ell')\in\Omega} \int_{\ell_1}^{\ell_2}\int_0^{\ell} |\eta(a,\ell,a',\ell')|\,\mathrm{d}a\mathrm{d}\ell < \infty, \tag{3.16}$$

$(\mathbf{A}_\eta^2)$
$$\eta \text{ is positive.}$$

Thanks to $(\mathbf{A}_\mu^1)$ and $(\mathbf{A}_\eta^1)$, both linear operators Q_μ and P_η are bounded from $\mathscr{X}$ into itself ([1, Lemma 7.1]). This allows us to define the following unbounded linear operators

$$\mathrm{V}_\alpha := \mathrm{T}_\alpha + \mathrm{Q}_\mu \qquad \text{on the domain} \qquad \mathrm{D}(\mathrm{V}_\alpha) = \mathrm{D}(\mathrm{T}_\alpha), \tag{3.17}$$

$$\mathrm{W}_\alpha := \mathrm{V}_\alpha + \mathrm{P}_\eta \qquad \text{on the domain} \qquad \mathrm{D}(\mathrm{W}_\alpha) = \mathrm{D}(\mathrm{T}_\alpha). \tag{3.18}$$

Lemma 3.2. *Suppose that $(\mathbf{A}_\mathrm{k}^1)$ holds and let $\alpha \geqslant 0$. Then the following assertions hold.*

(1) *If $(\mathbf{A}_\mu^1)$ holds, then V_α generates on $\mathscr{X}$ a C_0-semigroup $\mathbb{V}_\alpha = (\mathbb{V}_\alpha(t))_{t\geqslant 0}$. Furthermore, if $(\mathbf{A}_\mathrm{k}^2)$ and $(\mathbf{A}_\mu^2)$ hold, then $\mathbb{V}_\alpha = (\mathbb{V}_\alpha(t))_{t\geqslant 0}$ is positive in $\mathscr{X}$ and*

$$0 \leqslant \mathbb{V}_\alpha(t) - \mathbb{V}_0(t) \leqslant \mathbb{T}_\alpha(t) - \mathbb{T}_0(t), \qquad t \geqslant 0, \tag{3.19}$$

and
$$0 \leqslant \mathbb{V}_\alpha(t) \leqslant e^{-t\underline{\mu}}\mathbb{T}_\alpha(t), \qquad t \geqslant 0, \tag{3.20}$$

where
$$\underline{\mu} := \operatorname*{ess\,inf}_{(a,\ell)\in\Omega} \mu(a,l). \tag{3.21}$$

(2) *Suppose that $(\mathbf{A}_\mu^1)$ holds. If $(\mathbf{A}_\eta^1)$ holds, then W_α generates on $\mathscr{X}$ a C_0-semigroup $\mathbb{W}_\alpha = (\mathbb{W}_\alpha(t))_{t\geqslant 0}$. Furthermore, if $(\mathbf{A}_\mathrm{k}^2)$, $(\mathbf{A}_\mu^2)$ and $(\mathbf{A}_\eta^2)$ hold, then $\mathbb{W}_\alpha = (\mathbb{W}_\alpha(t))_{t\geqslant 0}$ is positive in $\mathscr{X}$ and*

$$0 \leqslant e^{-t\overline{\mu}}\mathbb{T}_\alpha(t) \leqslant \mathbb{W}_\alpha(t), \qquad t \geqslant 0, \tag{3.22}$$

where $\overline{\mu}$ is defined by (3.15).

Proof. (1). Firstly, the existence of C_0-semigroup $\mathbb{V}_\alpha = (\mathbb{V}_\alpha(t))_{t\geqslant 0}$ has already been proved (see [1, Theorem 6.1]). Furthermore, the Trotter product formula yields that

$$\mathbb{V}_\alpha(t)\varphi = \lim_{n\to\infty} \left(e^{\frac{t}{n}\mathrm{Q}_\mu}\mathbb{T}_\alpha\left(\tfrac{t}{n}\right)\right)^n \varphi, \qquad \varphi \in \mathscr{X}, \qquad t \geqslant 0. \tag{3.23}$$

Next, let $t > 0$ and let $\varphi \in \mathscr{X}$ be such that $\varphi \geqslant 0$. From [1, Theorem 4.4(2)] we get that

$$\mathbb{T}_\alpha(t)\varphi \geqslant \mathbb{T}_0(t)\varphi \geqslant 0, \tag{3.24}$$

which leads, by an easy induction, to

$$\left(e^{\frac{t}{n}\mathrm{Q}_\mu}\mathbb{T}_\alpha\left(\tfrac{t}{n}\right)\right)^n \varphi \geqslant \left(e^{\frac{t}{n}\mathrm{Q}_\mu}\mathbb{T}_0\left(\tfrac{t}{n}\right)\right)^n \varphi, \qquad n = 1, 2, \cdots$$

Passing to the limit $n \to \infty$ and then using (3.23) we obtain

$$\mathbb{V}_\alpha(t) \geqslant \mathbb{V}_0(t) \geqslant 0. \tag{3.25}$$

Both C_0-semigoups $\mathbb{T}_\alpha = (\mathbb{T}_\alpha(t))_{t\geqslant 0}$ and $\mathbb{V}_\alpha = (\mathbb{V}_\alpha(t))_{t\geqslant 0}$ are related by

$$\mathbb{V}_\alpha(t) = \mathbb{T}_\alpha(t) + \int_0^t \mathbb{V}_\alpha(t-s)\mathrm{Q}_\mu\mathbb{V}_\alpha(s)\mathrm{d}s,$$

and similarly for $\alpha = 0$,

$$\mathbb{V}_0(t) = \mathbb{T}_0(t) + \int_0^t \mathbb{V}_0(t-s)\mathrm{Q}_\mu\mathbb{V}_0(s)\mathrm{d}s,$$

and therefore

$$0 \leqslant \mathbb{V}_\alpha(t) - \mathbb{V}_0(t) = \mathbb{T}_\alpha(t) - \mathbb{T}_0(t) + \mathbb{U}(t), \tag{3.26}$$

where
$$\mathbb{U}(t) := \int_0^t \mathbb{V}_\alpha(t-s)\mathrm{Q}_\mu\mathbb{V}_\alpha(s)\mathrm{d}s - \int_0^t \mathbb{V}_0(t-s)\mathrm{Q}_\mu\mathbb{V}_0(s)\mathrm{d}s.$$

Now, the positivity of $(-\mathrm{Q}_\mu)$ (see $\left(\mathbf{A}_\mu^2\right)$) and then (3.24) and (3.25) lead to

$$\begin{aligned}
\mathbb{U}(t) &= \int_0^t \mathbb{V}_0(t-s)\left(-\mathrm{Q}_\mu\right)\mathbb{V}_0(s)\mathrm{d}s + \int_0^t \mathbb{V}_\alpha(t-s)\mathrm{Q}_\mu\mathbb{V}_\alpha(s)\mathrm{d}s \\
&\leqslant \int_0^t \mathbb{V}_\alpha(t-s)\left(-\mathrm{Q}_\mu\right)\mathbb{V}_\alpha(s)\mathrm{d}s + \int_0^t \mathbb{V}_\alpha(t-s)\mathrm{Q}_\mu\mathbb{V}_\alpha(s)\mathrm{d}s \\
&= 0,
\end{aligned}$$

which we combine with (3.26) to finally get (3.19).

On the other hand, for all real numbers x, we have

$$e^{x\mathrm{Q}_\mu} = \sum_{m\geqslant 0}\frac{1}{m!}\Big(-(x\mu)\mathbb{1}_{\mathscr{X}}\Big)^m = \left(\sum_{m\geqslant 0}\frac{1}{m!}\left(-x\mu\right)^m\right)\mathbb{1}_{\mathscr{X}} = e^{-x\mu}\mathbb{1}_{\mathscr{X}}$$

and therefore

$$e^{x\mathrm{Q}_\mu} \leqslant e^{-x\underline{\mu}}\mathbb{1}_{\mathscr{X}} \qquad \text{for all } x \geqslant 0. \tag{3.27}$$

The positivity of the operator $\mathbb{T}_\alpha(t)$ in $\mathscr{X}$ (see (3.24)) and (3.27) yield that

$$0 \leqslant \left(e^{\frac{t}{n}\mathrm{Q}_\mu}\mathbb{T}_\alpha\left(\tfrac{t}{n}\right)\right)^n \varphi \leqslant \left(e^{-\frac{t}{n}\underline{\mu}}\mathbb{T}_\alpha\left(\tfrac{t}{n}\right)\right)^n \varphi = e^{-t\underline{\mu}}\left(\mathbb{T}_\alpha\left(\tfrac{t}{n}\right)\right)^n \varphi = e^{-t\underline{\mu}}\mathbb{T}_\alpha(t)\varphi,$$

for all inters $n \geqslant 1$. Passing to the limit $n \to \infty$ and then using (3.24), the desired (3.20) follows and therefore $\mathbb{V}_\alpha = (\mathbb{V}_\alpha(t))_{t\geqslant 0}$ is positive in $\mathscr{X}$.

(2). Firstly, the existence of C_0-semigroup $\mathbb{W}_\alpha = (\mathbb{W}_\alpha(t))_{t\geqslant 0}$ has already been proved (see [1, Theorem 7.2]). Furthermore

$$\mathbb{W}_\alpha(t) \geqslant \mathbb{V}_\alpha(t) \geqslant 0, \qquad t \geqslant 0. \tag{3.28}$$

Next, from [1, Theorem 6.1(1)] we get that

$$\mathbb{V}_\alpha(t) \geqslant e^{-t\overline{\mu}}\mathbb{T}_\alpha(t) \geqslant 0, \qquad t \geqslant 0. \tag{3.29}$$

Finally, the desired (3.22) follows from (3.28) and (3.29). ∎

Corollary 3.3. *The type, $\omega_0(\mathbb{V}_0)$, of the C_0-semigroup $\mathbb{V}_0 = (\mathbb{V}_0(t))_{t\geqslant 0}$ satisfies*

$$\omega_0(\mathbb{V}_0) \leqslant -\underline{\mu}, \tag{3.30}$$

where $\underline{\mu}$ is defined by (3.21).

Proof. From (3.20)–(with $\alpha = 0$) together with (1.3) we get that

$$\omega_0(\mathbb{V}_0) \leqslant \omega_0(\mathbb{T}_0) - \underline{\mu}.$$

As the C_0-semigroup $\mathbb{T}_0 = (\mathbb{T}_0(t))_{t\geqslant 0}$ is contractive ([1, Lemma 4.2(3)]), then $\omega_0(\mathbb{T}_0) \leqslant 0$ which proves the desired (3.30) and ends the proof. ∎

4 Spectral Properties

The purpose of this section is to estimate the type, $\omega_0(\mathbb{W}_\alpha)$, of the full C_0-semigroup $\mathbb{W}_\alpha = (\mathbb{W}_\alpha(t))_{t\geqslant 0}$. Firstly, we consider the following linear operator

$$\mathcal{K}_{\alpha,\lambda}\psi = \mathrm{K}_\alpha(\varepsilon_\lambda\psi) \qquad \text{where} \qquad \varepsilon_\lambda(a) := e^{-\lambda a}, \tag{4.1}$$

whose explicit form is

$$\mathcal{K}_{\alpha,\lambda}\psi(\ell) := \alpha\int_{\ell_1}^{\ell_2}\int_0^{\ell'} e^{-\lambda a'}\mathrm{k}(\ell,a',\ell')\psi(\ell')\mathrm{d}a'\mathrm{d}\ell', \qquad \ell \in (\ell_1,\ell_2). \tag{4.2}$$

As we are going to see in this section, the operator $\mathcal{K}_{\alpha,\lambda}$ will play a crucial role. Its boundedness is given by

Lemma 4.1. *Let $\alpha \geqslant 0$. If $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)$ holds, then $\mathcal{K}_{\alpha,\lambda}$ $(\lambda > 0)$ is a bounded linear operator from $\mathscr{Y}$ into itself satisfying*

$$\left\|\mathcal{K}_{\alpha,\lambda}\psi\right\|_{\mathscr{Y}} \leqslant \frac{\alpha\overline{\kappa}}{\lambda}\|\psi\|_{\mathscr{Y}} \tag{4.3}$$

for all $\psi \in \mathscr{Y}$. Furthermore,

(1) *if $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)$ holds, then $\mathcal{K}_{\alpha,\lambda}$ is positive in $\mathscr{Y}$.*

(2) *if $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)'$ holds and $\alpha > 0$, then $\mathcal{K}_{\alpha,\lambda}$ is strongly positive in $\mathscr{Y}$.*

Proof. Let $\lambda > 0$ and let $\psi \in \mathscr{Y}$. Using (4.1) and then (3.3) we get that,

$$\left\|\mathcal{K}_{\alpha,\lambda}\psi\right\|_{\mathscr{Y}} \leqslant \alpha\overline{\kappa}\|\varepsilon_{\lambda}\psi\|_{\mathscr{X}} = \alpha\overline{\kappa}\int_{\ell_1}^{\ell_2}\left[\int_0^{\ell} e^{-\lambda a}\mathrm{d}a\right]|\psi(\ell)|\,\mathrm{d}\ell \leqslant \frac{\alpha\overline{\kappa}}{\lambda}\|\psi\|_{\mathscr{Y}}$$

which proves the desired (4.3) and therefore $\mathcal{K}_{\alpha,\lambda}$ is bounded from $\mathscr{Y}$ into itself because of $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)$. Finally, both points (1) and (2) are obvious. ■

Let κ be such that

$$\kappa(\ell) := \operatorname*{ess\,sup}_{(a',\ell')\in\Omega}|\mathrm{k}(\ell,a',\ell')|\,, \qquad \ell \in (\ell_1,\ell_2)\,, \tag{4.4}$$

and let us consider the following assumption

$$\left(\mathbf{A}_{\mathrm{k}}^{1}\right)' \qquad \kappa \in \mathscr{Y} \tag{4.5}$$

thanks to which, interesting properties of $\mathcal{K}_{\alpha,\lambda}$ can be given. The first one is

Lemma 4.2. *Let $\alpha \geqslant 0$. If $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)'$ holds, then $\mathcal{K}_{\alpha,\lambda}^{2}$ $(\lambda > 0)$ is a compact linear operator from $\mathscr{Y}$ into itself.*

Proof. Let $\lambda > 0$ and let $\psi \in \mathscr{Y}$ be such that $\psi \geqslant 0$. Then, for almost all $\ell \in (\ell_1,\ell_2)$, we have

$$\begin{aligned}\left|\mathcal{K}_{\alpha,\lambda}\psi(\ell)\right| &\leqslant \alpha\int_{\ell_1}^{\ell_2}\int_0^{\ell'} e^{-\lambda a'}\,|\mathrm{k}(\ell,a',\ell')|\,\psi(\ell')\mathrm{d}a'\mathrm{d}\ell' \\ &= \alpha\kappa(\ell)\int_{\ell_1}^{\ell_2}\left[\int_0^{\ell'} e^{-\lambda a'}\mathrm{d}a'\right]\psi(\ell')\mathrm{d}\ell' \\ &\leqslant \frac{\alpha}{\lambda}\kappa(\ell)\int_{\ell_1}^{\ell_2}\psi(\ell')\mathrm{d}\ell'\end{aligned}$$

and therefore

$$\left|\mathcal{K}_{\alpha,\lambda}\psi\right| \leqslant \frac{\alpha}{\lambda}\kappa\int_{\ell_1}^{\ell_2}\psi(\ell')\mathrm{d}\ell'. \tag{4.6}$$

Now $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)'$ yields that the right hand side of (4.6) is obviously a rank one operator in $\mathscr{Y}$ and thus weakly compact on $\mathscr{Y}$. Now Lemma 2.1(3) yields that $\mathcal{K}_{\alpha,\lambda}$ is a weakly compact operator on $\mathscr{Y}$ and therefore $\mathcal{K}_{\alpha,\lambda}^{2}$ is compact because of Lemma 2.1(1). ■

Remark 4.3. Assumption $\left(\mathbf{A}^1_{\mathrm{k}}\right)'$ is stronger than $\left(\mathbf{A}^1_{\mathrm{k}}\right)$ because of

$$\overline{\kappa} = \operatorname*{ess\,sup}_{(a',\ell')\in\Omega} \int_{\ell_1}^{\ell_2} |\mathrm{k}(\ell, a', \ell')| \,\mathrm{d}\ell \leqslant \int_{\ell_1}^{\ell_2} \kappa(\ell)\mathrm{d}\ell = \|\kappa\|_{\mathscr{Y}} < \infty.$$

Therefore, all the previous results related to $\left(\mathbf{A}^1_{\mathrm{k}}\right)$ still hold true whenever $\left(\mathbf{A}^1_{\mathrm{k}}\right)'$ holds true.

Now the spectrum $\sigma\left(\mathrm{T}_{\alpha}\right)$ of the generator T_{α} (defined by (3.4)) can be characterized as follows

Lemma 4.4. *Let* $\alpha > 0$. *Let* $\lambda \in \mathbb{C}$ *be such that* $\operatorname{Re}(\lambda) > 0$. *If* $\left(\mathbf{A}^1_{\mathrm{k}}\right)'$ *holds, then*

$$\lambda \in \sigma\left(\mathrm{T}_{\alpha}\right) \Longrightarrow 1 \in \sigma_p\left(\mathcal{K}^2_{\alpha,\lambda}\right). \tag{4.7}$$

Proof. Let us prove the converse of (4.7), *i.e.*,

$$1 \in \rho(\mathcal{K}^2_{\alpha,\lambda}) \Longrightarrow \lambda \in \rho\left(\mathrm{T}_{\alpha}\right). \tag{4.8}$$

Suppose then that $1 \in \rho(\mathcal{K}^2_{\alpha,\lambda})$ and let us divide the proof in some steps.

Step I. Firstly, using the boundedness of $\mathcal{K}_{\alpha,\lambda}$ (see Lemma 4.1) together with the boundedness of $(\lambda - \mathrm{T}_0)^{-1}$ (see [1, Lemma 4.2(1)]) and that of K_{α}, we get that $\left(\mathbb{1}_{\mathscr{Y}} + \mathcal{K}_{\alpha,\lambda}\right)\mathrm{K}_{\alpha}\left(\lambda - \mathrm{T}_0\right)^{-1}$ is a bounded linear operator from $\mathscr{X}$ into $\mathscr{Y}$ and therefore

$$\left\|\left(\mathbb{1}_{\mathscr{Y}} + \mathcal{K}_{\alpha,\lambda}\right)\mathrm{K}_{\alpha}\left(\lambda - \mathrm{T}_0\right)^{-1} g\right\|_{\mathscr{Y}} \leqslant C_\lambda \|g\|_{\mathscr{X}} \qquad \text{for all } g \in \mathscr{X},$$

where C_λ is a constant.

Next, let $g \in \mathscr{X}$. Since $1 \in \rho(\mathcal{K}^2_{\alpha,\lambda})$, then

$$\psi_\lambda := \left(\mathbb{1}_{\mathscr{Y}} - \mathcal{K}^2_{\alpha,\lambda}\right)^{-1}\left(\mathbb{1}_{\mathscr{Y}} + \mathcal{K}_{\alpha,\lambda}\right)\mathrm{K}_{\alpha}\left(\lambda - \mathrm{T}_0\right)^{-1} g, \tag{4.9}$$

belongs to $\mathscr{Y}$ because of

$$\|\psi_\lambda\|_{\mathscr{Y}} \leqslant C'_\lambda C_\lambda \|g\|_{\mathscr{X}} < \infty, \tag{4.10}$$

where $C'_\lambda = \left\|\left(\mathbb{1}_{\mathscr{Y}} - \mathcal{K}^2_{\alpha,\lambda}\right)^{-1}\right\|$. From (4.9), it follows that

$$\left(\mathbb{1}_{\mathscr{Y}} - \mathcal{K}_{\alpha,\lambda}\right)\psi_\lambda = \mathrm{K}_{\alpha}\left(\lambda - \mathrm{T}_0\right)^{-1} g,$$

and therefore

$$\psi_\lambda = \mathcal{K}_{\alpha,\lambda}\psi_\lambda + \mathrm{K}_{\alpha}\left(\lambda - \mathrm{T}_0\right)^{-1} g. \tag{4.11}$$

Step II. Let φ_λ be the following function

$$\varphi_\lambda(a,\ell) := \varepsilon_\lambda(a)\psi_\lambda(\ell) + (\lambda - \mathrm{T}_0)^{-1} g(a,\ell), \qquad (a,\ell) \in \Omega, \tag{4.12}$$

where ε_λ is defined in (4.1). We claim that $\varphi_\lambda \in \mathrm{D}(\mathrm{T}_{\alpha})$.

Firstly, integrating (4.12) and then using [1, Lemma 4.2(1)], we get that

$$\begin{aligned}
\|\varphi_\lambda\|_{\mathscr{X}} &\leqslant \int_{\ell_1}^{\ell_2}\left[\int_0^\ell e^{-\lambda a}\mathrm{d}a\right]|\psi_\lambda(\ell)|\,\mathrm{d}\ell + \left\|(\lambda - \mathrm{T}_0)^{-1}\,g\right\|_{\mathscr{X}} \\
&\leqslant \frac{1}{\lambda}\int_{\ell_1}^{\ell_2}|\psi_\lambda(\ell)|\,\mathrm{d}\ell + \frac{1}{\lambda}\|g\|_{\mathscr{X}} \\
&= \frac{1}{\lambda}\left(\|\psi_\lambda\|_{\mathscr{Y}} + \|g\|_{\mathscr{X}}\right),
\end{aligned}$$

and by (4.10),

$$\|\varphi_\lambda\|_{\mathscr{X}} \leqslant \frac{1}{\lambda}\left(C'_\lambda C_\lambda + 1\right)\|g\|_{\mathscr{X}}. \tag{4.13}$$

Similarly

$$\begin{aligned}
\left\|\frac{1}{\ell}\varphi_\lambda\right\|_{\mathscr{X}} &\leqslant \int_{\ell_1}^{\ell_2}\left[\frac{1}{\ell}\int_0^\ell e^{-\lambda a}\mathrm{d}a\right]|\psi_\lambda(\ell)|\,\mathrm{d}\ell + \left\|\frac{1}{\ell}\left(\lambda - \mathrm{T}_0\right)^{-1}g\right\|_{\mathscr{X}} \\
&\leqslant \int_{\ell_1}^{\ell_2}\left[\frac{1}{\ell}\int_0^\ell \mathrm{d}a\right]|\psi_\lambda(\ell)|\,\mathrm{d}\ell + \|g\|_{\mathscr{X}} \\
&= \|\psi_\lambda\|_{\mathscr{Y}} + \|g\|_{\mathscr{X}},
\end{aligned}$$

and by (4.10),

$$\left\|\frac{1}{\ell}\varphi_\lambda\right\|_{\mathscr{X}} \leqslant \left(C'_\lambda C_\lambda + 1\right)\|g\|_{\mathscr{X}}. \tag{4.14}$$

Deriving (4.12), we get that

$$\begin{aligned}
\frac{\partial\varphi_\lambda}{\partial a} &= \frac{\partial}{\partial a}\left(\varepsilon_\lambda\psi_\lambda\right) + \frac{\partial}{\partial a}\Big(\left(\lambda - \mathrm{T}_0\right)^{-1}g\Big) \\
&= -\lambda\varepsilon_\lambda\psi_\lambda - \mathrm{T}_0\left(\lambda - \mathrm{T}_0\right)^{-1}g \\
&= -\lambda\varepsilon_\lambda\psi_\lambda + \left(g - \lambda\left(\lambda - \mathrm{T}_0\right)^{-1}g\right) \\
&= -\lambda\left(\varepsilon_\lambda\psi_\lambda + \left(\lambda - \mathrm{T}_0\right)^{-1}g\right) + g
\end{aligned}$$

that is to say that

$$\frac{\partial\varphi_\lambda}{\partial a} = -\lambda\varphi_\lambda + g, \tag{4.15}$$

which leads to

$$\left\|\frac{\partial\varphi_\lambda}{\partial a}\right\|_{\mathscr{X}} \leqslant \lambda\|\varphi_\lambda\|_{\mathscr{X}} + \|g\|_{\mathscr{X}},$$

and by (4.13),

$$\left\|\frac{\partial\varphi_\lambda}{\partial a}\right\|_{\mathscr{X}} \leqslant \left(C'_\lambda C_\lambda + 2\right)\|g\|_{\mathscr{X}}. \tag{4.16}$$

Using again (4.12) we get that

$$\varphi_\lambda(0,\cdot) = \left(\varepsilon_\lambda\psi_\lambda + \left(\lambda - \mathrm{T}_0\right)^{-1}g\right)(0,\cdot) = \varepsilon_\lambda(0)\psi_\lambda(\cdot) + 0 = \psi_\lambda$$

and

$$\begin{aligned}
\mathrm{K}_\alpha \varphi_\lambda &= \mathrm{K}_\alpha (\varepsilon_\lambda \psi_\lambda) + \mathrm{K}_\alpha (\lambda - \mathrm{T}_0)^{-1} g \\
&= \mathcal{K}_{\alpha,\lambda} \psi_\lambda + \mathrm{K}_\alpha (\lambda - \mathrm{T}_0)^{-1} g \\
&= \psi_\lambda
\end{aligned}$$

because of (4.11) and therefore $\varphi_\lambda(0,\cdot) = \mathrm{K}_\alpha \varphi_\lambda$. This together with (4.13), (4.14) and (4.16) prove our claim, that is

$$\varphi_\lambda \in \mathrm{D}(\mathrm{T}_\alpha) \qquad \text{for all} \qquad g \in \mathscr{X}. \tag{4.17}$$

Step III. Due to (4.17) we can rewrite (4.15) as $(\lambda - \mathrm{T}_\alpha)\varphi_\lambda = g$ which implies that $\varphi_\lambda = (\lambda - \mathrm{T}_\alpha)^{-1} g$ (given by (4.12)). Using now (4.13), we get that

$$\left\|(\lambda - \mathrm{T}_\alpha)^{-1} g\right\|_{\mathscr{X}} = \|\varphi_\lambda\|_{\mathscr{X}} \leqslant \frac{1}{\lambda}(C_\lambda C'_\lambda + 1)\|g\|_{\mathscr{X}}$$

which proves that $(\lambda - \mathrm{T}_\alpha)^{-1}$ is a bounded linear operator from $\mathscr{X}$ into itself and therefore $\lambda \in \rho(\mathrm{T}_\alpha)$.

Hence, we have proved that if $1 \in \rho(\mathcal{K}^2_{\alpha,\lambda})$ then $\lambda \in \rho(\mathrm{T}_\alpha)$ and therefore (4.8) follows. Accordingly, the converse of (4.8), *i.e.,*

$$\lambda \in \sigma(\mathrm{T}_\alpha) \Longrightarrow 1 \in \sigma\left(\mathcal{K}^2_{\alpha,\lambda}\right)$$

also holds true. Finally, the compactness of $\mathcal{K}^2_{\alpha,\lambda}$ (Lemma 4.2) yields that $\sigma\left(\mathcal{K}^2_{\alpha,\lambda}\right) = \sigma_p\left(\mathcal{K}^2_{\alpha,\lambda}\right)$ and completes the proof. ■

Lemma 4.5. *Suppose that* $\left(\mathbf{A}^1_{\mathrm{k}}\right)'$ *and* $\left(\mathbf{A}^2_{\mathrm{k}}\right)'$ *hold and let* $\alpha > 0$. *Then the spectral radius* $\mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right)$ *of the linear operator* $\mathcal{K}_{\alpha,\lambda}$ $(\lambda > 0)$ *is positive. Furthermore*

$$\lambda \in (0\ ,\ \infty) \longrightarrow \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right) \tag{4.18}$$

is a continuous and strictly decreasing mapping.

Proof. For convenience, we divide the proof in some steps.

Step I. Let $\lambda > 0$. As $\mathcal{K}_{\alpha,\lambda}$ is a strongly positive operator (Lemma 4.1(2)) with a compact square (Lemma 4.2), then [3] yields that $\mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right) > 0$ and there exist a quasi-interior vector ψ_λ of $(\mathscr{Y})_+$ and a strictly positive functional $\psi^*_\lambda \in (\mathscr{Y}^*)_+$ such that

$$\mathcal{K}_{\alpha,\lambda}\psi_\lambda = \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right)\psi_\lambda \qquad \text{and} \qquad \mathcal{K}^*_{\alpha,\lambda}\psi^*_\lambda = \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right)\psi^*_\lambda$$

with $\|\psi_\lambda\|_{\mathscr{Y}} = \left\|\psi^*_\lambda\right\|_{\mathscr{Y}^*} = 1$, where $\mathcal{K}^*_{\alpha,\lambda}$ is the adjoint operator of $\mathcal{K}_{\alpha,\lambda}$.

Similarly, if $\delta > 0$ then $\mathrm{r}\left(\mathcal{K}_{\alpha,\delta}\right) > 0$ and there exist a quasi-interior vector ψ_δ of $(\mathscr{Y})_+$ and a strictly positive functional $\psi^*_\delta \in (\mathscr{Y}^*)_+$ such that,

$$\mathcal{K}_{\alpha,\delta}\psi_\delta = \mathrm{r}\left(\mathcal{K}_{\alpha,\delta}\right)\psi_\delta \qquad \text{and} \qquad \mathcal{K}^*_{\alpha,\delta}\psi^*_\delta = \mathrm{r}\left(\mathcal{K}_{\alpha,\delta}\right)\psi^*_\delta$$

with $\|\psi_\delta\|_{\mathscr{Y}} = \left\|\psi_\delta^*\right\|_{\mathscr{Y}^*} = 1$, where $\mathcal{K}_{\alpha,\delta}^*$ is the adjoint operator of $\mathcal{K}_{\alpha,\delta}$.

Step II. ((4.18) is a strictly decreasing mapping).

Let $\lambda > 0$ and $\delta > 0$. The previous step allows us to write that

$$\begin{aligned}
\left\langle \psi_\delta^*, \left(\mathcal{K}_{\alpha,\delta} - \mathcal{K}_{\alpha,\lambda}\right)\psi_\lambda \right\rangle &= \left\langle \psi_\delta^*, \mathcal{K}_{\alpha,\delta}\psi_\lambda \right\rangle - \left\langle \psi_\delta^*, \mathcal{K}_{\alpha,\lambda}\psi_\lambda \right\rangle \\
&= \left\langle \mathcal{K}_{\alpha,\delta}^*\psi_\delta^*, \psi_\lambda \right\rangle - \left\langle \psi_\delta^*, \mathcal{K}_{\alpha,\lambda}\psi_\lambda \right\rangle \\
&= \left\langle \mathrm{r}\left(\mathcal{K}_{\alpha,\delta}\right)\psi_\delta^*, \psi_\lambda \right\rangle - \left\langle \psi_\delta^*, \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right)\psi_\lambda \right\rangle \\
&= \mathrm{r}\left(\mathcal{K}_{\alpha,\delta}\right)\left\langle \psi_\delta^*, \psi_\lambda \right\rangle - \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right)\left\langle \psi_\delta^*, \psi_\lambda \right\rangle \\
&= \Big(\mathrm{r}\left(\mathcal{K}_{\alpha,\delta}\right) - \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right)\Big)\left\langle \psi_\delta^*, \psi_\lambda \right\rangle
\end{aligned}$$

which leads to

$$\mathrm{r}\left(\mathcal{K}_{\alpha,\delta}\right) - \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right) = \frac{\left\langle \psi_\delta^*, \left(\mathcal{K}_{\alpha,\delta} - \mathcal{K}_{\alpha,\lambda}\right)\psi_\lambda \right\rangle}{\left\langle \psi_\delta^*, \psi_\lambda \right\rangle}. \tag{4.19}$$

Next, using the explicit form (4.2) we get, for almost all $\ell \in (\ell_1, \ell_2)$, that

$$\left(\mathcal{K}_{\alpha,\delta} - \mathcal{K}_{\alpha,\lambda}\right)\psi_\lambda(\ell) = \alpha \int_{\ell_1}^{\ell_2}\int_0^{\ell'} \left(e^{-\delta a'} - e^{-\lambda a'}\right)\mathrm{k}(\ell, a', \ell')\psi_\lambda(\ell')\mathrm{d}a'\mathrm{d}\ell',$$

which implies, by virtue of $\left(\mathbf{A}_{\mathrm{k}}^1\right)'$ and then the fact that ψ_λ is a quasi-interior vector of $(\mathscr{Y})_+$, that

$$\left(\mathcal{K}_{\alpha,\delta} - \mathcal{K}_{\alpha,\lambda}\right)\psi_\lambda > 0, \qquad \text{whenever} \qquad \lambda > \delta,$$

and therefore

$$\frac{\left\langle \psi_\delta^*, \left(\mathcal{K}_{\alpha,\delta} - \mathcal{K}_{\alpha,\lambda}\right)\psi_\lambda \right\rangle}{\left\langle \psi_\delta^*, \psi_\lambda \right\rangle} > 0, \qquad \text{whenever} \qquad \lambda > \delta, \tag{4.20}$$

because ψ_δ^* is a strictly positive functional on $(\mathscr{Y})_+$. Finally, combining (4.19) and (4.20) we get that

$$\mathrm{r}\left(\mathcal{K}_{\alpha,\delta}\right) > \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right) \qquad \text{whenever} \qquad \lambda > \delta,$$

which proves that (4.18) is a strictly decreasing mapping.

Step III. ((4.18) is a continuous mapping).

Let $\lambda > 0$ and $\delta > 0$. From (4.19) we get that

$$\left|\mathrm{r}\left(\mathcal{K}_{\alpha,\delta}\right) - \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right)\right| \leqslant \frac{1}{\left\langle \psi_\delta^*, \psi_\lambda \right\rangle}\left\|\mathcal{K}_{\alpha,\delta}\psi_\lambda - \mathcal{K}_{\alpha,\lambda}\psi_\lambda\right\|_{\mathscr{Y}}. \tag{4.21}$$

Using the explicit form (4.2) and then (4.4) we get that

$$\begin{aligned}
\left\|\mathcal{K}_{\alpha,\delta}\psi_\lambda - \mathcal{K}_{\alpha,\lambda}\psi_\lambda\right\|_{\mathscr{Y}} &\leqslant \alpha \int_{\ell_1}^{\ell_2}\int_{\ell_1}^{\ell_2}\int_0^{\ell'} \left|e^{-\delta a'} - e^{-\lambda a'}\right| \mathrm{k}(\ell, a', \ell')\psi_\lambda(\ell')\mathrm{d}a'\mathrm{d}\ell'\mathrm{d}\ell \\
&\leqslant \alpha \int_{\ell_1}^{\ell_2}\int_{\ell_1}^{\ell_2}\int_0^{\ell'} \left|e^{-\delta a'} - e^{-\lambda a'}\right| \kappa(\ell)\psi_\lambda(\ell')\mathrm{d}a'\mathrm{d}\ell'\mathrm{d}\ell \\
&\leqslant \alpha \left[\int_{\ell_1}^{\ell_2} \kappa(\ell)\mathrm{d}\ell\right] \int_{\ell_1}^{\ell_2}\int_0^{\ell'} \left|e^{-\delta a'} - e^{-\lambda a'}\right| \psi_\lambda(\ell')\mathrm{d}a'\mathrm{d}\ell'
\end{aligned}$$

which leads, by virtue of $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)'$, to

$$\left\|\mathcal{K}_{\alpha,\delta}\psi_\lambda - \mathcal{K}_{\alpha,\lambda}\psi_\lambda\right\|_{\mathscr{Y}} \leqslant \alpha\|\kappa\|_{\mathscr{Y}} \int_{\ell_1}^{\ell_2}\left[\int_0^{\ell'} \left|e^{-\delta a'} - e^{-\lambda a'}\right| \mathrm{d}a'\right]\psi_\lambda(\ell')\mathrm{d}\ell'. \tag{4.22}$$

If $\delta > \lambda$, then (4.22) becomes

$$\begin{aligned}\left\|\mathcal{K}_{\alpha,\delta}\psi_\lambda - \mathcal{K}_{\alpha,\lambda}\psi_\lambda\right\|_{\mathscr{Y}} &\leqslant \alpha\|\kappa\|_{\mathscr{Y}} \int_{\ell_1}^{\ell_2}\left[\int_0^{\ell'} \left(e^{-\lambda a'} - e^{-\delta a'}\right) \mathrm{d}a'\right]\psi_\lambda(\ell')\mathrm{d}\ell' \\ &= \alpha\|\kappa\|_{\mathscr{Y}} \int_{\ell_1}^{\ell_2}\left[\frac{1-e^{-\delta\ell'}}{\delta} - \frac{1-e^{-\lambda\ell'}}{\lambda}\right]\psi_\lambda(\ell')\mathrm{d}\ell' \\ &\leqslant \frac{\alpha}{\lambda}\|\kappa\|_{\mathscr{Y}} \int_{\ell_1}^{\ell_2}\left[e^{-\lambda\ell'} - e^{-\delta\ell'}\right]\psi_\lambda(\ell')\mathrm{d}\ell'.\end{aligned}$$

Similarly, if $\delta < \lambda$, then (4.22) becomes

$$\left\|\mathcal{K}_{\alpha,\delta}\psi_\lambda - \mathcal{K}_{\alpha,\lambda}\psi_\lambda\right\|_{\mathscr{Y}} \leqslant \frac{\alpha}{\delta}\|\kappa\|_{\mathscr{Y}} \int_{\ell_1}^{\ell_2}\left[e^{-\delta\ell'} - e^{-\lambda\ell'}\right]\psi_\lambda(\ell')\mathrm{d}\ell'.$$

In both cases, Dominated Convergence Theorem leads to

$$\lim_{\delta\to\lambda}\left\|\mathcal{K}_{\alpha,\delta}\psi_\lambda - \mathcal{K}_{\alpha,\lambda}\psi_\lambda\right\|_{\mathscr{Y}} = 0,$$

and by (4.21),

$$\lim_{\delta\to\lambda}\left|\mathrm{r}\left(\mathcal{K}_{\alpha,\delta}\right) - \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right)\right| = 0,$$

which proves that the mapping (4.18) is continuous at $\lambda > 0$. ∎

Now the main result of this section can be announced as follows

Theorem 4.6. *Suppose that $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)'$, $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)'$, $\left(\mathbf{A}_{\mu}^{1}\right)$ and $\left(\mathbf{A}_{\mu}^{2}\right)$ hold and let $\alpha \geqslant 0$. Let $\lambda_\mu := \overline{\mu} - \underline{\mu}$ where $\overline{\mu}$ and $\underline{\mu}$ are respectively defined by* (3.15) *and* (3.21)*. If $\lambda_\mu > 0$ and the spectral radius $\mathrm{r}\left(\mathcal{K}_{\alpha,\lambda_\mu}\right) > 1$ then the type, $\omega_0\left(\mathbb{W}_\alpha\right)$, of the C_0-semigroup $\mathbb{W}_\alpha = \left(\mathbb{W}_\alpha(t)\right)_{t\geqslant 0}$ satisfies*

$$\omega_0\left(\mathbb{W}_\alpha\right) > -\underline{\mu}. \tag{4.23}$$

Proof. Firstly, from (4.3) we get that,

$$\lim_{\lambda\to\infty}\mathrm{r}\left(\mathcal{K}_{\alpha,\lambda}\right) \leqslant \lim_{\lambda\to\infty}\left\|\mathcal{K}_{\alpha,\lambda}\right\|_{\mathscr{L}(\mathscr{Y})} = 0. \tag{4.24}$$

Next, Lemma 4.5 says that (4.18) is a continuous and strictly decreasing mapping satisfying $\mathrm{r}\left(\mathcal{K}_{\alpha,\lambda_\mu}\right) > 1$ and (4.24). Now, we can say that there exists a unique λ_0 such that

$$\lambda_0 > \lambda_\mu > 0 \qquad \text{and} \qquad \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda_0}\right) = 1. \tag{4.25}$$

For convenience, we divide the rest of the proof in some steps.

Step I. Let $\lambda \in \sigma(\mathrm{T}_\alpha)$ be such that $\mathrm{Re}(\lambda) \geqslant \lambda_\mu$. Due to Lemma 4.4, there exists $\psi_\lambda \neq 0$ such that $\mathcal{K}^2_{\alpha,\lambda}\psi_\lambda = \psi_\lambda$. From (4.2) we can write that

$$\begin{aligned}\left|\mathcal{K}_{\alpha,\lambda}\psi_\lambda\right| &= \alpha \left| \int_{\ell_1}^{\ell_2}\int_0^{\ell'} e^{-\lambda a'}\mathrm{k}(\cdot,a',\ell')\psi_\lambda(\ell')\mathrm{d}a'\mathrm{d}\ell' \right| \\ &\leqslant \alpha \int_{\ell_1}^{\ell_2}\int_0^{\ell'} e^{-\mathrm{Re}(\lambda)a'}\mathrm{k}(\cdot,a',\ell')\left|\psi_\lambda(\ell')\right|\mathrm{d}a'\mathrm{d}\ell' \\ &= \mathcal{K}_{\alpha,\mathrm{Re}(\lambda)}\left|\psi_\lambda\right|\end{aligned}$$

which leads to,

$$|\psi_\lambda| = \left|\mathcal{K}^2_{\alpha,\lambda}\psi_\lambda\right| \leqslant \mathcal{K}_{\alpha,\mathrm{Re}(\lambda)}\left|\mathcal{K}_{\alpha,\lambda}\psi_\lambda\right| \leqslant \mathcal{K}^2_{\alpha,\mathrm{Re}(\lambda)}\left|\psi_\lambda\right|$$

and by induction

$$|\psi_\lambda| \leqslant \mathcal{K}^{2n}_{\alpha,\mathrm{Re}(\lambda)}\left|\psi_\lambda\right| \qquad n = 1, 2, \cdots$$

Hence $\mathrm{r}\left(\mathcal{K}_{\alpha,\mathrm{Re}(\lambda)}\right) \geqslant 1$. Now Lemma 4.5 leads to $\mathrm{Re}(\lambda) \leqslant \lambda_0$ and therefore

$$s(\mathrm{T}_\alpha) \leqslant \lambda_0. \tag{4.26}$$

Step II. As $\mathcal{K}_{\alpha,\lambda_0}$ is a strongly positive operator (Lemma 4.1(2)) with a compact square (Lemma 4.2), then there exists (see [3]) a quasi-interior vector ψ_λ of $(\mathscr{Y})_+$ such that

$$\mathcal{K}_{\alpha,\lambda_0}\psi_{\lambda_0} = \mathrm{r}\left(\mathcal{K}_{\alpha,\lambda_0}\right)\psi_{\lambda_0} = \psi_{\lambda_0}.$$

Let φ_{λ_0} be the following function

$$\varphi_{\lambda_0}(a,\ell) := \varepsilon_{\lambda_0}(a,\ell)\psi_{\lambda_0}(\ell), \qquad (a,\ell) \in \Omega,$$

which is nothing else than (4.12)–(with $\lambda_0 = \lambda$ and $g = 0$). So, using the proof of Lemma 4.4 (Step II), we infer that (4.17)–(with $\lambda_0 = \lambda$ and $g = 0$) holds true and therefore $\varphi_{\lambda_0} \in \mathrm{D}(\mathrm{T}_\alpha)$. Furthermore, (4.15)–(with $\lambda_0 = \lambda$ and $g = 0$) becomes $\mathrm{T}_\alpha\varphi_{\lambda_0} = \lambda_0\varphi_{\lambda_0}$ which leads to $\lambda_0 \in \sigma_p(\mathrm{T}_\alpha) \subset \sigma(\mathrm{T}_\alpha)$ and therefore

$$\lambda_0 \leqslant s(\mathrm{T}_\alpha). \tag{4.27}$$

Step III. Firstly, using (1.4) together with (4.26), (4.27) and then (4.25), we infer that

$$\omega_0(\mathbb{T}_\alpha) \geqslant s(\mathrm{T}_\alpha) = \lambda_0 > \lambda_\mu = \overline{\mu} - \underline{\mu} > 0. \tag{4.28}$$

Next, (3.22) leads to

$$\frac{1}{t}\ln\|\mathbb{W}_\alpha(t)\| \geqslant \frac{1}{t}\ln\|\mathbb{T}_\alpha(t)\| - \overline{\mu}, \qquad t \geqslant 0,$$

and at the limit $t \to \infty$, it follows that

$$\omega_0(\mathbb{W}_\alpha) \geqslant \omega_0(\mathbb{T}_\alpha) - \overline{\mu}. \tag{4.29}$$

Finally, combining (4.28) and (4.29) we get that

$$\omega_0(\mathbb{W}_\alpha) > \lambda_\mu - \overline{\mu} = -\underline{\mu}$$

which proves the desired (4.23) and completes the proof. ■

5 The essential type of the full semigroup

The aim of this section is to estimate the essential type, $\omega_{\text{ess}}\left(\mathbb{W}_{\alpha}\right)$, of the full C_0-semigroup $\mathbb{W}_{\alpha}=\left(\mathbb{W}_{\alpha}(t)\right)_{t\geqslant 0}$. This can be done after proving auxiliary results. So, let us start with the first auxiliary result

Lemma 5.1. *Suppose that* $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)'$, $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)$, $\left(\mathbf{A}_{\mu}^{1}\right)$ *and* $\left(\mathbf{A}_{\mu}^{2}\right)$ *hold and let* $\alpha\geqslant 0$*. Then, both* C_0-*semigroups* $\mathbb{V}_{\alpha}=\left(\mathbb{V}_{\alpha}(t)\right)_{t\geqslant 0}$ *and* $\mathbb{V}_{0}=\left(\mathbb{V}_{0}(t)\right)_{t\geqslant 0}$ *have the same essential type, i.e.,*

$$\omega_{\text{ess}}\left(\mathbb{V}_{\alpha}\right)=\omega_{\text{ess}}\left(\mathbb{V}_{0}\right). \tag{5.1}$$

Proof. Let t_0 be such that $\mathrm{t}_0>0$. Lemma 3.1 yields that

$$\mathbb{T}_{\alpha}(\mathrm{t}_0)-\mathbb{T}_{0}(\mathrm{t}_0)=\mathbb{M}_{\alpha}(\mathrm{t}_0)+\mathbb{N}_{\alpha}(\mathrm{t}_0). \tag{5.2}$$

Step I ($\mathbb{M}_{\alpha}(\mathrm{t}_0)$ is weakly compact).

Firstly, due to the positivity of $\mathbb{T}_{0}(\mathrm{t}_0)$ (see (3.6)) and then $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)$, it follows that $\mathbb{M}_{\alpha}(\mathrm{t}_0)$ is a positive operator.

Next, let $\varphi\in\mathscr{X}$ be such that $\varphi\geqslant 0$. Using (4.4) we can write, for almost all $(a,l)\in\Omega$, that

$$\mathbb{M}_{\alpha}(\mathrm{t}_0)\varphi(a,\ell)\leqslant\alpha\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\int_{\ell_1}^{\ell_2}\int_{0}^{\ell'}\mathbb{T}_{0}(\mathrm{t}_0-a)\varphi(a',\ell')\mathrm{d}a'\mathrm{d}\ell'$$

and by (3.6),

$$\mathbb{M}_{\alpha}(\mathrm{t}_0)\varphi(a,\ell)\leqslant\alpha\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\int_{\ell_1}^{\ell_2}\int_{0}^{\ell'}\chi(\mathrm{t}_0-a,a',\ell')\varphi(a'+a-\mathrm{t}_0,\ell')\mathrm{d}a'\mathrm{d}\ell'.$$

The change $x=a'+a-\mathrm{t}_0$ with $\mathrm{d}x=\mathrm{d}a'$ yields that

$$\mathbb{M}_{\alpha}(\mathrm{t}_0)\varphi(a,\ell)\leqslant\alpha\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\int_{\ell_1}^{\ell_2}\int_{a-\mathrm{t}_0}^{\ell'+a-\mathrm{t}_0}\chi(\mathrm{t}_0-a,x-a+\mathrm{t}_0,\ell')\varphi(x,\ell')\mathrm{d}x\mathrm{d}\ell'.$$

Due to (3.7) we get that $\chi(\mathrm{t}_0-a,x-a+\mathrm{t}_0,\ell')=1$ if and only if $x>0$ and therefore

$$\mathbb{M}_{\alpha}(\mathrm{t}_0)\varphi(a,\ell)\leqslant\alpha\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\int_{\ell_1}^{\ell_2}\int_{0}^{\ell'}\varphi(x,\ell')\mathrm{d}x\mathrm{d}\ell',$$

which can be written in the form

$$\mathbb{M}_{\alpha}(\mathrm{t}_0)\varphi\leqslant h\cdot\mathbb{1}\otimes\mathbb{1}\varphi, \tag{5.3}$$

where

$$\mathbb{1}\otimes\mathbb{1}\varphi:=\int_{\Omega}\varphi(a,l)\mathrm{d}a\mathrm{d}\ell\qquad\text{and}\qquad h(a,\ell):=\alpha\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell). \tag{5.4}$$

Using (3.10), we get that

$$\|h\|_{\mathscr{X}}=\alpha\int_{\ell_1}^{\ell_2}\int_{0}^{\ell}\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\mathrm{d}a\mathrm{d}\ell=\alpha\int_{\ell_1}^{\ell_2}\int_{0}^{\mathrm{t}_0}\kappa(\ell)\mathrm{d}a\mathrm{d}\ell=\alpha\mathrm{t}_0\|\kappa\|_{\mathscr{Y}} \tag{5.5}$$

which proves that $h \in \mathscr{X}$ because of $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)'$. Hence, the right hand side of (5.3) is a rank one operator in $\mathscr{X}$ and therefore $\mathbb{M}_\alpha(\mathrm{t}_0)\varphi$ is a weakly compact operator in $\mathscr{X}$ because of Lemma 2.1(2).

Step II ($\mathbb{N}_\alpha(\mathrm{t}_0)$ is weakly compact).

Firstly, due to the positivity of $\mathbb{T}_\alpha = (\mathbb{T}_\alpha(t))_{t\geqslant 0}$ (see [1, Theorem 4.4(2)]), the linear operator $\mathbb{N}_\alpha(\mathrm{t}_0)$ is positive.

Next, let $\varphi \in \mathscr{X}$ be such that $\varphi \geqslant 0$. Using (4.4) we can write, for almost all $(a,l) \in \Omega$, that

$$\mathbb{N}_\alpha(\mathrm{t}_0)\varphi(a,\ell) \leqslant \alpha^2\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\int_{\ell_1}^{\ell_2}\int_0^{\ell'}\int_{\ell_1}^{\ell_2}\int_0^{\ell''}\mathrm{k}(\ell',a'',\ell'') \\ \overline{\chi}(\mathrm{t}_0-a,a',\ell')\mathbb{T}_\alpha(\mathrm{t}_0-a-a')\varphi(a'',\ell'')\mathrm{d}a''\mathrm{d}\ell''\mathrm{d}a'\mathrm{d}\ell'.$$

The change $x = \mathrm{t}_0 - a - a'$ with $\mathrm{d}x = -\mathrm{d}a'$ yields that

$$\mathbb{N}_\alpha(\mathrm{t}_0)\varphi(a,\ell) \leqslant \alpha^2\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\int_{\ell_1}^{\ell_2}\int_0^{\ell''}\int_{\ell_1}^{\ell_2}\int_{\mathrm{t}_0-a-\ell'}^{\mathrm{t}_0-a}\mathrm{k}(\ell',a'',\ell'') \\ \overline{\chi}(\mathrm{t}_0-a,\mathrm{t}_0-a-x,\ell')\mathbb{T}_\alpha(x)\varphi(a'',\ell'')\mathrm{d}x\mathrm{d}\ell'\mathrm{d}a''\mathrm{d}\ell''.$$

Due to (3.10) we get that $\overline{\chi}(\mathrm{t}_0-a,\mathrm{t}_0-a-x,\ell') = 1$ if and only if $x>0$ and therefore

$$\mathbb{N}_\alpha(\mathrm{t}_0)\varphi(a,\ell) \leqslant \alpha^2\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\int_0^{\mathrm{t}_0}\int_{\ell_1}^{\ell_2}\int_0^{\ell''}\left[\int_{\ell_1}^{\ell_2}\mathrm{k}(\ell',a'',\ell'')\mathrm{d}\ell'\right] \\ \mathbb{T}_\alpha(x)\varphi(a'',\ell'')\mathrm{d}a''\mathrm{d}\ell''\mathrm{d}x$$

and by (3.2),

$$\begin{aligned}\mathbb{N}_\alpha(\mathrm{t}_0)\varphi(a,\ell) &\leqslant \alpha^2\overline{\kappa}\,\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\int_0^{\mathrm{t}_0}\int_{\ell_1}^{\ell_2}\int_0^{\ell''}\mathbb{T}_\alpha(x)\varphi(a'',\ell'')\mathrm{d}a''\mathrm{d}\ell''\mathrm{d}x \\ &= \alpha^2\overline{\kappa}\,\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\int_0^{\mathrm{t}_0}\|\mathbb{T}_\alpha(x)\varphi\|_{\mathscr{X}}\,\mathrm{d}x.\end{aligned}$$

From [1, Theorem 4.4], it follows that

$$\begin{aligned}\mathbb{N}_\alpha(\mathrm{t}_0)\varphi(a,\ell) &\leqslant \alpha^2\overline{\kappa}\,\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\int_0^{\mathrm{t}_0}e^{(\alpha\overline{\kappa})x}\|\varphi\|_{\mathscr{X}}\,\mathrm{d}x \\ &= \alpha\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\left(e^{(\alpha\overline{\kappa})\mathrm{t}_0}-1\right)\|\varphi\|_{\mathscr{X}} \\ &\leqslant \alpha e^{(\alpha\overline{\kappa})\mathrm{t}_0}\overline{\chi}(\mathrm{t}_0,a,\ell)\kappa(\ell)\|\varphi\|_{\mathscr{X}}\end{aligned}$$

which can be written in the form,

$$\mathbb{N}_\alpha(\mathrm{t}_0)\varphi \leqslant e^{(\alpha\overline{\kappa})\mathrm{t}_0}h\cdot\mathbb{1}\otimes\mathbb{1}\varphi, \tag{5.6}$$

where $\mathbb{1}\otimes\mathbb{1}\varphi$ and h are given by (5.4). Due to $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)'$, we infer that $e^{(\alpha\overline{\kappa})\mathrm{t}_0}$ is finite (see Remark 4.3) and therefore $e^{(\alpha\overline{\kappa})\mathrm{t}_0}h \in \mathscr{X}$ because of (5.5). Hence, the right hand side of (5.6) is a rank one operator in $\mathscr{X}$ and therefore $\mathbb{N}_{\alpha}(\mathrm{t}_0)$ is a weakly compact operator in $\mathscr{X}$ because of Lemma 2.1(2).

Step III (Proof of (5.1)).

Both previous steps yield that $\mathbb{T}_{\alpha}(\mathrm{t}_0) - \mathbb{T}_0(\mathrm{t}_0)$ a weakly compact operator in $\mathscr{X}$ because of (5.2). Therefore $\mathbb{V}_{\alpha}(\mathrm{t}_0) - \mathbb{V}_0(\mathrm{t}_0)$ is also a weakly compact operator in $\mathscr{X}$ because of (3.19) and Lemma 2.1(2). Finally, Lemma 2.2 leads to $\omega_{\mathrm{ess}}(\mathbb{V}_{\alpha}) = \omega_{\mathrm{ess}}(\mathbb{V}_0)$ which proves (5.1) and ends the proof. ∎

In the sequel, we need the following assumption

$$\left(\mathbf{A}_{\eta}^{1}\right)' \qquad \widehat{\eta} \in \mathscr{X}$$

where

$$\widehat{\eta}(a,\ell) := \sup_{(a',\ell')\in\Omega} |\eta(a,\ell,a',\ell')|, \qquad (a,\ell)\in\Omega. \tag{5.7}$$

Remark 5.2. The assumption $\left(\mathbf{A}_{\eta}^{1}\right)'$ is stronger that $\left(\mathbf{A}_{\eta}^{1}\right)$ because of

$$\overline{\eta} = \operatorname*{ess\,sup}_{(a',\ell')\in\Omega} \int_{\ell_1}^{\ell_2}\int_0^{\ell} |\eta(a,\ell,a',\ell')|\,\mathrm{d}a\mathrm{d}\ell \leqslant \int_{\ell_1}^{\ell_2}\int_0^{\ell} \widehat{\eta}(a,\ell)\mathrm{d}a\mathrm{d}\ell = \|\widehat{\eta}\|_{\mathscr{X}} < \infty.$$

This means that, if $\left(\mathbf{A}_{\eta}^{1}\right)'$ holds true then all the previous results related to $\left(\mathbf{A}_{\eta}^{1}\right)$ still hold true.

Lemma 5.3. *Suppose that $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)$, $\left(\mathbf{A}_{\mu}^{1}\right)$ and $\left(\mathbf{A}_{\eta}^{1}\right)'$ hold and let $\alpha \geqslant 0$. Then, both C_0-semigroup $\mathbb{W}_{\alpha} = (\mathbb{W}_{\alpha}(t))_{t\geqslant 0}$ and $\mathbb{V}_{\alpha} = (\mathbb{V}_{\alpha}(t))_{t\geqslant 0}$ have the same essential type, i.e.,*

$$\omega_{\mathrm{ess}}(\mathbb{W}_{\alpha}) = \omega_{\mathrm{ess}}(\mathbb{V}_{\alpha}). \tag{5.8}$$

Proof. Let $\varphi \in \mathscr{X}$ be such that $\varphi \geqslant 0$. From (3.14) we can write, for almost all $(a,\ell)\in\Omega$, that

$$\left|\mathrm{P}_{\eta}\varphi(a,\ell)\right| \leqslant \int_{\ell_1}^{\ell_2}\int_0^{\ell'} \eta(a,\ell,a',\ell')\,|\varphi(a',\ell')|\,\mathrm{d}a'\mathrm{d}\ell'$$

which leads, by virtue of (5.7), to

$$\left|\mathrm{P}_{\eta}\varphi(a,\ell)\right| \leqslant \widehat{\eta}(a,\ell)\int_{\ell_1}^{\ell_2}\int_0^{\ell'} \varphi(a',\ell')\mathrm{d}a'\mathrm{d}\ell'$$

and therefore

$$\left|\mathrm{P}_{\eta}\varphi\right| \leqslant \widehat{\eta}\cdot\mathbb{1}\otimes\mathbb{1}\varphi \tag{5.9}$$

where $\mathbb{1}\otimes\mathbb{1}\varphi$ is defined by (5.4). Now, $\left(\mathbf{A}_{\eta}^{1}\right)'$ yields that the right hand side of (5.9) is a rank one operator in $\mathscr{X}$ and therefore P_{η} is a weakly compact operator in $\mathscr{X}$ because of Lemma 2.1(3). Finally, Lemma 2.3 leads to $\omega_{\mathrm{ess}}(\mathbb{W}_{\alpha}) = \omega_{\mathrm{ess}}(\mathbb{V}_{\alpha})$ which proves (5.8) and ends the proof. ∎

Now we are able to state the aim of this section as follows

Theorem 5.4. *Suppose that* $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)'$, $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)$, $\left(\mathbf{A}_{\mu}^{1}\right)$, $\left(\mathbf{A}_{\mu}^{2}\right)$ *and* $\left(\mathbf{A}_{\eta}^{1}\right)'$ *hold and let* $\alpha \geqslant 0$. *Then the essential type,* $\omega_{\text{ess}}\left(\mathbb{W}_{\alpha}\right)$, *of the* C_0*−semigroup* $\mathbb{W}_{\alpha} = \left(\mathbb{W}_{\alpha}(t)\right)_{t\geqslant 0}$ *satisfies,*

$$\omega_{\text{ess}}\left(\mathbb{W}_{\alpha}\right) \leqslant -\underline{\mu}, \tag{5.10}$$

where $\underline{\mu}$ *is defined by* (3.21).

Proof. This follows from (5.1), (5.8) and (3.30). ■

6 Asynchronous Growth Property

This section deals with the *Asynchronous Growth Property* of the whole cell population. This can be done by descrying the asymptotic behavior of the full generated semigroup $\mathbb{W}_{\alpha} = \left(\mathbb{W}_{\alpha}(t)\right)_{t\geqslant 0}$ (like in Lemma 1.1) as follows

Theorem 6.1. *Suppose that* $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)'$, $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)$, $\left(\mathbf{A}_{\mu}^{1}\right)$, $\left(\mathbf{A}_{\mu}^{2}\right)$, $\left(\mathbf{A}_{\eta}^{1}\right)'$ *and* $\left(\mathbf{A}_{\eta}^{2}\right)$ *hold and let* $\alpha > 0$. *Let* $\lambda_\mu := \overline{\mu} - \underline{\mu}$ *where* $\overline{\mu}$ *and* $\underline{\mu}$ *are defined by* (3.15) *and* (3.21) *respectively. If* $\lambda_\mu > 0$ *and the spectral radius* $\mathrm{r}\left(\mathcal{K}_{\alpha,\lambda_\mu}\right) > 1$, *then there exist a rank one projector* $\mathbb{P}$ *into* $\mathscr{X}$ *and an* $\varepsilon > 0$ *such that: for any* $\eta \in (0,\varepsilon)$, *there exists* $M_\eta \geqslant 1$ *satisfying*

$$\left\| e^{-\omega_0(\mathbb{W}_\alpha)t}\mathbb{W}_{\alpha}(t) - \mathbb{P}\right\|_{\mathscr{L}(\mathscr{X})} \leqslant M_\eta e^{-\eta t}, \qquad t \geqslant 0.$$

Proof. The C_0−semigroup $\mathbb{W}_{\alpha} = \left(\mathbb{W}_{\alpha}(t)\right)_{t\geqslant 0}$ is positive and irreducible ([1, Theorem 7.2]) satisfying $\omega_{\text{ess}}\left(\mathbb{W}_{\alpha}\right) < \omega_0\left(\mathbb{W}_{\alpha}\right)$ because of (4.23) and (5.10). All the required conditions of Lemma 1.1 are now fulfilled. ■

Remark 6.2. Theorem 6.1 allows us to say that the whole cell population described by the full model (1.1)–(1.2) possesses the *Asynchronous Growth Property.* In other words, the unique solution

$$f(t) := \mathbb{W}_{\alpha}(t)f(0) \in \mathscr{X} \qquad t \geqslant 0$$

of the full model (1.1)–(1.2) fulfills

$$\left\| e^{-\omega_0(\mathbb{W}_\alpha)t}f(t) - \mathbb{P}f(0)\right\|_{\mathscr{X}} \leqslant M_\eta e^{-\eta t}\|f(0)\|_{\mathscr{X}} \qquad t \geqslant 0$$

for all initial data $f(0) \in X$.

7 Comments

Remark 7.1. The choice of the functional framework $\mathscr{X}$ $(= L^1(\Omega))$ was natural because $\|f(t)\|_{\mathscr{X}}$ denotes the number of cells at time $t \geqslant 0$. Nevertheless, we can easily extended this work to the case $p > 1$, *i.e.,*

$$\mathscr{X} := L^p(\Omega) \qquad \text{whose norm is} \qquad \|\varphi\|_{\mathscr{X}} := \left[\int_{\ell_1}^{\ell_2}\int_0^{\ell} |\varphi(a,\ell)|^p \,\mathrm{d}a\mathrm{d}\ell\right]^{\frac{1}{p}}$$

and $\mathscr{Y} := L^p(\ell_1,\ell_2)$ whose norm is $\|\psi\|_{\mathscr{Y}} := \left[\int_{\ell_1}^{\ell_2} |\psi(\ell)|^p \,\mathrm{d}\ell\right]^{\frac{1}{p}}.$

To this end, it suffices to replace (3.2) by

$$\overline{\kappa} = \left[\operatorname*{ess\,sup}_{(a',\ell')\in\Omega} \int_{\ell_1}^{\ell_2} |\mathrm{k}(\ell,a',\ell')|\,\mathrm{d}\ell\right]^{\frac{1}{p}} \left[\operatorname*{ess\,sup}_{\ell\in(\ell_1,\ell_2)} \int_{\ell_1}^{\ell_2}\int_0^{\ell'} |\mathrm{k}(\ell,a',\ell')|\,\mathrm{d}a'\mathrm{d}\ell'\right]^{\frac{1}{q}}$$

and (3.16) by

$$\overline{\eta} = \left[\sup_{(a',\ell')\in\Omega} \int_{\ell_1}^{\ell_2}\int_0^{\ell} |\eta(a,\ell,a',\ell')|\,\mathrm{d}a\mathrm{d}\ell\right]^{\frac{1}{p}} \left[\sup_{(a,\ell)\in\Omega} \int_{\ell_1}^{\ell_2}\int_0^{\ell'} |\eta(a,\ell,a',\ell')|\,\mathrm{d}a'\mathrm{d}\ell'\right]^{\frac{1}{q}}$$

where $q > 1$ is the conjugate of $p > 1$, *i.e.,* $\frac{1}{p} + \frac{1}{q} = 1$. The other assumptions remain unchanged.

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References

[1] M. Boulanouar. *Mathematical Analysis of Cycle Length-Age Structured Cell Population with Aggregate Transition Rule: Well-Posedness*. Preprint.

[2] G. Greiner. *Spectral Properties and Asymptotic behavior of Linear Transport Equation*. Math. Z., 185, 167–177, 1984.

[3] B. Pagter *Irreducible Compact Operators*. Math. Z. 192, pp.149-153. 1986.

[4] J. Voigt. *A Perturbation Theorem for the Essential Spectral Radius of Strongly Continuous Semigroups*. Mh. Math. 90. pp.153–161. 1980.

[5] J. Voigt. *Stability of the essential type of strongly continuous semigroups*. Trudy Mat. Inst. Steklov, Izbran. Voprosy., Mat.Fiz.Anal.,203, 469–477, 1994.

[6] Ph. Clément and al. *One-Parameter Semigroups*. North-Holland. Amerterdam, New York. 1987.