**Advances in Mathematical Sciences and Applications**
**Vol. 30, No. 1 (2021), pp. 105–127**

GAKKOTOSHO
TOKYO JAPAN

# A GEOMETRIC APPROACH TO PRICING MULTI-ASSET BARRIER OPTIONS

Estevão Rosalino Jr.
National Lab. for Scientific Computing - Systems and Control Dept., Petrópolis, RJ, Brazil
(E-mail: junior.estevao@gmail.com)

Juan B. R. Otazú
National Lab. for Scientific Computing - Systems and Control Dept., Petrópolis, RJ, Brazil
(E-mail: bladi_win@hotmail.com)

Allan J. da Silva
National Lab. for Scientific Computing - Systems and Control Dept., Petrópolis, RJ, Brazil
(E-mail: allan.jonathan@gmail.com)

and

Jack Baczynski
National Lab. for Scientific Computing - Systems and Control Dept., Petrópolis, RJ, Brazil
(E-mail: jack@lncc.br)

**Abstract.** We introduce multi-asset options with barrier of hyperplane type and, for a seminal class of these, closed-form expressions of no-arbitrage prices are obtained. An interesting aspect of such seminal structure is that it boils down to a well exploited financial object: a geometric basket of assets. This unveils the fact that general hyperplane structures naturally represent barrier formats and significantly expand the design possibilities of barrier options. In this work the risky assets evolve independently at first and, in the sequel, results for the dependent case are provided. We also provide an expression for a certain joint density function from which prices for more general barrier options could be obtained numerically.

Communicated by Editors; Received March 2, 2020
This work is supported by CNPq under the grant 300456/2019-0.
AMS Subject Classification: 60H05, 60J60, 91G20, 91G30, 93E20.
Keywords: Multi-Asset Barrier Option, No-arbitrage Pricing, Correlation, Hyperplane Barrier, Reflection Principle, Time-Change.

# 1 Introduction

European barrier options are a class of path dependent options that emerged as a significant product for hedging investments in equity and commodity markets since the late 1980s, largely in the over-the-counter markets. They are standard European options that involve a barrier constraint tailored for a certain behavior of the underlying asset along the option's lifetime. If this behavior is what the investor thinks will happen, then he may pay less buying the barrier option instead of its standard counterpart, obtaining the same result whenever his beliefs meet reality. Otherwise, the option's payoff cancels. We address the aforementioned subject matter, bringing into focus the multi-asset scenario.

Little work has been done to derive prices of path-dependent options (not necessarily of barrier type) in this sort of scenario. Quoting [21], an important reason for this is because the matter poses significant technical difficulties that do not appear in the single-asset case. Stepping works in this niche are [7] and [17], which consider the Black and Scholes world, and [24] - entering with jumps. A study on multi-asset Black-Scholes models with correlated underlyings is found in [12]. The work unveils the fact that certain conditions have to be satisfied in order to ensure existence of a valid pricing solution in such scenario, and that certain formulas that appear in the literature should be revisited for validation or not. When entering with models with more sophisticated structure (e.g. with stochastic covariance), most exotic options are priced numerically. An analytical result is that of [2] addressing the case of unrestricted correlations in a two-dimensional geometric Brownian model with a common stochastic volatility factor. There, closed-form expressions can be found for pricing two-asset barrier options (double-digital, spread and double-lookback options). Actually, the authors there reportedly claim that their work is the first analytical development in the multi-asset correlated framework.

The common feature of the multi-asset kind of options is to hedge the risk associated with a prescribed set of assets in a unified way. The investor can thus manage the risk of a hedging portfolio as a whole. In spite of the fact that we are dealing here with frictionless markets, a by-product of such options is avoiding multiple transaction costs replacing it by a single one. Now, a desired quality of multi-asset options, particularly multi-asset barrier options, is to capture the *joint* behavior of the assets involved. For instance, the price of a certain asset at a certain time may be viewed as "good" or admissible, or not, depending on what is happening with the price of some other asset or assets. This is not the case of the barrier options treated in [2], where the prices of the risky assets are tested by individual barriers.

In contrast, this notion is plainly captured by the options in this work. Their behavior is such that a sole asset cannot impose the option to be knocked out or not. Knock out will actually depend on the *joint* behavior of the assets. As we said above, there is no good or bad price that can be assigned beforehand to a certain asset. This condition will be determined, dynamically, by the behavior of the remaining assets. Namely, we introduce some sorts of European multi-asset up-and-out call options with barriers of hyperplane type. The hyperplane monitors the *discounted log price vector* of the underlying assets, i.e., the option cancels when this vector breaches the hyperplane. Alternatively, the barrier scheme can be viewed as an exponential modification of the hyperplane that now tests the discounted price vector of the underlyings. It is interesting reminding that an assortment of celebrated results addressing the scalar barriers with exponential shape, not in space, but in time, was obtained between 1994 and 2002, as can be observed in [4], [23], [14], [1].

The class of hyperplane barriers addressed herein is actually of a seminal type: the one generated by the vector $\overrightarrow{1} = (1, ...1) \in \mathcal{R}^n$. One could imagine spacial perturbations on this class producing a broad range of barrier options, which no-arbitrage prices could possibly be solved numerically or inferred via estimates stemming from options equipped with the above seminal class. A simple example is a class of hyperplane barriers now generated by $\overrightarrow{v} = (v_1, ...v_n) \in \mathcal{R}^n$, $v_i > 0$.

A quality of our seminal hyperplane monitoring scheme is that it actually refers to a well exploited financial object that interest practitioners: a geometric basket of assets. More precisely, saying that the option cancels whenever the discounted log price vector of assets breaches the hyperplane is the same as saying that the option cancels whenever the price of the geometric basket composed by such assets breaches the barrier ($B$) that tracks the price ($\beta$) of the risk-free asset (the value of the bank account), where

$$B(u) = c\beta(u), \qquad \beta(u) = \beta(0)e^{\int_0^u r(s)ds}, \tag{1}$$

$c > 0$, $\beta(0)$ is the initial value arbitrarily set for the bank account, $c\beta(0)$ is the barrier value at time zero and $r$ is the instantaneous interest rate. There is a one-to-one correspondence between the hyperplane barrier scheme and the one described above (this is not difficult to see and is shown ahead in the paper). Options equipped with the scalar barrier (1) are applied as building blocks for investment products like certificates in Germany, according to [2]. Moreover, two aspects may interest practitioners: (i) the idea of a barrier that tracks the risk-free asset is not strange in the context of the fixed income markets (see, e.g., [5]), although there, interest rates are naturally stochastic, whereas in the case of equity markets, they are predominantly deterministic (the rationale behind this barrier is fully described in [6]) and (*ii*) the scalar barrier (1) is constant if denominated in terms of the numeraire taken to be the share of the risk-free asset.

Now, the hyperplane format produces a multidimensional test while in the geometric basket format we have a scalar barrier test instead. One would correctly expect that a multidimensional test would be richer in information than a scalar test. Nevertheless, the two formats above are equivalent. The reason for this is that hyperplanes are particular and simple manifolds and indeed boil down to more strict information, scalar-equivalent.

So, in this paper, we present elegant closed-form pricing formulas for the exact prices of some sorts of European multi-asset up-and-out call options with barriers of seminal hyperplane type. Or else, we shall be solving the geometric basket pricing problem in the light of a multidimensional framework - a starting point for pricing or estimating more generic multidimensional barrier options. The stochastic underpinning tools are the risk-neutral pricing technique combined with the multidimensional version of the time change for martingales theorem (Knight's Theorem - see [3]), and the Generalised Reflection Principle for Brownian Motion devised by [21].

Models are à la Black and Scholes (BS) with time dependent volatility. Firstly, we impose the assets to be independent and, in the sequel, we assume dependence among the assets - which allows the investor to hedge more efficiently correlation risk. In the latter case, incomplete markets may arise, and consequently non-attainable derivatives. Still, our prices are good approximations, as we can see invoking the risk-minimizing approach, a technique that inherits concepts of the theory of optimal control ([13]).

BS models for pricing stock derivatives are still often used. This is so because the model is robust, simple and low-cost in terms of computations ([11], [16], [15], [22], [18], [9]). It is noteworthy that some works dealing with stochastic volatility focus on whether their solutions have indeed a significative better performance then the BS counterpart in applications with real market data in order to justify the use of more complex models ([20]).

Bearing in mind that pure analytical results are difficult, particularly on multivariate models (see [10], [2], [12]), the fast and ready to use pricing formulas we provide here can also serve as estimates for prices stemming from "nearby" models.

## 1.1 The Financial Pricing Problem and Findings

Let $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ be a probability space, with $\tilde{\mathbb{P}}$ a risk neutral probability measure.

### 1.1.1 Independent asset prices

On this probability space, consider a multidimensional market model where the risky assets prices evolve according to the stochastic differential equations (SDEs)

$$dS_i(u) = r(u)S_i(u)du + \sigma_i(u)S_i(u)d\tilde{W}_i(u), \quad i = 0, 2, \ldots, m, \;\; u \in [0, T], \tag{2}$$

where $\tilde{W} = (\tilde{W}_0, \tilde{W}_1, \ldots, \tilde{W}_m)$ is an $(m+1)$-dimensional $\tilde{\mathbb{P}}$-Brownian Motion and $(\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_t \subseteq \mathcal{F}$, a filtration for this Brownian Motion. The risk-free interest rate $r$ and the volatilities $\sigma_i$, $i = 0..., m$, are square integrable deterministic functions of time.
European multi-asset up-and-out call options with barriers of hyperplane type are considered, where the payoffs *per se*, i.e., without the barrier, are

$(i)$ on a stock $S_0$ that is not verified by the hyperplane barrier; however it is tested by the individual barrier

$$B_0(u) = c_0\beta(u) \tag{3}$$

where $c_0\beta(0)$ is the value of the barrier $B_0$ at time zero (note that pushing $c_0$ to infinity recovers the case of a payoff on $S_0$ without the barrier described in (3)), and

$(ii)$ on a geometric basket of stocks, involving the assets $S_1, ...S_m$.

Now define the discounted log price vector of the underlying assets by

$$\ln \bar{S}(u) = \big(\ln \bar{S}_1(u), ... \ln \bar{S}_m(u)\big) \tag{4}$$

where $\bar{S}_i(u) = \frac{S_i(u)}{\beta(u)}$ is the price of the $i^{th}$ asset subscribed on the numeraire taken to be the price of the risk-free asset $\beta$. If this vector breaches the hyperplane under concern at any time prior to $T$, the payoff renders null. This barrier is described in Section 2 and appears in all cases throughout the paper.

Section 3 provides closed form expressions for the exact prices of multi-asset up-and-out call barrier options according to items $(i)$ and $(ii)$ above. A class of joint density functions is also derived, from which prices of multi-asset up-and-out call barrier options with more generic payoffs than those of items $(i)$ and $(ii)$ can be derived numerically.

### 1.1.2 Dependent asset prices

We address the case of non-necessarily complete markets equipped with a risk neutral measure, where the risky assets are no longer independent on each other and the asset prices $S_1, ...S_m$ evolve according to

$$dS_i(t) = r(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d} \sigma_{ij}(t)\, d\tilde{W}_j(t), \quad i = 1, \dots, m, \quad t \in [0, T]. \tag{5}$$

We have that $m$ and $d$ are natural numbers, $(\tilde{W}_1(t), \dots, \tilde{W}_d(t))$ is a $d$-dimensional $\tilde{\mathbb{P}}$-Brownian Motion and a filtration $(\mathcal{F}_t)_{t\geq 0}$ is considered for this Brownian Motion. The risk-free interest rate $r$ and the components of the volatility matrix $\sigma(t) = (\sigma_{ij}(t))_{i=1,\dots,m,j=1,\dots,d}$ are square integrable deterministic functions of time. The payoff *per se*, i.e., without the barrier, is on a geometric basket of stocks, as in $(ii)$ above, and the barrier that tests the log price vector of the underlyings is a hyperplane, verbatim as above.

Section 4 provides closed form expressions for the exact prices of multi-asset up-and-out geometric basquet options (according to $(ii)$ above), addressing a bidimensional setup for the dynamics. It is noteworthy that a bidimensional setup is still good since, in practice, two-asset options are often traded. However, being aware of the importance of multi-asset options, we provide an expression for a certain class of joint density function from which prices can be derived numerically for options according to $(i)$, $(ii)$ or more generic ones.

# 2 The Hyperplane Barrier

The hyperplane barrier refers to a hyperplane that monitors the discounted log price vector of the underlyings. So, we are allowed to test if, over $[0, T]$, the vector $\ln \bar{S}(u) \equiv \big(\ln(\bar{S}_1(u)), \dots, \ln(\bar{S}_m(u))\big) \in \mathbb{R}^m$ breaches the hyperplane

$$\mathcal{H} = \{x \in \mathbb{R}^m \,|\, \vec{1} \cdot x = h\}, \tag{6}$$

where $h > 0$, $\vec{1} = (1, \dots, 1) \in \mathbb{R}^m$ and $z \cdot y$ denotes the usual inner product for $z, y \in \mathbb{R}^m$. A necessary and sufficient condition for the option to be exercised is having

$$\vec{1} \cdot \ln \bar{S}(u) \leq h, \ \forall\, u \in [0, T],$$

$\bar{S}(u) \equiv \big((\bar{S}_1(u)), \dots, (\bar{S}_m(u))\big)$.
This barrier scheme can also be viewed as an exponential modification of the hyperplane now monitoring the discounted price vector of the underlyings, i.e., we may test if, over $[0, T]$, $\bar{S}(u)$ breaches the surface

$$\mathcal{H}_e = \{y \in \mathbb{R}^m \,|\, \exists\; x \in \mathcal{H} \; s.t. \; y = e^x\},$$

$e^x = (e^{x_1}, ...e^{x_m})$. Defining the half space $\mathcal{S} = \{x \in \mathbb{R}^m \,|\, \vec{1} \cdot x \leq h\}$, we have that a necessary and sufficient condition for the option to be exercised is having

$$\bar{S}(u) \in \mathcal{S}_e = \{y \in \mathbb{R}^m \,|\, \exists\, x \in \mathcal{S} \ s.t.\ y = e^x\} \ \forall\, u \in [0, T].$$

Now, reminding that $\bar{S}_i(u) = \frac{S_i(u)}{\beta(u)}$ and $\ln \bar{S}(u) \equiv \big((\ln \bar{S}_1(u)), \ldots, (\ln \bar{S}_m(u))\big)$, and noticing that

$$\exp\left(\vec{1} \cdot \ln \bar{S}(u)\right) = \frac{\prod_{i=1}^m S_i(u)}{\beta^m(u)},$$

it follows that

$$\vec{1} \cdot \ln \bar{S}(u) \leq h \Leftrightarrow \prod_{i=1}^{m} (S_i(u))^{1/m} \leq c\beta(u),$$

where $c = e^{h/m}$. So, saying that the option cancels whenever the discounted log price vector of assets breaches the hyperplane (6) is the same as saying that the option cancels whenever the price of the geometric basket $\prod_{i=1}^m (S_i(u))^{1/m}$ breaches the barrier (B) that tracks the price ($\beta$) of the risk-free asset as given in (1). This equivalence, where a well exploited financial object appears, is due to the seminal $\overrightarrow{1}$-hyperplane format, which is the concern of this paper.

# 3 Pricing Hyperplane Barrier Options with Independent Assets

In this section, we obtain closed-form expressions for pricing up-and-out call options equipped with hyperplane barriers and independent risky assets, as described in Sections 1.1.1 and 2.

Before we skip to the main results, let us state an important lemma. Firstly, in order to have a zero-origin starting point, define $\eta^{t_0}(u) = \eta(t_0 + u)$, $0 \leq u \leq T - t_0$, where $\eta$ stands for $r$, $\sigma_0$, $\sigma_i$, $S$ and $\tilde{W}$. It follows that

$$dS_i^{t_0}(u) = r^{t_0}(u) S_i^{t_0}(u) du + \sigma_i^{t_0}(u) S_i^{t_0}(u) d\tilde{W}_i^{t_0}(u),$$

$S_i^{t_0}(0) = S_i(t_0)$, $i = 0, \ldots, m$. The $t_0$-discounted risky asset price processes

$$\bar{S}_i^{t_0}(u) = S_i^{t_0}(u)(\beta(t_0)/\beta(t_0 + u)) = S_i(t_0) \exp\{\hat{I}_i^{t_0}(u)\}$$

are martingales under $\tilde{\mathbb{P}}$, where

$$\hat{I}_i^{t_0}(u) = \int_0^u \sigma_i^{t_0}(s) d\tilde{W}_i^{t_0}(s) - \frac{1}{2} \int_0^u \left(\sigma_i^{t_0}(s)\right)^2 ds, \quad 0 \leq u \leq T - t_0,\ i = 0, \ldots, m. \tag{7}$$

Also define

$$\hat{M}_0^{t_0} = \max_{0 \leq u \leq T - t_0} \hat{I}_0^{t_0}(u), \quad \text{so that} \quad \max_{0 \leq u \leq T - t_0} \bar{S}_0^{t_0}(u) = S_0(t_0) \exp\{\hat{M}_0^{t_0}\}, \tag{8}$$

and

$$\hat{M}^{t_0} = \max\nolimits_{0 \leq u \leq T - t_0} \vec{1} \cdot \hat{I}^{t_0}(u) \quad \text{with} \quad \hat{I}^{t_0}(u) = \left(\hat{I}_1^{t_0}(u), \ldots, \hat{I}_m^{t_0}(u)\right). \tag{9}$$

To be used ahead, notice that

$$\begin{aligned} \{S_0(u) \leq B_0(u) \ \forall u \in [t_0, T]\} &= \{\bar{S}_0^{t_0}(u) \leq B_0(t_0) \ \forall u \in [0, T-t_0]\} \\ &= \{S_0(t_0) \exp\{\hat{M}_0^{t_0}\} \leq B_0(t_0)\}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \{\vec{1} \cdot \ln \bar{S}(u) \leq h \ \forall \ u \in [t_0, T]\} &= \{\vec{1} \cdot \ln \bar{S}^{t_0}(u) \leq h \ \forall \ u \in [0, T-t_0]\} \\ &= \{\vec{1} \cdot \hat{I}^{t_0}(u) \leq \bar{h} \ \forall \ u \in [0, T-t_0]\} \\ &= \left\{ \max_{0 \leq u \leq T-t_0} \vec{1} \cdot \hat{I}^{t_0}(u) \leq \bar{h} \ \forall \ u \in [0, T-t_0] \right\} \\ &= \{\hat{M}^{t_0} \leq \bar{h}\}, \end{aligned} \tag{11}$$

where $\bar{h} = h - \vec{1} \cdot \ln S(t_0)$.

We now derive the expression for a certain joint probability function, from which our pricing result stems.

**Lemma 3.1.** *Let us consider an arbitrary but fixed time $t_0 \in [0, T)$ and assume that*

$$\int_{t_0}^{t} \sigma_i^2(u)du = \int_{t_0}^{t} \sigma_j^2(u)du := Q(t), \quad i \neq j \neq 0, \quad \forall \ t \in [t_0, T]. \tag{12}$$

*Then, under $\tilde{\mathbb{P}}$, the joint density function of $\left(\hat{M}^{t_0}, \vec{1} \cdot \hat{I}^{t_0}(T-t_0)\right)$ is given by*

$$\tilde{f}_{\hat{M}^{t_0}, \vec{1} \cdot \hat{I}^{t_0}(T-t_0)}(x, y) = \begin{cases} \frac{2(2x-y)}{m\,Q(T)\sqrt{2\pi\,m\,Q(T)}} \exp\left\{-\frac{1}{2}y - \frac{m}{8}Q(T) - \frac{(2x-y)^2}{2\,m\,Q(T)}\right\}, y \leq x, \ x > 0; \\ 0, \quad \textit{otherwise}. \end{cases} \tag{13}$$

*Proof.* Firstly, we obtain a joint density function of the pair $\left(\hat{M}^{t_0}, \vec{1} \cdot \hat{I}^{t_0}(T-t_0)\right)$ with respect to an auxiliary measure $\hat{\mathbb{P}}$, under which the process $\hat{I}^{t_0}$ is an $m$-dimensional martingale. So, relying on the square integrability of $\sigma_i(u)$, $i = 1, \ldots, m$, define the Radon-Nikodým derivative process

$$Z^{t_0}(u) = \exp\left\{ \sum_{i=1}^{m} \int_0^u \frac{\sigma_i^{t_0}(s)}{2} d\tilde{W}_i^{t_0}(s) - \frac{1}{8} \sum_{i=1}^{m} \int_0^u \left(\sigma_i^{t_0}(s)\right)^2 ds \right\}, \tag{14}$$

$0 \leq u \leq T - t_0$, as well as the new measure $\hat{\mathbb{P}}(A) = \int_A Z^{t_0}(T-t_0)d\tilde{\mathbb{P}}$, $A \in \mathcal{F}$, under which (we invoke Girsanov's Theorem - see, e.g., [19]) $\hat{W}(u) = (\hat{W}_1(u), \ldots, \hat{W}_m(u))$ with $\hat{W}_i(u) = \tilde{W}_i^{t_0}(u) - \frac{1}{2}\int_0^u \sigma_i^{t_0}(s)ds$, $i = 1, \ldots, m$, is an $m$-dimensional Brownian Motion and $\hat{I}^{t_0}(u) = (\hat{I}_1^{t_0}(u), \ldots, \hat{I}_m^{t_0}(u))$, $0 \leq u \leq T - t_0$, is an $m$-dimensional continuous Martingale, where

$$\hat{I}_i^{t_0}(u) = \int_0^u \sigma_i^{t_0}(s) d\hat{W}_i(s), \quad i = 1, \ldots, m.$$

By virtue of the time-change for Martingales theorem - Knight's Theorem (see, e.g. [3]) and bearing in mind the condition (12), we may further express $\hat{I}^{t_0}$ as

$$\hat{I}^{t_0}(u) = \mathcal{W}\left(h^{t_0}(u)\right), \quad 0 \leq u \leq T - t_0, \tag{15}$$

where $\mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_m)$ is a m-dimensional Brownian Motion under $\hat{\mathbb{P}}$, and

$$h^{t_0}(u) = \left[\hat{I}_i^{t_0}, \hat{I}_i^{t_0}\right](u) = \int_0^u \left(\sigma_i^{t_0}(s)\right)^2 ds$$

is the quadratic variation of $\hat{I}_i^{t_0}(u)$ for all $i = 1, \ldots, m$. Moreover, since $h^{t_0}(u)$ is continuous and increasing, (9) gives us that $\hat{M}^{t_0} = \max_{0 \leq s \leq h^{t_0}(T-t_0)} \vec{1} \cdot \mathcal{W}(s)$.

This, (15) and Proposition 3.25 in [21] allow us to state the following version of the generalised Reflection Principle extended for continuous-time Martingales whose components have null covariation and the same deterministic quadratic variation function:

$$\hat{\mathbb{P}}\left\{\hat{M}^{t_0} \geq x, \vec{1} \cdot \hat{I}^{t_0}(T - t_0) \leq y\right\} = \hat{\mathbb{P}}\left\{\vec{1} \cdot \hat{I}^{t_0}(T - t_0) \geq 2x - y\right\}, \quad y \leq x,\, x > 0,$$

or else,

$$\int_x^{+\infty} \int_{-\infty}^y \hat{f}_{\hat{M}^{t_0}, \vec{1} \cdot \hat{I}^{t_0}(T-t_0)}(w, v)\, dv\, dw = \hat{\mathbb{P}}\left\{\vec{1} \cdot \hat{I}^{t_0}(T - t_0) \geq 2x - y\right\}, \quad y \leq x,\, x > 0, \tag{16}$$

where $\hat{f}_{\hat{M}^{t_0}, \vec{1} \cdot \hat{I}^{t_0}(T-t_0)}$ is the joint density function of $\left(\hat{M}^{t_0}, \vec{1} \cdot \hat{I}^{t_0}(T - t_0)\right)$ under $\hat{\mathbb{P}}$. Differentiating (16) twice, first with respect to $x$ and then with respect to $y$, and bearing in mind that $\vec{1} \cdot \hat{I}^{t_0}(T - t_0)$ is normally distributed with zero mean and variance $m.h^{t_0}(T - t_0)$, we obtain

$$\hat{f}_{\hat{M}^{t_0}, \vec{1} \cdot \hat{I}^{t_0}(T-t_0)}(x, y) = \frac{2(2x - y)}{m.h^{t_0}(T - t_0)\sqrt{2\pi m h^{t_0}(T - t_0)}} \exp\left\{\frac{-(2x - y)^2}{2mh^{t_0}(T - t_0)}\right\}, \tag{17}$$

$y \leq x$, $x > 0$. On the other hand,

$$\begin{aligned}
Z^{t_0}(T - t_0) &= \exp\left\{\sum_{i=1}^m \left(\int_0^{T-t_0} \frac{\sigma_i^{t_0}(s)}{2}\left(d\hat{W}_i^{t_0}(s) + \frac{1}{2}\sigma_i^{t_0}(s)ds\right) - \frac{1}{8}\int_0^{T-t_0} \left(\sigma_i^{t_0}(s)\right)^2 ds\right)\right\} \\
&= \exp\left\{\frac{1}{2}\sum_{i=1}^m \hat{I}_i^{t_0}(T - t_0) + \frac{1}{4}\sum_{i=1}^m h^{t_0}(T - t_0) - \frac{1}{8}\sum_{i=1}^m h^{t_0}(T - t_0)\right\} \\
&= \exp\left\{\frac{1}{2}\left(\vec{1} \cdot \hat{I}^{t_0}(T - t_0)\right) + \frac{1}{8}\sum_{i=1}^m h^{t_0}(T - t_0)\right\} \\
&= \exp\left\{\frac{1}{2}\left(\vec{1} \cdot \hat{I}^{t_0}(T - t_0)\right) + \frac{m}{8}h^{t_0}(T - t_0)\right\},
\end{aligned}$$

so we may write

$$\tilde{\mathbb{P}}\left\{\hat{M}^{t_0} \leq x, \vec{1} \cdot \hat{I}^{t_0}(T - t_0) \leq y\right\} = \tilde{E}\left[\mathbb{I}_{\left\{\hat{M}^{t_0} \leq x, \vec{1} \cdot \hat{I}^{t_0}(T-t_0) \leq y\right\}}\right]$$

$$
\begin{aligned}
&= \hat{E}\left[e^{\left\{-\frac{1}{2}\left(\vec{1}\cdot\hat{I}^{t_0}(T-t_0)\right)-\frac{m}{8}h^{t_0}(T-t_0)\right\}}\mathbb{I}_{\left\{\hat{M}^{t_0}\le x,\vec{1}\cdot\hat{I}^{t_0}(T-t_0)\le y\right\}}\right]\\
&= \int_{-\infty}^{y}\int_{-\infty}^{x} e^{\left\{-\frac{1}{2}v-\frac{m}{8}h^{t_0}(T-t_0)\right\}}\cdot \hat{f}_{\hat{M}^{t_0},\vec{1}\cdot\hat{I}^{t_0}(T-t_0)}(w,v)\,dw\,dv.
\end{aligned} \tag{18}
$$

Note that

$$
h^{t_0}(T-t_0) = \int_0^{T-t_0} (\sigma_i(s+t_0))^2\,ds = \int_{t_0}^{T} (\sigma_i(u))^2\,du = Q(T).
$$

So, differentiating (18) gives us the desired result. □

## 3.1 Payoff on the Stock $S_0$

This up-and-out call option is defined in item $(i)$ of Section 1.1.1. Accordingly, its payoff reads

$$
\begin{aligned}
H(T) &= (S_0(T)-K)^{+}\mathbb{I}_{\{S_0(u)\le B_0(u),\ \forall\ u\in[0,T]\}}\,\mathbb{I}_{\left\{\prod_{i=1}^{m}(S_i(u))^{1/m}\le c\beta(u),\ \forall\ u\in[0,T]\right\}}\\
&= (S_0(T)-K)^{+}\mathbb{I}_{\{S_0(u)\le B_0(u),\ \forall\ u\in[0,T]\}}\,\mathbb{I}_{\{\vec{1}\cdot\ln\bar{S}(u)\le h,\ \forall\ u\in[0,T]\}},
\end{aligned} \tag{19}
$$

where $c > 0$, $0 < K < B_0(T)$ is the strike price, $T > 0$ is the expiration time, $\vec{1} = (1,\ldots,1)\in\mathbb{R}^m$ and $h = m\,ln\,c$.

We wish to determine the no-arbitrage price for this option at time $t_0\in[0,T]$ arbitrarily fixed, which, by virtue of [8]), reads as

$$
H(t_0) = \tilde{E}\left[H(T)\,\beta(t_0)/\beta(T)\mid\mathcal{F}(t_0)\right]. \tag{20}
$$

So, invoking (10) and (11), the $t_0$-discounted payoff reads

$$
\begin{aligned}
H(T)\frac{\beta(t_0)}{\beta(T)} &= \left(\frac{\beta(t_0)}{\beta(T)}S_0(T)-\frac{\beta(t_0)}{\beta(T)}K\right)^{+}\mathbb{I}_{\{S_0(u)\le B_0(u)\ \forall\ u\in[t_0,T]\}}\,\mathbb{I}_{\{\vec{1}\cdot\ln\bar{S}(u)\le h\ \forall\ u\in[t_0,T]\}}\\
&= \left(\bar{S}_0^{t_0}(T-t_0)-K(t_0)\right)^{+}\mathbb{I}_{\{S_0(t_0)\exp\{\hat{M}_0^{t_0}\}\le B_0(t_0)\}}\mathbb{I}_{\{\hat{M}^{t_0}\le\bar{h}\}}\\
&= \left(S_0(t_0)\exp\{\hat{I}_0^{t_0}(T-t_0)\}-K(t_0)\right)\mathbb{I}_{\left\{\hat{I}_0^{t_0}(T-t_0)\ge k_0(t_0)\,,\,\hat{M}_0^{t_0}\le b_0(t_0)\right\}}\mathbb{I}_{\left\{\hat{M}^{t_0}\le\bar{h}\right\}}
\end{aligned} \tag{21}
$$

with $K(t_0) = K\frac{\beta(t_0)}{\beta(T)}$, $k_0(t_0) = \ln\left(\frac{K(t_0)}{S_0(t_0)}\right)$ and $b_0(t_0) = \ln\left(\frac{B_0(t_0)}{S_0(t_0)}\right)$.

A simple and scalar version of Lemma 3.1 is needed.

**Lemma 3.2.** *Let $t_0\in[0,T)$ be an arbitrary but fixed time. Under $\tilde{\mathbb{P}}$, the joint density function of $\left(\hat{M}_0^{t_0},\hat{I}_0^{t_0}(T-t_0)\right)$ is given by*

$$
\tilde{f}_{\hat{M}_0^{t_0},\hat{I}_0^{t_0}(T-t_0)}(x_0,y_0) = \begin{cases}\frac{2(2x_0-y_0)}{Q_0(T)\sqrt{2\pi Q_0(T)}}\exp\left\{-\frac{1}{2}y_0-\frac{1}{8}Q_0(T)-\frac{(2x_0-y_0)^2}{2Q_0(T)}\right\}, & \\ \qquad\qquad y_0\le x_0,\ x_0>0, & \\ 0,\ \textit{otherwise}, & \end{cases} \tag{22}
$$

*where $Q_0(T) = \int_{t_0}^{T}\sigma_0^2(u)du$.*

The pricing result is the content of the following theorem.

**Theorem 3.1.** *Consider a market with a money market account and risky assets whose prices $S_i$, $i = 0, \ldots, m$, evolve according to (2). Also consider a multi-asset up-and-out call option with payoff on the stock $S_0$ with strike price $K$ and time of expiration $T > 0$, as given by (19). A moving barrier is set for $S_0$, which tracks the price of the risk-free asset as in (3). Another barrier which also tracks the price of the risk-free asset, namely, $B = c\beta$, as in (1), monitors a geometric basket composed of stocks $S_i$, $i = 1, \ldots, m$. Equivalently, in lieu of $B$ and the geometric basket, consider the hyperplane barrier $\{x \in \mathbb{R}^m \mid \vec{1} \cdot x = h\}$, $h = m \ln c$, testing the discounted log price vector $(\ln \bar{S}_1, \ldots, \ln \bar{S}_m)$. Let $t_0 \in [0, T)$ be the prescribed pricing time of interest and assume that condition* (12) *holds. If the option has not knocked out prior to $t_0$, then its no-arbitrage price is given by*

$$\begin{aligned} H(t_0) &= \{S_0(t_0)[N(z^{0,+}_{t_0,a_0}) - N(z^{0,+}_{t_0,c_0})] - K(t_0)[N(z^{0,-}_{t_0,a_0}) - N(z^{0,-}_{t_0,c_0})] \\ &\quad - B_0(t_0)[N(z^{0,+}_{t_0,h_0}) - N(z^{0,+}_{t_0,d_0})] + K(t_0)c_0[N(z^{0,-}_{t_0,h_0}) - N(z^{0,-}_{t_0,d_0})]\} \\ &\quad \cdot \left\{ N(z^{+}_{t_0,\bar{h}}) - e^{-\bar{h}} N(z^{-}_{t_0,\bar{h}}) \right\}, \end{aligned} \tag{23}$$

*where $S_0(t_0)$ is the price of the risky asset $S_0$ observed at time $t_0$, $K(t_0) = \frac{\beta(t_0)}{\beta(T)} K$, $Q_0(T) = \int_{t_0}^{T} \sigma_0^2(u) du$, $z^{0,\pm}_{t_0,s} = \frac{1}{\sqrt{Q_0(T)}} \left[\ln s \pm \frac{1}{2} Q_0(T)\right]$, $Q(T) = \int_{t_0}^{T} \sigma_i^2(u) du$ for any $i \in \{1, 2, \ldots, m\}$, $z^{\pm}_{t_0,s} = \frac{1}{\sqrt{m\,Q(T)}} \left[\frac{m\,Q(T)}{2} \pm s\right]$, $N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2}} dy$, $a_0 = S_0(t_0)/K(t_0)$, $c_0 = S_0(t_0)/B_0(t_0)$, $h_0 = B_0^2(t_0)/(K(t_0)S_0(t_0))$, $d_0 = B_0(t_0)/S_0(t_0)$ and $\bar{h} = h - \vec{1} \cdot \ln S(t) = h - \sum_{i=1}^{m} \ln S_i(t_0)$.*

*Proof.* Bearing in mind that $S_0(t_0)$ is $\mathcal{F}(t_0)$-measurable, $\hat{I}_0^{t_0}(T - t_0)$, $\hat{M}_0^{t_0}$ and $\hat{M}^{t_0}$ are independent of $\mathcal{F}(t_0)$, and the pair $(\hat{I}_0^{t_0}(T - t_0), \hat{M}_0^{t_0})$ is independent of $\hat{M}^{t_0}$, it follows, from (20) and (21), that

$$\begin{aligned} H(t_0) &= \tilde{E}\left[(S_0(t_0)e^{\{\hat{I}_0^{t_0}(T-t_0)\}} - K(t_0))\, \mathbb{I}_{\{\hat{I}_0^{t_0}(T-t_0) \geq k_0(t_0), \hat{M}_0^{t_0} \leq b_0(t_0)\}}\, \mathbb{I}_{\{\hat{M}^{t_0} \leq \bar{h}\}} \mid \mathcal{F}(t_0)\right] \\ &= \tilde{E}\left[(S_0(t_0)e^{\{\hat{I}_0^{t_0}(T-t_0)\}} - K(t_0))\, \mathbb{I}_{\{\hat{I}_0^{t_0}(T-t_0) \geq k_0(t_0), \hat{M}_0^{t_0} \leq b_0(t_0)\}} \mid \mathcal{F}(t_0)\right] \\ &\quad \cdot \tilde{E}\left[\mathbb{I}_{\{\hat{M}^{t_0} \leq \bar{h}\}} \mid \mathcal{F}(t_0)\right] \\ &= (I) \cdot (II), \end{aligned}$$

where

$$(I) = \int_{k_0(t_0)}^{b_0(t_0)} \int_{y_0^+}^{b_0(t_0)} (S_0(t_0)e_0^y - K(t_0))\, \tilde{f}_{\hat{M}_0^{t_0}, \hat{I}_0^{t_0}(T-t_0)}(x_0, y_0)\, dx_0\, dy_0,$$

and

$$(II) = \int_{-\infty}^{\bar{h}} \int_{x^+}^{\bar{h}} \tilde{f}_{\hat{M}^{t_0}, \bar{1} \cdot \hat{I}^{t_0}(T-t_0)}(x, y)\, dx\, dy$$

where $\tilde{f}_{\hat{M}_0^{t_0}, \hat{I}_0^{t_0}(T-t_0)}$ and $\tilde{f}_{\hat{M}^{t_0}, \bar{1} \cdot \hat{I}^{t_0}(T-t_0)}$ are given by (22) and (13) respectively, and $v^+ = \max\{v, 0\}$. The pricing formula (23) follows after some algebraic manipulation. $\square$

## 3.2 Payoff on a geometric basket of stocks

This up-and-out call option is defined in item (ii) of Section 1.1.1 and its payoff reads

$$\begin{aligned}\mathcal{H}(T) &= \left(\prod_{i=1}^{m}(S_i(T))^{1/m}-K\right)^{+}\mathbb{I}_{\left\{\prod_{i=1}^{m}(S_i(u))^{1/m}\le c\beta(u),\ \forall\ u\in[0,T]\right\}}\\ &= \left(\prod_{i=1}^{m}(S_i(T))^{1/m}-K\right)^{+}\mathbb{I}_{\{\vec{1}\cdot\ln\bar{S}(u)\le h\ \forall\ u\in[0,T]\}},\end{aligned}\tag{24}$$

where parameters are as defined in Section 3.1. Reminding (11), and that $\bar{S}_i^t(T-t_0)=S_i(T)\,(\beta(t_0)/\beta(T))$ and $\bar{S}_i^{t_0}(u)=S_i(t_0)\exp\{\hat{I}_i^{t_0}(u)\}$, the $t_0$-discounted payoff reads

$$\begin{aligned}\mathcal{H}(T)\frac{\beta(t_0)}{\beta(T)} &= \left(\prod_{i=1}^{m}\left(\bar{S}_i^{t_0}(T-t_0)\right)^{1/m}-K(t_0)\right)^{+}\mathbb{I}_{\{\vec{1}\cdot\ln\bar{S}(u)\le h\ \forall\ u\in[t_0,T]\}}\\ &= \left(\prod_{i=1}^{m}\left(S_i(t_0)\exp\{\hat{I}_i^{t_0}(T-t_0)\}\right)^{1/m}-K(t_0)\right)^{+}\mathbb{I}_{\{\hat{M}^{t_0}\le\bar{h}\}}\\ &= \left(\mathcal{S}(t_0)\exp\left\{\frac{\vec{1}\cdot\hat{I}^{t_0}(T-t_0)}{m}\right\}-K(t_0)\right)\mathbb{I}_{\{\vec{1}\cdot\hat{I}^{t_0}(T-t_0)\ge k(t_0)\}}\mathbb{I}_{\{\hat{M}^{t_0}\le\bar{h}\}},\end{aligned}\tag{25}$$

where $\mathcal{S}(t_0)=\left(\prod_{i=1}^{m}S_i(t_0)\right)^{1/m}$, $K(t_0)=K\,(\beta(t_0)/\beta(T))$ and $k(t_0)=m\ln\left(K(t_0)/\mathcal{S}(t_0)\right)$.

Thus, the no-arbitrage price at $t_0\in[0,T)$ of this basket option is

$$\begin{aligned}\mathcal{H}(t_0) &= \tilde{E}\left[\mathcal{H}(T)\frac{\beta(t_0)}{\beta(T)}\mid\mathcal{F}(t_0)\right]\\ &= \tilde{E}\left[\left(\mathcal{S}(t_0)\exp\left\{\frac{\vec{1}\cdot\hat{I}^{t_0}(T-t_0)}{m}\right\}-K(t_0)\right)\mathbb{I}_{\{\vec{1}\cdot\hat{I}^{t_0}(T-t_0)\ge k(t_0)\}}\mathbb{I}_{\{\hat{M}^{t_0}\le\bar{h}\}}\mid\mathcal{F}(t_0)\right].\end{aligned}\tag{26}$$

We again rely on our underpinning Lemma 3.1 to obtain the following pricing result.

**Theorem 3.2.** *Consider a market with a money market account and risky assets whose prices $S_i$, $i=1,\dots,m$, evolve according to (2). Also consider a multi-asset up-and-out call option with payoff on a geometric basket of stocks with strike price $K$ and time of expiration $T>0$, as given by (24). The barrier that tracks the price of the risk-free asset, namely $B=c\beta$, as in (1), monitors that same geometric basket. Equivalently, in lieu of $B$ and the monitored geometric basket, consider the hyperplane barrier $\{x\in\mathbb{R}^m\mid\vec{1}\cdot x=h\}$, $h=m\ln c$, testing the discounted log price vector $(\ln\bar{S}_1,\dots\ln\bar{S}_m)$. Let $t_0\in[0,T)$ be the prescribed pricing time of interest and assume that condition* (12) *holds. If the option has* *not knocked out prior to $t_0$, then its no-arbitrage price is given by*

$$\begin{aligned}\mathcal{H}(t_0) &= \mathcal{S}(t_0)e^{\left(\frac{1-m}{2m}\right)Q(t_0)}\left\{N(z^+_{t_0,\bar{h}-Q(t_0)}) - N(z^+_{t_0,k(t_0)-Q(t_0)})\right\} \\ &\quad -K(t_0)\left\{N(z^+_{t_0,\bar{h}}) - N(z^+_{t_0,k(t_0)})\right\} \\ &\quad -\mathcal{S}(t_0)e^{\left(\frac{2-m}{m}\right)\bar{h}+\left(\frac{1-m}{2m}\right)Q(t_0)}\left\{N(z^-_{t_0,\bar{h}+Q(t_0)}) - N(z^+_{t_0,k(t_0)-2\bar{h}-Q(t_0)})\right\}, \\ &\quad +K(t_0)e^{-\bar{h}}\left\{N(z^-_{t_0,\bar{h}}) - N(z^+_{t_0,k(t_0)-2\bar{h}})\right\} \end{aligned} \tag{27}$$

*where $S(t_0) = (S_1(t_0), \ldots, S_m(t_0))$ is the vector price of the risky assets observed at time $t_0$, $\mathcal{S}(t_0) = \left(\prod_{i=1}^m S_i(t_0)\right)^{1/m}$, $K(t_0) = \frac{\beta(t_0)}{\beta(T)}K$, $k(t_0) = m\ln\left(K(t_0)/\mathcal{S}(t_0)\right)$, $\bar{h} = h - \vec{1}\cdot\ln S(t_0)$, $Q(T) = \int_{t_0}^T \sigma_i^2(u)du$ for any $i \in \{1, 2, \ldots, m\}$, $z^\pm_{t_0,s} = \frac{1}{\sqrt{m\,Q(T)}}\left[\frac{m\,Q(T)}{2} \pm s\right]$, $N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^z e^{-\frac{y^2}{2}}dy$.*

*Proof.* Bearing in mind that $\mathcal{S}(t_0)$ is $\mathcal{F}(t_0)$-measurable and $\vec{1}\cdot\hat{I}^{t_0}(T-t_0)$ and $\hat{M}^{t_0}$ are independent of $\mathcal{F}(t_0)$, it follows, from (26), that

$$\begin{aligned}\mathcal{H}(t_0) &= \tilde{E}\left[\left(\mathcal{S}(t_0)e^{\vec{1}\cdot\hat{I}^{t_0}(T-t_0)/m} - K(t_0)\right)\mathbb{I}_{\{\vec{1}\cdot\hat{I}^{t_0}(T-t_0)\geq k(t_0)\}}\mathbb{I}_{\{\hat{M}^{t_0}\leq\bar{h}\}} \mid \mathcal{F}(t_0)\right] \\ &= \int_{k(t_0)}^{\bar{h}}\int_{y^+}^{\bar{h}}\left(\mathcal{S}(t_0)e^{y/m} - K(t_0)\right)\tilde{f}_{\hat{M}^{t_0},\vec{1}\cdot\hat{I}^{t_0}(T-t_0)}(x,y)\,dx\,dy,\end{aligned}$$

where $\tilde{f}_{\hat{M}^{t_0},\vec{1}\cdot\hat{I}^{t_0}(T-t_0)}$ is given by (13) and $y^+ = \max\{y, 0\}$.

Some algebraic manipulation leads us to the pricing formula (27). □

### 3.3 Pricing More General Payoffs

Every up-and-out call barrier option whose payoff can be written as a function of the random variables $\hat{M}^{t_0}$ and $\vec{1}\cdot\hat{I}^{t_0}(T-t_0)$, with $\hat{I}_i^{t_0}$ and $\hat{M}^{t_0}$ given by (7) and (9), respectively, can have its no-arbitrage price computed at least numerically, at the time $t_0 \in [0, T]$, by using the joint density function provided in the underpinning Lemma 3.1.

## 4 Pricing Hyperplane Barrier Options with Dependent Assets

In what follows we deal with scenarios where the risky assets are not independent amongst themselves. Complying with Sections 1.1.2 and 2, a hyperplane up-and-out call option with payoff on a geometric basket of stocks is addressed. We start with a motivating exercise addressing the bidimensional market model with a specific dependence format where the closed form expression for the price is obtained. We then extend the ideas to the multi-asset case.

## 4.1 The Bidimensional Case

Let us now consider two dependent risky asset price processes $\mathrm{S}_1$ and $\mathrm{S}_2$, i.e., two assets that depend on each other. More specifically, under the risk-neutral measure $\tilde{\mathbb{P}}$, the dynamics evolve according to the following SDEs

$$d\mathrm{S}_1(t) = r(t)\mathrm{S}_1(t)dt + \sigma_1(t)\mathrm{S}_1(t)d\tilde{\mathrm{W}}_1(t) + \sigma_2(t)\mathrm{S}_1(t)d\tilde{\mathrm{W}}_2(t), \tag{28}$$

$$d\mathrm{S}_2(t) = r(t)\mathrm{S}_2(t)dt + \sigma_2(t)\mathrm{S}_2(t)d\tilde{\mathrm{W}}_1(t) - \sigma_1(t)\mathrm{S}_2(t)d\tilde{\mathrm{W}}_2(t), \tag{29}$$

where $t \in [0,T]$ and $(\tilde{\mathrm{W}}_1(t), \tilde{\mathrm{W}}_2(t))$ is a two-dimensional $\tilde{\mathbb{P}}$-Brownian Motion. The volatilities $\sigma_1(t)$, $\sigma_2(t)$ and the risk-free interest rate $r$ are deterministic functions of time subject to the mild constraint of square-integrability. Note that a particular structure for the disturbances is used: only $\sigma_1$ and $\sigma_2$ appears, instead of the general framework using $\sigma_{11}$, $\sigma_{12}$, $\sigma_{21}$ and $\sigma_{22}$.

Let $t_0 \in [0,T)$ be an arbitrarily fixed time. The $t_0$-discounted risky asset price processes are $\tilde{\mathbb{P}}$-martingales and follow the SDEs

$$\begin{aligned} d\bar{\mathrm{S}}_1^{t_0}(u) &= \sigma_1^{t_0}(u)\bar{\mathrm{S}}_1^{t_0}(u)d\tilde{\mathrm{W}}_1^{t_0}(u) + \sigma_2^{t_0}(u)\bar{\mathrm{S}}_1^{t_0}(u)d\tilde{\mathrm{W}}_2^{t_0}(u), \\ d\bar{\mathrm{S}}_2^{t_0}(u) &= \sigma_2^{t_0}(u)\bar{\mathrm{S}}_2^{t_0}(u)d\tilde{\mathrm{W}}_1^{t_0}(u) - \sigma_1^{t_0}(u)\bar{\mathrm{S}}_2^{t_0}(u)d\tilde{\mathrm{W}}_2^{t_0}(u), \end{aligned} \tag{30}$$

$0 \leq u \leq T - t_0$. The solutions for (30) are

$$\bar{\mathrm{S}}_1^{t_0}(u) \;=\; \mathrm{S}_1(t_0)\exp\left\{\hat{\mathrm{I}}_1^{t_0}(u)\right\} \quad \text{and} \quad \bar{\mathrm{S}}_2^{t_0}(u) \;=\; \mathrm{S}_2(t_0)\exp\left\{\hat{\mathrm{I}}_2^{t_0}(u)\right\},$$

where

$$\hat{\mathrm{I}}_1^{t_0}(u) = \int_0^u \sigma_1^{t_0}(s)d\tilde{\mathrm{W}}_1^{t_0}(s) + \int_0^u \sigma_2^{t_0}(s)d\tilde{\mathrm{W}}_2^{t_0}(s) - \frac{1}{2}\int_0^u \left((\sigma_1^{t_0}(s))^2 + (\sigma_2^{t_0}(s))^2\right)ds$$

and

$$\hat{\mathrm{I}}_2^{t_0}(u) = \int_0^u \sigma_2^{t_0}(s)d\tilde{\mathrm{W}}_1^{t_0}(s) - \int_0^u \sigma_1^{t_0}(s)d\tilde{\mathrm{W}}_2^{t_0}(s) - \frac{1}{2}\int_0^u \left((\sigma_1^{t_0}(s))^2 + (\sigma_2^{t_0}(s))^2\right)ds.$$

Relying on the square integrability of $\sigma_1(u)$ and $\sigma_2(u)$, we define the Radon-Nikodým derivative process

$$Z^{t_0}(u) = \exp\left\{-\int_0^u \tilde{\Theta}(s)\cdot d\tilde{\mathrm{W}}^{t_0}(s) - \frac{1}{2}\int_0^u \|\tilde{\Theta}(s)\|^2 ds\right\}, \tag{31}$$

$0 \leq u \leq T - t_0$, where $\tilde{\Theta}(s) = (\tilde{\Theta}_1(s), \tilde{\Theta}_2(s))$ is such that

$$\tilde{\Theta}_1(s) = \frac{-(\sigma_1^{t_0}(s) + \sigma_2^{t_0}(s))}{2} \quad \text{and} \quad \tilde{\Theta}_2(s) = \frac{-(\sigma_2^{t_0}(s) - \sigma_{t_0}^{t}(s))}{2},$$

and $\|\tilde{\Theta}(s)\|$ denotes the Euclidean norm

$$\|\tilde{\Theta}(s)\| = \left(\tilde{\Theta}_1^2(s) + \tilde{\Theta}_2^2(s)\right)^{1/2}.$$

The Itô integral in (31) is

$$\int_0^u \tilde{\Theta}(s)\cdot d\tilde{\mathrm{W}}^{t_0}(s) = \int_0^u \tilde{\Theta}_1(s)d\tilde{\mathrm{W}}_1^{t_0}(s) + \int_0^u \tilde{\Theta}_2(s)d\tilde{\mathrm{W}}_2^{t_0}(s).$$

We define the new measure $\bar{\mathbb{P}}(A) = \int_A Z^{t_0}(T-t_0)d\tilde{\mathbb{P}}$, $A \in \mathcal{F}$, under which (we invoke the multidimensional version of Girsanov's Theorem) $\bar{\mathrm{W}}(u) = (\bar{\mathrm{W}}_1(u), \bar{\mathrm{W}}_2(u))$ is a two-dimensional Brownian Motion with

$$\bar{\mathrm{W}}_1(u) = \tilde{\mathrm{W}}_1^{t_0}(u) + \int_0^u \tilde{\Theta}_1(s)ds \qquad \text{and} \qquad \bar{\mathrm{W}}_2(u) = \tilde{\mathrm{W}}_2^{t_0}(u) + \int_0^u \tilde{\Theta}_2(s)ds. \tag{32}$$

With respect to the new measure $\bar{\mathbb{P}}$, the processes $\hat{\mathrm{I}}_1^{t_0}$ and $\hat{\mathrm{I}}_2^{t_0}$ are martingales given by

$$\hat{\mathrm{I}}_1^{t_0}(u) = \int_0^u \sigma_1^{t_0}(s)d\bar{\mathrm{W}}_1(s) + \int_0^u \sigma_2^{t_0}(s)d\bar{\mathrm{W}}_2(s), \tag{33}$$

and

$$\hat{\mathrm{I}}_2^{t_0}(u) = \int_0^u \sigma_2^{t_0}(s)d\bar{\mathrm{W}}_1(s) - \int_0^u \sigma_1^{t_0}(s)d\bar{\mathrm{W}}_2(s), \tag{34}$$

while, from (31) and (32), we have that

$$Z^{t_0}(u) = \exp\left\{\frac{1}{2}\left(\hat{\mathrm{I}}_1^{t_0}(u) + \hat{\mathrm{I}}_2^{t_0}(u)\right) + \frac{1}{4}\int_0^u \left((\sigma_1^{t_0}(s))^2 + (\sigma_2^{t_0}(s))^2\right)ds\right\}. \tag{35}$$

So,

$$\begin{aligned} d\hat{\mathrm{I}}_1^{t_0}(u)d\hat{\mathrm{I}}_2^{t_0}(u) &= \sigma_1^{t_0}(u)\sigma_2^{t_0}(u)(d\bar{\mathrm{W}}_1(u))^2 - (\sigma_1^{t_0}(u))^2 d\bar{\mathrm{W}}_2(u)d\bar{\mathrm{W}}_1(u) \\ &\quad +(\sigma_2^{t_0}(u))^2 d\bar{\mathrm{W}}_2(u)d\bar{\mathrm{W}}_1(u) - \sigma_1^{t_0}(u)\sigma_2^{t_0}(u)(d\bar{\mathrm{W}}_2(u))^2 \\ &= \sigma_1^{t_0}(u)\sigma_2^{t_0}(u)du - \sigma_1^{t_0}(u)\sigma_2^{t_0}(u)du = 0, \end{aligned} \tag{36}$$

$$(d\hat{\mathrm{I}}_i^{t_0}(u))^2 = \left((\sigma_1^{t_0}(u))^2 + (\sigma_2^{t_0}(u))^2\right)du \quad \text{i=1,2}, \tag{37}$$

or else, from (36) and (37) we may conclude that the covariation process between $\hat{\mathrm{I}}_1^{t_0}(u)$, $\hat{\mathrm{I}}_2^{t_0}(u)$ and the quadratic variation process of $\hat{\mathrm{I}}_i^{t_0}(u)$, $i = 1, 2$, read as

$$\left[\hat{\mathrm{I}}_1^{t_0}, \hat{\mathrm{I}}_2^{t_0}\right](u) = 0 \qquad \text{and} \qquad \left[\hat{\mathrm{I}}_i^{t_0}, \hat{\mathrm{I}}_i^{t_0}\right](u) = \int_0^u \left((\sigma_1^{t_0}(s))^2 + (\sigma_2^{t_0}(s))^2\right)ds = A(u). \tag{38}$$

By virtue of (38), the multivariate version of the Time-Change for Martingale Theorem (Knight Theorem - see e.g. [3]), and reminding that $\hat{\mathrm{I}}_1^{t_0}$ and $\hat{\mathrm{I}}_2^{t_0}$ have null covariation, we may further express

$$\left(\hat{\mathrm{I}}_1^{t_0}(u), \hat{\mathrm{I}}_2^{t_0}(u)\right) = \left(\bar{\mathcal{W}}_1(A(u)), \bar{\mathcal{W}}_2(A(u))\right), \quad 0 \le u \le T - t_0, \tag{39}$$

where $(\bar{\mathcal{W}}_1(u), \bar{\mathcal{W}}_2(u))$ is a two-dimensional $\bar{\mathbb{P}}$-Brownian Motion.

Since $A(u)$ is increasing and continuous we have that

$$\bar{\mathcal{M}}^{t_0} = \max_{0\le u\le T-t_0}\left(\hat{\mathrm{I}}_1^{t_0}(u) + \hat{\mathrm{I}}_2^{t_0}(u)\right) = \max_{0\le s\le A^{t_0}(T-t_0)}\left(\bar{\mathcal{W}}_1(s) + \bar{\mathcal{W}}_2(s)\right).$$

This, (39) and Proposition 3.25 in [21] allow us to state the following version of the generalised Reflection Principle extended for continuous-time Martingales whose components have null covariation and the same deterministic quadratic variation function (in this case given by the function $A$ in (38)):

$$\bar{\mathbb{P}}\left\{\bar{\mathcal{M}}^{t_0} \geq x, \hat{\mathrm{I}}_1^{t_0}(T-t_0)+\hat{\mathrm{I}}_2^{t_0}(T-t_0) \leq y\right\} = \bar{\mathbb{P}}\left\{\hat{\mathrm{I}}_1^{t_0}(T-t_0)+\hat{\mathrm{I}}_2^{t_0}(T-t_0) \geq 2x-y\right\}, \tag{40}$$

$y \leq x$, $x > 0$. Differentiating (40) and bearing in mind that $\hat{\mathrm{I}}_1^{t_0}(T-t_0)+\hat{\mathrm{I}}_2^{t_0}(T-t_0)$ is normally distributed with zero mean and variance $2\,A(T-t_0)$, we obtain

$$\bar{f}_{\bar{\mathcal{M}}^{t_0},\hat{\mathrm{I}}_1^{t_0}+\hat{\mathrm{I}}_2^{t_0}}(x,y) = \begin{cases} \frac{(2x-y)}{\sqrt{2}A(T-t_0)\sqrt{2\pi A(T-t_0)}} \exp\left\{-\frac{(2x-y)^2}{4A(T-t_0)}\right\}, & y \leq x, \;\; x > 0; \\ 0, & \text{otherwise}, \end{cases} \tag{41}$$

which is the joint density function of the pair $(\bar{\mathcal{M}}^{t_0}, \hat{\mathrm{I}}_1^{t_0}+\hat{\mathrm{I}}_2^{t_0})$ under $\bar{\mathbb{P}}$.

We are now ready to state the following result.

**Lemma 4.1.** *Under $\tilde{\mathbb{P}}$, the joint density function of*

$$\left(\bar{\mathcal{M}}^{t_0}, \hat{\mathrm{I}}_1^{t_0}(T-t_0)+\hat{\mathrm{I}}_2^{t_0}(T-t_0)\right)$$

*is given by*

$$\tilde{f}_{\bar{\mathcal{M}}^{t_0},\hat{\mathrm{I}}_1^{t_0}+\hat{\mathrm{I}}_2^{t_0}}(x,y) = \begin{cases} \frac{(2x-y)}{\sqrt{2}\mathbf{A}(T)\sqrt{2\pi\mathbf{A}(T)}} \exp\left\{-\frac{1}{2}y-\frac{1}{4}\mathbf{A}(T)-\frac{(2x-y)^2}{4\mathbf{A}(T)}\right\}, & y \leq x, \;\; x > 0; \\ 0, & \textit{otherwise}, \end{cases} \tag{42}$$

*where* $\mathbf{A}(T) = \int_{t_0}^{T}(\sigma_1^2(s)+\sigma_2^2(s))\,ds$.

*Proof.* Using the previous expressions (35), (38) and (41) we may write

$$\begin{aligned} \tilde{\mathbb{P}}\left\{\bar{\mathcal{M}}^{t_0} \leq x, \hat{\mathrm{I}}_1^{t_0}(T-t_0)+\hat{\mathrm{I}}_2^{t_0}(T-t_0) \leq y\right\} &= \tilde{E}\left[\mathbb{I}_{\left\{\bar{\mathcal{M}}^{t_0}\leq x,\hat{\mathrm{I}}_1^{t_0}(T-t_0)+\hat{\mathrm{I}}_2^{t_0}(T-t_0)\leq y\right\}}\right] \\ &= \bar{E}\left[\frac{1}{Z^{t_0}(T-t_0)}\mathbb{I}_{\left\{\bar{\mathcal{M}}^{t_0}\leq x,\hat{\mathrm{I}}_1^{t_0}(T-t_0)+\hat{\mathrm{I}}_2^{t_0}(T-t_0)\leq y\right\}}\right] \\ &= \int_{-\infty}^{y}\int_{-\infty}^{x} e^{\left\{-\frac{1}{2}v-\frac{1}{4}A(T-t_0)\right\}}\bar{f}_{\bar{\mathcal{M}}^{t_0},\hat{\mathrm{I}}_1^{t_0}+\hat{\mathrm{I}}_2^{t_0}}(u,v)du\,dv. \end{aligned}$$

Differentiating the above expression gives us the desired result. □

### 4.1.1 Pricing a two-dimensional geometric basket barrier option

Let us illustrate an application of Lemma 4.1 by computing the price of an up-and-out option with payoff on a geometric basket of two stocks. Its payoff is

$$\begin{aligned}\mathbf{H}(T) &= \left((S_1(T)S_2(T))^{1/2} - K\right)^+ \mathbb{I}_{\{\vec{1}\cdot\ln\bar{S}(u)\le h\ \forall\ u\in[0,T]\}}\\ &= \left((S_1(T)S_2(T))^{1/2} - K\right)^+ \mathbb{I}_{\left\{(S_1(T)S_2(T))^{1/2}\le c\beta(u),\ \forall\ u\in[0,T]\right\}},\end{aligned} \tag{43}$$

where $c = e^{h/m}$. The $t_0$-discounted payoff reads as

$$\begin{aligned}\bar{\mathbf{H}}^{t_0}(T) &= \frac{\beta(t_0)}{\beta(T)}\mathbf{H}(T)\\ &= \left(\left(\bar{S}_1^{t_0}(T-t_0)\bar{S}_2^{t_0}(T-t_0)\right)^{1/2} - K(t_0)\right)^+ \mathbb{I}_{\{\vec{1}\cdot\ln\bar{S}(u)\le h\ \forall\ u\in[t_0,T]\}}\\ &= \left((S_1(t_0)S_2(t_0))^{1/2}\left(e^{\hat{I}_1^{t_0}(T-t_0)+\hat{I}_2^{t_0}(T-t_0)}\right)^{1/2} - K(t_0)\right)^+ \mathbb{I}_{\{\vec{1}\cdot\ln\bar{S}(u)\le h\ \forall\ u\in[t_0,T]\}}\end{aligned} \tag{44}$$

where $\bar{S}_i^{t_0}(T-t_0) = S_i(T)\left(\beta(t_0)/\beta(T)\right)$ for $i = 1, 2$, and $K(t_0) = K\left(\beta(t_0)/\beta(T)\right)$. In this case we have that the following sets are equivalent:

$$\begin{aligned}\{\vec{1}\cdot\ln\bar{S}(u)\le h\ \forall\ u\in[t_0,T]\} &= \{\vec{1}\cdot\ln\bar{S}^{t_0}(u)\le h\ \forall\ u\in[0,T-t_0]\}\\ &= \{\hat{I}_1(u)+\hat{I}_2(u)\le\bar{h}\ \forall\ u\in[0,T-t_0]\}\\ &= \{\bar{\mathcal{M}}^{t_0}\le\bar{h}\},\end{aligned}$$

where

$$\bar{h} = h - (\ln S_1(t_0) + \ln S_2(t_0)) \quad\text{and}\quad \bar{\mathcal{M}}^{t_0} = \max_{0\le u\le T-t_0}\left(\hat{\mathrm{I}}_1^{t_0}(u) + \hat{\mathrm{I}}_2^{t_0}(u)\right).$$

So, we can write (44) as

$$\begin{aligned}\bar{\mathbf{H}}^{t_0}(T) &= \left(\mathbf{S}(\mathbf{t_0})\,e^{\left\{\frac{\hat{I}_1^{t_0}(T-t_0)+\hat{I}_2^{t_0}(T-t_0)}{2}\right\}} - K(t_0)\right)^+ \mathbb{I}_{\{\bar{\mathcal{M}}^{t_0}\le\bar{h}\}}\\ &= \left(\mathbf{S}(\mathbf{t_0})\,e^{\left\{\frac{\hat{I}_1^{t_0}(T-t_0)+\hat{I}_2^{t_0}(T-t_0)}{2}\right\}} - K(t_0)\right) \mathbb{I}_{\left\{\hat{I}_1^{t_0}(T-t_0)+\hat{I}_2^{t_0}(T-t_0)\ge\mathbf{k},\,\bar{\mathcal{M}}^{t_0}\le\bar{h}\right\}},\end{aligned} \tag{45}$$

where $\mathbf{S}(\mathbf{t_0}) = (S_1(t_0)S_2(t_0))^{1/2}$ and $\mathbf{k} = 2\ln\left(\frac{K(t_0)}{\mathbf{S}(\mathbf{t_0})}\right)$, and the no-arbitrage price of this option at $t_0 \in [0, T]$ is

$$\mathbf{H}(t_0) = \tilde{E}\left[\bar{\mathbf{H}}^{t_0}(T) \mid \mathcal{F}(t_0)\right]. \tag{46}$$

**Theorem 4.1.** *Consider a market with a money market account and two risky assets whose prices $S_i$, $i = 1, 2$, depend on each other as they evolve according to (28) and (29). Also consider a two-asset up-and-out call option with payoff on a geometric basket of stocks as given by (43), where the hyperplane barrier $\{x \in \mathbb{R}^m \mid \vec{1}\cdot x = h\}$ tests the* *discounted log price vector of the risky assets. Equivalently, in lieu of this monitoring scheme, consider the barrier that tracks the price of the risk-free asset, namely $B = c\beta$ as in (1), $c = e^{h/m}$, monitoring the geometric basket afore mentioned. Let $t_0 \in [0, T)$ be the prescribed pricing time of interest. If the derivative has not knocked out prior to that time, then its no-arbitrage price is given by*

$$\begin{aligned}\mathbf{H}(t_0) &= \mathbf{S}(t_0)e^{\left(\frac{-1}{4}\right)\mathbf{A}(T)}\left\{N(z^+_{t_0,\bar{h}-\mathbf{A}(T)}) - N(z^+_{t_0,\mathbf{k}-\mathbf{A}(T)})\right\} - K(t_0)\left\{N(\delta^+_{t_0,\bar{h}}) - N(\delta^+_{t_0,\mathbf{k}})\right\}\\ &\quad + K(t_0)e^{-\bar{h}}\left\{N(\delta^+_{t_0,-\bar{h}}) - N(\delta^+_{t_0,\mathbf{k}-2\bar{h}})\right\}\\ &\quad - \mathbf{S}(t_0)e^{\left(\frac{-1}{4}\right)\mathbf{A}(T)\bar{h}} \cdot \left\{N(z^-_{t_0,\bar{h}+\mathbf{A}(T)}) - N(z^-_{t_0,2\bar{h}+\mathbf{A}(T)-\mathbf{k}})\right\},\end{aligned} \tag{47}$$

*where $S(t_0) = (S_1(t_0), S_2(t_0))$ is the vector price of the risky assets observed at time $t_0$,* $\mathbf{S(t_0)} = (S_1(t_0)S_2(t_0))^{1/2}$, $K(t_0) = \frac{\beta(t_0)}{\beta(T)}K$, $\mathbf{k} = 2\ln\left(\frac{K(t_0)}{\mathbf{S(t_0)}}\right)$, $\bar{h} = h - (\ln S_1(t_0) + \ln S_2(t_0))$, $\mathbf{A}(T) = \int_{t_0}^{T}(\sigma_1^2(s) + \sigma_2^2(s))\,ds$, $z^{\pm}_{t_0,s} = \frac{1}{\sqrt{2\,\mathbf{A}(T)}}[\mathbf{A}(T) \pm s]$, $\delta^{\pm}_{t_0,s} = \frac{1}{\sqrt{2\,\mathbf{A}(T)}}[s \pm \mathbf{A}(T)]$, $N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2}}\,dy$.

*Proof.* Bearing in mind that $\mathbf{S}(t_0)$ is $\mathcal{F}(t_0)$-measurable and $\hat{I}_1^{t_0}(T-t_0) + \hat{I}_2^{t_0}(T-t_0)$ and $\bar{\mathcal{M}}^{t_0}$ are independent of $\mathcal{F}(t_0)$, it follows, from (46) and (45), that

$$\begin{aligned}\mathbf{H}(t_0) &= \tilde{E}\left[\left(\mathbf{S}(t_0)e^{\left\{\frac{\hat{I}_1^{t_0}(T-t_0)+\hat{I}_2^{t_0}(T-t_0)}{2}\right\}} - K(t_0)\right)\mathbb{I}_{\{\hat{I}_1^{t_0}(T-t_0)+\hat{I}_2^{t_0}(T-t_0)\geq\mathbf{k},\,\bar{\mathcal{M}}^{t_0}\leq\bar{h}\}} \mid \mathcal{F}(t_0)\right]\\ &= \int_k^{\bar{h}}\int_{y^+}^{\bar{h}}\left(\mathbf{S}(t_0)e^{y/m} - K(t_0)\right)\tilde{f}_{\bar{\mathcal{M}}^{t_0},\hat{I}_1^{t_0}+\hat{I}_2^{t_0}}(x,y)\,dx\,dy,\end{aligned}$$

where $\tilde{f}_{\bar{\mathcal{M}}^{t_0},\hat{I}_1^{t_0}+\hat{I}_2^{t_0}}$ is given by (42) and $y^+ = \max\{y, 0\}$. The pricing formula (47) follows after some algebraic manipulation. □

## 4.2 Pricing More General Barrier Options in the Multidimensional Case

Let us now consider a non-necessarily complete market equipped with a risk-neutral measure $\tilde{\mathbb{P}}$, and $m$ risky assets whose prices evolve according to SDEs as those given by (5). As before, the discounted prices of the risky assets, with $t_0$ set as starting point, are $\tilde{\mathbb{P}}$-martingales given by

$$d\bar{\mathrm{S}}_i^{t_0}(u) = \bar{\mathrm{S}}_i^{t_0}(u)\sum_{j=1}^{d}\sigma_{ij}^{t_0}(u)\,d\tilde{\mathrm{W}}_j^{t_0}(u), \quad i = 1,\ldots,m, \quad u \in [0, T-t_0], \tag{48}$$

where $m$ and $d$ are positive integers, $(\tilde{\mathrm{W}}_1(u),\ldots,\tilde{\mathrm{W}}_d(u))$ is a $d$-dimensional $\tilde{\mathbb{P}}$-Brownian Motion and the components of the volatility matrix $\sigma(u) = (\sigma_{ij}(u))_{i=1,\ldots,m,j=1,\ldots,d}$ are square integrable deterministic functions of time.

The solutions for the SDEs in (48) are $\tilde{\mathbb{P}}$-geometric Brownian Motions given by

$$\bar{\mathrm{S}}_i^{t_0}(u) = \bar{\mathrm{S}}_i(t_0)\exp\left\{\hat{\mathrm{I}}_i^{t_0}(u)\right\}, \quad i = 1, \dots, m, \quad u \in [0, T - t_0],$$

where

$$d\hat{\mathrm{I}}_i^{t_0}(u) = \sum_{j=1}^{d} \sigma_{ij}^{t_0}(u)\, d\tilde{\mathrm{W}}_j^{t_0}(u) - \frac{1}{2}\sum_{j=1}^{d}\left(\sigma_{ij}^{t_0}(u)\right)^2 du, \quad i = 1, \dots, m, \quad u \in [0, T - t_0]. \tag{49}$$

We require having a solution $\tilde{\Theta}(u) = \left(\tilde{\Theta}_1(u), ..., \tilde{\Theta}_d(u)\right)$ for

$$\sigma x = -\frac{1}{2}D(\sigma\sigma^T), \;\; x \in \mathbb{R}^d, \tag{50}$$

where $\sigma = \left(\sigma_{ij}^{t_0}(u)\right)_{i=1,\dots,m,\, j=1,\dots,d}$ and $D(A)$ stands for the vector whose entries are those of the diagonal of $A$, for a square matrix $A$.

In order to obtain a joint density function for $(\bar{\mathcal{M}}^{t_0}, \vec{1}\cdot\hat{\mathrm{I}}^{t_0}(T - t_0))$ with

$$\vec{1}\cdot\hat{\mathrm{I}}^{t_0}(T - t_0) = \sum_{i=1}^{m}\hat{\mathrm{I}}_i^{t_0}(T - t_0) \quad \text{ and } \quad \bar{\mathcal{M}}^{t_0} = \max_{0 \le u \le T - t_0}\sum_{i=1}^{m}\hat{\mathrm{I}}_i^{t_0}(u), \tag{51}$$

we construct an auxiliary measure $\bar{\mathbb{P}}$ under which

$$\hat{\mathrm{I}}^{t_0}(u) = \left(\hat{\mathrm{I}}_1^{t_0}(u), \hat{\mathrm{I}}_2^{t_0}(u), \dots, \hat{\mathrm{I}}_m^{t_0}(u)\right),$$

is an $m$-dimensional martingale. Since we assumed that a solution $\tilde{\Theta}$ for (50) exists, we may write (49) as

$$d\hat{\mathrm{I}}_i^{t_0}(u) = \sum_{j=1}^{d}\sigma_{ij}^{t_0}(u)\,\left[\tilde{\Theta}_j(u)du + d\tilde{\mathrm{W}}_j^{t_0}(u)\right], \quad i = 1, \dots, m, \quad u \in [0, T - t_0]. \tag{52}$$

Note that if $m = d$ and the determinant of the volatility matrix is not null, existence of $\tilde{\Theta}(u)$ is ensured.

We then define the Radon-Nikodým derivative process

$$\mathcal{Z}^{t_0}(u) = \exp\left\{-\int_0^u \tilde{\Theta}(s)\cdot d\tilde{\mathrm{W}}^{t_0}(s) - \frac{1}{2}\int_0^u \|\tilde{\Theta}(s)\|^2 ds\right\}, \quad 0 \le u \le T - t_0, \tag{53}$$

and the new measure

$$\bar{\mathbb{P}}(A) = \int_A \mathcal{Z}^{t_0}(T - t_0)d\tilde{\mathbb{P}}, \quad A \in \mathcal{F},$$

where the Itô integral in (53) is

$$\int_0^u \tilde{\Theta}(s)\cdot d\tilde{\mathrm{W}}^{t_0}(s) = \int_0^u \tilde{\Theta}_1(s)d\tilde{\mathrm{W}}_1^{t_0}(s) + \cdots + \int_0^u \tilde{\Theta}_d(s)d\tilde{\mathrm{W}}_d^{t_0}(s),$$

and $\|\tilde{\Theta}(s)\|$ denotes the Euclidean norm

$$\|\tilde{\Theta}(s)\| = \left(\tilde{\Theta}_1^2(s) + \cdots + \tilde{\Theta}_d^2(s)\right)^{1/2}.$$

Under $\bar{\mathbb{P}}$ (we invoke the multidimensional version of Girsanov's Theorem) $\bar{\mathrm{W}}(u) = (\bar{\mathrm{W}}_1(u), \ldots, \bar{\mathrm{W}}_d(u))$ with

$$d\bar{\mathrm{W}}_j(u) = d\tilde{\mathrm{W}}_j^{t_0}(u) + \tilde{\Theta}_j(u)du, \qquad j = 1, 2, \ldots, d, \tag{54}$$

is a $d$-dimensional Brownian Motion. (52) and (54) tell us that the processes $\hat{\mathrm{I}}_i^{t_0}$, $i = 1, \ldots, m$, are continuous $\bar{\mathbb{P}}$-martingales whose dynamics are

$$d\hat{\mathrm{I}}_i^{t_0}(u) = \sum_{j=1}^{d} \sigma_{ij}^{t_0}(u)\, d\bar{\mathrm{W}}_j^{t_0}(u), \quad u \in [0, T - t_0].$$

The differential forms of the covariation and the quadratic variation associated to $\hat{\mathrm{I}}^{t_0}(u) = \left(\hat{\mathrm{I}}_1^{t_0}(u), \hat{\mathrm{I}}_2^{t_0}(u), \ldots, \hat{\mathrm{I}}_m^{t_0}(u)\right)$ are

$$d\hat{\mathrm{I}}_i^{t_0}(u)\, d\hat{\mathrm{I}}_j^{t_0}(u) = \left(\sum_{k=1}^{d} \sigma_{ik}^{t_0}(u)\sigma_{jk}^{t_0}(u)\right) du \quad for \quad i \neq j$$

and

$$\left(d\hat{\mathrm{I}}_i^{t_0}(u)\right)^2 = \sum_{k=1}^{d} \left(\sigma_{ik}^{t_0}(u)\right)^2 du,$$

respectively.

Now, imposing

**(a)** $\int_0^u \left(\sum_{k=1}^d \sigma_{ik}^{t_0}(s)\sigma_{jk}^{t_0}(s)\right) ds = 0$ for $i \neq j$;

**(b)** $\int_0^u \left(\sum_{k=1}^d \left(\sigma_{ik}^{t_0}(s)\right)^2\right) ds = \mathcal{A}(u)$ for all $i = 1, \ldots, m$,

the entries of the $m$-dimensional $\bar{\mathbb{P}}$-martingale $\hat{\mathrm{I}}^{t_0}(u)$ will exhibit null covariation, i.e., $\left[\hat{\mathrm{I}}_i^{t_0}, \hat{\mathrm{I}}_j^{t_0}\right](u) = 0 \, for \, i \neq j$, and the same quadratic variation, i.e., $\left[\hat{\mathrm{I}}_i^{t_0}, \hat{\mathrm{I}}_i^{t_0}\right](u) = \mathcal{A}(u)$ for all $i = 1, \ldots, m$.

Hence,

$$\left(\hat{\mathrm{I}}_1^{t_0}(u), \ldots, \hat{\mathrm{I}}_m^{t_0}(u)\right) = \left(\bar{\mathcal{W}}_1\left(\mathcal{A}(u)\right), \ldots, \bar{\mathcal{W}}_m\left(\mathcal{A}(u)\right)\right), \quad 0 \leq u \leq T - t_0, \tag{55}$$

where $(\bar{\mathcal{W}}_1(u), \ldots, \bar{\mathcal{W}}_m(u))$ is an $m$-dimensional $\bar{\mathbb{P}}$-Brownian Motion.

Since $\mathcal{A}(u)$ is increasing and continuous we have that

$$\begin{aligned}
\bar{\mathcal{M}}^{t_0} &= \max_{0 \leq u \leq T - t_0} \left(\hat{\mathrm{I}}_1^{t_0}(u) + \cdots + \hat{\mathrm{I}}_m^{t_0}(u)\right) \\
&= \max_{0 \leq u \leq T - t_0} \left(\bar{\mathcal{W}}_1\left(\mathcal{A}(u)\right) + \cdots + \bar{\mathcal{W}}_m\left(\mathcal{A}(u)\right)\right) \\
&= \max_{0 \leq s \leq \mathcal{A}(T - t_0)} \left(\bar{\mathcal{W}}_1(s) + \cdots + \bar{\mathcal{W}}_m(s)\right).
\end{aligned}$$

This, (55) and Proposition 3.25 in [21] allow us to state the following version of the generalised Reflection Principle extended for continuous-time Martingales whose components have null covariation and the same deterministic quadratic variation function:

$$\bar{\mathbb{P}}\left\{\bar{\mathcal{M}}^{t_0} \geq x, \hat{\mathrm{I}}_1^{t_0}(T-t_0) + \cdots + \hat{\mathrm{I}}_m^{t_0}(T-t_0) \leq y\right\} =$$

$$\bar{\mathbb{P}}\left\{\hat{\mathrm{I}}_1^{t_0}(T-t_0) + \cdots + \hat{\mathrm{I}}_m^{t_0}(T-t_0) \geq 2x - y\right\},\ y \leq x,\ x > 0. \tag{56}$$

Differentiating (56) and bearing in mind that $\hat{\mathrm{I}}_1^{t_0}(T-t_0) + \cdots + \hat{\mathrm{I}}_m^{t_0}(T-t_0)$ is normally distributed with zero mean and variance $m\,\mathcal{A}(T-t_0)$, we obtain

$$\bar{f}_{\bar{\mathcal{M}}^{t_0}, \hat{\mathrm{I}}_1^{t_0}+\cdots+\hat{\mathrm{I}}_m^{t_0}}(x,y) = \begin{cases} \frac{2(2x-y)}{m\sqrt{m}\mathcal{A}(T-t_0)\sqrt{2\pi\mathcal{A}(T-t_0)}} \exp\left\{-\frac{(2x-y)^2}{2m\mathcal{A}(T-t_0)}\right\},\ y \leq x,\ x > 0 \\ \\ 0, \qquad\qquad \text{otherwise}, \end{cases} \tag{57}$$

which is the joint density function of the pair $(\bar{\mathcal{M}}^{t_0}, \hat{\mathrm{I}}_1^{t_0} + \cdots + \hat{\mathrm{I}}_m^{t_0})$ under $\bar{\mathbb{P}}$. To work out the density of $(\bar{\mathcal{M}}^{t_0}, \hat{\mathrm{I}}_1^{t_0} + \cdots + \hat{\mathrm{I}}_m^{t_0})$ under $\tilde{\mathbb{P}}$ we may write

$$\tilde{\mathbb{P}}\left\{\bar{\mathcal{M}}^{t_0} \leq x, \hat{\mathrm{I}}_1^{t_0}(T-t_0) + \cdots + \hat{\mathrm{I}}_m^{t_0}(T-t_0) \leq y\right\} = \tilde{E}\left[\mathbb{I}_{\left\{\bar{\mathcal{M}}^{t_0} \leq x, \hat{\mathrm{I}}_1^{t_0}(T-t_0)+\cdots+\hat{\mathrm{I}}_m^{t_0}(T-t_0) \leq y\right\}}\right]$$

$$= \bar{E}\left[\frac{1}{\mathcal{Z}^{t_0}(T-t_0)}\mathbb{I}_{\left\{\bar{\mathcal{M}}^{t_0} \leq x, \hat{\mathrm{I}}_1^{t_0}(T-t_0)+\cdots+\hat{\mathrm{I}}_m^{t_0}(T-t_0) \leq y\right\}}\right]. \tag{58}$$

Differentiating (58) twice, with respect to $y$ and $x$, we may write

$$\tilde{f}_{\bar{\mathcal{M}}^{t_0}, \hat{\mathrm{I}}_1^{t_0}+\cdots+\hat{\mathrm{I}}_m^{t_0}}(x,y) = \frac{\partial^2}{\partial x \partial y}\bar{E}\left[\frac{1}{\mathcal{Z}^{t_0}(T-t_0)}\mathbb{I}_{\left\{\bar{\mathcal{M}}^{t_0} \leq x, \hat{\mathrm{I}}_1^{t_0}(T-t_0)+\cdots+\hat{\mathrm{I}}_m^{t_0}(T-t_0) \leq y\right\}}\right], \tag{59}$$

where $\tilde{f}_{\bar{\mathcal{M}}^{t_0}, \hat{\mathrm{I}}_1^{t_0}+\cdots+\hat{\mathrm{I}}_m^{t_0}}$ is the joint density function of the pair $(\bar{\mathcal{M}}^{t_0}, \hat{\mathrm{I}}_1^{t_0} + \cdots + \hat{\mathrm{I}}_m^{t_0})$ under $\tilde{\mathbb{P}}$.

### 4.2.1 Pricing More General Payoffs

Hence, every attainable multi-asset up-and-out call barrier option whose payoff can be written as a function of the random variables $\bar{\mathcal{M}}^{t_0}$ and $\hat{\mathrm{I}}_1^{t_0} + \cdots + \hat{\mathrm{I}}_m^{t_0}$, with $\hat{\mathrm{I}}_1^{t_0}$ following (49) and $\bar{\mathcal{M}}^{t_0}$ as given in (51), may also have its no-arbitrage price computed numerically, at the time $t_0 \in [0,T]$, via the right side of (59) in conjunction with $\tilde{E}\left[\cdot|\mathcal{F}_{t_0}\right]$. For non-attainable multi-asset barrier options, $\tilde{E}\left[\cdot|\mathcal{F}_{t_0}\right]$ is still a good approximation for the price. To see this, we can rely on the called risk minimizing approach, where a certain class of strategies evolving on $[0,T]$ are defined in conjunction with a risk function to be minimized. The concepts of this technique are inherited from the theory of optimal control ([13]).

# 5 Conclusion

This work contributes, theoretically, to the financial scenario with closed-form expressions of no-arbitrage prices for European multi-asset up-and-out call options with barriers of hyperplane type, as they recover verbatim standard results as particular cases.

Such hyperplane geometric format, we believe, is being introduced in the literature for the first time. The barrier scheme consists of a hyperplane that monitors the discounted log price vector of the underlying assets, or else, an exponential modification of the hyperplane testing the discounted price vector of the underlyings.

A quality of the hyperplane barrier is that it naturally captures the joint behavior of the assets involved. The class of hyperplanes we address in this paper is generated by the vector with entries equal to one. Other classes naturally appear via relaxing this seminal condition and should be the concern of future studies. Other forms, e.g., nonlinear barrier surfaces, are also in view. It is of interest noticing that the barrier scheme associated to the seminal class of hyperplanes as specified above boils down to a scalar barrier test on a popular financial product, namely, a geometric basket of stocks.

Multivariate models à la Black and Scholes with time dependent volatilities are used, representing, at first, independent assets. We were also able to obtain, analytically, prices adopting a certain two-dimensional market model for assets that depend on each other, as well as an expression for a certain joint density function which boils down to calculating, numerically, the prices of more general barrier options - including non-attainable ones, in the multidimensional scenario. It is worth pointing out that pure analytical results are difficult, particularly on multivariate models, not to say the problem identified in [12] concerning dependent assets. So, the formulas we provide here can also serve as estimates for prices stemming from "nearby" models.

# References


[1] A. W. Kolkiewicz (2002) Pricing and Hedging More General Barrier Options, *Journal of Computational Finance* **5**, 1–26.

[2] B. Goetz, M. Escobar & R. Zagst (2018) Two asset-barrier option under stochastic volatility, *pplied Mathematical Finance* **24(6)**, 520–546.

[3] D. Revuz & M. Yor (2004) *Continuous Martingales and Brownian Motion.* Berlin: Springer.

[4] D. R. Rich (1994) The Mathematical Foundations of Barrier Option-pricing Theory, *Advances in Futures and Options Research* **7**, 267–311.

[5] E. Briys & F. Varenne (1997) Valuing risky fixed rate debt: an extension, *Journal of Financial and Quantitative Analysis* **32**, 239–248.

[6] E. Rosalino Jr, A.J. da Silva, J. Baczynski & D. L eão (2018) Pricing and Hedging Barrier Options, *Applied Stochastic Models in Business and Industry*, 1–14.

[7] G. Tristan (2011) Some Sequential Boundary Crossing Results from Geometric Brownian Motion and Their Applications in Financial Engineering, *International Scholarly Research Notices: Applied Mathematics*, 1–21.

[8] J. M. Harrison & S.R. Pliska (1981) Martingales and Stochastic Integrals in Theory of Continuous Trading, *Stochastic Processes and their Applications* **11**, 215–260.

[9] J. Ma & C. Xu(2010) An Efficient Control Variate Method For Pricing Variance Derivatives, *Journal of Computational and Applied Mathematics* **235**, 108–119.

[10] J.V.K. Rombouts & L. Stentoft(2011) Multivariate option pricing with time varying volatility and correlations, *Journal of Banking & Finance* **35**, 2267–2281.

[11] L. Jódar, P. Sevilla-Peris, J.C. Cortés & R. Sala(2005) A newdirect method for solving the Black-Scholes equation, *Applied Mathematics Letters* **18**, 29–32.

[12] M. Contreras, A. Llanquihu'en & M. Villena(2016) On the Solution of the Multi-Asset Black-Scholes Model: Correlations, Eigenvalues and Geometry, *Journal of Mathematical Finance* **6**, 562–579.

[13] M. Schweizer (1994) Risk-Minimizing Hedging Strategies under Restricted Information, *Mathematical Finance* **4**, 327–342.

[14] N. Kunitomo & M. Ikeda (1992) Pricing Options with Curved Boundaries, *Mathematical Finance* **2**, 275–298.

[15] R. Company, E. Navarro, J. R. Pintos & E. Ponsoda(2008) Numerical solution of linear and nonlinear Black-Scholes option pricing equations, *An International Journal Computers & Mathematics with Applications* **56**, 813–821.

[16] R. Company, A.L. González & L. Jódar(2006) Numerical solution of modified Black-Scholes equation pricing stock options with discrete dividend, *Mathematical and Computer Modelling* **44**, 1058–1068.

[17] R. Company, V. Egorova, L. J'odar & F. Soleymani (2017) Computing Stable Numerical Solutions for Multidimensional American Option Pricing Problems: a Semi-discretization Approach. arxiv: 1701.08545.

[18] R. Jordan & C. Tier(2016) Asymptotic Approximations For Pricing Derivatives Under Mean-Reverting Processes, *International Journal of Theoretical and Applied Finance* **19**, 1650030.

[19] S. E. Shreve (2004) *Stochastic Calculus for Finance II: Continuous-time models.* New York: Springer.

[20] S. Kim & C. Lee(2014) On the Importance of the Traders' Rules for Pricing Options: Evidence From Intraday Data, *Asia-Pacific Journal of Financial Studies* **43**, 873–894.

[21] S. Muirhead (2011) Pricing Multi-asset Barrier Options Using the Generalised Reflection Principle, *Honours Thesis*. University of Melbourne: Department of Mathematics and Statistics.

[22] T. Yamamoto & K. Miyazaki(2011) Deterministic volatility models and dynamics of option returns, *JSIAM Letters* **3**, 57–60.

[23] Y. K. Kwok, L. Wu & H. Yu (1998) Pricing Multi-asset Options with an External Barrier, *International Journal of Theoretical and Applied Finance* **1**, 523–541.

[24] Z. Gao, X. Wang & M. Ha (2016) Multi-asset Option Pricing in an Uncertain Financial Market with Jump Risk, *Journal of Uncertainty Analysis and Applications* **4**, 1–11.