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MATHEMATICAL ANALYSIS OF CYCLE LENGTH-AGE STRUCTURED CELL POPULATION WITH AGGREGATE TRANSITION RULE: WELL-POSEDNESS

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Abstract. In this work, we analyze a mathematical model of a structured cell population. Each cell is distinguished by its cycle length and by its age. The daughter cells are correlated to the whole cell population thanks to the *Aggregate Transition Rule*. We prove then that this mathematical model is governed by a C_0−semigroup and we investigate some of its properties.

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1 Introduction

In this work, we are concerned with a mathematical model of structured cell population in which each cell is distinguished by its cell cycle length ℓ and by its age a. If $f = f(t, a, \ell)$ denotes, at time t, the cell density with respect to the age a and the cell cycle length ℓ, then the cell population is governed by the following equation

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial a} - \mu f + \int_{\ell_1}^{\ell}\int_0^{\ell'} \eta(a, \ell, a', \ell') f(t, a', \ell') \mathrm{d}a' \mathrm{d}\ell', \tag{1.1}$$

where $\mu = \mu(a, \ell)$ stands for the *cell mortality rate* and $\eta(a, \ell, a', \ell')$ denotes the *transition rate* in which cells change their cell cycle length from ℓ' to ℓ and their age from a' to a.

The cell cycle length ℓ ($0 \leqslant \ell_1 < \ell < \ell_2 \leqslant \infty$) of each cell, is the time between its birth and its division, where ℓ_1 and ℓ_2 are fixed. The age a of each cell is null ($a = 0$) at birth and equals to its cell cycle length ℓ ($a = \ell$) at division. Between birth and division, we always have $0 \leqslant a \leqslant \ell$.

The master equation (1.1) must be endowed with boundary conditions describing, at each mitosis, how daughter cells are related to other cells. To this end, we propose two biological approaches whose first one is

$(\mathcal{A}_1)$: *we suppose, at each mitosis, that a daughter cell is always related to his mother cell.*

This approach means that there exists a correlation $\theta = \theta(\ell, \ell')$ between the cell cycle length, ℓ, of a daughter cell and that of his mother, ℓ'. This one, called *Biological Transition Rule*, is mathematically described by

$$f(t, 0, \ell) = \beta \int_{\ell_1}^{\ell_2} \theta(\ell, \ell') f(t, \ell', \ell') \mathrm{d}\ell', \qquad \ell \in (\ell_1, \ell_2), \tag{1.2}$$

where $\beta \geqslant 0$ denotes the *average number* of daughter cells per mitosis.

The model (1.1), (1.2) has already been studied in [10] but only when $0 < \ell_1 < \ell_2 < \infty$ and then in [4, 5, 6] when $0 \leqslant \ell_1 < \ell_2 < \infty$. The limit case, $0 = \ell_1 < \ell < \ell_2 = \infty$, has also been considered in [2, 3]. We have then proved that the model (1.1), (1.2) is governed by a C_0-semigroup and then we have investigated its *Asynchronous Growth property*.

The second approach is,

$(\mathcal{A}_2)$: *we suppose, at each mitosis, a daughter cell is viewed as a fraction of the whole cell population.*

This approach means that there exists a correlation $\mathrm{k} = \mathrm{k}(\ell, a', \ell')$ between a daughter cell and the rest of the cell population. We mathematically describe this correlation (which we call : *Aggregate Transition Rule*) by

$$f(t, 0, \ell) = \alpha \int_{\ell_1}^{\ell_2}\int_0^{\ell'} \mathrm{k}(\ell, a', \ell') f(t, a', \ell') \mathrm{d}a' \mathrm{d}\ell', \qquad \ell \in (\ell_1, \ell_2). \tag{1.3}$$

We call $\alpha \geqslant 0$, the *fractional number* of daughter cells viable per mitosis.

When the cell population is without mitosis, *i.e.,* $\alpha = \beta = 0$, then both approaches ($\mathcal{A}_1$) and ($\mathcal{A}_2$) are the same and therefore (1.2) and (1.3) are reduced to

$$f(t, 0, \ell) = 0, \qquad \ell \in (\ell_1, \ell_2), \tag{1.4}$$

which is biologically irrelevant.

However, at each mitosis, both approaches ($\mathcal{A}_1$) and ($\mathcal{A}_2$) are fundamentally different whenever $\alpha \neq 0$ and $\beta \neq 0$, and therefore, the mathematical treatment of the boundary condition (1.2) (see [2, 3, 4, 5, 6, 10]) can not anyway be applied, in this work, to the boundary condition (1.3).

To the best of our knowledge, the boundary conditions (1.3) have never been considered in connection with the master equation (1.1). This obviously shows the novelty of the full model (1.1), (1.3) and motivates this work.

Our challenge, in this work, is to study the well posedness of the full model (1.1), (1.3). So, in Section 2, we recall some well-known theoretical results that we will use throughout this work. In Section 3, we define the mathematical setting related to the full model (1.1), (1.3), and we prove some properties of the linear operator defined by the right hand side of (1.3).

In Section 4, we firstly prove a useful trace result which allows us to define all unbounded linear operators of this work. Next, according to suitable assumptions on the kernel k, we prove that the unperturbed model (1.1)–(with $\mu = \eta = 0$), (1.3) is governed by a C_0-semigroup. The explicit form of that semigroup (see Section 5) will be very useful to describe the asynchronous exponential growth related to the full model (1.1), (1.3). This one will be the subject of a forthcoming work ([7]).

In Section 6, we introduce the first linear perturbation and we prove that the unperturbed model (1.1)–(with $\eta = 0$), (1.3) is governed by a C_0-semigroup. Finally, in Section 7, we introduce the second linear perturbation and we prove that the full model (1.1), (1.3) can be inferred as a linear perturbation of the unperturbed model (1.1)–(with $\mu = \eta = 0$), (1.3) (see Section 4) or (1.1)–(with $\eta = 0$), (1.3) (see Section 6). In both cases, we prove that the full model (1.1), (1.3) is governed by a C_0-semigroup. For the positivity and irreducibility of the generated semigroup are also investigated. We end this work by some useful remarks.

For the novelty of this subject and for the mathematical background we refer the readers [1, 8, 9].

2 Mathematical tools

This section deals with some well-known theoretical results that we will use throughout this work. Let then $\mathscr{X}$ be a Banach space whose norm is $\|\cdot\|_{\mathscr{X}}$.

Lemma 2.1 ([9, Th.II.3.8]). *Let* $(\mathrm{U}, \mathrm{D}(\mathrm{U}))$ *be a closed and densely defined linear operator on* $\mathscr{X}$. *Suppose that there exist* $\omega \in \mathbb{R}$ *and* $M \geqslant 1$ *such that,* $(\omega, \infty) \subset \rho(\mathrm{U})$ *(resolvent set) and if* $\lambda > \omega$, *then for all* $x \in \mathscr{X}$

$$\left\|(\lambda - \mathrm{U})^{-n} x\right\|_{\mathscr{X}} \leqslant M (\lambda - \omega)^{-n} \|x\|_{\mathscr{X}} \qquad n = 1, 2, \cdots .$$

Then U *generates on* $\mathscr{X}$ *a* C_0-*semigroup* $\mathbb{U}=(\mathbb{U}(t))_{t\geqslant 0}$ *satisfying, for* $x \in \mathscr{X}$,

$$\|\mathbb{U}(t)x\|_{\mathscr{X}} \leqslant Me^{\omega t}\|x\|_{\mathscr{X}} \qquad t \geqslant 0, \tag{2.1}$$

and,
$$\mathbb{U}(t)x = \lim_{n\to\infty}\left(\frac{n}{t}\left(\frac{n}{t} - \mathrm{U}\right)^{-1}\right)^n x \qquad t \geqslant 0. \tag{2.2}$$

Lemma 2.2 ([9, Th.III.1.3]). *Let* $(\mathrm{U}, \mathrm{D}(\mathrm{U}))$ *be the infinitesimal generator of a* C_0-*semigroup* $\mathbb{U} = (\mathbb{U}(t))_{t\geqslant 0}$, *on* $\mathscr{X}$, *satisfying* (2.1) *and let* B *be a bounded linear operator from* $\mathscr{X}$ *into itself. Then* $(\mathrm{U} + \mathrm{B}, \mathrm{D}(\mathrm{U}))$ *generates, on* $\mathscr{X}$, *a* C_0-*semigroup* $\mathbb{V} = (\mathbb{V}(t))_{t\geqslant 0}$ *satisfying, for all* $x \in \mathscr{X}$,

$$\|\mathbb{V}(t)x\|_{\mathscr{X}} \leqslant Me^{\left(\omega+M\|B\|\right)t}\|x\|_{\mathscr{X}} \qquad t \geqslant 0, \tag{2.3}$$

and,
$$\mathbb{V}(t)x = \lim_{n\to\infty}\left(e^{\frac{t}{n}B}\mathbb{U}\left(\tfrac{t}{n}\right)\right)^n x \qquad t \geqslant 0. \tag{2.4}$$

Lemma 2.3 ([9, Th.II.2.7]). *Let* $(\mathrm{U}, \mathrm{D}(\mathrm{U}))$ *be the infinitesimal generator of a* C_0-*semigroup of contractions, on* $\mathscr{X}$, *and let* B *be a linear dissipative operator such that,* $\mathrm{D}(\mathrm{U}) \subset \mathrm{D}(\mathrm{B}) \subset \mathscr{X}$. *Suppose that there exist* $0 \leqslant a < 1$ *and* $b \geqslant 0$ *such that*

$$\|\mathrm{B}x\|_X \leqslant a\|\mathrm{U}x\|_X + b\|x\|_X \qquad x \in \mathrm{D}(\mathrm{U}).$$

Then $(\mathrm{U} + \mathrm{B}, \mathrm{D}(\mathrm{U}))$ *generates, on* $\mathscr{X}$, *a* C_0-*semigroup of contractions.*

Now, we suppose that $\mathscr{X}$ is a Banach lattice space.

Lemma 2.4 ([8, Prop.7.1 and 7.6]). *Let* $(\mathrm{U}, \mathrm{D}(\mathrm{U}))$ *be the infinitesimal generator of a* C_0-*semigroup* $\mathbb{U} = (\mathbb{U}(t))_{t\geqslant 0}$ *on* $\mathscr{X}$. *Then*

(1) $\mathbb{U} = (\mathbb{U}(t))_{t\geqslant 0}$ *is positive if and only if* $(\lambda - \mathrm{U})^{-1}$ *is positive for large* λ.

(2) *Suppose that* $\mathbb{U} = (\mathbb{U}(t))_{t\geqslant 0}$ *is positive. Then* $\mathbb{U} = (\mathbb{U}(t))_{t\geqslant 0}$ *is irreducible if and only if* $(\lambda - \mathrm{U})^{-1}$ *is irreducible for large* λ.

3 Preparative Results

Let ℓ_1 and ℓ_2 be such that $0 \leqslant \ell_1 < \ell_2 \leqslant \infty$ and let

$$\Omega := \Big\{(a, \ell) \quad : \quad 0 < a < \ell \qquad \text{and} \qquad \ell_1 < \ell < \ell_2\Big\}.$$

Let $\mathscr{X}$ and $\mathscr{Y}$ be the following Banach spaces

$$\mathscr{X} := L^1(\Omega) \qquad \text{whose norm is} \qquad \|\varphi\|_{\mathscr{X}} := \int_{\ell_1}^{\ell_2}\int_0^{\ell} |\varphi(a, \ell)| \, \mathrm{d}a\mathrm{d}\ell$$

and $\mathscr{Y} := L^1(\ell_1, \ell_2)$ whose norm is $\|\psi\|_{\mathscr{Y}} := \int_{\ell_1}^{\ell_2} |\psi(\ell)|\,\mathrm{d}\ell.$

Since the boundary condition (1.3) we can define the following linear operator

$$\mathrm{K}_\alpha \varphi(\ell) := \alpha \int_{\ell_1}^{\ell_2}\int_0^{\ell'} \mathrm{k}(\ell, a', \ell')\varphi(a', \ell')\mathrm{d}a'\mathrm{d}\ell', \qquad \ell \in (\ell_1, \ell_2), \tag{3.1}$$

where $\alpha \geqslant 0$ stands for the fractional number of daughter cells viable per mitosis while $\mathrm{k} = \mathrm{k}(\ell, a', \ell')$ denotes the correlation between daughter cells and the whole cell population. The kernel k is subject to the following assumptions

$$\left(\mathbf{A}_{\mathrm{k}}^1\right) \qquad \overline{\kappa} := \operatorname*{ess\,sup}_{(a',\ell')\in\Omega} \int_{\ell_1}^{\ell_2} |\mathrm{k}(\ell, a', \ell')|\,\mathrm{d}\ell < \infty, \tag{3.2}$$

$$\left(\mathbf{A}_{\mathrm{k}}^2\right) \qquad \mathrm{k} \text{ is positive,}$$

$$\left(\mathbf{A}_{\mathrm{k}}^2\right)' \qquad \mathrm{k}(\ell, a', \ell') > 0 \quad \text{for a.e.} \quad (\ell, a', \ell') \in (\ell_1, \ell_2) \times \Omega.$$

Some properties of the operator K_α can be formulated as follows,

Lemma 3.1. *Let $\alpha \geqslant 0$. If $\left(\mathbf{A}_{\mathrm{k}}^1\right)$ holds, then K_α is a bounded linear operator from $\mathscr{X}$ into $\mathscr{Y}$ satisfying, for all $\varphi \in \mathscr{X}$,*

$$\|\mathrm{K}_\alpha \varphi\|_{\mathscr{Y}} \leqslant \alpha\overline{\kappa}\|\varphi\|_{\mathscr{X}} \tag{3.3}$$

where $\overline{\kappa}$ is defined by (3.2)*. Furthermore,*

(1) *if $\left(\mathbf{A}_{\mathrm{k}}^2\right)$ holds, then K_α is positive in $\mathscr{Y}$.*

(2) *if $\left(\mathbf{A}_{\mathrm{k}}^2\right)'$ holds and $\alpha > 0$, then K_α is strongly positive in $\mathscr{Y}$ $\big($i.e., if $\varphi \geqslant 0$ and $\varphi \neq 0$, then $\mathrm{K}_\alpha\varphi(\ell) > 0$ for almost all $\ell \in (\ell_1, \ell_2)\big)$.*

Proof. Let $\varphi \in \mathscr{X}$. Since (3.1), we can write that

$$\begin{aligned}
\|\mathrm{K}_\alpha \varphi\|_{\mathscr{Y}} &= \alpha \int_{\ell_1}^{\ell_2} \left| \int_{\ell_1}^{\ell_2}\int_0^{\ell'} \mathrm{k}(\ell, a', \ell')\varphi(a', \ell')\mathrm{d}a'\mathrm{d}\ell' \right| \mathrm{d}\ell \\
&\leqslant \alpha \int_{\ell_1}^{\ell_2}\int_0^{\ell'} \left[\int_{\ell_1}^{\ell_2} |\mathrm{k}(\ell, a', \ell')|\,\mathrm{d}\ell\right] |\varphi(a', \ell')|\,\mathrm{d}a'\mathrm{d}\ell' \\
&\leqslant \alpha \left[\operatorname*{ess\,sup}_{(a',\ell')\in\Omega} \int_{\ell_1}^{\ell_2} |\mathrm{k}(\ell, a', \ell')|\,\mathrm{d}\ell\right] \int_{\ell_1}^{\ell_2}\int_0^{\ell'} |\varphi(a', \ell')|\,\mathrm{d}a'\mathrm{d}\ell' \\
&= \alpha \qquad\qquad \overline{\kappa} \qquad\qquad\qquad \|\varphi\|_{\mathscr{X}}
\end{aligned}$$

which proves (3.3) and therefore K_α is bounded from $\mathscr{X}$ into $\mathscr{Y}$ because of $\left(\mathbf{A}_{\mathrm{k}}^1\right)$. Finally, both assertions (1) and (2) are obvious. ■

Let $\mathcal{K}_{\alpha,\lambda}$ $(\lambda > 0)$ be the following linear operator

$$\mathcal{K}_{\alpha,\lambda}\varphi(a,\ell) = e^{-\lambda a}\mathrm{K}_{\alpha}\varphi(\ell), \qquad (a,\ell) \in \Omega, \tag{3.4}$$

whose explicit form is

$$\mathcal{K}_{\alpha,\lambda}\varphi(a,\ell) := \alpha e^{-\lambda a}\int_{\ell_1}^{\ell_2}\int_0^{\ell'} \mathrm{k}(\ell,a',\ell')\varphi(a',\ell')\mathrm{d}a'\mathrm{d}\ell', \qquad (a,\ell) \in \Omega. \tag{3.5}$$

As we are going to see in the sequel, the operator $\mathcal{K}_{\alpha,\lambda}$ will play an important role. Therefore, it is natural to give some of its properties.

Lemma 3.2. *Let $\alpha \geqslant 0$. If $\left(\mathbf{A}_{\mathrm{k}}^1\right)$ holds, then $\mathcal{K}_{\alpha,\lambda}$ $(\lambda > 0)$ is a bounded linear operator from $\mathscr{X}$ into itself satisfying, for all $\varphi \in \mathscr{X}$,*

$$\left\|\mathcal{K}_{\alpha,\lambda}\varphi\right\|_{\mathscr{X}} \leqslant \frac{\alpha\overline{\kappa}}{\lambda}\|\varphi\|_{\mathscr{X}} \tag{3.6}$$

and

$$\left\|\frac{1}{\ell}\mathcal{K}_{\alpha,\lambda}\varphi\right\|_{\mathscr{X}} \leqslant \alpha\overline{\kappa}\|\varphi\|_{\mathscr{X}} \tag{3.7}$$

where $\overline{\kappa}$ is defined by (3.2)*. Furthermore,*

(1) *if $\left(\mathbf{A}_{\mathrm{k}}^2\right)$ holds, then $\mathcal{K}_{\alpha,\lambda}$ is positive in $\mathscr{X}$.*

(2) *if $\left(\mathbf{A}_{\mathrm{k}}^2\right)'$ holds and $\alpha > 0$, then $\mathcal{K}_{\alpha,\lambda}$ is strongly positive in $\mathscr{X}$.*

Proof. Let $\lambda > 0$ and let $\varphi \in \mathscr{X}$. From (3.4) we can write that

$$\left\|\mathcal{K}_{\alpha,\lambda}\varphi\right\|_{\mathscr{X}} = \int_{\ell_1}^{\ell_2}\left[\int_0^{\ell} e^{-\lambda a}\mathrm{d}a\right]|\mathrm{K}_{\alpha}\varphi(\ell)|\,\mathrm{d}\ell \leqslant \frac{1}{\lambda}\|\mathrm{K}_{\alpha}\varphi\|_{\mathscr{Y}}$$

which leads, by virtue of (3.3), to the desired (3.6) and therefore $\mathcal{K}_{\alpha,\lambda}$ is bounded from $\mathscr{X}$ into itself because of $\left(\mathbf{A}_{\mathrm{k}}^1\right)$. Similarly,

$$\left\|\frac{1}{\ell}\mathcal{K}_{\alpha,\lambda}\varphi\right\|_{\mathscr{X}} = \int_{\ell_1}^{\ell_2}\left[\frac{1}{\ell}\int_0^{\ell} e^{-\lambda a}\mathrm{d}a\right]|\mathrm{K}_{\alpha}\varphi(\ell)|\,\mathrm{d}\ell \leqslant \|\mathrm{K}_{\alpha}\varphi\|_{\mathscr{Y}}$$

which leads, by virtue of (3.3), to the desired (3.7). Finally, both assertions (1) and (2) are obvious. ∎

4 The model (1.1)–(with $\mu = \eta = 0$), (1.3)

In this section we focus our attention on the model (1.1), (1.3) without cell mortality (*i.e.,* $\mu = 0$) and without cell transition (*i.e.,* $\eta = 0$). More precisely, we will prove that the unperturbed model (1.1)–(with $\mu = \eta = 0$), (1.3) is governed by a C_0−semigroup on $\mathscr{X}$. Let then T_{α} be the following unbounded linear operator

$$\begin{aligned} \mathrm{T}_{\alpha}\varphi &:= -\frac{\partial\varphi}{\partial a} \qquad \text{on the domain} \\ \mathrm{D}(\mathrm{T}_{\alpha}) &:= \left\{\varphi \in \mathscr{W} \quad : \quad \varphi(0,\cdot) = \mathrm{K}_{\alpha}\varphi\right\} \end{aligned} \tag{4.1}$$

where K_α is the linear operator defined by (3.1) and

$$\mathscr{W} := \left\{ \varphi \in \mathscr{X} \quad : \quad \frac{\partial \varphi}{\partial a} \in \mathscr{X} \quad \text{and} \quad \frac{1}{\ell}\varphi \in \mathscr{X} \right\}$$

normed by
$$\|\varphi\|_{\mathscr{W}} := \|\varphi\|_{\mathscr{X}} + \left\| \frac{\partial \varphi}{\partial a} \right\|_{\mathscr{X}} + \left\| \frac{1}{\ell}\varphi \right\|_{\mathscr{X}}.$$

Lemma 4.1. *Let $\alpha \geqslant 0$. The unbounded linear operator T_α is well defined.*

Proof. Let us prove that the definition of the domain $\mathrm{D}(\mathrm{T}_\alpha)$ makes a sense. So, as the linear operator K_α is acting in $\mathscr{Y}$ (Lemma 3.1), then it suffices to prove that $\varphi(0,\cdot) \in \mathscr{Y}$ whenever $\varphi \in \mathscr{W}$.

Let $\varphi \in \mathscr{W}$. For almost all $(a,\ell) \in \Omega$ we have,

$$\begin{aligned}
|\varphi(0,\ell)| &= |\varphi(a,\ell)| - \int_0^a \frac{\partial\,|\varphi|}{\partial a}(s,\ell)\mathrm{d}s \\
&= |\varphi(a,\ell)| - \int_0^a \left(\operatorname{sgn}\varphi \frac{\partial \varphi}{\partial a} \right)(s,\ell)\mathrm{d}s \\
&\leqslant |\varphi(a,\ell)| + \int_0^\ell \left| \frac{\partial \varphi}{\partial a}(s,\ell) \right| \mathrm{d}s.
\end{aligned}$$

Integrating this with respect to $a \in (0,\ell)$ and then with respect to $\ell \in (\ell_1,\ell_2)$,

$$\int_{\ell_1}^{\ell_2} |\varphi(0,\ell)|\,\mathrm{d}\ell \leqslant \int_{\ell_1}^{\ell_2}\int_0^\ell \frac{1}{\ell}\,|\varphi(a,\ell)|\,\mathrm{d}a\mathrm{d}\ell + \int_{\ell_1}^{\ell_2}\int_0^\ell \left| \frac{\partial \varphi}{\partial a}(s,\ell) \right| \mathrm{d}s\mathrm{d}\ell$$

which can be written in the form

$$\|\varphi(0,\cdot)\|_{\mathscr{Y}} \leqslant \left\| \frac{1}{\ell}\varphi \right\|_{\mathscr{X}} + \left\| \frac{\partial \varphi}{\partial a} \right\|_{\mathscr{X}} \leqslant \|\varphi\|_{\mathscr{W}} < \infty$$

and therefore $\varphi(0,\cdot) \in \mathscr{Y}$. ■

Before we study the unbounded linear operator T_α we need to state some properties of

$$\begin{aligned}
\mathrm{T}_0\varphi &:= -\frac{\partial \varphi}{\partial a} \qquad \text{on the domain} \\
\mathrm{D}(\mathrm{T}_0) &:= \left\{ \varphi \in \mathscr{W} \quad : \quad \varphi(0,\cdot) = 0 \right\}
\end{aligned} \tag{4.2}$$

which is nothing else but a special case of T_α (see (4.1)) corresponding to $\alpha = 0$ (the case of no mitosis). From Lemma 4.1 (with $\alpha = 0$) we obviously infer that T_0 is well defined. Now useful properties of T_0 are given as follows

Lemma 4.2. *The following assertions hold.*

(1) *Let* $\lambda > 0$. *Then* $(\lambda - \mathrm{T}_0)^{-1}$ *is a bounded linear operator from* $\mathscr{X}$ *into itself satisfying, for all* $g \in \mathscr{X}$,

$$\left\|(\lambda - \mathrm{T}_0)^{-1} g\right\|_{\mathscr{X}} \leqslant \frac{1}{\lambda}\|g\|_{\mathscr{X}} \tag{4.3}$$

and
$$\left\|\frac{1}{\ell}(\lambda - \mathrm{T}_0)^{-1} g\right\|_{\mathscr{X}} \leqslant \|g\|_{\mathscr{X}}. \tag{4.4}$$

(2) *Let* $\lambda > 0$. *Then* $(\lambda - \mathrm{T}_0)^{-1}$ *is strictly positive in* $\mathscr{X}$ $\big($*i.e., if* $g \geqslant 0$ *and* $g \neq 0$ *then* $(\lambda - \mathrm{T}_0)^{-1} g \geqslant 0$ *and* $(\lambda - \mathrm{T}_0)^{-1} g \neq 0\big)$.

(3) T_0 *generates, on* $\mathscr{X}$, *a positive* C_0-*semigroup* $\mathbb{T}_0 = (\mathbb{T}_0(t))_{t\geqslant 0}$ *of contractions, i.e., for all* $\varphi \in \mathscr{X}$,

$$\|\mathbb{T}_0(t)\varphi\|_{\mathscr{X}} \leqslant \|\varphi\|_{\mathscr{X}} \qquad t \geqslant 0. \tag{4.5}$$

(4) *The* C_0-*semigroup* $\mathbb{T}_0 = (\mathbb{T}_0(t))_{t\geqslant 0}$ *is defined, for all* $\varphi \in \mathscr{X}$, *by*

$$\mathbb{T}_0(t)\varphi(a,\ell) := \chi(t,a,\ell)\,\varphi(a-t,\ell), \qquad (a,\ell) \in \Omega, \tag{4.6}$$

where
$$\chi(t,a,\ell) := \begin{cases} 1 & \text{if} \quad a \geqslant t, \\ 0 & \text{if} \quad a < t. \end{cases} \tag{4.7}$$

Proof. (1). Let $\lambda > 0$ and let $g \in \mathscr{X}$. Easy computations show that

$$(\lambda - \mathrm{T}_0)^{-1} g(a,\ell) = \int_0^a e^{-\lambda(a-a')} g(a',\ell)\mathrm{d}a', \qquad (a,\ell) \in \Omega. \tag{4.8}$$

Accordingly,

$$\begin{aligned}\left\|(\lambda - \mathrm{T}_0)^{-1} g\right\|_{\mathscr{X}} &= \int_{\ell_1}^{\ell_2}\int_0^{\ell}\left|\int_0^a e^{-\lambda(a-a')} g(a',\ell)\mathrm{d}a'\right|\mathrm{d}a\mathrm{d}\ell \\ &\leqslant \int_{\ell_1}^{\ell_2}\left\{\int_0^{\ell}\left[e^{-\lambda a}\right]\left[\int_0^a e^{\lambda a'}\,|g(a',\ell)|\,\mathrm{d}a'\right]\mathrm{d}a\right\}\mathrm{d}\ell.\end{aligned}$$

Integrating by parts the term inside the brackets, we get that

$$\begin{aligned}\left\|(\lambda - \mathrm{T}_0)^{-1} g\right\|_{\mathscr{X}} &\leqslant \frac{1}{\lambda}\int_{\ell_1}^{\ell_2}\left\{-\int_0^{\ell} e^{-\lambda(\ell-a')}\,|g(a',\ell)|\,\mathrm{d}a' + \int_0^{\ell}|g(a,\ell)|\,\mathrm{d}a\right\}\mathrm{d}\ell \\ &\leqslant \frac{1}{\lambda}\int_{\ell_1}^{\ell_2}\int_0^{\ell}|g(a,\ell)|\,\mathrm{d}a\mathrm{d}\ell \\ &= \frac{1}{\lambda}\|g\|_{\mathscr{X}}\end{aligned}$$

which proves (4.3) and leads to the boundedness of $(\lambda - \mathrm{T}_0)^{-1}$.

On the other hand

$$\begin{aligned}\left\|\frac{1}{\ell}\left(\lambda-\mathrm{T}_{0}\right)^{-1}g\right\|_{\mathscr{X}} &\leqslant \int_{\ell_1}^{\ell_2}\int_0^{\ell}\frac{1}{\ell}\left[\int_0^a e^{-\lambda(a-a')}\left|g(a',\ell)\right|\mathrm{d}a'\right]\mathrm{d}a\mathrm{d}\ell\\ &\leqslant \int_{\ell_1}^{\ell_2}\int_0^{\ell}\frac{1}{\ell}\left[\int_0^{\ell}\left|g(a',\ell)\right|\mathrm{d}a'\right]\mathrm{d}a\mathrm{d}\ell\\ &= \int_{\ell_1}^{\ell_2}\left[\frac{1}{\ell}\int_0^{\ell}\mathrm{d}a\right]\left[\int_0^{\ell}\left|g(a',\ell)\right|\mathrm{d}a'\right]\mathrm{d}\ell\\ &= \int_{\ell_1}^{\ell_2}\int_0^{\ell}\left|g(a',\ell)\right|\mathrm{d}a\mathrm{d}\ell\end{aligned}$$

which proves (4.4).

(2). This point easily follows from the explicit form (4.8).

(3). Point (1) yields that $(0\ ,\ \infty)\subset\rho\left(\mathrm{T}_{0}\right)$ (resolvent set of T_{0}) and if $\lambda>0$ then $(\lambda-\mathrm{T}_{0})$ is closed and so is $\mathrm{T}_{0}=\lambda-(\lambda-\mathrm{T}_{0})$. Also T_{0} is densely defined because of $\mathcal{C}_c(\Omega)\subset\mathrm{D}(\mathrm{T}_{0})\subset\mathscr{X}$ $\big(\mathcal{C}_c(\Omega)$ is the subspace of all continuous functions with compact support in $\Omega\big)$.

Let $\lambda>0$. Using (4.3) we infer by induction that, for all $g\in\mathscr{X}$,

$$\left\|\left(\lambda-\mathrm{T}_{0}\right)^{-n}g\right\|_{\mathscr{X}}\leqslant\frac{1}{\lambda^n}\|g\|_{\mathscr{X}},\qquad n=1,2,....$$

Now, all the required conditions of Lemma 2.1 are fulfilled and therefore T_{0} generates, on $\mathscr{X}$, a C_0-semigroup $\mathbb{T}_{0}=(\mathbb{T}_{0}(t))_{t\geqslant0}$ satisfying (4.5). The positivity of $\mathbb{T}_{0}=(\mathbb{T}_{0}(t))_{t\geqslant0}$ follows from that of $\left(\lambda-\mathrm{T}_{0}\right)^{-1}$ (Point (2)) because of Lemma 2.4(1).

(4). This point is easy to check. ■

Now, the resolvent operator of T_{α} can be computed as follows

Lemma 4.3. *Let $\alpha\geqslant0$ and let $\lambda>\alpha\overline{\kappa}$ where $\overline{\kappa}$ is defined by* (3.2). *If $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)$ holds, then $\left(\lambda-\mathrm{T}_{\alpha}\right)^{-1}$ is a bounded linear operator from $\mathscr{X}$ into itself and*

$$\left(\lambda-\mathrm{T}_{\alpha}\right)^{-1}=\left(\mathbb{1}_{\mathscr{X}}-\mathcal{K}_{\alpha,\lambda}\right)^{-1}\left(\lambda-\mathrm{T}_{0}\right)^{-1}\tag{4.9}$$

where $\mathcal{K}_{\alpha,\lambda}$ is defined by (3.5) *and $\mathbb{1}_{\mathscr{X}}$ denotes the identity operator in $\mathscr{X}$. Furthermore,*

(1) *if $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)$ holds, then $\left(\lambda-\mathrm{T}_{\alpha}\right)^{-1}$ is positive in $\mathscr{X}$ and*

$$\left(\lambda-\mathrm{T}_{\alpha}\right)^{-1}\geqslant\left(\lambda-\mathrm{T}_{0}\right)^{-1}\geqslant0.\tag{4.10}$$

(2) *if $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)'$ holds and $\alpha>0$, then $\left(\lambda-\mathrm{T}_{\alpha}\right)^{-1}$ is strongly positive in $\mathscr{X}$.*

Proof. Let $\lambda>\alpha\overline{\kappa}$ and let $f\in\left(\mathbb{1}_{\mathscr{X}}-\mathcal{K}_{\alpha,\lambda}\right)(\mathrm{D}(\mathrm{T}_{\alpha}))$. There exists $\varphi\in\mathrm{D}(\mathrm{T}_{\alpha})$ such that $f=\left(\mathbb{1}_{\mathscr{X}}-\mathcal{K}_{\alpha,\lambda}\right)\varphi$. Using (3.6) and then (3.7) we get that

$$\|f\|_{\mathscr{X}}\leqslant\left(1+\frac{\alpha\overline{\kappa}}{\lambda}\right)\|\varphi\|_{\mathscr{X}}<\infty$$

and
$$\left\|\frac{1}{\ell}f\right\|_{\mathscr{X}} \leqslant \left\|\frac{1}{\ell}\varphi\right\|_{\mathscr{X}} + \alpha\overline{\kappa}\|\varphi\|_{\mathscr{X}} < \infty$$

and therefore $f \in \mathscr{X}$ and $\frac{1}{\ell}f \in \mathscr{X}$. Using (3.4) we can write that

$$\frac{\partial f}{\partial a} = \frac{\partial}{\partial a}\Big(\varphi - \mathcal{K}_{\alpha,\lambda}\varphi\Big) = \frac{\partial \varphi}{\partial a} + \lambda\mathcal{K}_{\alpha,\lambda}\varphi$$

which leads, by virtue of (3.6), to

$$\left\|\frac{\partial f}{\partial a}\right\|_{\mathscr{X}} \leqslant \left\|\frac{\partial \varphi}{\partial a}\right\|_{\mathscr{X}} + \alpha\overline{\kappa}\|\varphi\|_{\mathscr{X}} < \infty$$

and therefore $\frac{\partial f}{\partial a} \in \mathscr{X}$. Using again (3.4) we can write that

$$f(0,\cdot) = \big(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\big)\varphi(0,\cdot) = \varphi(0,\cdot) - \mathrm{K}_{\alpha}\varphi = 0$$

and therefore $f(0,\cdot) = 0$. Hence, we have proved that

$$f \in \mathscr{X}, \qquad \frac{1}{\ell}f \in \mathscr{X}, \qquad \frac{\partial f}{\partial a} \in \mathscr{X} \qquad \text{and} \qquad f(0,\cdot) = 0$$

which means that $f \in \mathrm{D}(\mathrm{T}_0)$ (given by (4.2)) and therefore

$$\big(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\big)\left(\mathrm{D}(\mathrm{T}_\alpha)\right) \subset \mathrm{D}(\mathrm{T}_0). \tag{4.11}$$

Let $\varphi \in \mathrm{D}(\mathrm{T}_\alpha)$. Then (4.11) leads to $\left(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\right)\varphi \in \mathrm{D}(\mathrm{T}_0)$ and therefore

$$\begin{aligned}
(\lambda - \mathrm{T}_0)\left(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\right)\varphi &= \lambda\left(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\right)\varphi + \frac{\partial}{\partial a}\left(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\right)\varphi \\
&= \lambda\varphi - \lambda\mathcal{K}_{\alpha,\lambda}\varphi + \frac{\partial\varphi}{\partial a} + \lambda\mathcal{K}_{\alpha,\lambda}\varphi \\
&= (\lambda - \mathrm{T}_\alpha)\,\varphi,
\end{aligned}$$

whence
$$(\lambda - \mathrm{T}_0)\left(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\right) = (\lambda - \mathrm{T}_\alpha). \tag{4.12}$$

Finally, using Lemma 4.2(1) together with (3.6) we get that the left hand side of (4.12) is an invertible operator whose inverse is

$$\left(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\right)^{-1}(\lambda - \mathrm{T}_0)^{-1} = (\lambda - \mathrm{T}_\alpha)^{-1} \tag{4.13}$$

which proves (4.9) and leads to the boundedness of $(\lambda - \mathrm{T}_\alpha)^{-1}$ from $\mathscr{X}$ into itself.

(1). Using (4.13) together with (3.6) and then both Lemma 3.2(1) and Lemma 4.2(2) we can write that

$$(\lambda - \mathrm{T}_\alpha)^{-1} = \sum_{n=0}^{\infty}\mathcal{K}_{\alpha,\lambda}^{n}(\lambda - \mathrm{T}_0)^{-1} \geqslant \mathcal{K}_{\alpha,\lambda}^{m}(\lambda - \mathrm{T}_0)^{-1} \geqslant 0, \qquad m = 0, 1, 2, \cdots$$

Putting $m = 0$ we get (4.10) and therefore $(\lambda - \mathrm{T}_\alpha)^{-1}$ is a positive operator.

(2). Similarly, putting $m = 1$ we get that

$$(\lambda - \mathrm{T}_\alpha)^{-1} \geqslant \mathcal{K}_{\alpha,\lambda} (\lambda - \mathrm{T}_0)^{-1} \geqslant 0.$$

The strict positivity of $(\lambda - \mathrm{T}_0)^{-1}$ (Lemma 4.2(2)) and then the strong positivity of $\mathcal{K}_{\alpha,\lambda}$ (Lemma 3.2(2)) imply that $\mathcal{K}_{\alpha,\lambda} (\lambda - \mathrm{T}_0)^{-1}$ is strongly positive in $\mathscr{X}$ and so is $(\lambda - \mathrm{T}_\alpha)^{-1}$. ■

Now, the main result of this section can be announced as follows,

Theorem 4.4. *Let $\alpha \geqslant 0$. If $\left(\mathbf{A}_{\mathrm{k}}^1\right)$ holds, then T_α generates, on $\mathscr{X}$, a C_0–semigroup $\mathbb{T}_\alpha = (\mathbb{T}_\alpha(t))_{t \geqslant 0}$ satisfying, for all $\varphi \in \mathscr{X}$,*

$$\|\mathbb{T}_\alpha(t)\varphi\|_{\mathscr{X}} \leqslant e^{(\alpha\overline{\kappa})t} \|\varphi\|_{\mathscr{X}} \qquad t \geqslant 0, \tag{4.14}$$

where $\overline{\kappa}$ is defined by (3.2). *Furthermore,*

(1) *if $\left(\mathbf{A}_{\mathrm{k}}^2\right)$ holds, then $\mathbb{T}_\alpha = (\mathbb{T}_\alpha(t))_{t \geqslant 0}$ is positive in $\mathscr{X}$ and*

$$\mathbb{T}_\alpha(t) \geqslant \mathbb{T}_0(t) \geqslant 0, \qquad t \geqslant 0. \tag{4.15}$$

(2) *if $\left(\mathbf{A}_{\mathrm{k}}^2\right)'$ holds and $\alpha > 0$, then $\mathbb{T}_\alpha = (\mathbb{T}_\alpha(t))_{t \geqslant 0}$ is irreducible in $\mathscr{X}$.*

Proof. Lemma 4.3 yields that $(\alpha\overline{\kappa}\,, \infty) \subset \rho(\mathrm{T}_\alpha)$ (resolvent set of T_α) and if $\lambda > \alpha\overline{\kappa}$ then $(\lambda - \mathrm{T}_\alpha)$ is closed and so is $\mathrm{T}_\alpha = \lambda - (\lambda - \mathrm{T}_\alpha)$. Also T_α is densely defined because of $\mathcal{C}_c(\Omega) \subset \mathrm{D}(\mathrm{T}_\alpha) \subset \mathscr{X}$ ($\mathcal{C}_c(\Omega)$ is the subspace of all continuous functions with compact support in Ω).

Let $\lambda > \alpha\overline{\kappa}$ and let $g \in \mathscr{X}$. Using (4.9) and then (3.6) and (4.3) we get that

$$\begin{aligned}
\left\|(\lambda - \mathrm{T}_\alpha)^{-1} g\right\|_{\mathscr{X}} &\leqslant \left\|\left(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\right)^{-1}\right\|_{\mathscr{L}(\mathscr{X})} \left\|(\lambda - \mathrm{T}_0)^{-1} g\right\|_{\mathscr{X}} \\
&\leqslant \frac{1}{1 - \dfrac{\alpha\overline{\kappa}}{\lambda}} \frac{1}{\lambda} \|g\|_{\mathscr{X}} \\
&= \frac{1}{\lambda - \alpha\overline{\kappa}} \|g\|_{\mathscr{X}}
\end{aligned}$$

and by induction

$$\left\|(\lambda - \mathrm{T}_\alpha)^{-n} g\right\|_{\mathscr{X}} \leqslant \frac{1}{(\lambda - \alpha\overline{\kappa})^n} \|g\|_{\mathscr{X}} \qquad n = 1, 2, ...$$

Now, all the required conditions of Lemma 2.1 are fulfilled and therefore T_α generates, on $\mathscr{X}$, a C_0–semigroup $\mathbb{T}_\alpha = (\mathbb{T}_\alpha(t))_{t \geqslant 0}$ satisfying (4.14).

(1). Firstly, the positivity of $\mathbb{T}_\alpha = (\mathbb{T}_\alpha(t))_{t \geqslant 0}$ follows from that of $(\lambda - \mathrm{T}_\alpha)^{-1}$ (Lemma 4.3(1)) because of Lemma 2.4(1).

Next, let $\lambda > \alpha\overline{\kappa}$ and let $g \in \mathscr{X}$ be such that $g \geqslant 0$. Using (4.10) it follows, by induction, that

$$\left[\lambda\left(\lambda - \mathrm{T}_{\alpha}\right)^{-1}\right]^{n} g \geqslant \left[\lambda\left(\lambda - \mathrm{T}_{0}\right)^{-1}\right]^{n} g \geqslant 0, \qquad n = 1, 2, \cdots$$

Let $t > 0$. Putting $\lambda = \frac{n}{t}$ (n large enough), we get that

$$\left[\frac{n}{t}\left(\frac{n}{t} - \mathrm{T}_{\alpha}\right)^{-1}\right]^{n} g \geqslant \left[\frac{n}{t}\left(\frac{n}{t} - \mathrm{T}_{0}\right)^{-1}\right]^{n} g \geqslant 0,$$

and at the limit $n \to \infty$, the desired (4.15) follows because of (2.2).

(2). This point follows from Lemma 4.3(2) because of Lemma 2.4(2). ■

5 The explicit form of $\mathbb{T}_{\alpha} = (\mathbb{T}_{\alpha}(t))_{t\geqslant 0}$

In this section we are concerned with the explicit form of the C_0−semigroup $\mathbb{T}_{\alpha} = (\mathbb{T}_{\alpha}(t))_{t\geqslant 0}$ whose existence has already been proved in Theorem 4.4. This explicit form will be very useful to describe the asynchronous exponential growth related to the full model (1.1), (1.3), and which will be the subject of a forthcoming work ([7]).

Let $\Delta := (-\infty, 0) \times (\ell_1, \ell_2)$ and let $\mathscr{X}_{\omega}$ ($\omega \geqslant 0$) be the weighted Banach space

$$\mathscr{X}_{\omega} := L^1(\Delta, e^{\omega x}) \quad \text{whose norm is} \quad \|h\|_{\omega} = \int_{\ell_1}^{\ell_2}\int_{-\infty}^{0} |h(x,\ell)|\, e^{\omega x}\mathrm{d}x\mathrm{d}\ell,$$

and let us prove some auxiliary results. The first one concerns the linear operator $\overline{\mathbb{T}}_{0}(t)$ ($t \geqslant 0$) defined by

$$\overline{\mathbb{T}}_{0}(t)h(a,\ell) := \overline{\chi}(t,a,\ell)h(a-t,\ell), \qquad (a,\ell) \in \Omega, \tag{5.1}$$

where
$$\overline{\chi}(t,a,\ell) := \begin{cases} 1 & \text{if} \quad a < t, \\ 0 & \text{if} \quad a \geqslant t. \end{cases} \tag{5.2}$$

Lemma 5.1. *$\overline{\mathbb{T}}_{0}(t)$ ($t \geqslant 0$) is a bounded linear operator from $\mathscr{X}_{\omega}$ ($\omega \geqslant 0$) into $\mathscr{X}$ satisfying, for all $h \in \mathscr{X}_{\omega}$,*

$$\overline{\mathbb{T}}_{0}(0)h = 0 \qquad \textit{and} \qquad \lim_{t\downarrow 0}\left\|\overline{\mathbb{T}}_{0}(t)h\right\|_{\mathscr{X}} = 0. \tag{5.3}$$

Furthermore, if $\omega > 0$ then

$$\int_{0}^{\infty} e^{-\omega t}\left\|\overline{\mathbb{T}}_{0}(t)h\right\|_{\mathscr{X}}\mathrm{d}t \leqslant \frac{1}{\omega}\|h\|_{\omega}. \tag{5.4}$$

Proof. Let $t \geqslant 0$. Let $\omega \geqslant 0$ and let $h \in \mathscr{X}_{\omega}$. Then

$$\left\|\overline{\mathbb{T}}_{0}(t)h\right\|_{\mathscr{X}} = \int_{\ell_1}^{\ell_2}\int_{0}^{\ell} \overline{\chi}(t,a,\ell)\,|h(a-t,\ell)|\,\mathrm{d}a\mathrm{d}\ell$$

$$= \int_{\ell_1}^{\ell_2}\int_{-t}^{\ell-t} \overline{\chi}(t,s+t,\ell)\,|h(s,\ell)|\,\mathrm{d}s\mathrm{d}\ell.$$

Using (5.2) we get that $\overline{\chi}(t,s+t,\ell)=1$ if and only if $s<0$ and therefore

$$\left\|\overline{\mathbb{T}}_{0}(t)h\right\|_{\mathscr{X}} = \int_{\ell_1}^{\ell_2}\int_{-t}^{0} |h(s,\ell)|\,\mathrm{d}s\mathrm{d}\ell \leqslant e^{\omega t}\int_{\ell_1}^{\ell_2}\int_{-t}^{0} |h(s,\ell)|\,e^{\omega s}\mathrm{d}s\mathrm{d}\ell$$

which obviously leads to (5.3). This again leads to

$$\left\|\overline{\mathbb{T}}_{0}(t)h\right\|_{\mathscr{X}} \leqslant e^{\omega t}\int_{\ell_1}^{\ell_2}\int_{-\infty}^{0} |h(s,\ell)|\,e^{\omega s}\mathrm{d}s\mathrm{d}\ell = e^{\omega t}\|h\|_{\omega}$$

which proves the boundedness of $\overline{\mathbb{T}}_{0}(t)$ from $\mathscr{X}_\omega$ into $\mathscr{X}$.

However, if $\omega>0$ then

$$\begin{aligned}\int_0^\infty e^{-\omega t}\left\|\overline{\mathbb{T}}_{0}(t)h\right\|_{\mathscr{X}}\mathrm{d}t &= \int_0^\infty e^{-\omega t}\left[\int_{\ell_1}^{\ell_2}\int_0^{\ell} \overline{\chi}(t,a,\ell)\,|h(a-t,\ell)|\,\mathrm{d}a\mathrm{d}\ell\right]\mathrm{d}t\\ &= \int_{\ell_1}^{\ell_2}\int_0^{\ell}\left[\int_0^\infty e^{-\omega t}\overline{\chi}(t,a,\ell)\,|h(a-t,\ell)|\,\mathrm{d}t\right]\mathrm{d}a\mathrm{d}\ell\\ &= \int_{\ell_1}^{\ell_2}\int_0^{\ell}\left[\int_{-\infty}^{a} e^{\omega(s-a)}\overline{\chi}(a-s,a,\ell)\,|h(s,\ell)|\,\mathrm{d}s\right]\mathrm{d}a\mathrm{d}\ell.\end{aligned}$$

Using (5.2) we get that $\overline{\chi}\,(a-s,a,\ell)=1$ if and only if $s<0$ and therefore

$$\begin{aligned}\int_0^\infty e^{-\omega t}\left\|\overline{\mathbb{T}}_{0}(t)h\right\|_{\mathscr{X}}\mathrm{d}t &= \int_{\ell_1}^{\ell_2}\int_0^{\ell}\left[\int_{-\infty}^{0} e^{\omega(s-a)}\,|h(s,\ell)|\,\mathrm{d}s\right]\mathrm{d}a\mathrm{d}\ell\\ &= \int_{\ell_1}^{\ell_2}\int_{-\infty}^{0}\left[\int_0^{\ell} e^{-\omega a}\mathrm{d}a\right]|h(s,\ell)|\,e^{\omega s}\mathrm{d}s\mathrm{d}\ell\\ &\leqslant \frac{1}{\omega}\int_{\ell_1}^{\ell_2}\int_{-\infty}^{0}|h(s,\ell)|\,e^{\omega s}\mathrm{d}s\mathrm{d}\ell\\ &= \frac{1}{\omega}\|h\|_{\omega}\end{aligned}$$

which proves (5.4). ■

The second auxiliary result concerns both linear operators

$$\mathrm{M}_{\alpha}\varphi(x,\ell) := \alpha\int_{\ell_1}^{\ell_2}\int_0^{\ell} \mathrm{k}(\ell,a',\ell')\mathbb{T}_{0}(-x)\varphi(a',\ell')\mathrm{d}a'\mathrm{d}\ell' \quad (x,\ell)\in\Delta, \tag{5.5}$$

and

$$\overline{\mathrm{M}}_{\alpha}h(x,\ell) := \alpha\int_{\ell_1}^{\ell_2}\int_0^{\ell} \mathrm{k}(\ell,a',\ell')\overline{\mathbb{T}}_{0}(-x)h(a',\ell')\mathrm{d}a'\mathrm{d}\ell' \quad (x,\ell)\in\Delta, \tag{5.6}$$

where $\mathbb{T}_{0}(t)$ and $\overline{\mathbb{T}}_{0}(t)$ are respectively defined by (4.6) and (5.1). Note that (5.5) and (5.6) are the explicit forms of

$$\mathrm{M}_{\alpha}\varphi(x,\ell) = \mathrm{K}_{\alpha}\left(\mathbb{T}_{0}(-x)\varphi\right)(\ell) \quad\text{and}\quad \overline{\mathrm{M}}_{\alpha}h(x,\ell) = \mathrm{K}_{\alpha}\left(\overline{\mathbb{T}}_{0}(-x)h\right)(\ell). \tag{5.7}$$

Lemma 5.2. *Let $\alpha \geqslant 0$. If $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)$ holds, then*

(1) M_{α} *is a bounded linear operator from $\mathscr{X}$ into $\mathscr{X}_\omega$ $(\omega > 0)$ satisfying, for all $\varphi \in \mathscr{X}$,*

$$\|\mathrm{M}_{\alpha}\varphi\|_{\omega} \leqslant \frac{\alpha\overline{\kappa}}{\omega}\|\varphi\|_{\mathscr{X}}, \tag{5.8}$$

where $\overline{\kappa}$ is defined by (3.2).

(2) $\overline{\mathrm{M}}_{\alpha}$ *is a bounded linear operator from $\mathscr{X}_\omega$ $(\omega > 0)$ into itself satisfying, for all $h \in \mathscr{X}_\omega$,*

$$\left\|\overline{\mathrm{M}}_{\alpha}h\right\|_{\omega} \leqslant \frac{\alpha\overline{\kappa}}{\omega}\|h\|_{\omega}. \tag{5.9}$$

Proof. Let $\omega > 0$. Using (5.7) we can write, for all $\varphi \in \mathscr{X}$, that

$$\begin{aligned}
\|\mathrm{M}_{\alpha}\varphi\|_{\omega} &= \int_{\ell_1}^{\ell_2}\left[\int_{-\infty}^{0}\left|\mathrm{K}_{\alpha}\left(\mathbb{T}_0(-x)\varphi\right)(\ell)\right| e^{\omega x}\mathrm{d}x\right]\mathrm{d}\ell \\
&= \int_{\ell_1}^{\ell_2}\left[\int_{0}^{\infty}\left|\mathrm{K}_{\alpha}\left(\mathbb{T}_0(t)\varphi\right)(\ell)\right| e^{-\omega t}\mathrm{d}t\right]\mathrm{d}\ell \\
&= \int_{0}^{\infty}\left[\int_{\ell_1}^{\ell_2}\left|\mathrm{K}_{\alpha}\left(\mathbb{T}_0(t)\varphi\right)(\ell)\right|\mathrm{d}\ell\right] e^{-\omega t}\mathrm{d}t \\
&= \int_{0}^{\infty}\left\|\mathrm{K}_{\alpha}\left(\mathbb{T}_0(t)\varphi\right)\right\|_{\mathscr{Y}} e^{-\omega t}\mathrm{d}t
\end{aligned}$$

which leads, by virtue of (3.3) and then (4.5), to

$$\|\mathrm{M}_{\alpha}\varphi\|_{\omega} \leqslant \alpha\overline{\kappa}\int_{0}^{\infty}\|\mathbb{T}_0(t)\varphi\|_{\mathscr{X}} e^{-\omega t}\mathrm{d}t \leqslant \alpha\overline{\kappa}\left[\int_{0}^{\infty}e^{-\omega t}\mathrm{d}t\right]\|\varphi\|_{\mathscr{X}} = \frac{\alpha\overline{\kappa}}{\omega}\|\varphi\|_{\mathscr{X}}$$

which proves (5.8) and leads to the boundedness of M_{α} because of $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)$.
Similarly, using (5.7) we can write, for all $h \in \mathscr{X}_\omega$, that

$$\begin{aligned}
\left\|\overline{\mathrm{M}}_{\alpha}h\right\|_{\omega} &= \int_{\ell_1}^{\ell_2}\left[\int_{-\infty}^{0}\left|\mathrm{K}_{\alpha}\left(\overline{\mathbb{T}}_0(-x)h\right)(\ell)\right| e^{\omega x}\mathrm{d}x\right]\mathrm{d}\ell \\
&= \int_{\ell_1}^{\ell_2}\left[\int_{0}^{\infty}\left|\mathrm{K}_{\alpha}\left(\overline{\mathbb{T}}_0(t)h\right)(\ell)\right| e^{-\omega t}\mathrm{d}t\right]\mathrm{d}\ell \\
&= \int_{0}^{\infty}e^{-\omega t}\left[\int_{\ell_1}^{\ell_2}\left|\mathrm{K}_{\alpha}\left(\overline{\mathbb{T}}_0(t)h\right)(\ell)\right|\mathrm{d}\ell\right]\mathrm{d}t \\
&= \int_{0}^{\infty}e^{-\omega t}\left\|\mathrm{K}_{\alpha}\left(\overline{\mathbb{T}}_0(t)h\right)\right\|_{\mathscr{Y}}\mathrm{d}t
\end{aligned}$$

which leads, by virtue of (3.3) and then (5.4), to

$$\left\|\overline{\mathrm{M}}_{\alpha}h\right\|_{\omega} \leqslant \alpha\overline{\kappa}\int_{0}^{\infty}e^{-\omega t}\left\|\overline{\mathbb{T}}_0(t)h\right\|_{\mathscr{X}}\mathrm{d}t \leqslant \frac{\alpha\overline{\kappa}}{\omega}\|h\|_{\omega}$$

which proves (5.9) and leads to the boundedness of $\overline{\mathrm{M}}_{\alpha}$ because of $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)$. ∎

The last auxiliary result is given as follows

Lemma 5.3. *Let $\alpha \geqslant 0$ and let $\omega > \alpha\overline{\kappa}$ where $\overline{\kappa}$ is defined by* (3.2)*. Let $\mathbb{A}_\alpha(t)$ $(t \geqslant 0)$ be the following linear operator*

$$\mathbb{A}_\alpha(t) := \overline{\mathbb{T}}_0(t)\big(\mathbb{1}_\omega - \overline{\mathrm{M}}_\alpha\big)^{-1}\mathrm{M}_\alpha \tag{5.10}$$

where $\overline{\mathbb{T}}_0(t)$, M_α and $\overline{\mathrm{M}}_\alpha$ are respectively defined by (5.1), (5.5) *and* (5.6) *while $\mathbb{1}_\omega$ denotes the identity operator in $\mathscr{X}_\omega$.*

If $\left(\mathbf{A}_{\mathrm{k}}^1\right)$ holds, then $\mathbb{A}_\alpha(t)$ $(t \geqslant 0)$ is a bounded linear operator from $\mathscr{X}$ into itself satisfying, for all $\varphi \in \mathscr{X}$,

$$\mathbb{A}_\alpha(0)\varphi = 0 \qquad \text{and} \qquad \lim_{t\downarrow 0}\|\mathbb{A}_\alpha(t)\varphi\|_{\mathscr{X}} = 0, \tag{5.11}$$

and,
$$\mathbb{A}_\alpha(t)\varphi(a,\ell) = \overline{\chi}(t,a,\ell)\mathrm{K}_\alpha\big(\mathbb{T}_0(t-a)\varphi + \mathbb{A}_\alpha(t-a)\varphi\big)(\ell), \tag{5.12}$$

for almost all $(a,\ell) \in \Omega$, where $\overline{\chi}$ is given by (5.2)*. Furthermore*

$$\|\mathbb{A}_\alpha(t)\varphi\|_{\mathscr{X}} \leqslant \alpha\overline{\kappa}\int_0^t \|\mathbb{T}_0(s)\varphi + \mathbb{A}_\alpha(s)\varphi\|_{\mathscr{X}}\,\mathrm{d}s, \quad t \geqslant 0, \tag{5.13}$$

$$\int_0^\infty e^{-\lambda t}\mathbb{A}_\alpha(t)\varphi\mathrm{d}t = \big(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\big)^{-1}\mathcal{K}_{\alpha,\lambda}\left(\lambda - \mathrm{T}_0\right)^{-1}\varphi, \quad \lambda > \alpha\overline{\kappa}, \tag{5.14}$$

where $\mathbb{T}_0(s)$, T_0 and $\mathcal{K}_{\alpha,\lambda}$ are respectively defined by (4.6), (4.2) *and* (3.5)*.*

Proof. Let $\omega > \alpha\overline{\kappa}$. Due to (5.9) we infer that $\big(\mathbb{1}_\omega - \overline{\mathrm{M}}_\alpha\big)^{-1}$ is a bounded linear operator from $\mathscr{X}_\omega$ into itself. This boundedness together with that of $\overline{\mathbb{T}}_0(t)$ $(t \geqslant 0)$ (Lemma 5.1) and that of M_α (Lemma 5.2(1)) lead to the desired boundedness of the operator $\mathbb{A}_\alpha(t)$ $(t \geqslant 0)$ from $\mathscr{X}$ into itself. Furthermore, if $\varphi \in \mathscr{X}$ then $h := \big(\mathbb{1}_\omega - \overline{\mathrm{M}}_\alpha\big)^{-1}\mathrm{M}_\alpha\varphi \in \mathscr{X}_\omega$ and $\overline{\mathbb{T}}_0(t)h = \mathbb{A}_\alpha(t)\varphi$ and therefore

$$\mathbb{A}_\alpha(0)\varphi = \overline{\mathbb{T}}_0(0)h \qquad \text{and} \qquad \lim_{t\downarrow 0}\|\mathbb{A}_\alpha(t)\varphi\|_{\mathscr{X}} = \lim_{t\downarrow 0}\left\|\overline{\mathbb{T}}_0(t)h\right\|_{\mathscr{X}}$$

which proves (5.11) because of (5.3).

Let $t \geqslant 0$ and let $\varphi \in \mathscr{X}$. Using (5.1) we get, for almost all $(a,\ell) \in \Omega$, that

$$\begin{aligned}
\mathbb{A}_\alpha(t)\varphi(a,\ell) &= \overline{\chi}(t,a,\ell)\left(\mathbb{1}_\omega - \overline{\mathrm{M}}_\alpha\right)^{-1}\mathrm{M}_\alpha\varphi(a-t,\ell)\\
&= \overline{\chi}(t,a,\ell)\Big[\mathbb{1}_\omega + \overline{\mathrm{M}}_\alpha\left(\mathbb{1}_\omega - \overline{\mathrm{M}}_\alpha\right)^{-1}\Big]\mathrm{M}_\alpha\varphi(a-t,\ell)\\
&= \overline{\chi}(t,a,\ell)\Big[\mathrm{M}_\alpha\varphi(a-t,\ell) + \overline{\mathrm{M}}_\alpha\left(\mathbb{1}_\omega - \overline{\mathrm{M}}_\alpha\right)^{-1}\mathrm{M}_\alpha\varphi(a-t,\ell)\Big]
\end{aligned}$$

which leads, by virtue of (5.7), to

$$\begin{aligned}
\mathbb{A}_\alpha(t)\varphi(a,\ell) &= \overline{\chi}(t,a,\ell)\Big[\mathrm{K}_\alpha\Big(\mathbb{T}_0(t-a)\varphi\Big) + \mathrm{K}_\alpha\Big(\overline{\mathbb{T}}_0(t-a)\big(\mathbb{1}_\omega - \overline{\mathrm{M}}_\alpha\big)^{-1}\mathrm{M}_\alpha\varphi\Big)\Big](\ell)\\
&= \overline{\chi}(t,a,\ell)\Big[\mathrm{K}_\alpha\Big(\mathbb{T}_0(t-a)\varphi\Big) + \mathrm{K}_\alpha\Big(\mathbb{A}_\alpha(t-a)\varphi\Big)\Big]\,(\ell)
\end{aligned}$$

$$= \overline{\chi}(t,a,\ell)\mathrm{K}_{\alpha}\Big(\mathbb{T}_{0}(t-a)\varphi + \mathbb{A}_{\alpha}(t-a)\varphi\Big)(\ell)$$

which is nothing else but the desired (5.12).

Since (5.12), we can write that

$$\begin{aligned}\|\mathbb{A}_{\alpha}(t)\varphi\|_{\mathscr{X}} &= \int_{\ell_1}^{\ell_2}\left[\int_0^{\ell}\overline{\chi}(t,a,\ell)\left|\mathrm{K}_{\alpha}\Big(\mathbb{T}_{0}(t-a)\varphi + \mathbb{A}_{\alpha}(t-a)\varphi\Big)(\ell)\right|\mathrm{d}a\right]\mathrm{d}\ell\\ &= \int_{\ell_1}^{\ell_2}\left[\int_{t-\ell}^{t}\overline{\chi}(t,t-s,\ell)\left|\mathrm{K}_{\alpha}\Big(\mathbb{T}_{0}(s)\varphi + \mathbb{A}_{\alpha}(s)\varphi\Big)(\ell)\right|\mathrm{d}s\right]\mathrm{d}\ell.\end{aligned}$$

Due to (5.2) we get that $\overline{\chi}\,(t,t-s,\ell)=1$ if and only if $s>0$ and therefore

$$\begin{aligned}\|\mathbb{A}_{\alpha}(t)\varphi\|_{\mathscr{X}} &= \int_{\ell_1}^{\ell_2}\left[\int_0^{t}\left|\mathrm{K}_{\alpha}\Big(\mathbb{T}_{0}(s)\varphi + \mathbb{A}_{\alpha}(s)\varphi\Big)(\ell)\right|\mathrm{d}s\right]\mathrm{d}\ell\\ &= \int_0^{t}\left[\int_{\ell_1}^{\ell_2}\left|\mathrm{K}_{\alpha}\Big(\mathbb{T}_{0}(s)\varphi + \mathbb{A}_{\alpha}(s)\varphi\Big)(\ell)\right|\mathrm{d}\ell\right]\mathrm{d}s\\ &= \int_0^{t}\left\|\mathrm{K}_{\alpha}\Big(\mathbb{T}_{0}(s)\varphi + \mathbb{A}_{\alpha}(s)\varphi\Big)\right\|_{\mathscr{Y}}\mathrm{d}s.\end{aligned}$$

Now (3.3) yields that

$$\|\mathbb{A}_{\alpha}(t)\varphi\|_{\mathscr{X}} \leqslant \alpha\overline{\kappa}\int_0^t \|\mathbb{T}_{0}(s)\varphi + \mathbb{A}_{\alpha}(s)\varphi\|_{\mathscr{X}}\,\mathrm{d}s$$

which is nothing else but the desired (5.13).

Let $\lambda > \alpha\overline{\kappa}$. Using again (5.12) we can write, for almost all $(a,\ell)\in\Omega$, that,

$$\begin{aligned}\int_0^{\infty}e^{-\lambda t}\mathbb{A}_{\alpha}(t)\varphi(a,\ell)\mathrm{d}t &= \int_0^{\infty}e^{-\lambda t}\overline{\chi}(t,a,\ell)\mathrm{K}_{\alpha}\Big(\mathbb{T}_{0}(t-a)\varphi + \mathbb{A}_{\alpha}(t-a)\varphi\Big)(\ell)\mathrm{d}t\\ &= \int_{-a}^{\infty}e^{-\lambda(s+a)}\overline{\chi}(s+a,a,\ell)\mathrm{K}_{\alpha}\Big(\mathbb{T}_{0}(s)\varphi + \mathbb{A}_{\alpha}(s)\varphi\Big)(\ell)\mathrm{d}s.\end{aligned}$$

Due to (5.2) we get that $\overline{\chi}(s+a,a,\ell)=1$ if and only if $s>0$ and therefore

$$\begin{aligned}\int_0^{\infty}e^{-\lambda t}\mathbb{A}_{\alpha}(t)\varphi(a,\ell)\mathrm{d}t &= \int_0^{\infty}e^{-\lambda(s+a)}\mathrm{K}_{\alpha}\Big(\mathbb{T}_{0}(s)\varphi + \mathbb{A}_{\alpha}(s)\varphi\Big)(\ell)\mathrm{d}s\\ &= e^{-\lambda a}\mathrm{K}_{\alpha}\Big(\int_0^{\infty}e^{-\lambda s}\mathbb{T}_{0}(s)\varphi\mathrm{d}s + \int_0^{\infty}e^{-\lambda s}\mathbb{A}_{\alpha}(s)\varphi\mathrm{d}s\Big)(\ell).\end{aligned}$$

This leads, by virtue of (3.4) and then Lemma 4.2(3), to

$$\int_0^{\infty}e^{-\lambda t}\mathbb{A}_{\alpha}(t)\varphi\mathrm{d}t(a,\ell) = \mathcal{K}_{\alpha,\lambda}\left(\left(\lambda-\mathrm{T}_{0}\right)^{-1}\varphi + \int_0^{\infty}e^{-\lambda s}\mathbb{A}_{\alpha}(s)\varphi\mathrm{d}s\right)(a,\ell),$$

that is to say that,

$$\left(\int_0^{\infty}e^{-\lambda t}\mathbb{A}_{\alpha}(t)\varphi\mathrm{d}t\right)(a,\ell) = \mathcal{K}_{\alpha,\lambda}\left(\left(\lambda-\mathrm{T}_{0}\right)^{-1}\varphi + \int_0^{\infty}e^{-\lambda s}\mathbb{A}_{\alpha}(s)\varphi\mathrm{d}s\right)(a,\ell),$$

for almost all $(a,\ell)\in\Omega$, and therefore,

$$\int_0^\infty e^{-\lambda t}\mathbb{A}_\alpha(t)\varphi \mathrm{d}t=\mathcal{K}_{\alpha,\lambda}\left(\lambda-\mathrm{T}_0\right)^{-1}\varphi+\mathcal{K}_{\alpha,\lambda}\left(\int_0^\infty e^{-\lambda s}\mathbb{A}_\alpha(s)\varphi \mathrm{d}s\right).$$

Finally,

$$\left(\mathbb{1}_{\mathscr{X}}-\mathcal{K}_{\alpha,\lambda}\right)\int_0^\infty e^{-\lambda t}\mathbb{A}_\alpha(t)\varphi \mathrm{d}t=\mathcal{K}_{\alpha,\lambda}\left(\lambda-\mathrm{T}_0\right)^{-1}\varphi$$

which leads to (5.14) because of (3.6). ■

Now, the aim of this section can be announced as follows

Theorem 5.4. *Let $\alpha\geqslant 0$. If $\left(\mathbf{A}^1_{\mathrm{k}}\right)$ holds, then the generated C_0-semigroup $\mathbb{T}_\alpha=(\mathbb{T}_\alpha(t))_{t\geqslant 0}$ $\left(\text{Theorem 4.4}\right)$ is given by*

$$\mathbb{T}_\alpha(t):=\mathbb{T}_0(t)+\mathbb{A}_\alpha(t),\qquad\qquad t\geqslant 0, \tag{5.15}$$

where $\mathbb{T}_0(t)$ and $\mathbb{A}_\alpha(t)$ are respectively defined by (4.6) *and* (5.10).

Proof. For convenience, we divide the proof in several steps.

Step I. Let $\mathbb{S}_\alpha(t)$ $(t\geqslant 0)$ be the following linear operator

$$\mathbb{S}_\alpha(t):=\mathbb{T}_0(t)+\mathbb{A}_\alpha(t). \tag{5.16}$$

We claim that $\mathbb{S}_\alpha=(\mathbb{S}_\alpha(t))_{t\geqslant 0}$ is a C_0-semigroup, on $\mathscr{X}$, satisfying

$$\|\mathbb{S}_\alpha(t)\varphi\|_{\mathscr{X}}\leqslant e^{(\alpha\overline{\kappa})t}\|\varphi\|_{\mathscr{X}}\qquad\qquad t\geqslant 0, \tag{5.17}$$

for all $\varphi\in\mathscr{X}$.

Due to Lemma 4.2(3) together with Lemma 5.3, we infer that $\mathbb{S}_\alpha(t)$ $(t\geqslant 0)$ is a bounded linear operator from $\mathscr{X}$ into itself satisfying, for all $\varphi\in\mathscr{X}$,

$$\mathbb{S}_\alpha(0)\varphi=\mathbb{T}_0(0)\varphi+\mathbb{A}_\alpha(0)\varphi=\varphi$$

and
$$\lim_{t\downarrow 0}\|\mathbb{S}_\alpha(t)\varphi-\varphi\|_{\mathscr{X}}\leqslant\lim_{t\downarrow 0}\|\mathbb{T}_0(t)\varphi-\varphi\|_{\mathscr{X}}+\lim_{t\downarrow 0}\|\mathbb{A}_\alpha(t)\varphi\|_{\mathscr{X}}=0.$$

It remains to prove that

$$\mathbb{S}_\alpha(t)\mathbb{S}_\alpha(s)=\mathbb{S}_\alpha(t+s),\qquad\text{for all}\quad t,s\geqslant 0. \tag{5.18}$$

Step II. Let $t\geqslant 0$ and $s\geqslant 0$ and let $\overline{\mathbb{S}}_\alpha(t,s)$ be

$$\overline{\mathbb{S}}_\alpha(t,s):=\mathbb{S}_\alpha(t)\mathbb{S}_\alpha(s)-\mathbb{S}_\alpha(t+s), \tag{5.19}$$

which can be easily simplified in

$$\overline{\mathbb{S}}_\alpha(t,s)=\mathbb{T}_0(t)\mathbb{A}_\alpha(s)+\mathbb{A}_\alpha(t)\mathbb{S}_\alpha(s)\varphi-\mathbb{A}_\alpha(t+s). \tag{5.20}$$

Let $\varphi \in \mathscr{X}$. Combining then (5.16) and (5.12), it follows that

$$\mathbb{A}_{\alpha}(t)\varphi(a,\ell) = \overline{\chi}(t,a,\ell)\mathrm{K}_{\alpha}\big(\mathbb{S}_{\alpha}(t-a)\varphi\big)(\ell), \tag{5.21}$$

for almost all $(a,\ell) \in \Omega$, and therefore,

$$\mathbb{A}_{\alpha}(t+s)\varphi\,(a,\ell) = \overline{\chi}\,(t+s,a,\ell)\,\mathrm{K}_{\alpha}\big(\mathbb{S}_{\alpha}(t+s-a)\varphi\big)(\ell), \tag{5.22}$$

and
$$\mathbb{A}_{\alpha}(t)\mathbb{S}_{\alpha}(s)\varphi(a,\ell) = \overline{\chi}\,(t,a,\ell)\,\mathrm{K}_{\alpha}\big(\mathbb{S}_{\alpha}(t-a)\mathbb{S}_{\alpha}(s)\varphi\big)(\ell). \tag{5.23}$$

On the other hand, using (5.21) and (4.6) we get that

$$\begin{aligned}\mathbb{T}_{0}(t)\mathbb{A}_{\alpha}(s)\varphi(a,\ell) &= \chi\,(t,a,\ell)\,\overline{\chi}\,(s,a-t,\ell)\,\mathrm{K}_{\alpha}\big(\mathbb{S}_{\alpha}(t+s-a)\varphi\big)(\ell)\\ &= \Big[\overline{\chi}\,(t+s,a,\ell) - \overline{\chi}(t,a,\ell)\Big]\mathrm{K}_{\alpha}\big(\mathbb{S}_{\alpha}(t+s-a)\varphi\big)(\ell),\end{aligned} \tag{5.24}$$

for almost all $(a,\ell) \in \Omega$. Putting (5.22), (5.23) and (5.24) in (5.20) we infer

$$\overline{\mathbb{S}}_{\alpha}(t,s)\varphi(a,\ell) = \overline{\chi}\,(t,a,\ell)\,\mathrm{K}_{\alpha}\,\big(\big[\mathbb{S}_{\alpha}(t-a)\mathbb{S}_{\alpha}(s) - \mathbb{S}_{\alpha}(t+s-a)\big]\varphi\big)\,(\ell)$$

which can be written, by virtue of (5.19), in the form

$$\overline{\mathbb{S}}_{\alpha}(t,s)\varphi(a,\ell) = \overline{\chi}\,(t,a,\ell)\,\mathrm{K}_{\alpha}\big(\overline{\mathbb{S}}_{\alpha}(t-a,s)\varphi\big)(\ell) \qquad \text{a.e. } (a,\ell) \in \Omega.$$

Integrating it, we get that

$$\begin{aligned}\big\|\overline{\mathbb{S}}_{\alpha}(t,s)\varphi\big\|_{\mathscr{X}} &= \int_{\ell_1}^{\ell_2}\bigg[\int_0^{\ell} \overline{\chi}\,(t,a,\ell)\,\big|\mathrm{K}_{\alpha}\big(\overline{\mathbb{S}}_{\alpha}(t-a,s)\varphi\big)(\ell)\big|\,\mathrm{d}a\bigg]\,\mathrm{d}\ell\\ &= \int_{\ell_1}^{\ell_2}\bigg[\int_{t-\ell}^{t} \overline{\chi}\,(t,t-x,\ell)\,\big|\mathrm{K}_{\alpha}\big(\overline{\mathbb{S}}_{\alpha}(x,s)\varphi\big)(\ell)\big|\,\mathrm{d}x\bigg]\,\mathrm{d}\ell.\end{aligned}$$

Due to (5.2) we get that $\overline{\chi}\,(t,t-x,\ell) = 1$ if and only if $x > 0$ and therefore

$$\begin{aligned}\big\|\overline{\mathbb{S}}_{\alpha}(t,s)\varphi\big\|_{\mathscr{X}} &= \int_{\ell_1}^{\ell_2}\bigg[\int_0^{t} \big|\mathrm{K}_{\alpha}\,\big(\overline{\mathbb{S}}_{\alpha}(x,s)\varphi\big)\,(\ell)\big|\,\mathrm{d}\ell\bigg]\,\mathrm{d}x\\ &= \int_0^{t}\bigg[\int_{\ell_1}^{\ell_2} \big|\mathrm{K}_{\alpha}\,\big(\overline{\mathbb{S}}_{\alpha}(x,s)\varphi\big)\,(\ell)\big|\,\mathrm{d}x\bigg]\,\mathrm{d}\ell\\ &= \int_0^{t} \big\|\mathrm{K}_{\alpha}\,\big(\overline{\mathbb{S}}_{\alpha}(x,s)\varphi\big)\big\|_{\mathscr{Y}}\,\mathrm{d}x\end{aligned}$$

and by (3.3), it follows that

$$\big\|\overline{\mathbb{S}}_{\alpha}(t,s)\varphi\big\|_{\mathscr{X}} \leqslant \alpha\overline{\kappa}\int_0^{t} \big\|\overline{\mathbb{S}}_{\alpha}(x,s)\varphi\big\|_{\mathscr{X}}\,\mathrm{d}x.$$

Grönwall Lemma yields that $\big\|\overline{\mathbb{S}}_{\alpha}(t,s)\varphi\big\|_{\mathscr{X}} = 0$ which leads to $\overline{\mathbb{S}}_{\alpha}(t,s) = 0$ and therefore (5.18) holds. Our claim, that $(\mathbb{S}_{\alpha}(t))_{t\geqslant 0}$ is a C_0-semigroup on $\mathscr{X}$, is now proved.

Finally, using (5.16) together with, on one hand (4.5) and on the other hand (5.13), we get that

$$\|\mathbb{S}_\alpha(t)\varphi\|_{\mathscr{X}} \leqslant \|\varphi\|_{\mathscr{X}} + \|\mathbb{A}_\alpha(t)\varphi\|_{\mathscr{X}}$$

and
$$\|\mathbb{A}_\alpha(t)\varphi\|_{\mathscr{X}} \leqslant \alpha\overline{\kappa}\int_0^t \|\mathbb{S}_\alpha(x)\varphi\|_{\mathscr{X}}\,\mathrm{d}x$$

which leads to

$$\|\mathbb{S}_\alpha(t)\varphi\|_{\mathscr{X}} \leqslant \|\varphi\|_{\mathscr{X}} + \alpha\overline{\kappa}\int_0^t \|\mathbb{S}_\alpha(x)\varphi\|_{\mathscr{X}}\,\mathrm{d}x.$$

Now Grönwall Lemma proves (5.17).

Step III. Let S_α be the generator of the C_0-semigroup $\mathbb{S}_\alpha = (\mathbb{S}_\alpha(t))_{t\geqslant 0}$.

Firstly, (5.17) means that $(\alpha\overline{\kappa}, \infty) \subset \rho(\mathrm{S}_\alpha)$ (resolvent set of S_α). So, let $\lambda > \alpha\overline{\kappa}$ and let $\varphi \in \mathscr{X}$. Using (5.16) we get that

$$\begin{aligned}(\lambda - \mathrm{S}_\alpha)^{-1}\varphi &= \int_0^\infty e^{-\lambda t}\mathbb{S}_\alpha(t)\varphi\mathrm{d}t\\ &= \int_0^\infty e^{-\lambda t}\mathbb{T}_0(t)\varphi\mathrm{d}t + \int_0^\infty e^{-\lambda t}\mathbb{A}_\alpha(t)\varphi\mathrm{d}t\end{aligned}$$

which leads, by virtue of Lemma 4.2(3) and then (5.14), to

$$\begin{aligned}(\lambda - \mathrm{S}_\alpha)^{-1}\varphi &= (\lambda - \mathrm{T}_0)^{-1}\varphi + \left(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\right)^{-1}\mathcal{K}_{\alpha,\lambda}(\lambda - \mathrm{T}_0)^{-1}\varphi\\ &= \left(\mathbb{1}_{\mathscr{X}} + \left(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\right)^{-1}\mathcal{K}_{\alpha,\lambda}\right)(\lambda - \mathrm{T}_0)^{-1}\varphi\\ &= \left(\mathbb{1}_{\mathscr{X}} - \mathcal{K}_{\alpha,\lambda}\right)^{-1}(\lambda - \mathrm{T}_0)^{-1}\varphi,\end{aligned}$$

and by (4.9),

$$(\lambda - \mathrm{S}_\alpha)^{-1}\varphi = (\lambda - \mathrm{T}_\alpha)^{-1}\varphi,$$

and therefore $\mathrm{S}_\alpha = \mathrm{T}_\alpha$. Now, the uniqueness of the generated semigroup yields that $\mathbb{T}_\alpha(t) = \mathbb{S}_\alpha(t)$ for all $t \geqslant 0$ which proves the desired (5.15) and achieves the proof. ■

6 The model (1.1)–(with $\eta = 0$), (1.3)

In this section we are concerned with the model (1.1), (1.3) without cell transition (*i.e.,* $\eta = 0$). We will prove that the unperturbed model (1.1)–(with $\eta = 0$), (1.3) is governed by a C_0-semigroup on $\mathscr{X}$. Let then V_α be the following unbounded linear operator

$$\mathrm{V}_\alpha := \mathrm{T}_\alpha + \mathrm{Q}_\mu \qquad \text{on} \qquad \mathrm{D}(\mathrm{V}_\alpha) = \mathrm{D}(\mathrm{T}_\alpha) \tag{6.1}$$

with,
$$\mathrm{Q}_\mu\varphi := -\mu\varphi,$$

where μ is subject to the following relevant biological assumptions

$(\mathbf{A}^1_\mu)$
$$\overline{\mu} := \operatorname*{ess\,sup}_{(a,\ell)\in\Omega} |\mu(a,\ell)| < \infty, \tag{6.2}$$

$(\mathbf{A}^2_\mu)$ $\qquad \mu$ is positive.

Theorem 6.1. *Suppose that* $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)$ *holds and let* $\alpha \geqslant 0$. *If* $\left(\mathbf{A}_{\mu}^{1}\right)$ *holds, then* V_{α} *generates, on* $\mathscr{X}$, *a* C_0*−semigroup* $\mathbb{V}_{\alpha} = (\mathbb{V}_{\alpha}(t))_{t\geqslant 0}$. *Furthermore*

(1) *if* $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)$ *and* $\left(\mathbf{A}_{\mu}^{2}\right)$ *hold, then* $\mathbb{V}_{\alpha} = (\mathbb{V}_{\alpha}(t))_{t\geqslant 0}$ *is positive in* $\mathscr{X}$ *and*

$$\mathbb{V}_{\alpha}(t) \geqslant e^{-t\overline{\mu}}\mathbb{T}_{\alpha}(t) \geqslant 0, \qquad\qquad t \geqslant 0, \tag{6.3}$$

where $\overline{\mu}$ *is defined by* (6.2).

(2) *if* $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)'$ *and* $\left(\mathbf{A}_{\mu}^{2}\right)$ *hold and* $\alpha > 0$, *then* $\mathbb{V}_{\alpha} = (\mathbb{V}_{\alpha}(t))_{t\geqslant 0}$ *is irreducible.*

Proof. Obviously, $\left(\mathbf{A}_{\mu}^{1}\right)$ yields that Q_{μ} is a bounded linear operator from $\mathscr{X}$ into itself. Accordingly, V_{α} appears as a bounded linear perturbation of the generator T_{α} (Theorem 4.4) and therefore V_{α} generates, by virtue of Lemma 2.2, a C_0−semigroup $\mathbb{V}_{\alpha} = (\mathbb{V}_{\alpha}(t))_{t\geqslant 0}$ satisfying (2.4), *i.e.*,

$$\mathbb{V}_{\alpha}(t)\varphi = \lim_{n\to\infty}\left(e^{\frac{t}{n}\mathrm{Q}_{\mu}}\mathbb{T}_{\alpha}\left(\tfrac{t}{n}\right)\right)^{n}\varphi, \qquad \varphi \in \mathscr{X}, \qquad t \geqslant 0. \tag{6.4}$$

(1). Let $\varphi \in \mathscr{X}$ be such that $\varphi \geqslant 0$. Then

$$e^{x\mathrm{Q}_{\mu}} = \sum_{m\geqslant 0}\frac{1}{m!}\left(-x\mu\mathbb{1}_{\mathscr{X}}\right)^{m} = \left(\sum_{m\geqslant 0}\frac{1}{m!}\left(-x\mu\right)^{m}\right)\mathbb{1}_{\mathscr{X}} = e^{-x\mu}\mathbb{1}_{\mathscr{X}}$$

and therefore

$$e^{x\mathrm{Q}_{\mu}} \geqslant e^{-x\overline{\mu}}\mathbb{1}_{\mathscr{X}} \qquad \text{for all } x \geqslant 0. \tag{6.5}$$

Let $t > 0$. The positivity of $\mathbb{T}_{\alpha} = (\mathbb{T}_{\alpha}(t))_{t\geqslant 0}$ (Theorem 4.4(1)) together with (6.5) yields, for all inters $n \geqslant 1$, that

$$\left(e^{\frac{t}{n}\mathrm{Q}_{\mu}}\mathbb{T}_{\alpha}\left(\tfrac{t}{n}\right)\right)^{n}\varphi \geqslant \left(e^{-\frac{t}{n}\overline{\mu}}\mathbb{T}_{\alpha}\left(\tfrac{t}{n}\right)\right)^{n}\varphi = e^{-t\overline{\mu}}\left(\mathbb{T}_{\alpha}\left(\tfrac{t}{n}\right)\right)^{n}\varphi = e^{-t\overline{\mu}}\mathbb{T}_{\alpha}(t)\varphi \geqslant 0.$$

Passing to the limit $n \to \infty$ and using (6.4), the desired (6.3) immediately follows and therefore $\mathbb{V}_{\alpha} = (\mathbb{V}_{\alpha}(t))_{t\geqslant 0}$ is positive in $\mathscr{X}$.

(2). The irreducibility of the C_0−semigroup $\mathbb{V}_{\alpha} = (\mathbb{V}_{\alpha}(t))_{t\geqslant 0}$ follows from that of the C_0−semigroup $\mathbb{T}_{\alpha} = (\mathbb{T}_{\alpha}(t))_{t\geqslant 0}$ (Theorem 4.4(2)) because of (6.3). ∎

7 The full model (1.1), (1.3)

The aim of this section is to prove that the full model (1.1), (1.3) is governed by a C_0−semigroup. To this end, we remark that the full model (1.1), (1.3) can be viewed as a linear perturbation of the model (1.1)–($\eta = 0$), (1.3) by the following linear operator

$$\mathrm{P}_{\eta}\varphi(a,\ell) := \int_{\ell_1}^{\ell}\int_{0}^{\ell'}\eta(a,\ell,a',\ell')\varphi(a',\ell')\mathrm{d}a'\mathrm{d}\ell', \qquad (a,\ell) \in \Omega,$$

where η is subject to the following relevant biological assumptions,

$$\left(\mathbf{A}_{\eta}^{1}\right) \qquad \overline{\eta} := \sup_{(a',\ell')\in\Omega} \int_{\ell_1}^{\ell_2}\int_{0}^{\ell} |\eta(a,\ell,a',\ell')| \, \mathrm{d}a\mathrm{d}\ell < \infty, \tag{7.1}$$

$$\left(\mathbf{A}_{\eta}^{2}\right) \qquad \eta \text{ is positive.}$$

So, we have

Lemma 7.1. *If $\left(\mathbf{A}_{\eta}^{1}\right)$ holds, then P_{η} is a bounded linear operator from $\mathscr{X}$ into itself satisfying, for all $\varphi \in \mathscr{X}$,*

$$\left\|\mathrm{P}_{\eta}\varphi\right\|_{\mathscr{X}} \leqslant \overline{\eta}\|\varphi\|_{\mathscr{X}} \tag{7.2}$$

where $\overline{\eta}$ is given by (7.1)*. Moreover if $\left(\mathbf{A}_{\eta}^{2}\right)$ holds, then P_{η} is positive in $\mathscr{X}$.*

Proof. For all $\varphi \in \mathscr{X}$ we have

$$\begin{aligned}
\left\|\mathrm{P}_{\eta}\varphi\right\|_{\mathscr{X}} &= \int_{\ell_1}^{\ell_2}\int_{0}^{\ell} \left| \int_{\ell_1}^{\ell}\int_{0}^{\ell'} \eta(a,\ell,a',\ell')\varphi(a',\ell')\mathrm{d}a'\mathrm{d}\ell' \right| \mathrm{d}a\mathrm{d}\ell \\
&\leqslant \int_{\ell_1}^{\ell_2}\int_{0}^{\ell} \left[\int_{\ell_1}^{\ell_2}\int_{0}^{\ell'} |\eta(a,\ell,a',\ell')|\,|\varphi(a',\ell')|\,\mathrm{d}a'\mathrm{d}\ell' \right] \mathrm{d}a\mathrm{d}\ell \\
&= \int_{\ell_1}^{\ell_2}\int_{0}^{\ell'} \left[\int_{\ell_1}^{\ell_2}\int_{0}^{\ell} |\eta(a,\ell,a',\ell')|\,\mathrm{d}a\mathrm{d}\ell \right] |\varphi(a',\ell')|\,\mathrm{d}a'\mathrm{d}\ell' \\
&\leqslant \left[\sup_{(a',\ell')\in\Omega} \int_{\ell_1}^{\ell_2}\int_{0}^{\ell} |\eta(a,\ell,a',\ell')|\,\mathrm{d}a\mathrm{d}\ell \right] \int_{\ell_1}^{\ell_2}\int_{0}^{\ell'} |\varphi(a',\ell')|\,\mathrm{d}a'\mathrm{d}\ell' \\
&= \qquad\qquad \overline{\eta} \qquad\qquad\qquad \|\varphi\|_{\mathscr{X}}
\end{aligned}$$

which proves (7.2) and therefore P_{η} is bounded because of $\left(\mathbf{A}_{\eta}^{1}\right)$. Finally, the positivity of P_{η} is obvious. ■

Now, let W_{α} be the following unbounded linear operator

$$\mathrm{W}_{\alpha} := \mathrm{V}_{\alpha} + \mathrm{P}_{\eta} \qquad \text{on} \qquad \mathrm{D}(\mathrm{W}_{\alpha}) = \mathrm{D}(\mathrm{T}_{\alpha}) \tag{7.3}$$

which can also be defined by

$$\mathrm{W}_{\alpha} = \mathrm{T}_{\alpha} + \mathrm{Q}_{\mu} + \mathrm{P}_{\eta} \qquad \text{on} \qquad \mathrm{D}(\mathrm{W}_{\alpha}) = \mathrm{D}(\mathrm{T}_{\alpha}) \tag{7.4}$$

because of (6.1). Now, the aim of this section can be announced as follows

Theorem 7.2. *Suppose that $\left(\mathbf{A}_{\mathrm{k}}^{1}\right)$ and $\left(\mathbf{A}_{\mu}^{1}\right)$ hold and let $\alpha \geqslant 0$. If $\left(\mathbf{A}_{\eta}^{1}\right)$ holds, then W_{α} generates, on $\mathscr{X}$, a C_0-semigroup $\mathbb{W}_{\alpha} = (\mathbb{W}_{\alpha}(t))_{t\geqslant 0}$. Furthermore*

(1) *if $\left(\mathbf{A}_{\mathrm{k}}^{2}\right)$, $\left(\mathbf{A}_{\mu}^{2}\right)$ and $\left(\mathbf{A}_{\eta}^{2}\right)$ hold, then $(\mathbb{W}_{\alpha}(t))_{t\geqslant 0}$ is positive in $\mathscr{X}$ and*

$$\mathbb{W}_{\alpha}(t) \geqslant \mathbb{V}_{\alpha}(t) \geqslant 0, \qquad t \geqslant 0. \tag{7.5}$$

(2) *if* $\left(\mathbf{A}^2_{\mathrm{k}}\right)'$, $\left(\mathbf{A}^2_{\mu}\right)$ *and* $\left(\mathbf{A}^2_{\eta}\right)$ *hold and* $\alpha > 0$, *then* $(\mathbb{T}_{\alpha}(t))_{t\geqslant 0}$ *is irreducible.*

Proof. Obviously (7.3) allows us to say that W_{α} is a perturbation of the generator V_{α} (Theorem 6.1) by the bounded linear operator P_{η} (Lemma 7.1). From Lemma 2.2 we infer that W_{α} generates a C_0−semigroup $\mathbb{W}_{\alpha} = (\mathbb{W}_{\alpha}(t))_{t\geqslant 0}$ satisfying (2.4), *i.e.,* for all $\varphi \in \mathscr{X}$,

$$\mathbb{W}_{\alpha}(t)\varphi = \lim_{n\to\infty}\left(e^{\frac{t}{n}\mathrm{P}_{\eta}}\mathbb{V}_{\alpha}\left(\tfrac{t}{n}\right)\right)^n\varphi, \qquad t \geqslant 0. \tag{7.6}$$

(1). Let $\varphi \in \mathscr{X}$ be such that $\varphi \geqslant 0$. Firstly, using the positivity of P_{η} (Lemma 7.1), we can write that

$$e^{x\mathrm{P}_{\eta}}\varphi = \sum_{m\geqslant 0}\frac{1}{m!}\left(x\mathrm{P}_{\eta}\right)^m\varphi \geqslant \varphi, \qquad \text{for all } x \geqslant 0. \tag{7.7}$$

Let $t > 0$. The positivity of $\mathbb{V}_{\alpha} = (\mathbb{V}_{\alpha}(t))_{t\geqslant 0}$ (Theorem 6.1(1)) together with (7.7) yield that

$$\left(e^{\frac{t}{n}\mathrm{P}_{\eta}}\mathbb{V}_{\alpha}\left(\tfrac{t}{n}\right)\right)^n\varphi \geqslant \left(\mathbb{V}_{\alpha}\left(\tfrac{t}{n}\right)\right)^n\varphi = \mathbb{V}_{\alpha}(t)\varphi \geqslant 0, \qquad n = 1, 2, \cdots$$

and at the limit $n \to \infty$ we finally get, by virtue of (7.6), the desired (7.5) and therefore $\mathbb{W}_{\alpha} = (\mathbb{W}_{\alpha}(t))_{t\geqslant 0}$ is positive in $\mathscr{X}$.

(2). The irreducibility of the C_0−semigroup $\mathbb{W}_{\alpha} = (\mathbb{W}_{\alpha}(t))_{t\geqslant 0}$ follows from that of $\mathbb{V}_{\alpha} = (\mathbb{V}_{\alpha}(t))_{t\geqslant 0}$ (Theorem 6.1(2)) because of (7.5). ∎

In the context of Theorem 7.2, the unbounded linear operator W_{α} (defined also by (7.4)) appears as a bounded linear perturbation of the generator T_{α} (Theorem 4.4) by the bounded linear operator $\mathrm{Q}_{\mu} + \mathrm{P}_{\eta}$. According to Lemma 2.2, we infer then that W_{α} generates, on $\mathscr{X}$, the same C_0−semigroup $\mathbb{W}_{\alpha} = (\mathbb{W}_{\alpha}(t))_{t\geqslant 0}$ satisfying (2.3) *i.e.,* for all $\varphi \in \mathscr{X}$,

$$\left\|\mathbb{W}_{\alpha}(t)\varphi\right\|_{\mathscr{X}} \leqslant e^{\left(\alpha\overline{\kappa}+\left\|\mathrm{Q}_{\mu}+\mathrm{P}_{\eta}\right\|\right)t}\left\|\varphi\right\|_{\mathscr{X}} \qquad t \geqslant 0,$$

which is obviously an imprecise inequality. Therefore, in order to improve it, we need the following assumption

$$\left(\mathbf{A}_{\mu-\eta}\right) \qquad\qquad \overline{\eta} \leqslant \underline{\mu}$$

provided that $\left(\mathbf{A}^2_{\mu}\right)$ holds true, where $\overline{\eta}$ is given by (7.1) and

$$\underline{\mu} := \operatorname*{ess\,inf}_{(a,\ell)\in\Omega}\mu(a,\ell). \tag{7.8}$$

The assumption $\left(\mathbf{A}_{\mu-\eta}\right)$ describes mathematically a relevant biological phenomenon. It says that the number of died cells is greater than the number of cells changing their cell cycle length and their age. Now we have

Theorem 7.3. *Suppose that* $\left(\mathbf{A}^1_{\mathrm{k}}\right)$, $\left(\mathbf{A}^1_{\mu}\right)$ *and* $\left(\mathbf{A}^1_{\eta}\right)$ *hold and let* $\alpha \geqslant 0$. *If* $\left(\mathbf{A}^2_{\mu}\right)$ *and* $\left(\mathbf{A}_{\mu-\eta}\right)$ *hold, then the* C_0−*semigroup* $\mathbb{W}_{\alpha} = (\mathbb{W}_{\alpha}(t))_{t\geqslant 0}$ *satisfies, for all* $\varphi \in \mathscr{X}$,

$$\left\|\mathbb{W}_{\alpha}(t)\varphi\right\|_{\mathscr{X}} \leqslant e^{(\alpha\overline{\kappa})t}\left\|\varphi\right\|_{\mathscr{X}} \qquad t \geqslant 0, \tag{7.9}$$

where $\overline{\kappa}$ *is defined by* (3.2).

Proof. Firstly, the existence of the C_0-semigroup $\mathbb{W}_\alpha = (\mathbb{W}_\alpha(t))_{t\geqslant 0}$ is already proved in Theorem 7.2. It remains then to prove (7.9).

Step I. Let $\big(\overline{\mathrm{T}}_\alpha, \mathrm{D}\big(\overline{\mathrm{T}}_\alpha\big)\big)$ be the following unbounded linear operator

$$\overline{\mathrm{T}}_\alpha := \mathrm{T}_\alpha - (\alpha\overline{\kappa})\,\mathbb{1}_{\mathscr{X}} \qquad \text{on} \qquad \mathrm{D}\big(\overline{\mathrm{T}}_\alpha\big) = \mathrm{D}(\mathrm{T}_\alpha). \tag{7.10}$$

As $\overline{\mathrm{T}}_\alpha$ is a bounded linear perturbation of the generator T_α (Theorem 4.4) by the bounded linear operator $(\alpha\overline{\kappa})\,\mathbb{1}_{\mathscr{X}}$, it follows then that $\overline{\mathrm{T}}_\alpha$ generates, on $\mathscr{X}$, the C_0-semigroup $\overline{\mathbb{T}}_\alpha = \big(\overline{\mathbb{T}}_\alpha(t)\big)_{t\geqslant 0} = \big(e^{-(\alpha\overline{\kappa})t}\mathbb{T}_\alpha(t)\big)_{t\geqslant 0}$ satisfying, for all $\varphi \in \mathscr{X}$,

$$\big\|\overline{\mathbb{T}}_\alpha(t)\varphi\big\|_{\mathscr{X}} = e^{-(\alpha\overline{\kappa})t}\|\mathbb{T}_\alpha(t)\varphi\|_{\mathscr{X}} \leqslant e^{-(\alpha\overline{\kappa})t}e^{(\alpha\overline{\kappa})t}\|\varphi\|_{\mathscr{X}} = \|\varphi\|_{\mathscr{X}} \qquad\qquad t \geqslant 0,$$

which means that the C_0-semigroup $\overline{\mathbb{T}}_\alpha = \big(\overline{\mathbb{T}}_\alpha(t)\big)_{t\geqslant 0}$ is of contractions and therefore $\overline{\mathrm{T}}_\alpha$ is a dissipative generator.

Step II. For all $\varphi \in \mathscr{X}$ we have

$$\begin{aligned}
\big\langle \operatorname{sgn}\varphi, \big(\mathrm{Q}_\mu + \mathrm{P}_\eta\big)\,\varphi\big\rangle &= \int_{\ell_1}^{\ell_2}\int_0^{\ell} \operatorname{sgn}\varphi(a,\ell)\Big(\mathrm{Q}_\mu\varphi(a,\ell) + \mathrm{P}_\eta\varphi(a,\ell)\Big)\mathrm{d}a\mathrm{d}\ell \\
&\leqslant -\int_{\ell_1}^{\ell_2}\int_0^{\ell} \mu(a,\ell)\,|\varphi(a,\ell)|\,\mathrm{d}a\mathrm{d}\ell + \big\|\mathrm{P}_\eta\varphi\big\|_{\mathscr{X}} \\
&\leqslant -\underline{\mu}\|\varphi\|_{\mathscr{X}} + \big\|\mathrm{P}_\eta\varphi\big\|_{\mathscr{X}}
\end{aligned}$$

which leads, by virtue of (7.2) and then $\big(\mathbf{A}_{\mu-\eta}\big)$, to

$$\big\langle \operatorname{sgn}\varphi, \big(\mathrm{Q}_\mu + \mathrm{P}_\eta\big)\,\varphi\big\rangle \leqslant \big(-\underline{\mu} + \overline{\eta}\big)\,\|\varphi\|_{\mathscr{X}} \leqslant 0$$

and therefore $\mathrm{Q}_\mu + \mathrm{P}_\eta$ is a dissipative operator.

Step III. The boundedness of $\mathrm{Q}_\mu + \mathrm{P}_\eta$ allows us to write that,

$$\begin{aligned}
&\mathrm{D}\big(\overline{\mathrm{T}}_\alpha\big) \subset \mathscr{X} = \mathrm{D}\big(\mathrm{Q}_\mu + \mathrm{P}_\eta\big) \quad \text{and for all} \quad \varphi \in \mathrm{D}\big(\overline{\mathrm{T}}_\alpha\big),\\
&\big\|\big(\mathrm{Q}_\mu + \mathrm{P}_\eta\big)\,\varphi\big\|_{\mathscr{X}} \leqslant 0\cdot\big\|\overline{\mathrm{T}}_\alpha\varphi\big\|_{\mathscr{X}} + \big\|\mathrm{Q}_\mu + \mathrm{P}_\eta\big\|_{\mathscr{L}(\mathscr{X})}\|\varphi\|_{\mathscr{X}}.
\end{aligned}$$

Accordingly, the following unbounded linear operator

$$\overline{\mathrm{W}}_\alpha := \overline{\mathrm{T}}_\alpha + \mathrm{Q}_\mu + \mathrm{P}_\eta \qquad \text{on} \qquad \mathrm{D}\big(\overline{\mathrm{W}}_\alpha\big) = \mathrm{D}\big(\overline{\mathrm{T}}_\alpha\big) \tag{7.11}$$

appears as a 0-bounded linear dissipative perturbation (Step II) of the dissipative generator $\overline{\mathrm{T}}_\alpha$ (Step I). Accordingly, all the required conditions of Lemma 2.3 are fulfilled and therefore $\overline{\mathrm{W}}_\alpha$ generates, on $\mathscr{X}$, a C_0-semigroup $\overline{\mathbb{W}}_\alpha = \big(\overline{\mathbb{W}}_\alpha(t)\big)_{t\geqslant 0}$ of contractions *i.e.*, satisfying for all $\varphi \in \mathscr{X}$,

$$\big\|\overline{\mathbb{W}}_\alpha(t)\varphi\big\|_{\mathscr{X}} \leqslant \|\varphi\|_{\mathscr{X}} \qquad\qquad t \geqslant 0. \tag{7.12}$$

Finally, according to (7.4) together with (7.10) and (7.11), we infer that

$$\mathrm{W}_\alpha = \mathrm{T}_\alpha + \mathrm{Q}_\mu + \mathrm{P}_\eta = \overline{\mathrm{T}}_\alpha + (\alpha\overline{\kappa})\,\mathbb{1}_{\mathscr{X}} + \mathrm{Q}_\mu + \mathrm{P}_\eta = \overline{\mathrm{W}}_\alpha + (\alpha\overline{\kappa})\,\mathbb{1}_{\mathscr{X}}$$

which means that W_α is a bounded linear perturbation of the generator $\overline{\mathrm{W}}_\alpha$ by the bounded linear operator $(\alpha\overline{\kappa})\,\mathbb{1}_{\mathscr{X}}$. Hence W_α generates, on $\mathscr{X}$, the C_0−semigroup $\mathbb{W}_\alpha = (\mathbb{W}_\alpha(t))_{t\geqslant 0} = \left(e^{(\alpha\overline{\kappa})t}\overline{\mathbb{W}}_\alpha(t)\right)_{t\geqslant 0}$ satisfying, for all $\varphi \in \mathscr{X}$,

$$\|\mathbb{W}_\alpha(t)\varphi\|_{\mathscr{X}} = e^{(\alpha\overline{\kappa})t}\left\|\overline{\mathbb{W}}_\alpha(t)\varphi\right\|_{\mathscr{X}} \leqslant e^{(\alpha\overline{\kappa})t}\|\varphi\|_{\mathscr{X}} \qquad t \geqslant 0$$

because of (7.12). The proof is now achieved. ■

Remark 7.4. Theorem 7.3 means that the full model (1.1), (1.3), *i.e.,*

$$\begin{cases} \dfrac{\partial f}{\partial t} = -\dfrac{\partial f}{\partial a} - \mu f + \displaystyle\int_{\ell_1}^{\ell}\int_0^{\ell'} \eta(a,\ell,a',\ell')f(t,a',\ell')\mathrm{d}a'\mathrm{d}\ell', \\ f(t,0,\ell) = \alpha \displaystyle\int_{\ell_1}^{\ell_2}\int_0^{\ell'} \mathrm{k}(\ell,a',\ell')f(t,a',\ell')\mathrm{d}a'\mathrm{d}\ell', \\ f(0,a,\ell) = \varphi(a,\ell), \end{cases}$$

is well posed in $\mathscr{X}$. Namely, for all initial data $f(0) \in \mathscr{X}$ there exists a unique solution

$$f(t) = \mathbb{W}_\alpha(t)f(0) \in \mathscr{X} \qquad \text{satisfying} \qquad \|f(t)\|_{\mathscr{X}} \leqslant e^{(\alpha\overline{\kappa})t}\|f(0)\|_{\mathscr{X}}. \tag{7.13}$$

8 Comments

Remark 8.1. The choice of the functional framework $\mathscr{X}$ $(= L^1(\Omega))$ was natural because $\|f(t)\|_{\mathscr{X}}$ denotes the cell number at time $t \geqslant 0$. Nevertheless, we can easily extended this work to the case $p > 1$, *i.e.,*

$$\mathscr{X} := L^p(\Omega) \qquad \text{whose norm is} \qquad \|\varphi\|_{\mathscr{X}} := \left[\int_{\ell_1}^{\ell_2}\int_0^{\ell} |\varphi(a,\ell)|^p\,\mathrm{d}a\mathrm{d}\ell\right]^{\frac{1}{p}}$$

and $$\mathscr{Y} := L^p(\ell_1,\ell_2) \qquad \text{whose norm is} \qquad \|\psi\|_{\mathscr{Y}} := \left[\int_{\ell_1}^{\ell_2} |\psi(\ell)|^p\,\mathrm{d}\ell\right]^{\frac{1}{p}}.$$

To this end, it suffices to replace (3.2) by

$$\overline{\kappa} = \left[\operatorname*{ess\,sup}_{(a',\ell')\in\Omega}\int_{\ell_1}^{\ell_2} |\mathrm{k}(\ell,a',\ell')|\,\mathrm{d}\ell\right]^{\frac{1}{p}}\left[\operatorname*{ess\,sup}_{\ell\in(\ell_1,\ell_2)}\int_{\ell_1}^{\ell_2}\int_0^{\ell'} |\mathrm{k}(\ell,a',\ell')|\,\mathrm{d}a'\mathrm{d}\ell'\right]^{\frac{1}{q}}$$

and (7.1) by

$$\overline{\eta} = \left[\sup_{(a',\ell')\in\Omega}\int_{\ell_1}^{\ell_2}\int_0^{\ell} |\eta(a,\ell,a',\ell')|\,\mathrm{d}a\mathrm{d}\ell\right]^{\frac{1}{p}}\left[\sup_{(a,\ell)\in\Omega}\int_{\ell_1}^{\ell_2}\int_0^{\ell'} |\eta(a,\ell,a',\ell')|\,\mathrm{d}a'\mathrm{d}\ell'\right]^{\frac{1}{q}}$$

where $q > 1$ is the conjugate of $p > 1$, *i.e.,* $\frac{1}{p} + \frac{1}{q} = 1$. The other assumptions remain unchanged.

Remark 8.2. If $\alpha = 0$ (*i.e.,* the case of no mitosis) then (7.13) leads to

$$\|f(t)\|_{\mathscr{X}} \leqslant \|f(0)\|_{\mathscr{X}}$$

which means that the number of cells does not increase and therefore no biological interest can be observed in this case. However, if $\alpha \neq 0$ then (7.13) means that the number of cells increases which leads us to naturally ask the following question : what happens when the cell density is increasing? This question is answered in [7].

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