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ON DIVERGENCE OF PRODUCTS OF IRREGULAR SCALAR AND SOLENOIDAL VECTOR FIELDS

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Abstract. We construct an irregular pair satisfying a divergence equation. By using the convex integration method, it can be proved that there exists a pair which violates the chain rule of differentiation. In contrast, such a violation cannot occur when we consider a more regular pair. In the end, we obtain an interpretation of these results from a microscopic point of view. The mechanism of constructing an irregular pair becomes clear by considering a time-independent case.

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1 Introduction

1.1 Introduction and main results

In this paper, we consider the following divergence equation

$$\begin{cases} \operatorname{div}(\rho u) = F \\ \\ \operatorname{div} u = 0 \end{cases} \quad \text{in} \quad \mathbb{T}^d \tag{1.1}$$

where ρ is a real-valued function, u is a vector-valued function, F is a given real-valued function and $\mathbb{T}^d$ is defined by $\mathbb{R}^d/\mathbb{Z}^d$. We construct a pair (ρ, u) of (1.1) which violates the chain rule of differentiation. We construct such a pair by adjusting the convex integration method of Modena and Székelyhidi Jr.[17], where they constructed several weak solutions of a linear transport equation. Since there is no time dependence, it seems that the mechanism of finding such a solution is clearer than that of [17]. This construction yields the following result.

Assumption (A).

$$\frac{1}{p} + \frac{1}{\tilde{p}} > 1 + \frac{1}{d-1}$$

where $p \in (1, \infty)$ and $\tilde{p} \in [1, \infty)$.

Theorem 1.1. *Assume* (A)*. Let* $\varepsilon > 0$, $\rho_0 \in C^\infty(\mathbb{T}^d)$ *and* $F \in C^\infty(\mathbb{T}^d)$ *with* $F \not\equiv 0$ *and* $\int_{\mathbb{T}^d} F dx = 0$*. Then there exist* $\rho \in L^p(\mathbb{T}^d)$ *and* $u \in L^{p'}(\mathbb{T}^d) \cap W^{1,\tilde{p}}(\mathbb{T}^d)$ *such that* (ρ, u) *satisfies* (1.1) *in the sense of distributions and*

$$\|\rho - \rho_0\|_{L^p(\mathbb{T}^d)} < \ \varepsilon.$$

Here, p' *is the conjugate exponent of* p*, i.e.* $\frac{1}{p} + \frac{1}{p'} = 1$.

From Theorem 1.1, it becomes possible to investigate the reason which causes the violation of the chain rule. We expect that a solution of (1.1) satisfies the following equation formally.

$$\begin{cases} \operatorname{div}(\Phi(\rho) u) = \varphi(\rho) F \\ \\ \operatorname{div} u = 0 \end{cases} \quad \text{in} \quad \mathbb{T}^d \tag{1.2}$$

where $\varphi \in C_0^\infty(\mathbb{R})$ and $\Phi(\tau) = \int_{-\infty}^{\tau} \varphi(\xi) d\xi$. However, this expectation is not true. This fact is proved in the following theorem.

Theorem 1.2. *Assume* (A)*. Let* $F \in C^\infty(\mathbb{T}^d)$ *with* $F \not\equiv 0$ *and* $\int_{\mathbb{T}^d} F dx = 0$*. Then there exist* $\varphi \in C_0^\infty(\mathbb{R})$ *with* $\varphi \not\equiv 0$, $\rho \in L^p(\mathbb{T}^d)$ *and* $u \in L^{p'}(\mathbb{T}^d) \cap W^{1,\tilde{p}}(\mathbb{T}^d)$ *such that* (ρ, u) *satisfies* (1.1) *in the sense of distributions but does not satisfy* (1.2).

This theorem asserts that the solution constructed in this way does not satisfy the chain rule. On the other hand, it is possible to understand this result from the perspective of renormalized solutions. A solution of (1.1) is called renormalized if it satisfies (1.2) for all $\varphi \in C_0^\infty(\mathbb{R})$. By this definition, Theorem 1.2 asserts that the solution constructed in this way is not renormalized.

In contrast, a solution of (1.1) is always renormalized when we consider a more regular pair (ρ, u). This corresponds the results of DiPerna and Lions for transport equations [11].

Assumption (B).

$$\frac{1}{p} + \frac{1}{\tilde{p}} \leq 1$$

where $p \in (1, \infty)$ and $\tilde{p} \in [1, \infty)$.

Theorem 1.3. *Assume* (B). *Let $F \in C^\infty(\mathbb{T}^d)$ with $F \not\equiv 0$ and $\int_{\mathbb{T}^d} F dx = 0$. Suppose that $\rho \in L^p(\mathbb{T}^d)$ and $u \in L^{p'}(\mathbb{T}^d) \cap W^{1,\tilde{p}}(\mathbb{T}^d)$ satisfy* (1.1) *in the sense of distributions. Then (ρ, u) satisfies* (1.2) *for any $\varphi \in C_0^\infty(\mathbb{R})$.*

Furthermore, these results can be interpreted from a microscopic point of view. A method based on the kinetic theory to construct a global solution of first order quasi-linear equations was introduced in [3, 12, 13]. The key idea is that integrating a microscopic quantity gives an expression for a macroscopic quantity. They consider motion of microscopic quantity, and then averaging yields macroscopic quantity. The close relation between a microscopic description and a macroscopic description is essential. A similar microscopic point of view has been applied to the entropy criterion of more general multi-dimensional scalar conservation laws [15] and nonlinear elliptic equations [2]. We consider the equation (1.1) as a macroscopic equation via the following form:

$$\chi_\rho(x, \xi) = \begin{cases} 1 & (\,\xi \leq \rho(x)\,)\,, \\ 0 & (\,\xi > \rho(x)\,)\,. \end{cases}$$

Then the microscopic equation is the following:

$$\begin{cases} \mathrm{div}_x\,(\chi_\rho u) = F\partial_\xi \chi_\rho \\ \\ \mathrm{div}_x\ u = 0 \end{cases} \quad \text{in} \quad \mathbb{T}^d \times \mathbb{R}. \tag{1.3}$$

Weak solutions of (1.3) correspond to renormalized solutions of (1.1). Hence, we obtain a reason behind non-renormalized solutions.

Corollary 1.4. *Assume* (A). *Then there exist $\rho \in L^p(\mathbb{T}^d)$ and $u \in L^{p'}(\mathbb{T}^d) \cap W^{1,\tilde{p}}(\mathbb{T}^d)$ such that (ρ, u) satisfies* (1.1) *in the sense of distributions but does not satisfy* (1.3).

$Z = \mathrm{div}\,(\chi_\rho u) - F\partial_\xi \chi_\rho (\not\equiv 0)$ has a great effect on the non-uniqueness because it is an obstruction to the renormalization. This result is consistent with the results of DiPerna and Lions [11]. In a similar way, it is also possible to describe Theorem 1.3 for appropriate test functions.

1.2 Literature overview

Recently the convex integration method has been widely used in the process of constructing weak solutions of fluid equations. The convex integration method was used by DeLellis and Székelyhidi Jr.[8, 9, 10] in order to solve Onsager's conjecture. They constructed non-unique weak solutions to the Euler equation which do not obey the energy conservation law. Such results have been gradually improved, and recently Onsager's conjecture was proved rigorously in [5]. This method played a critical role in constructing non-unique weak solutions not only to the Euler equation but also to the Navier-Stokes equation [6]. In a similar way, Modena and Székelyhidi Jr.[17] constructed non-unique weak solutions of the following transport equation

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho u) = 0 \\ \\ \mathrm{div}\ u = 0 \end{cases} \qquad \text{in} \quad [0,T] \times \mathbb{T}^d. \tag{1.4}$$

In this direction, the study of the relation between non-uniqueness and regularity has been further developed [18, 16, 7]. Especially, the critical space is studied by [7] under less time-regularity of ρ and u. On the other hand, a long time ago DiPerna and Lions [11] obtained the unique existence of a weak solution under an irregular u for a transport equation which is more general than (1.4). Their solution is called a renormalized solution. Their paper contained the results of the unique and renormalized solution. A remarkable point of their work is that they proved the uniqueness by using the renormalization. To secure consistency of the results above, the solution constructed by Modena and Székelyhidi Jr. [17] should not be renormalized because the renormalization implies the uniqueness.

This paper is organized as follows. In Section 2 we present the facts on the torus. They are needed in constructing and estimating the solution. Section 3 is devoted to the construction of the solution of (1.1). We show a key lemma which enables us to construct the solution by an inductive method. Using this lemma, we prove Theorem 1.1. In Section 4 we consider the renormalization. We show that commutator estimates play an important role for the renormalization. In Section 5 we prove that Theorem 1.1 implies Theorem 1.2. Renormalization can be characterized by the microscopic point of view.

Notation. We list some notations for scalar functions. We use the same notation for vector-valued functions.

- Let X be a functional space on the torus whose elements are integrable on the torus. Then we denote by X_{ave} the set $\{f \in X : \int_{\mathbb{T}^d} f dx = 0\}$. For example, $C^\infty_{\mathrm{ave}}(\mathbb{T}^d) = \{f \in C^\infty(\mathbb{T}^d) : \int_{\mathbb{T}^d} f dx = 0\}$.
- For $g\colon \mathbb{T}^d \to \mathbb{R}$ and $\lambda \in \mathbb{N}$, g_λ is defined by

$$g_\lambda(x) = g(\lambda x).$$

- We denote by $C(A_1, \ldots, A_m)$ a constant which depends on the numbers $A_1, \ldots, A_m$. The exact value denoted by C may change from line to line.

2 Basic estimates and tools

Here, we list some estimations in order to prove theorems in this paper. The proofs of these estimations are based on the paper of Modena and Székelyhidi Jr.[16, 17]. We state propositions without proofs here. The only difference is the introduction of a projective operator in constructing an anti-divergence operator.

2.1 Improved Hölder inequality

Roughly speaking, this proposition says that a product of a function by a scaled one is bounded above by the original product and a good behaved part. It makes possible to control the norm of a solution constructed in the following section.

Proposition 2.1. ([17, Lemma 2.1]) Let $\lambda \in \mathbb{N}$ and $f, g \in C^\infty(\mathbb{T}^d)$. Then for every $p \in [1, \infty]$,

$$\|fg_\lambda\|_{L^p(\mathbb{T}^d)} \ \le \ \|f\|_{L^p(\mathbb{T}^d)}\|g\|_{L^p(\mathbb{T}^d)} + \frac{C}{\lambda^{1/p}}\|f\|_{C^1(\mathbb{T}^d)}\|g\|_{L^p(\mathbb{T}^d)}$$

where C is a constant independent of λ, f and g.

This proposition is proved by dividing $\mathbb{T}^d$ into small cubes of edge $1/\lambda$ and using the mean value theorem.

2.2 Anti-divergence operators

In this section, we introduce anti-divergence operators. The estimates on these operators are essentially same as those of Modena and Székelyhidi Jr.[17]. We give slightly different expression of operators by using the Helmholtz projection.

For $f \in C^\infty_{\text{ave}}(\mathbb{T}^d)$ there exists a unique $u \in C^\infty_{\text{ave}}(\mathbb{T}^d)$ such that $\Delta u = f$. So, the operator $\Delta^{-1}\colon C^\infty_{\text{ave}}(\mathbb{T}^d) \to C^\infty_{\text{ave}}(\mathbb{T}^d)$ is well defined. We write the standard anti-divergence operator $\nabla\Delta^{-1}\colon C^\infty_{\text{ave}}(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ by K. Then it satisfies $\operatorname{div}(Kf) = f$. In addition, K satisfies the following estimates.

Proposition 2.2. ([17, Lemma 2.2]) Let K be the standard anti-divergence operator

$$\nabla\Delta^{-1}\colon C^\infty_{\text{ave}}(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d;\mathbb{R}^d).$$

Then for every $k \in \mathbb{N}$, $p \in [1, \infty]$ and $g \in C^\infty_{\text{ave}}(\mathbb{T}^d)$,

$$\|D^k(Kg)\|_{L^p(\mathbb{T}^d)} \le C\|D^k g\|_{L^p(\mathbb{T}^d)}. \tag{2.1}$$

Moreover, for every $\lambda \in \mathbb{N}$

$$\|D^k(Kg_\lambda)\|_{L^p(\mathbb{T}^d)} \le C'\lambda^{k-1}\|D^k g\|_{L^p(\mathbb{T}^d)}. \tag{2.2}$$

Here, C and C' are constants independent of g and λ.

Notice that $\operatorname{div} u \in C^\infty_{\text{ave}}(\mathbb{T}^d)$ if $u \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$. Therefore, by using the standard anti-divergence operator, we write P for the projection $\text{Id} - K\text{div}$, where Id is the identity operator. This projection maps $C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ onto a subspace of $C^\infty(\mathbb{T}^d;\mathbb{R}^d)$ whose elements are divergence-free vector fields. This projection is often called the Helmholtz projection. We have $\operatorname{div}(Pu) = 0$ for any $u \in C^\infty(\mathbb{T}^d;\mathbb{R}^d)$.

Using these operators, we define the improved anti-divergence operator S for a product of functions. For $f, g \in C^\infty(\mathbb{T}^d)$ with $g, fg \in C^\infty_{\text{ave}}(\mathbb{T}^d)$, S is defined by $S(fg) = K(fg) + P(fKg)$. From the definition, we have $\operatorname{div}(Sfg) = fg$. The word “improved” comes from the following result. This proposition says that S gives good estimates for a product of a function by a scaled one.

Proposition 2.3. *Let K be the standard anti-divergence operator*

$$\nabla\Delta^{-1} \colon C^\infty_{\text{ave}}(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d;\mathbb{R}^d),$$

$P = \text{Id} - K\text{div}$ and $\lambda \in \mathbb{N}$. For $f, g \in C^\infty(\mathbb{T}^d)$ with $g, fg_\lambda \in C^\infty_{\text{ave}}(\mathbb{T}^d)$, set $S(fg_\lambda) = K(fg_\lambda) + P(fKg_\lambda)$. Then for every $k \in \mathbb{N}$ and $p \in [1,\infty]$,

$$\|D^k S(fg_\lambda)\|_{L^p(\mathbb{T}^d)} \le C\lambda^{k-1}\|f\|_{C^{k+1}(\mathbb{T}^d)}\|g\|_{W^{k,p}(\mathbb{T}^d)}$$

where C is a constant independent of λ, f and g.

Remark 2.4. The same estimates were obtained by Modena and Székelyhidi Jr.[17, Lemma 2.3] by replacing $S(fg_\lambda)$ with $fKg_\lambda - K(\nabla f \cdot Kg_\lambda)$. However, it turns out that these quantities are same. Indeed,

$$\begin{aligned} S(fg_\lambda) &= K(fg_\lambda) + P(fKg_\lambda) \\ &= K(fg_\lambda) + fKg_\lambda - K\text{div}(fKg_\lambda) \\ &= K(fg_\lambda) + fKg_\lambda - K(\nabla f \cdot Kg_\lambda + f\text{div}(Kg_\lambda)) \\ &= K(fg_\lambda) + fKg_\lambda - K(\nabla f \cdot Kg_\lambda) - K(fg_\lambda) \\ &= fKg_\lambda - K(\nabla f \cdot Kg_\lambda). \end{aligned}$$

In the same way, we are able to find u such that $\operatorname{div} u = f \cdot g_\lambda$ if f and g are vector fields.

2.3 Mikado density and Mikado flow

Here, we recall the result of Mikado from the paper of Modena and Székelyhidi Jr.[17]. We construct a solution by adding these terms step by step in an inductive way.

Proposition 2.5. (cf. [17, Proposition 4.1 and Remark 4.2]) Assume (A). Then there exist universal constants $M > 0$ and $\gamma > 0$ such that the following holds: For every $\mu > 2d$ and $j \in \{1, \dots, d\}$, there exist $\Theta^j_\mu \in C^\infty_{\text{ave}}(\mathbb{T}^d)$ and $W^j_\mu \in C^\infty_{\text{ave}}(\mathbb{T}^d;\mathbb{R}^d)$ such that the following properties hold.

(i)
$$\text{div}W^j_\mu = \text{div}(\Theta^j_\mu W^j_\mu) = 0.$$

(ii)
$$\frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d} \Theta^j_\mu W^j_\mu = e_j$$

where $\{e_1, \ldots, e_d\}$ is the standard basis in $\mathbb{R}^d$.

(iii)
$$\sum_{j=1}^{d} \|\Theta_\mu^j\|_{L^p(\mathbb{T}^d)}, \sum_{j=1}^{d} \|W_\mu^j\|_{L^{p'}(\mathbb{T}^d)}, \sum_{j=1}^{d} \|\Theta_\mu^j W_\mu^j\|_{L^1(\mathbb{T}^d)} \leq M/2. \tag{2.3}$$

(iv)
$$\|\Theta_\mu^j\|_{L^1(\mathbb{T}^d)}, \|W_\mu^j\|_{L^1(\mathbb{T}^d)}, \|W_\mu^j\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} \leq M\mu^{-\gamma}. \tag{2.4}$$

(v) For $j \neq k$,
$$\mathrm{supp}\Theta_\mu^j = \mathrm{supp}W_\mu^j \quad \text{and} \quad \mathrm{supp}\Theta_\mu^j \cap \mathrm{supp}W_\mu^k = \emptyset.$$

We call Θ_μ^j and W_μ^j Mikado density and Mikado flow, respectively.

3 Construction of irregular pairs for a divergence equation

In this section, we prove Theorem 1.1. In order to construct a solution satisfying the properties of Theorem 1.1, we consider the approximate solution:

$$\begin{cases} \mathrm{div}\,(\rho u) - F = -\mathrm{div} R \\ \mathrm{div}\, u = 0 \end{cases} \quad \text{in} \quad \mathbb{T}^d. \tag{3.1}$$

R is considered as an error term. Lemma 3.1 plays an important role in the estimates of a solution. By this lemma, it turns out that the error goes to zero in an inductive method.

This construction of irregular pairs is based on the idea of Modena and Székelyhidi Jr. [17]. Our method is essentially similar with theirs, but we do not have to consider the time-dependence in this case. So, we adjust their proof for our setting.

3.1 First step for an inductive method

The following lemma is essential. We prove this by using the Mikado density and the Mikado flow, which are introduced in Section 2.

Lemma 3.1. *Assume* (A). *Then there exists a universal constant $M > 0$ such that the following holds: If δ and η are positive numbers and if a smooth triplet (ρ_0, u_0, R_0) with $R_0 \not\equiv 0$ satisfies the equation* (3.1)*, then there exists another smooth triplet (ρ_1, u_1, R_1) satisfying the equation* (3.1) *such that*

$$\begin{aligned} \|\rho_1 - \rho_0\|_{L^p(\mathbb{T}^d)} &\leq M\eta \|R_0\|_{L^1(\mathbb{T}^d)}^{1/p}, \\ \|u_1 - u_0\|_{L^{p'}(\mathbb{T}^d)} &\leq M\eta^{-1} \|R_0\|_{L^1(\mathbb{T}^d)}^{1/p'}, \\ \|u_1 - u_0\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} &\leq \delta, \\ \|R_1\|_{L^1(\mathbb{T}^d)} &\leq \delta. \end{aligned}$$

Proof. We divide the proof into five steps. After constructing the triplet (ρ_1, u_1, R_1) in the first step, we show that this satisfies the estimates in this lemma.

First step (The definitions of ρ_1, u_1 and R_1)

Let $\lambda \in \mathbb{N}$ and $\mu > 2d$ be parameters. They will be determined later, being based on (3.2), (3.4), (3.9) and (3.14). Proposition 2.5 is applied to this μ. We define the triplet (ρ_1, u_1, R_1) by

$$\begin{aligned}
\rho_1 &= \rho_0 + \theta_0,\\
u_1 &= u_0 + w_0 + w_{\mathrm{c}},\\
R_1 &= -(R^{\mathrm{quad}} + R^{\mathrm{h}} + R^{\mathrm{linear}} + R^{\mathrm{corr}}).
\end{aligned}$$

Here,

$$\begin{aligned}
\theta_0 &= \eta \sum_{j=1}^{d} h_j \ \mathrm{sign}(R_{0,j}) \ |R_{0,j}|^{1/p} \ (\Theta_\mu^j)_\lambda,\\
w_0 &= \eta^{-1} \sum_{j=1}^{d} h_j \ |R_{0,j}|^{1/p'} \ (W_\mu^j)_\lambda,\\
w_{\mathrm{c}} &= -\eta^{-1} \sum_{j=1}^{d} S\left(\nabla (h_j \ |R_{0,j}|^{1/p'}) \cdot \ (W_\mu^j)_\lambda \right),\\
R^{\mathrm{quad}} &= \sum_{j=1}^{d} S\left(\nabla (h_j^2 R_{0,j}) \cdot ((\Theta_\mu^j W_\mu^j)_\lambda - e_j) \right),\\
R^{\mathrm{h}} &= \sum_{j=1}^{d} (h_j^2 - 1) R_{0,j} e_j,\\
R^{\mathrm{linear}} &= \rho_0 w_0 + \theta_0 u_0,\\
R^{\mathrm{corr}} &= \rho_0 w_{\mathrm{c}} + \theta_0 w_{\mathrm{c}}
\end{aligned}$$

where we write

$$R_0 = \sum_{j=1}^{d} R_{0,j} e_j$$

using the standard basis in $\mathbb{R}^d$ and take the following cut-off function $h_j \in C^\infty(\mathbb{T}^d)$ with $|h_j| \leq 1$ and

$$h_j(x) = \begin{cases} 0 & \text{if } |R_{0,j}(x)| \leq \delta/8d,\\ 1 & \text{if } |R_{0,j}(x)| \geq \delta/4d. \end{cases}$$

Note that w_{c} and R^{quad} are well defined. In fact, by the construction of the Mikado density and the Mikado flow, W_μ^j and $\Theta_\mu^j W_\mu^j - e_j$ have zero-mean value. In addition, $\nabla (h_j \ |R_{0,j}|^{1/p'}) \cdot \ (W_\mu^j)_\lambda$ and $\nabla (h_j^2 R_{0,j}) \cdot ((\Theta_\mu^j W_\mu^j)_\lambda - e_j)$ have divergence forms because the divergence of the Mikado is zero.

By this definition of (ρ_1, u_1, R_1),

$$\begin{aligned}
-\mathrm{div} R_1 =& \sum_{j=1}^{d} \nabla(h_j^2 R_{0,j}) \cdot ((\Theta_\mu^j W_\mu^j)_\lambda - e_j) + \mathrm{div}\left(\sum_{j=1}^{d} (h_j^2 - 1) R_{0,j} e_j\right) \\
& + \mathrm{div}(\rho_0 w_0 + \theta_0 u_0 + \rho_0 w_{\mathrm{c}} + \theta_0 w_{\mathrm{c}}) \\
=& \mathrm{div}\left(\sum_{j=1}^{d} h_j^2 R_{0,j} \left((\Theta_\mu^j W_\mu^j)_\lambda - e_j\right)\right) + \mathrm{div}\left(\sum_{j=1}^{d} (h_j^2 - 1) R_{0,j} e_j\right) \\
& + \mathrm{div}(\rho_0 w_0 + \theta_0 u_0 + \rho_0 w_{\mathrm{c}} + \theta_0 w_{\mathrm{c}}) \\
=& \mathrm{div}\left(-R_0 + \sum_{j=1}^{d} h_j^2 R_{0,j} (\Theta_\mu^j W_\mu^j)_\lambda + \rho_0 w_0 + \theta_0 u_0 + \rho_0 w_{\mathrm{c}} + \theta_0 w_{\mathrm{c}}\right) \\
=& -\mathrm{div} R_0 + \mathrm{div}(\theta_0 w_0 + \rho_0 w_0 + \theta_0 u_0 + \rho_0 w_{\mathrm{c}} + \theta_0 w_{\mathrm{c}}) \\
=& \mathrm{div}(\rho_0 u_0) - F + \mathrm{div}(\theta_0 w_0 + \rho_0 w_0 + \theta_0 u_0 + \rho_0 w_{\mathrm{c}} + \theta_0 w_{\mathrm{c}}) \\
=& \mathrm{div}(\rho_0 + \theta_0)(u_0 + w_0 + w_{\mathrm{c}}) - F \\
=& \mathrm{div}(\rho_1 u_1) - F.
\end{aligned}$$

Forth equality from the last line is by (v) of Proposition 2.5. Hence, the triplet (ρ_1, u_1, R_1) satisfies (3.1).

Second step (The estimates of $\|\rho_1 - \rho_0\|_{L^p(\mathbb{T}^d)}$)

By the definition of ρ_1,

$$\begin{aligned}
\|\rho_1 - \rho_0\|_{L^p(\mathbb{T}^d)} &= \|\theta_0\|_{L^p(\mathbb{T}^d)} \\
&\leq \eta \sum_{j=1}^{d} \|h_j \ \mathrm{sign}(R_{0,j}) \ |R_{0,j}|^{1/p} \ (\Theta_\mu^j)_\lambda\|_{L^p(\mathbb{T}^d)}.
\end{aligned}$$

Proposition 2.1 gives

$$\begin{aligned}
\|\rho_1 - \rho_0\|_{L^p(\mathbb{T}^d)} \leq& \eta \sum_{j=1}^{d} \Big\{ \|h_j \ \mathrm{sign}(R_{0,j}) \ |R_{0,j}|^{1/p}\|_{L^p(\mathbb{T}^d)} \|\Theta_\mu^j\|_{L^p(\mathbb{T}^d)} \\
& + \frac{C(p)}{\lambda^{1/p}} \|h_j \ \mathrm{sign}(R_{0,j}) \ |R_{0,j}|^{1/p}\|_{C^1(\mathbb{T}^d)} \|\Theta_\mu^j\|_{L^p(\mathbb{T}^d)} \Big\}.
\end{aligned}$$

Recalling the definition of h_j, notice that

$$\|h_j \ \mathrm{sign}(R_{0,j}) \ |R_{0,j}|^{1/p}\|_{L^p(\mathbb{T}^d)} \leq \||R_{0,j}|^{1/p}\|_{L^p(\mathbb{T}^d)} \leq \|R_0\|_{L^1(\mathbb{T}^d)}^{1/p}$$

and that

$$\|h_j \ \mathrm{sign}(R_{0,j}) \ |R_{0,j}|^{1/p}\|_{C^1(\mathbb{T}^d)} \leq C(\delta, \|R_0\|_{C^1(\mathbb{T}^d)}).$$

In addition, using (2.3), we have

$$\begin{aligned}\|\rho_1-\rho_0\|_{L^p(\mathbb{T}^d)} &\le \eta\|R_0\|^{1/p}_{L^1(\mathbb{T}^d)}\sum_{j=1}^d\|\Theta^j_\mu\|_{L^p(\mathbb{T}^d)}+\eta\,\frac{C(p)}{\lambda^{1/p}}C(\delta,\|R_0\|_{C^1(\mathbb{T}^d)})\sum_{j=1}^d\|\Theta^j_\mu\|_{L^p(\mathbb{T}^d)}\\ &\le \frac{M}{2}\eta\|R_0\|^{1/p}_{L^1(\mathbb{T}^d)}+\frac{C(\eta,\delta,\|R_0\|_{C^1(\mathbb{T}^d)})}{\lambda^{1/p}}. \end{aligned}\tag{3.2}$$

If λ is chosen large enough, we get

$$\|\rho_1-\rho_0\|_{L^p(\mathbb{T}^d)}\le M\eta\|R_0\|^{1/p}_{L^1(\mathbb{T}^d)}.$$

Third step (The estimates of $\|u_1-u_0\|_{L^{p'}(\mathbb{T}^d)}$)

By the definition of u_1,

$$\|u_1-u_0\|_{L^{p'}(\mathbb{T}^d)}\le\|w_0\|_{L^{p'}(\mathbb{T}^d)}+\|w_{\mathrm{c}}\|_{L^{p'}(\mathbb{T}^d)}.$$

Proposition 2.1 gives

$$\begin{aligned}\|w_0\|_{L^{p'}(\mathbb{T}^d)} &\le \eta^{-1}\sum_{j=1}^d\|h_j\ |R_{0,j}|^{1/p'}\ (W^j_\mu)_\lambda\|_{L^{p'}(\mathbb{T}^d)}\\ &\le \eta^{-1}\sum_{j=1}^d\left\{\|h_j\ |R_{0,j}|^{1/p'}\|_{L^{p'}(\mathbb{T}^d)}\|W^j_\mu\|_{L^{p'}(\mathbb{T}^d)}+\frac{C(p')}{\lambda^{1/p'}}\ \|h_j\ |R_{0,j}|^{1/p'}\|_{C^1(\mathbb{T}^d)}\|W^j_\mu\|_{L^{p'}(\mathbb{T}^d)}\right\}.\end{aligned}$$

Recalling the definition of h_j, notice that

$$\|h_j\ |R_{0,j}|^{1/p'}\|_{L^{p'}(\mathbb{T}^d)}\le\||R_{0,j}|^{1/p'}\|_{L^{p'}(\mathbb{T}^d)}\le\|R_0\|^{1/p'}_{L^1(\mathbb{T}^d)}$$

and that

$$\|h_j\ |R_{0,j}|^{1/p'}\|_{C^1(\mathbb{T}^d)}\le C(\delta,\|R_0\|_{C^1(\mathbb{T}^d)}).$$

In addition, using (2.3), we have

$$\begin{aligned}\|w_0\|_{L^{p'}(\mathbb{T}^d)} &\le \eta^{-1}\|R_0\|^{1/p'}_{L^1(\mathbb{T}^d)}\sum_{j=1}^d\|W^j_\mu\|_{L^{p'}(\mathbb{T}^d)}+\eta^{-1}\frac{C(p')}{\lambda^{1/p'}}\ C(\delta,\|R_0\|_{C^1(\mathbb{T}^d)})\sum_{j=1}^d\|W^j_\mu\|_{L^{p'}(\mathbb{T}^d)}\\ &\le \frac{M}{2}\eta^{-1}\|R_0\|^{1/p'}_{L^1(\mathbb{T}^d)}+\frac{C(\eta,\delta,\|R_0\|_{C^1(\mathbb{T}^d)})}{\lambda^{1/p'}}.\end{aligned}$$

On the other hand, Proposition 2.3 gives

$$\begin{aligned}\|w_{\mathrm{c}}\|_{L^{p'}(\mathbb{T}^d)} &\le \eta^{-1}\sum_{j=1}^d\|S\left(\nabla(h_j\ |R_{0,j}|^{1/p'})\cdot\ (W^j_\mu)_\lambda\right)\|_{L^{p'}(\mathbb{T}^d)}\\ &\le \eta^{-1}\sum_{j=1}^d C(p)\lambda^{-1}\|\nabla(h_j\ |R_{0,j}|^{1/p'})\|_{C^1(\mathbb{T}^d)}\|W^j_\mu\|_{L^{p'}(\mathbb{T}^d)}\\ &\le \eta^{-1}\lambda^{-1}C(\delta,\|R_0\|_{C^2(\mathbb{T}^d)})\sum_{j=1}^d\|W^j_\mu\|_{L^{p'}(\mathbb{T}^d)}\\ &\le \frac{C(\eta,\delta,\|R_0\|_{C^2(\mathbb{T}^d)})}{\lambda}.\end{aligned}\tag{3.3}$$

Hence,

$$\begin{aligned}\|u_1-u_0\|_{L^{p'}(\mathbb{T}^d)} \leq &\frac{M}{2}\eta^{-1}\|R_0\|_{L^1(\mathbb{T}^d)}^{1/p'} \\ &+\frac{C(\eta,\delta,\|R_0\|_{C^1(\mathbb{T}^d)})}{\lambda^{1/p'}}+\frac{C(\eta,\delta,\|R_0\|_{C^2(\mathbb{T}^d)})}{\lambda}.\end{aligned} \tag{3.4}$$

If λ is chosen large enough, we get

$$\|u_1-u_0\|_{L^{p'}(\mathbb{T}^d)} \leq M\eta^{-1}\|R_0\|_{L^1(\mathbb{T}^d)}^{1/p'}.$$

Forth step (The estimates of $\|u_1-u_0\|_{W^{1,\tilde{p}}(\mathbb{T}^d)}$)

By the definition of u_1,

$$\|u_1-u_0\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} \leq \|w_0\|_{W^{1,\tilde{p}}(\mathbb{T}^d)}+\|w_{\mathrm{c}}\|_{W^{1,\tilde{p}}(\mathbb{T}^d)}.$$

We start to estimate $\|w_0\|_{W^{1,\tilde{p}}(\mathbb{T}^d)}$. Recalling the definition of h_j, we have the pointwise estimate:

$$\begin{aligned}|w_0| &\leq \eta^{-1}\sum_{j=1}^d |h_j\ |R_{0,j}|^{1/p'}|\ |(W_\mu^j)_\lambda| \\ &\leq \eta^{-1}C(\delta,\|R_0\|_{C^1(\mathbb{T}^d)})\sum_{j=1}^d |(W_\mu^j)_\lambda|.\end{aligned}$$

Hence, using (2.4),

$$\begin{aligned}\|w_0\|_{L^{\tilde{p}}(\mathbb{T}^d)} &\leq C(\eta,\delta,\|R_0\|_{C^1(\mathbb{T}^d)})\sum_{j=1}^d \|(W_\mu^j)_\lambda\|_{L^{\tilde{p}}(\mathbb{T}^d)} \\ &\leq C(\eta,\delta,\|R_0\|_{C^1(\mathbb{T}^d)})\sum_{j=1}^d \|W_\mu^j\|_{L^{\tilde{p}}(\mathbb{T}^d)} \\ &\leq C(\eta,\delta,\|R_0\|_{C^1(\mathbb{T}^d)})\mu^{-\gamma}.\end{aligned} \tag{3.5}$$

Furthermore, it holds that

$$Dw_0(x)=\eta^{-1}\sum_{j=1}^d\left\{\nabla\left(h_j(x)|R_{0,j}(x)|^{1/p'}\right)\otimes W_\mu^j(\lambda x)+h_j(x)|R_{0,j}(x)|^{1/p'}\lambda DW_\mu^j(\lambda x)\right\}.$$

So, we have the pointwise estimate:

$$|Dw_0(x)| \leq \eta^{-1}C(\delta,\|R_0\|_{C^1(\mathbb{T}^d)})\sum_{j=1}^d\left\{|W_\mu^j(\lambda x)|+\lambda|DW_\mu^j(\lambda x)|\right\}.$$

Therefore, by using (2.4) and the condition $\lambda \geq 1$,

$$\begin{aligned}\|Dw_0\|_{L^{\tilde{p}}(\mathbb{T}^d)} &\leq C(\eta, \delta, \|R_0\|_{C^1(\mathbb{T}^d)}) \sum_{j=1}^{d} \left\{ \|W^j_\mu\|_{L^{\tilde{p}}(\mathbb{T}^d)} + \lambda \|DW^j_\mu\|_{L^{\tilde{p}}(\mathbb{T}^d)} \right\} \\ &\leq C(\eta, \delta, \|R_0\|_{C^1(\mathbb{T}^d)}) \lambda \mu^{-\gamma}. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\|w_0\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} \leq C(\eta, \delta, \|R_0\|_{C^1(\mathbb{T}^d)}) \lambda \mu^{-\gamma}. \tag{3.7}$$

Next, we estimate $\|w_{\mathrm{c}}\|_{W^{1,\tilde{p}}(\mathbb{T}^d)}$. Proposition 2.3 and (2.4) give

$$\begin{aligned}\|w_{\mathrm{c}}\|_{L^{\tilde{p}}(\mathbb{T}^d)} &\leq \eta^{-1} \sum_{j=1}^{d} \left\| S\left(\nabla (h_j \ |R_{0,j}|^{1/p'}) \cdot \ (W^j_\mu)_\lambda \right) \right\|_{L^{\tilde{p}}(\mathbb{T}^d)} \\ &\leq \eta^{-1} \sum_{j=1}^{d} C\lambda^{-1} \|\nabla (h_j \ |R_{0,j}|^{1/p'})\|_{C^1(\mathbb{T}^d)} \|W^j_\mu\|_{L^{\tilde{p}}(\mathbb{T}^d)} \\ &\leq \eta^{-1} \lambda^{-1} C(\delta, \|R_0\|_{C^2(\mathbb{T}^d)}) \sum_{j=1}^{d} \|W^j_\mu\|_{L^{\tilde{p}}(\mathbb{T}^d)} \\ &\leq C(\eta, \delta, \|R_0\|_{C^2(\mathbb{T}^d)}) \ \lambda^{-1} \mu^{-\gamma} \end{aligned}$$

and

$$\begin{aligned}\|Dw_{\mathrm{c}}\|_{L^{\tilde{p}}(\mathbb{T}^d)} &\leq \eta^{-1} \sum_{j=1}^{d} \left\| DS\left(\nabla (h_j \ |R_{0,j}|^{1/p'}) \cdot \ (W^j_\mu)_\lambda \right) \right\|_{L^{\tilde{p}}(\mathbb{T}^d)} \\ &\leq \eta^{-1} \sum_{j=1}^{d} C\lambda^{0} \|\nabla (h_j \ |R_{0,j}|^{1/p'})\|_{C^2(\mathbb{T}^d)} \|W^j_\mu\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} \\ &\leq \eta^{-1} C(\delta, \|R_0\|_{C^3(\mathbb{T}^d)}) \sum_{j=1}^{d} \|W^j_\mu\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} \\ &\leq C(\eta, \delta, \|R_0\|_{C^3(\mathbb{T}^d)}) \ \mu^{-\gamma}. \end{aligned}$$

Hence,

$$\|w_{\mathrm{c}}\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} \leq C(\eta, \delta, \|R_0\|_{C^3(\mathbb{T}^d)}) \ \mu^{-\gamma}. \tag{3.8}$$

From (3.7) and (3.8),

$$\|u_1 - u_0\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} \leq C(\eta, \delta, \|R_0\|_{C^1(\mathbb{T}^d)}) \lambda \mu^{-\gamma} + C(\eta, \delta, \|R_0\|_{C^3(\mathbb{T}^d)}) \ \mu^{-\gamma}. \tag{3.9}$$

If μ is chosen of the form $\mu = \lambda^\beta$ with $\beta > 1/\gamma$ and λ is chosen large enough, we get

$$\|u_1 - u_0\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} \leq \delta.$$

Fifth step (The estimates of $\|R_1\|_{L^1(\mathbb{T}^d)}$)

We estimate R^{quad}, R^{h}, R^{linear} and R^{corr}, respectively.

Proposition 2.3 and (2.3) give

$$\begin{aligned}\|R^{\text{quad}}\|_{L^1(\mathbb{T}^d)} &\le \sum_{j=1}^{d} \left\|S\left(\nabla(h_j^2 R_{0,j})\cdot((\Theta_\mu^j W_\mu^j)_\lambda - e_j)\right)\right\| \\ &\le \sum_{j=1}^{d} C\lambda^{-1}\|\nabla(h_j^2 R_{0,j})\|_{C^1(\mathbb{T}^d)}\|(\Theta_\mu^j W_\mu^j - e_j)\|_{L^1(\mathbb{T}^d)} \\ &\le \lambda^{-1} C(\delta, \|R_0\|_{C^2(\mathbb{T}^d)}) \sum_{j=1}^{d} \|(\Theta_\mu^j W_\mu^j - e_j)\|_{L^1(\mathbb{T}^d)} \\ &\le \frac{C(\delta, \|R_0\|_{C^2(\mathbb{T}^d)})}{\lambda}. \end{aligned} \tag{3.10}$$

By the definition of h_j, $R^{\text{h}} \neq 0$ only when $|R_{0,j}(x)| \le \delta/4d$. So, we have the pointwise estimate:

$$|R^{\text{h}}| \le \sum_{j=1}^{d} |h_j^2 - 1||R_{0,j}| \le d \cdot 2 \cdot \frac{\delta}{4d} = \frac{\delta}{2}.$$

Hence,

$$\|R^{\text{h}}\|_{L^1(\mathbb{T}^d)} \le \frac{\delta}{2}. \tag{3.11}$$

Using (2.4), we have

$$\begin{aligned}\|R^{\text{linear}}\|_{L^1(\mathbb{T}^d)} &\le \|\rho_0 w_0\|_{L^1(\mathbb{T}^d)} + \|\theta_0 u_0\|_{L^1(\mathbb{T}^d)} \\ &\le \|\rho_0\|_{L^\infty(\mathbb{T}^d)}\|w_0\|_{L^1(\mathbb{T}^d)} + \|u_0\|_{L^\infty(\mathbb{T}^d)}\|\theta_0\|_{L^1(\mathbb{T}^d)} \\ &\le \|\rho_0\|_{L^\infty(\mathbb{T}^d)}\ \eta^{-1}\sum_{j=1}^{d} \||R_{0,j}|^{1/p'}\|_{L^\infty(\mathbb{T}^d)}\|W_\mu^j\|_{L^1(\mathbb{T}^d)} \\ &\quad + \|u_0\|_{L^\infty(\mathbb{T}^d)}\ \eta\sum_{j=1}^{d} \||R_{0,j}|^{1/p}\|_{L^\infty(\mathbb{T}^d)}\|\Theta_\mu^j\|_{L^1(\mathbb{T}^d)} \\ &\le C(\eta, \|\rho_0\|_{L^\infty(\mathbb{T}^d)}, \|R_0\|_{L^\infty(\mathbb{T}^d)})\ \mu^{-\gamma} \\ &\quad + C(\eta, \|u_0\|_{L^\infty(\mathbb{T}^d)}, \|R_0\|_{L^\infty(\mathbb{T}^d)})\ \mu^{-\gamma} \\ &\le C(\eta, \|\rho_0\|_{L^\infty(\mathbb{T}^d)}, \|u_0\|_{L^\infty(\mathbb{T}^d)}, \|R_0\|_{L^\infty(\mathbb{T}^d)})\ \mu^{-\gamma}. \end{aligned} \tag{3.12}$$

Using (3.2) and (3.3), we have

$$
\begin{aligned}
\|R^{\mathrm{corr}}\|_{L^1(\mathbb{T}^d)} \leq & \|\rho_0 w_{\mathrm{c}}\|_{L^1(\mathbb{T}^d)} + \|\theta_0 w_{\mathrm{c}}\|_{L^1(\mathbb{T}^d)} \\
\leq & \|\rho_0\|_{L^\infty(\mathbb{T}^d)} \|w_{\mathrm{c}}\|_{L^1(\mathbb{T}^d)} + \|\theta_0\|_{L^p(\mathbb{T}^d)} \|w_{\mathrm{c}}\|_{L^{p'}(\mathbb{T}^d)} \\
\leq & \|\rho_0\|_{L^\infty(\mathbb{T}^d)} C \|w_{\mathrm{c}}\|_{L^{p'}(\mathbb{T}^d)} + \|\theta_0\|_{L^p(\mathbb{T}^d)} \|w_{\mathrm{c}}\|_{L^{p'}(\mathbb{T}^d)} \\
\leq & \frac{C(\eta, \delta, \|\rho_0\|_{L^\infty(\mathbb{T}^d)}, \|R_0\|_{C^2(\mathbb{T}^d)})}{\lambda} \\
& + \left(\frac{M}{2} \eta \|R_0\|_{L^1(\mathbb{T}^d)}^{1/p} + \frac{C(\eta, \delta, \|R_0\|_{C^1(\mathbb{T}^d)})}{\lambda^{1/p}} \right) \cdot \frac{C(\eta, \delta, \|R_0\|_{C^2(\mathbb{T}^d)})}{\lambda} \\
\leq & \frac{C(\eta, \delta, \|\rho_0\|_{L^\infty(\mathbb{T}^d)}, \|R_0\|_{C^2(\mathbb{T}^d)})}{\lambda}.
\end{aligned}
\tag{3.13}
$$

From (3.10), (3.11), (3.12) and (3.13),

$$
\begin{aligned}
\|R_1\|_{L^1(\mathbb{T}^d)} \leq & \frac{\delta}{2} + C(\eta, \|\rho_0\|_{L^\infty(\mathbb{T}^d)}, \|u_0\|_{L^\infty(\mathbb{T}^d)}, \|R_0\|_{L^\infty(\mathbb{T}^d)})\ \mu^{-\gamma} \\
& + \frac{C(\eta, \delta, \|\rho_0\|_{L^\infty(\mathbb{T}^d)}, \|R_0\|_{C^2(\mathbb{T}^d)})}{\lambda}.
\end{aligned}
\tag{3.14}
$$

If μ is chosen of the form $\mu = \lambda^\beta$ with $\beta > 1/\gamma$ and λ is chosen large enough, we get

$$
\|R_1\|_{L^1(\mathbb{T}^d)} \leq \delta.
$$

Thus, we have completed the proof of Lemma 3.1.

□

3.2 Construction of an approximate solution by an inductive method

From Lemma 3.1, we obtain Theorem 1.1. We construct a solution by applying Lemma 3.1 repeatedly. Lemma 3.1 guarantees the convergence of the approximate sequence to a solution of (1.1).

Proof of Theorem 1.1. Take $u_0 \in C^\infty(\mathbb{T}^d; \mathbb{R}^d)$ with div $u_0 = 0$ and set

$$
R_0 = -\rho_0 u_0 + KF.
$$

It is well defined because $F \in C^\infty_{\mathrm{ave}}(\mathbb{T}^d)$. Then the triplet (ρ_0, u_0, R_0) satisfies (3.1). If $R_0 = 0$, we have nothing to prove. Let $R_0 \neq 0$. Fix η with

$$
\eta < \frac{\varepsilon}{M(\|R_0\|_{L^1(\mathbb{T}^d)}^{1/p} + \sum_{n=1}^\infty 2^{-n/p})}.
$$

Starting from (ρ_0, u_0, R_0), we apply Lemma 3.1 repeatedly. In the n-th step, we apply Lemma 3.1 for the above η and $\delta = 2^{-n}$. As a result, we get the sequences $\{\rho_n\}$, $\{u_n\}$

and $\{R_n\}$ such that for $n \geq 1$

$$\begin{aligned}
\|\rho_{n+1} - \rho_n\|_{L^p(\mathbb{T}^d)} &\leq M\eta\, 2^{-n/p}, \\
\|u_{n+1} - u_n\|_{L^{p'}(\mathbb{T}^d)} &\leq M\eta^{-1}\, 2^{-n/p}, \\
\|u_{n+1} - u_n\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} &\leq 2^{-(n+1)/p}, \\
\|R_{n+1}\|_{L^1(\mathbb{T}^d)} &\leq 2^{-(n+1)/p}.
\end{aligned}$$

Therefore, there exist $\rho \in L^p(\mathbb{T}^d)$ and $u \in L^{p'}(\mathbb{T}^d) \cap W^{1,\tilde{p}}(\mathbb{T}^d)$ such that

$$\begin{aligned}
\|\rho - \rho_n\|_{L^p(\mathbb{T}^d)} &\to 0, \\
\|u - u_n\|_{L^{p'}(\mathbb{T}^d)} &\to 0, \\
\|u - u_n\|_{W^{1,\tilde{p}}(\mathbb{T}^d)} &\to 0, \\
\|R_n\|_{L^1(\mathbb{T}^d)} &\to 0
\end{aligned}$$

as $n \to \infty$. Hence, (ρ, u) satisfies (1.1) in the sense of distributions. Moreover, by the construction,

$$\begin{aligned}
\|\rho - \rho_0\|_{L^p(\mathbb{T}^d)} &\leq \sum_{n=0}^{\infty} \|\rho_{n+1} - \rho_n\|_{L^p(\mathbb{T}^d)} \\
&\leq M\eta \|R_0\|_{L^1(\mathbb{T}^d)}^{1/p} + M\eta \sum_{n=1}^{\infty} 2^{-n/p} \\
&< \varepsilon.
\end{aligned}$$

This completes the proof. □

4 Regular pairs

In this section, we prove Theorem 1.3 after proving commutator estimates. By the commutator estimates, we can prove the result, which is more general than Theorem 1.3.

4.1 Commutator estimates

We prepare commutator estimates [11, Lemma II.1] for a periodic setting. We impose the divegence-free condition for simplicity.

Lemma 4.1. (cf. [11, Lemma II.1]) Assume (B). Let $\rho \in L^p(\mathbb{T}^d)$ and $u \in L^{p'}(\mathbb{T}^d) \cap W^{1,\tilde{p}}(\mathbb{T}^d)$ with div $u = 0$. Let η_ε be Friedrich's mollifier for $\varepsilon > 0$. Put

$$r_\varepsilon(\rho, u) = \operatorname{div}((\rho * \eta_\varepsilon)u) - (\operatorname{div}(\rho u)) * \eta_\varepsilon.$$

Then r_ε goes to 0 as ε goes to 0 in $L^\alpha(\mathbb{T}^d)$ where $\frac{1}{\alpha} = \frac{1}{p} + \frac{1}{\tilde{p}}$.

Proof. We give the proof for the reader's convenience. The proof is essentially found in [11]. First, we estimate $r_\varepsilon(\rho,u)$. We observe that

$$\begin{aligned}(\operatorname{div}(\rho u)) * \eta_\varepsilon &= \int_{\mathbb{T}^d} (\operatorname{div}_y(\rho(y)u(y)))\eta_\varepsilon(x-y)dy \\ &= -\int_{\mathbb{T}^d} \rho(y)u(y)\cdot\nabla_y\eta_\varepsilon(x-y)dy.\end{aligned}$$

Therefore,

$$\begin{aligned}(r_\varepsilon(\rho,u))(x) &= \operatorname{div}_x\left(\int_{\mathbb{T}^d}\rho(y)\eta_\varepsilon(x-y)dy\, u(x)\right) + \int_{\mathbb{T}^d}\rho(y)u(y)\cdot\nabla_y\eta_\varepsilon(x-y)dy \\ &= \int_{\mathbb{T}^d}\{\rho(y)\operatorname{div}_x\left(\eta_\varepsilon(x-y)u(x)\right) + \rho(y)u(y)\cdot\nabla_y\eta_\varepsilon(x-y)\}dy.\end{aligned}$$

Since div $u=0$,

$$(r_\varepsilon(\rho,u))(x) = \int_{\mathbb{T}^d}\rho(y)\left\{u(x)-u(y)\cdot\nabla_x\eta_\varepsilon(x-y)\right\}dy.$$

Notice that $\operatorname{supp}\eta_\varepsilon\subset\{x:|x|\le\tilde{C}\varepsilon\}$ for some $\tilde{C}$ and that $\|\nabla\eta\|_{L^\infty(\mathbb{T}^d)}$ is bounded. Then, it holds that

$$\begin{aligned}|(r_\varepsilon(\rho,u))(x)| &= \left|\int_{|x-y|\le\tilde{C}\varepsilon}\rho(y)\frac{u(x)-u(y)}{\varepsilon}\cdot\varepsilon\nabla_x\eta_\varepsilon(x-y)dy\right| \\ &\le C\int_{|z|\le\tilde{C}}|\rho(x+\varepsilon z)|\frac{|u(x+\varepsilon z)-u(x)|}{\varepsilon}dz\end{aligned}$$

Hence,

$$\begin{aligned}\|r_\varepsilon(\rho,u)\|^\alpha_{L^\alpha(\mathbb{T}^d)} &\le C^\alpha\int_{\mathbb{T}^d}\left|\int_{|z|\le\tilde{C}}|\rho(x+\varepsilon z)|\frac{|u(x+\varepsilon z)-u(x)|}{\varepsilon}dz\right|^\alpha dx \\ &\le C^\alpha\int_{\mathbb{T}^d}\int_{|z|\le\tilde{C}}\left|\rho(x+\varepsilon z)\frac{u(x+\varepsilon z)-u(x)}{\varepsilon}\right|^\alpha dzdx\end{aligned}$$

From this, we obtain

$$\|r_\varepsilon(\rho,u)\|_{L^\alpha(\mathbb{T}^d)} \le C\left\{\int_{\mathbb{T}^d}\int_{|z|\le\tilde{C}}|\rho(x+\varepsilon z)|^p dzdx\right\}^{1/p}\left\{\int_{\mathbb{T}^d}\int_{|z|\le\tilde{C}}\left|\frac{u(x+\varepsilon z)-u(x)}{\varepsilon}\right|^{\tilde{p}}dzdx\right\}^{1/\tilde{p}}.$$

Observe that

$$|u(x+\varepsilon z)-u(x)|\le\varepsilon\int_0^1|Du(x+t\varepsilon z)|dt|z|.$$

Then,

$$\begin{aligned}\int_{\mathbb{T}^d}\int_{|z|\le\tilde{C}}\left|\frac{u(x+\varepsilon z)-u(x)}{\varepsilon}\right|^{\tilde{p}}dzdx &\le \int_{\mathbb{T}^d}\int_{|z|\le\tilde{C}}\int_0^1|Du(x+t\varepsilon z)|^{\tilde{p}}dt|z|^{\tilde{p}}dzdx \\ &= \int_{|z|\le\tilde{C}}\|Du\|^{\tilde{p}}_{L^{\tilde{p}}(\mathbb{T}^d)}|z|^{\tilde{p}}dz \\ &= C\|Du\|^{\tilde{p}}_{L^{\tilde{p}}(\mathbb{T}^d)}.\end{aligned}$$

This yields

$$\|r_\varepsilon(\rho,u)\|_{L^\alpha(\mathbb{T}^d)} \le C\|\rho\|_{L^p(\mathbb{T}^d)}\|Du\|_{L^{\tilde{p}}(\mathbb{T}^d)}.$$

From density and this equality, we conclude that $r_\varepsilon \to 0$ as $\varepsilon \to 0$. Indeed, $r_\varepsilon \to 0$ as $\varepsilon \to 0$ if ρ and u are smooth. □

Corollary 4.2. *Assume* (B). *Let $F \in C^\infty(\mathbb{T}^d)$ with $F \not\equiv 0$ and $\int_{\mathbb{T}^d} F dx = 0$. Suppose that $\rho \in L^p(\mathbb{T}^d)$ and $u \in L^{p'}(\mathbb{T}^d) \cap W^{1,\tilde{p}}(\mathbb{T}^d)$ satisfy* (1.1) *in the sense of distributions. Let η_ε be Friedrich's mollifier for $\varepsilon > 0$. Put $\rho_\varepsilon = \rho * \eta_\varepsilon$. Then ρ_ε satisfies*

$$\mathrm{div}\,(\rho_\varepsilon u) = F_\varepsilon + \tilde{r}_\varepsilon$$

where $\tilde{r}_\varepsilon$ goes to 0 and F_ε goes to F as ε goes to 0 in $L^\alpha(\mathbb{T}^d)$ and α is given by $\frac{1}{\alpha} = \frac{1}{p} + \frac{1}{\tilde{p}}$.

Proof. From (1.1),

$$(\mathrm{div}(\rho u)) * \eta_\varepsilon = F * \eta_\varepsilon.$$

By using Lemma 4.1, we have

$$\mathrm{div}(\rho_\varepsilon u) = F * \eta_\varepsilon + r_\varepsilon(\rho,u).$$

Therefore, if we take $F_\varepsilon = F * \eta_\varepsilon$ and $\tilde{r}_\varepsilon = r_\varepsilon(\rho,u)$, they satisfy the desired properties. □

4.2 Regularity for renormalization

Here, we prove the result, which is more general than Theorem 1.3. The proof of Theorem 1.3 is a consequence of the following.

Theorem 4.3. *Assume* (B). *Let $F \in C^\infty(\mathbb{T}^d)$ with $F \not\equiv 0$ and $\int_{\mathbb{T}^d} F dx = 0$. Suppose that $\rho \in L^p(\mathbb{T}^d)$ and $u \in L^{p'}(\mathbb{T}^d) \cap W^{1,\tilde{p}}(\mathbb{T}^d)$ satisfy* (1.1) *in the sense of distributions. Then (ρ,u) satisfies*

$$\mathrm{div}((\theta\circ\rho)u) = (\theta'\circ\rho)F$$

for any $\theta \in C^2(\mathbb{R})$ with $\theta', \theta'' \in L^\infty(\mathbb{R})$.

Proof. From (1.1) and Corollary 4.2, we have

$$\begin{aligned}\mathrm{div}((\theta\circ\rho_\varepsilon)u) &= (\theta'\circ\rho_\varepsilon)\nabla\rho_\varepsilon\cdot u\\ &= (\theta'\circ\rho_\varepsilon)\mathrm{div}(\rho_\varepsilon u)\\ &= (\theta'\circ\rho_\varepsilon)F_\varepsilon + (\theta'\circ\rho_\varepsilon)\tilde{r}_\varepsilon\end{aligned}$$

Notice that $\theta'\circ\rho_\varepsilon$ is bounded independent of ε. Since $\tilde{r}_\varepsilon \to 0$ as $\varepsilon \to 0$ in $L^\alpha(\mathbb{T}^d)$, the second term converges to 0 as ε goes to 0 in $L^\alpha(\mathbb{T}^d)$. Observe that

$$(\theta'\circ\rho_\varepsilon)F_\varepsilon - (\theta'\circ\rho)F = (\theta'\circ\rho_\varepsilon - \theta'\circ\rho)F_\varepsilon + (\theta'\circ\rho)(F_\varepsilon - F)$$

and that

$$\begin{aligned}\|\theta' \circ \rho_\varepsilon - \theta' \circ \rho\|_{L^\alpha(\mathbb{T}^d)} &= \left\| \int_{\rho(x)}^{\rho_\varepsilon(x)} \theta''(\xi) d\xi \right\|_{L^\alpha(\mathbb{T}^d)} \\ &\leq \|\theta''\|_{L^\infty(\mathbb{R})} \|\rho_\varepsilon - \rho\|_{L^\alpha(\mathbb{T}^d)}.\end{aligned}$$

Then, the first term converges to $(\theta' \circ \rho)F$ as ε goes to 0 in $L^\alpha(\mathbb{T}^d)$. Since $\theta \circ \rho_\varepsilon \to \theta \circ \rho$ a.e., we obtain

$$\operatorname{div}((\theta \circ \rho)u) = (\theta' \circ \rho)F.$$

□

5 Microscopic viewpoint

In this section, we prove Theorem 1.2. After proving the existence of a non-renormalized solution, we summarize the results from the microscopic point of view.

5.1 Existence of a non-renormalized solution

Here, we prove that Theorem 1.1 implies Theorem 1.2.

Proof of Theorem 1.2. Take $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi \not\equiv 0$ and

$$\varphi(\xi)\xi \geq 0$$

for all $\xi \in \mathbb{R}$. Set $\alpha_\varphi = \int_{\mathbb{T}^d} \varphi(F)F$, then $\alpha_\varphi > 0$. Furthermore, fix ε with

$$0 < \varepsilon < \frac{\alpha_\varphi}{2\|\varphi'\|_{L^\infty(\mathbb{R})}\|F\|_{L^{p'}(\mathbb{T}^d)}}.$$

By applying Theorem 1.1 to this ε, $\rho_0 = F$ and a given F, there exist $\rho \in L^p(\mathbb{T}^d)$ and $u \in L^{p'}(\mathbb{T}^d) \cap W^{1,\tilde{p}}(\mathbb{T}^d)$ such that (ρ, u) satisfies (1.1) and

$$\|\rho - F\|_{L^p(\mathbb{T}^d)} < \varepsilon.$$

Notice that

$$\varphi(\rho(x)) - \varphi(F(x)) = \int_{F(x)}^{\rho(x)} \varphi'(\xi) d\xi.$$

Then, it holds that

$$\begin{aligned}\left| \int_{\mathbb{T}^d} (\varphi(\rho) - \varphi(F))F \right| &\leq \int_{\mathbb{T}^d} |\varphi(\rho) - \varphi(F)||F| \\ &\leq \|\varphi'\|_{L^\infty(\mathbb{R})} \int_{\mathbb{T}^d} |\rho - F||F| \\ &\leq \|\varphi'\|_{L^\infty(\mathbb{R})} \ \|\rho - F\|_{L^p(\mathbb{T}^d)} \ \|F\|_{L^{p'}(\mathbb{T}^d)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{T}^d} \varphi(\rho)F &= \int_{\mathbb{T}^d} \varphi(F)F + \int_{\mathbb{T}^d} (\varphi(\rho) - \varphi(F))F \\ &\geq \int_{\mathbb{T}^d} \varphi(F)F - \left| \int_{\mathbb{T}^d} (\varphi(\rho) - \varphi(F))F \right| \\ &\geq \int_{\mathbb{T}^d} \varphi(F)F - \|\varphi'\|_{L^\infty(\mathbb{R})} \|\rho - F\|_{L^p(\mathbb{T}^d)} \|F\|_{L^{p'}(\mathbb{T}^d)} \\ &> \alpha_\varphi - \alpha_\varphi/2 > 0.\end{aligned}$$

On the other hand, we have $\int_{\mathbb{T}^d} \operatorname{div}(\Phi(\rho)u) = 0$. Hence, this (ρ, u) does not satisfy (1.2). Thus, we have completed the proof. $\square$

Theorem 5.1. *For ε and F under the same assumption as Theorem* 1.2*, suppose that $\varphi \in C_0^\infty(\mathbb{R})$ with the following properties:*

$$\begin{aligned}&\varphi \not\equiv 0, \\ &\varphi(\xi)\xi \geq 0 \qquad \textit{for all} \quad \xi \in \mathbb{R}, \\ &\varepsilon < \frac{\int_{\mathbb{T}^d} \varphi(F)F}{\|\varphi'\|_{L^\infty(\mathbb{R})} \|F\|_{L^{p'}(\mathbb{T}^d)}}.\end{aligned}$$

Then, the same argument as above is valid. Hence, there exist several φ in Theorem 1.2*.*

5.2 Kinetic interpretation of a non-renormalized solution

We introduce the relation between a renormalized solution of (1.1) and a weak solution of (1.3). Consider a weak solution of (1.3). It holds that

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}} \{\chi_\rho(x,\xi)u(x) \cdot \nabla_x \phi(x,\xi) - F(x)\chi_\rho(x,\xi)\partial_\xi \phi(x,\xi)\} d\xi dx = 0$$

for all $\phi \in C_0^\infty(\mathbb{T}^d \times \mathbb{R})$. In particular, if a test function is chosen of the form $\phi(x,\xi) = \varphi(\xi)\psi(x)$, then it holds that

$$\begin{aligned}0 &= \int_{\mathbb{T}^d} \int_{\mathbb{R}} \{\chi_\rho(x,\xi)u(x) \cdot \nabla_x(\varphi(\xi)\psi(x)) - F(x)\chi_\rho(x,\xi)\partial_\xi(\varphi(\xi)\psi(x))\} d\xi dx \\ &= \int_{\mathbb{T}^d} \int_{\mathbb{R}} \{\chi_\rho(x,\xi)u(x) \cdot \varphi(\xi)\nabla_x\psi(x) - F(x)\chi_\rho(x,\xi)(\partial_\xi\varphi(\xi))\psi(x)\} d\xi dx \\ &= \int_{\mathbb{T}^d} \{u(x) \cdot \nabla_x\psi(x) \int_{-\infty}^{\rho(x)} \varphi(\xi) d\xi - F(x)\psi(x) \int_{-\infty}^{\rho(x)} (\partial_\xi\varphi(\xi)) d\xi\} dx \\ &= \int_{\mathbb{T}^d} \{\Phi(\rho(x))u(x) \cdot \nabla_x\psi(x) - \varphi(\rho(x))F(x)\psi(x)\} dx\end{aligned}$$

for all $\varphi(\xi) \in C_0^\infty(\mathbb{R})$ and $\psi(x) \in C_0^\infty(\mathbb{T}^d)$. Therefore, a weak solution of (1.3) is a renormalized solution of (1.1), but not vice versa. This calculation shows that a solution of (1.1) in Theorem 1.2 cannot give a solution of (1.3) and hence we obtain Corollary1.4.

We consider the non-zero term $Z = \operatorname{div}(\chi_\rho u) - F\partial_\xi \chi_\rho$ as a factor causing the non-renormalization. It remains to be investigated what kind of properties this Z has.

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