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ON THE HELMHOLTZ DECOMPOSITIONS OF VECTOR FIELDS OF BOUNDED MEAN OSCILLATION AND IN REAL HARDY SPACES OVER THE HALF SPACE

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Abstract. This paper is concerned with the Helmholtz decompositions of vector fields of bounded mean oscillation over the half space and vector fields in real Hardy spaces over the half space. It proves the Helmholtz decomposition for vector fields of bounded mean oscillation over the half space whereas a partial Helmholtz decomposition for vector fields in real Hardy spaces over the half space. Meanwhile, it also establishes two sets of theories of real Hardy spaces over the half space which are compatible with the theory of Miyachi (1990).

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1 Introduction

In this paper, we investigate the Helmholtz decompositions of vector fields of bounded mean oscillation over the half space and vector fields in real Hardy spaces over the half space. The subject of studying Helmholtz decompositions asks the standard question whether a vector field, in some specific function spaces over some specific domains, can be decomposed into the direct sum of a solenoidal subspace and a subspace which is exactly a gradient field. The reason why we are interested in this subject is due to the well known fact that Helmholtz decomposition plays an important role in constructing mild solutions of the Navier-Stokes equations.

Helmholtz decompositions are widely studied for vector fields of L^p spaces over many kinds of different domains when $1 < p < \infty$. For example, we have the result that for every open domain $\Omega \subset \mathbb{R}^n$ the Helmholtz decomposition holds for vector fields of $L^2(\Omega)$. When p does not equal to 2, we also know that the Helmholtz decompositions of vector fields of L^p spaces hold for some domains while there exists other domains where the Helmholtz decompositions of vector fields of L^p spaces fail to hold, e.g. see [4]. Although problems when p does not equal to 2 are much more difficult than the case when p equals to 2, we still had various results. However, this subject is poorly studied for vector fields of other function spaces. In the case for vector fields of bounded mean oscillation and vector fields in real Hardy spaces, we only have a single piece of result, obtained by Miyakawa [8], states that the Helmholtz decompositions of vector fields of bounded mean oscillation over $\mathbb{R}^n$ and vector fields in real Hardy spaces over $\mathbb{R}^n$ hold. This lack of study is due to the fact that the theories of real Hardy spaces and BMO spaces over domains other than $\mathbb{R}^n$ are harder to deal with and moreover, the proper definitions of the space of vector fields of bounded mean oscillation and the space of vector fields in real Hardy spaces over other domains are not known perfectly. The purpose of this paper seeks to extend the result of Miyakawa [8] from $\mathbb{R}^n$ to $\mathbb{R}^n_+ = \{(\mathbf{x}', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} | x_n > 0\}$. In the meantime, we show that our definitions of the space of vector fields of bounded mean oscillation over $\mathbb{R}^n_+$ and the space of vector fields in real Hardy spaces over $\mathbb{R}^n_+$ are valid, in the sense that they admit a duality relation.

In order to define the space of vector fields of bounded mean oscillation over $\mathbb{R}^n_+$, we need to define two types of BMO spaces over $\mathbb{R}^n_+$ firstly, one corresponds to the function space for the tangent direction while the other one corresponds to the function space for the normal direction. The BMO space over $\mathbb{R}^n_+$ for the tangent direction we define is the space $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$. In Section 5, we prove that $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ is equivalent to $BMO(\mathbb{R}^n_+) := r_{\mathbb{R}^n_+} BMO$, the restriction of functions of BMO to $\mathbb{R}^n_+$. The BMO space over $\mathbb{R}^n_+$ for the normal direction we define is the space $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$. In [1], it is proved that $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$ is equivalent to $BMO_M(\mathbb{R}^n_+)$ where $BMO_M(\mathbb{R}^n_+)$ is the BMO space defined by Miyachi in [7]. Therefore the space of vector fields of bounded mean oscillation over $\mathbb{R}^n_+$, denoted by $\mathbf{X}$, can be defined as $\mathbf{X} := (BMO(\mathbb{R}^n_+))^{n-1} \times BMO_M(\mathbb{R}^n_+)$. The first main theorem of this paper reads as follows. Let $\mathbf{n}$ be the exterior unit normal of the boundary of $\mathbb{R}^n_+$, i.e. $\mathbf{n} = (0, 0, -1)$ so that the inner product $\mathbf{v} \cdot \mathbf{n}$ denotes the normal trace to $\partial\mathbb{R}^n_+$ of a vector field $\mathbf{v}$ on $\mathbb{R}^n_+$.

Theorem 1.1. *Let $\mathbf{X}$ be the space of vector fields of bounded mean oscillation over the* *half space $\mathbb{R}^n_+$, then $\mathbf{X}$ admits the Helmholtz decomposition*

$$\mathbf{X} = \mathbf{X}_\sigma \oplus \mathbf{X}_\pi$$

with the Helmholtz projection $\mathbb{P}_{\mathbb{R}^n_+}$ where

$$\begin{aligned}\mathbf{X}_\sigma &= \{\, \mathbf{v} \in \mathbf{X} \mid \operatorname{div} \mathbf{v} = 0 \;\; \textit{in } \mathbb{R}^n_+ \;\; \& \;\; \mathbf{v}\cdot\mathbf{n} = 0 \;\; \textit{on } \partial\mathbb{R}^n_+ \,\},\\ \mathbf{X}_\pi &= \{\, \nabla p \in \mathbf{X} \mid p \in L^1_{loc}(\overline{\mathbb{R}^n_+}) \,\}.\end{aligned}$$

The key idea of the proof of Theorem 1.1 is to consider extension and restriction. When Miyakawa [8] established the Helmholtz decomposition of vector fields of bounded mean oscillation over $\mathbb{R}^n$ and vector fields in real Hardy spaces over $\mathbb{R}^n$, he considered the Helmholtz projection $\mathbb{P}$ where $\mathbb{P}_{i,j} := \delta_{i,j} + R_iR_j$ and R_i is the i-th Riesz transform for $1 \le i, j \le n$. Here we make use of this idea. We define our projection by $\mathbb{P}_{\mathbb{R}^n_+} := r_{\mathbb{R}^n_+}\mathbb{P}E$ where E is the extension operator which extends vectors in $\mathbf{X}$ to vectors in BMO and $r_{\mathbb{R}^n_+}$ is the restriction operator which restricts vectors in BMO back to vectors in $\mathbf{X}$. Then we prove that our projection $\mathbb{P}_{\mathbb{R}^n_+}$ is actually a bounded linear map from $\mathbf{X}$ to $\mathbf{X}$. Hence through this projection we have a natural decomposition of our space $\mathbf{X}$ of the form

$$\mathbf{X} = \mathbb{P}_{\mathbb{R}^n_+}\mathbf{X} \oplus (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}.$$

Then we prove that the subspace $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{X}$ is actually the solenoidal part and the subspace $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$ is actually the gradient part. As for the trace problem, we can make use of the theory of Temam [10] since $\mathbf{X} \subset \mathbf{L}^2_{loc}(\overline{\mathbb{R}^n_+})$. Notice that the space $\mathbf{X}$ is not a proper Banach space due to the fact that the BMO-type norm is just a seminorm. Therefore, in order to avoid any ambiguity, we mean the Helmholtz decomposition not for $\mathbf{X}$ in the usual sense but for the quotient space $\mathbf{X}/(\mathbb{R}^{n-1} \times \{0\})$. Here we direct the readers to Section 2 for the precise definitions of the extension E, the restriction $r_{\mathbb{R}^n_+}$, the space $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ and the space $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$.

By similar ideas as above, we need to define two types of real Hardy spaces over $\mathbb{R}^n_+$ in order to define the space of vector fields in real Hardy spaces over $\mathbb{R}^n_+$. For the real Hardy space over $\mathbb{R}^n_+$ in the tangent direction, denoted by $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$, is defined to be the restriction of all even functions in the real Hardy space over $\mathbb{R}^n$ to the half space $\mathbb{R}^n_+$. For the real Hardy space over $\mathbb{R}^n_+$ in the normal direction, denoted by $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, is defined to be the restriction of all odd functions in the real Hardy space over $\mathbb{R}^n$ to the half space $\mathbb{R}^n_+$. In Section 5, we also prove that $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ is equivalent to $\mathscr{H}^1_M(\mathbb{R}^n_+)$ where $\mathscr{H}^1_M(\mathbb{R}^n_+)$ is the real Hardy space defined by Miyachi in [7]. Hence the space of vector fields in real Hardy spaces over $\mathbb{R}^n_+$, denoted by $\mathbf{Y}$, can be defined as $\mathbf{Y} := (\mathscr{H}^1_{even}(\mathbb{R}^n_+))^{n-1} \times \mathscr{H}^1_M(\mathbb{R}^n_+)$. Let $\mathbf{Y}_\sigma = \{\, \mathbf{v} \in \mathbf{Y} \mid \operatorname{div} \mathbf{v} = 0 \;\; \text{in } \mathbb{R}^n_+ \;\; \& \;\; \mathbf{v}\cdot\mathbf{n} = 0 \;\; \text{on } \partial\mathbb{R}^n_+ \,\}$, the second main theorem in this paper reads as follows.

Theorem 1.2. *Let $\mathbf{Y}$ be the vector field in real Hardy spaces over the half space $\mathbb{R}^n_+$, then $\mathbf{Y}$ admits a decomposition of the form*

$$\mathbf{Y} = \mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} \oplus \mathbf{Y}_\pi$$

with a bounded linear projection $\mathbb{P}_{\mathbb{R}^n_+} : \mathbf{Y} \to \mathbf{Y}$ where

$$\begin{aligned}\mathbf{Y}_\sigma \subset \mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} &\subset \{\, \mathbf{v} \in \mathbf{Y} \mid \operatorname{div} \mathbf{v} = 0 \;\; \textit{in } \; \mathbb{R}^n_+ \,\},\\ \mathbf{Y}_\pi &= \{\, \nabla p \in \mathbf{Y} \mid p \in L^1_{loc}(\overline{\mathbb{R}^n_+}) \,\}.\end{aligned}$$

Similar to the proof of Theorem 1.1, we consider the same projection $\mathbb{P}_{\mathbb{R}^n_+} := r_{\mathbb{R}^n_+}\mathbb{P}E$ and we prove that $\mathbb{P}_{\mathbb{R}^n_+}$ is also a bounded linear map from $\mathbf{Y}$ to $\mathbf{Y}$. Using the same idea, we can see that $\mathbf{Y}$ also admits a natural decomposition of the form

$$\mathbf{Y} = \mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} \oplus (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}.$$

Although the later theory is basically the same as the previous case for vector fields of bounded mean oscillation, in this case we do not know how to solve the trace problem. Hence for the subspace $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y}$ we can only say that it is divergence free, we cannot say that it is the right solenoidal part in the Helmholtz decomposition. We have no problems in characterising the subspace $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$. $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$ is the right gradient part, just like the previous case. For the precise definitions of the spaces $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, we direct the readers to Section 2. Notice that if we can solve the trace problem, then this decomposition turns into the full Helmholtz decomposition immediately. Hence for this decomposition, we call it a partial Helmholtz decomposition.

By the standard theory of real Hardy spaces, we can see that the space of vector fields of bounded mean oscillation over $\mathbb{R}^n$ is exactly the dual space of the space of vector fields in real Hardy spaces $\mathscr{H}^1(\mathbb{R}^n)$. In order to make the theory over $\mathbb{R}^n_+$ to be compatible with the theory over $\mathbb{R}^n$, it is necessary to consider the relation between the spaces $\mathbf{X}$ and $\mathbf{Y}$. Fortunately, we have a positive answer to this question.

Theorem 1.3. *Suppose $\mathbf{v} \in \mathbf{X}$. Then the linear functional l defined on $\mathbf{Y}$ by*

$$l(\mathbf{u}) = \int_{\mathbb{R}^n_+} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x}$$

for $\mathbf{u} \in \mathbf{Y}$ is a bounded linear functional which satisfies $||l|| \leq c \cdot ||\mathbf{v}||_{\mathbf{X}}$ with some constant c. Conversely, every bounded linear functional on $\mathbf{Y}$ can be written in the form of

$$l(\mathbf{u}) = \int_{\mathbb{R}^n_+} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} \quad \textit{for all } \mathbf{u} \in \mathbf{Y}$$

with $\mathbf{v} \in \mathbf{X}$ and $||\mathbf{v}||_{\mathbf{X}} \leq c \cdot ||l||$ with some constant c. Here $||l||$ means the norm of l as a bounded linear functional on $\mathbf{Y}$.

In short, the above theorem states the simple fact that $\mathbf{X}$ is the dual space of $\mathbf{Y}$. To prove the above theorem, we prove that $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ is the dual space of $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$ is the dual space of $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. The key idea in showing these two duality relations is again to consider extensions and restrictions. By the theories in the previous part, we see that the even extension of elements in $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ produce elements in $\mathscr{H}^1(\mathbb{R}^n)$ and the odd extension of elements in $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ also produce elements in $\mathscr{H}^1(\mathbb{R}^n)$. Since elements in $\mathscr{H}^1(\mathbb{R}^n)$ admit atomic decompositions, by taking the restrictions we can get the half space version of atomic decompositions of elements in $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. Then by similar arguments of Fefferman and Stein [3] in proving that BMO is the dual space of $\mathscr{H}^1(\mathbb{R}^n)$, we can prove the two duality relations concerning $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. The proof of Theorem 1.3 establishes two sets of complete theories for our two types of real Hardy spaces over $\mathbb{R}^n_+$. These two sets of theories are indeed compatible with the theory of Miyachi [7] where he established the theory of real Hardy spaces over arbitrary open subsets of $\mathbb{R}^n$. As a result, Theorem 1.3 verifies the validity of the definitions of $\mathbf{X}$ and $\mathbf{Y}$.

In the work of Miyakawa [8], he also found the fact that the dual operator of the whole space Helmholtz projection $\mathbb{P}$ is indeed $\mathbb{P}$ itself. In this paper we also investigate the dual operator of our half space Helmholtz projection $\mathbb{P}_{\mathbb{R}^n_+}$ and we obtain the following result.

Theorem 1.4. *The dual operator of* $\mathbb{P}_{\mathbb{R}^n_+} : \mathbf{Y} \to \mathbf{Y}$ *is* $\mathbb{P}_{\mathbb{R}^n_+}$ *itself as a map from* $\mathbf{X}$ *to* $\mathbf{X}$, *i.e.* ${\mathbb{P}_{\mathbb{R}^n_+}}^* = \mathbb{P}_{\mathbb{R}^n_+}$ *as a map from* $\mathbf{X}$ *to* $\mathbf{X}$.

The key idea lies in the proof of Theorem 1.3. This theorem can be easily deduced by simply considering the dual operators of E, $\mathbb{P}$ and $r_{\mathbb{R}^n_+}$. By making use of this theorem, we can further deduce the following important corollary.

Corollary 1.5. $\mathbf{X}_\sigma = {\mathbf{Y}_\pi}^\perp$ *and* $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} = {\mathbf{X}_\pi}^\perp$.

Notice that here because we do not know how to take the trace of elements in $\mathbf{Y}$ properly, we can only say that $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y}$ is the annihilator of $\mathbf{X}_\pi$. If the trace problem is settled, this relation turns into $\mathbf{Y}_\sigma = {\mathbf{X}_\pi}^\perp$ immediately.

This paper is organised as follow. In section 2, we give out the basic definitions. In section 3, we investigate the Helmholtz decomposition of $\mathbf{X}$. In section 4, we investigate the Helmholtz decomposition of $\mathbf{Y}$. In section 5, we study the duality relationship between $\mathbf{X}$ and $\mathbf{Y}$. In section 6, we study the dual operator of our Helmholtz projection $\mathbb{P}_{\mathbb{R}^n_+} : \mathbf{Y} \to \mathbf{Y}$.

2 Definitions and Notations

Let $\mathbb{R}^{\mathrm{n}}_+ := \{\mathbf{x} \in \mathbb{R}^n | x_n > 0\}$ be the half space where x_n here is the n-th component of $\mathbf{x}$ and let $\partial\mathbb{R}^{\mathrm{n}}_+ := \{\mathbf{x} \in \mathbb{R}^n | x_n = 0\}$ be the boundary of the half space $\mathbb{R}^{\mathrm{n}}_+$. The space $L^1_{loc}(\mathbb{R}^{\mathrm{n}}_+)$ is defined in the usual way as the set $\{f : \mathbb{R}^n_+ \to \mathbb{R}$ measurable $|\|f\|_{L^1(\Omega)} < \infty$ for any open subsets $\Omega \subset\subset \mathbb{R}^{\mathrm{n}}_+\}$ and $\mathbf{L}^1_{loc}(\mathbb{R}^{\mathrm{n}}_+) := (L^1_{loc}(\mathbb{R}^{\mathrm{n}}_+))^n$.

Definition 2.1. *Let* $f \in L^1_{loc}(\mathbb{R}^n_+)$ *and* $B_r(\mathbf{x})$ *be the open ball of radius* r *centered at* $\mathbf{x}$, *we define three types of BMO-type seminorms as the following:*

- $[f]_{BMO^\infty(\mathbb{R}^n_+)} := \sup\limits_{B \subset \mathbb{R}^n_+} \frac{1}{|B|}\int_B |f(\mathbf{y}) - f_B| \,\mathrm{d}\mathbf{y}$

  where $f_B := \frac{1}{|B|}\int_B f(\mathbf{y})\,\mathrm{d}\mathbf{y}$ *and* B *is an open ball.*

- $[f]_{b^\infty(\mathbb{R}^n_+)} := \sup\limits_{\substack{r>0 \\ \mathbf{x}\in\partial\mathbb{R}^n_+}} \frac{1}{|B_r(\mathbf{x})\cap\mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y})|\,\mathrm{d}\mathbf{y}$.

- $[f]_{ba^\infty(\mathbb{R}^n_+)} := \sup\limits_{\substack{r>0 \\ \mathbf{x}\in\partial\mathbb{R}^n_+}} \frac{1}{|B_r(\mathbf{x})\cap\mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y}) - f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+}|\,\mathrm{d}\mathbf{y}$

  where $f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} := \frac{1}{|B_r(\mathbf{x})\cap\mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} f(\mathbf{y})\,\mathrm{d}\mathbf{y}$.

The seminorm $[\,\cdot\,]_{b^\infty(\mathbb{R}^n_+)}$ is already introduced in [1] with a more general form. In [1], the definition of this seminorm is of the form $[\,\cdot\,]_{b^\nu p(\Omega)}$ where ν could be any real number including ∞ and $p \in [1, \infty)$. In our case, when ν is equal to ∞ and $p = 1$, an easy check quickly shows that this seminorm is indeed a norm. Therefore it is unambiguous to replace $[\,\cdot\,]_{b^\infty(\mathbb{R}^n_+)}$ by $||\cdot||_{b^\infty(\mathbb{R}^n_+)}$.

Definition 2.2. *We define two types of BMO spaces over the half space $\mathbb{R}^n_+$ in the following way:*

- $BMO_b^{\infty,\infty}(\mathbb{R}^n_+) := \{f \in L^1_{loc}(\mathbb{R}^n_+) \mid ||f||_{BMO_b^{\infty,\infty}(\mathbb{R}^n_+)} < \infty\}$

  where $||f||_{BMO_b^{\infty,\infty}(\mathbb{R}^n_+)} := [f]_{BMO^\infty(\mathbb{R}^n_+)} + ||f||_{b^\infty(\mathbb{R}^n_+)}$.

- $BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+) := \{f \in L^1_{loc}(\mathbb{R}^n_+) \mid [f]_{BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)} < \infty\}$

  where $[f]_{BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)} := [f]_{BMO^\infty(\mathbb{R}^n_+)} + [f]_{ba^\infty(\mathbb{R}^n_+)}$.

Since $||\cdot||_{b^\infty(\mathbb{R}^n_+)}$ is indeed a norm, $||\cdot||_{BMO_b^{\infty,\infty}(\mathbb{R}^n_+)}$ is also a norm. However, $[\,\cdot\,]_{BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)}$ is simply a seminorm.

Definition 2.3. *The space of vector fields of bounded mean oscillation over the half space $\mathbb{R}^n_+$ is defined in the following way:*

$$\mathbf{X}(\mathbb{R}^n_+, \mathbb{R}^n) := \{(\mathbf{v}', v^n) \mid \mathbf{v}' \in (BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+))^{n-1}, v^n \in BMO_b^{\infty,\infty}(\mathbb{R}^n_+)\}$$

where $\mathbf{v}' := (v^1, \dots, v^{n-1})$ and $\mathbf{v} := (v^1, \dots, v^{n-1}, v^n)$. We define the seminorm $[\,\cdot\,]_{\mathbf{X}}$ on the space of vector fields $\mathbf{X}(\mathbb{R}^n_+, \mathbb{R}^n)$ as follow:

$$[\mathbf{v}]_{\mathbf{X}} := \sum_{i=1}^{n-1} [v^i]_{BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)} + ||v^n||_{BMO_b^{\infty,\infty}(\mathbb{R}^n_+)}.$$

From now on, without any ambiguity, we shall denote $(\mathbf{X}, [\,\cdot\,]_{\mathbf{X}})$ simply by $\mathbf{X}$ for abbreviation.

Next we would like to define two extension operators which extend functions over the half space $\mathbb{R}^n_+$ to functions over the whole space $\mathbb{R}^n$.

Definition 2.4. *Let $f : \mathbb{R}^n_+ \to \mathbb{R}$, we say that $E_{odd}\, f : \mathbb{R}^n \to \mathbb{R}$ is the odd extension of f if*

$$E_{odd}\, f(\mathbf{x}', x_n) = \begin{cases} f(\mathbf{x}', x_n) & \text{if } x_n > 0, \\ -f(\mathbf{x}', -x_n) & \text{if } x_n < 0. \end{cases}$$

a.e. (almost everywhere).

Definition 2.5. *Let $f : \mathbb{R}^n_+ \to \mathbb{R}$, we say that $E_{even}\, f : \mathbb{R}^n \to \mathbb{R}$ is the even extension of f if*

$$E_{even}\, f(\mathbf{x}', x_n) = \begin{cases} f(\mathbf{x}', x_n) & \text{if } x_n > 0, \\ f(\mathbf{x}', -x_n) & \text{if } x_n < 0. \end{cases}$$

a.e. (almost everywhere).

Based on these two definitions of extension, we are able to define an extension operator for vector fields of functions over the half space $\mathbb{R}^n_+$.

Definition 2.6. *Let $f^i : \mathbb{R}^n_+ \to \mathbb{R}$ for $1 \leq i \leq n$ and let $\mathbf{f} = (f^1, \dots, f^{n-1}, f^n)$, we define the extension of f by*

$$E\mathbf{f} = \begin{cases} (E\mathbf{f})^i := E_{even}\, f^i & \text{for } 1 \leq i \leq n-1, \\ (E\mathbf{f})^n := E_{odd}\, f^n. \end{cases}$$

After we defined the extension operator, we shall now define the restriction operator, for functions and vector field of functions.

Definition 2.7. *The restriction operator is defined as follow in two cases:*

- *Let $f : \mathbb{R}^n \to \mathbb{R}$, we define the restriction $r_{\mathbb{R}^n_+} f$ by $r_{\mathbb{R}^n_+} f := f\,|_{\mathbb{R}^n_+} : \mathbb{R}^n_+ \to \mathbb{R}^n$.*
- *Let $\mathbf{f} = (f^1, \dots, f^{n-1}, f^n)$ and $f^i : \mathbb{R}^n \to \mathbb{R}$ with $1 \leq i \leq n$, we define the i-th component of the restriction $r_{\mathbb{R}^n_+}\mathbf{f}$ by $(r_{\mathbb{R}^n_+}\mathbf{f})^i := r_{\mathbb{R}^n_+} f^i$.*

Now we have done enough preparations for defining our vector field of real Hardy space $\mathscr{H}^1$ over $\mathbb{R}^n_+$.

Definition 2.8. *We define two types of real Hardy space $\mathscr{H}^1$ over the half space $\mathbb{R}^n_+$ in the following way:*

- $\mathscr{H}^1_{odd}(\mathbb{R}^n_+) := \{f \in L^1(\mathbb{R}^n_+) \mid ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} < \infty\}$

  where $||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} := ||\sup_{t>0} |r_{\mathbb{R}^n_+}\, e^{t\Delta}\, E_{odd}\, f|\,(\mathbf{x})\,||_{L^1(\mathbb{R}^n_+)}$.
- $\mathscr{H}^1_{even}(\mathbb{R}^n_+) := \{f \in L^1(\mathbb{R}^n_+) \mid ||f||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)} < \infty\}$

  where $||f||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)} := ||\sup_{t>0} |r_{\mathbb{R}^n_+}\, e^{t\Delta}\, E_{even}\, f|\,(\mathbf{x})\,||_{L^1(\mathbb{R}^n_+)}$.

Here $e^{t\Delta}$ is the heat semigroup. In other words, $(e^{t\Delta} f)(\mathbf{x}) = \int_{\mathbb{R}^n} G_t(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\mathrm{d}\mathbf{y}$ where $G_t(\mathbf{x}) = \frac{1}{(4\pi t)^n} e^{-\frac{|\mathbf{x}|^2}{4t}}$ denotes the heat kernel. We also write as $(G_t * f)(\mathbf{x})$ by using the notation of convolution.

Definition 2.9. *The space of vector fields in real Hardy spaces over the half space $\mathbb{R}^n_+$ is defined in the following way:*

$$\mathbf{Y}(\mathbb{R}^n_+, \mathbb{R}^n) := \{(\mathbf{u}', u^n) \mid \mathbf{u}' \in (\mathscr{H}^1_{even}(\mathbb{R}^n_+))^{n-1}, u^n \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)\}$$

where $\mathbf{u}' := (u^1, \dots, u^{n-1})$ and $\mathbf{u} := (u^1, \dots, u^{n-1}, u^n)$. We define the norm $||\cdot||_{\mathbf{Y}}$ on $\mathbf{Y}$ by :

$$||\mathbf{u}||_{\mathbf{Y}} := \sum_{i=1}^{n-1} ||u^i||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)} + ||u^n||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}.$$

From now on, without any ambiguity, we shall denote $(\mathbf{Y}, ||\cdot||_{\mathbf{Y}})$ *simply by* $\mathbf{Y}$ *for abbreviation.*

Definition 2.10. *We define* $\mathbb{P}$ *by* $(\mathbb{P})_{ij} := \delta_{ij} + R_i R_j$ *with* $1 \leq i, j \leq n$ *where* R_i *is the* i*-th Riesz transform.*

Here $\mathbb{P}$ is an $n \times n$ matrix whose entries are transforms. This $\mathbb{P}$ is exactly the Helmholtz projection established by Miyakawa in [8].

Definition 2.11. *We define the half space projection operator* $\mathbb{P}_{\mathbb{R}^n_+}$ *by* $\mathbb{P}_{\mathbb{R}^n_+} := r_{\mathbb{R}^n_+} \mathbb{P} E$*, that means for* $\mathbf{v} \in \mathbf{X}$ *(or* $\mathbf{Y}$*) we have that* $\mathbb{P}_{\mathbb{R}^n_+} \mathbf{v} := r_{\mathbb{R}^n_+} \mathbb{P} E \mathbf{v}$*.*

Before we end this section, let us recall the real Hardy space and the BMO space defined by Miyachi in [7] when the domain $\Omega = \mathbb{R}^n_+$ and $p = 1$. Let $\varphi \in C_0^\infty(B(0,1))$ such that $\int_{\mathbb{R}^n} \varphi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1$. For $\mathbf{x} \in \mathbb{R}^n_+$, let $d_{\mathbb{R}^n_+}(\mathbf{x}) := \mathrm{dist}(\mathbf{x}, (\mathbb{R}^n_+)^c)$.

Definition 2.12. *We denote by* $\mathscr{H}^1_M(\mathbb{R}^n_+)$ *the set of those* $f \in L^1(\mathbb{R}^n_+)$ *such that* $|| \sup\limits_{0<t<d_{\mathbb{R}^n_+}(\mathbf{x})} |\varphi_t * f|(\mathbf{x})||_{L^1(\mathbb{R}^n_+)} < \infty$*.*

Definition 2.13. *Let* $f \in L^1_{loc}(\mathbb{R}^n_+)$*, we say* $f \in BMO_M(\mathbb{R}^n_+)$ *if* $||f||_{BMO_M(\mathbb{R}^n_+)} := [f]_{BMO(\mathbb{R}^n_+)} + [f]_{b(\mathbb{R}^n_+)} < \infty$ *where*

$$[f]_{BMO(\mathbb{R}^n_+)} := \sup \left\{ \frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} |f - f_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \mid B_{2r}(\mathbf{x}) \subset \mathbb{R}^n_+ \right\},$$
$$[f]_{b(\mathbb{R}^n_+)} := \sup \left\{ \frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} |f|\,\mathrm{d}\mathbf{y} \mid B_{2r}(\mathbf{x}) \subset \mathbb{R}^n_+ \text{ and } B_{5r}(\mathbf{x}) \cap (\mathbb{R}^n_+)^c \neq \emptyset \right\}.$$

3 Helmholtz decomposition of vector field of bounded mean oscillation over the half space

3.1 Boundedness of projection $\mathbb{P}_{\mathbb{R}^n_+}$ from $\mathbf{X}$ to $\mathbf{X}$

Let $\mathbf{v} \in \mathbf{X}$ and $\mathbb{P}_{\mathbb{R}^n_+} \mathbf{v} := r_{\mathbb{R}^n_+} \mathbb{P} E \mathbf{v}$.

Lemma 3.1. *Let* $f \in BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$*, then we have that* $E_{odd} f \in BMO(\mathbb{R}^n, \mathbb{R})$ *and there exists a constant* C *which only depends on* n *such that*

$$[E_{odd} f]_{BMO} \leq C \cdot ||f||_{BMO_b^{\infty,\infty}(\mathbb{R}^n_+)}.$$

Proof. This lemma has already been established in [1, Lemma 7]. $\square$

Lemma 3.2. *Let* $f \in BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)$*, then we have that* $E_{even} f \in BMO(\mathbb{R}^n, \mathbb{R})$ *and there exists a constant* C *which only depends on* n *such that*

$$[E_{even} f]_{BMO} \leq C \cdot [f]_{BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)}.$$

Proof. For simplicity let us denote $E_{even}f$ by $\tilde{f}$, let $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$. If $B_r(\mathbf{x}) \subset \mathbb{R}^n_+$ or $B_r(\mathbf{x}) \subset (\mathbb{R}^n_+)^c$, we can easily verify that

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \le [f]_{BMO^\infty(\mathbb{R}^n_+)}.$$

(1).If $B_r(\mathbf{x}) \cap \partial\mathbb{R}^n_+ \neq \varnothing$ and $\mathbf{x} \in \partial\mathbb{R}^n_+$, then due to the fact that $\tilde{f}$ is even with respect to x_n, we have

$$\begin{aligned}\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} &\le \frac{2}{|B_r(\mathbf{x}) \cap \mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y}) - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y}\\ &\le \frac{2}{|B_r(\mathbf{x}) \cap \mathbb{R}^n_+|}\Big(\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y}) - f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+}|\,\mathrm{d}\mathbf{y}\\ &+ \int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y}\Big)\cdots\cdots(*1).\end{aligned}$$

Here $f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} := \frac{1}{|B_r(\mathbf{x})\cap\mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} f(\mathbf{y})\,\mathrm{d}\mathbf{y}$. By simple check we can further notice that

$$\tilde{f}_{B_r(\mathbf{x})} = \frac{1}{|B_r(\mathbf{x}) \cap \mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} f(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

Therefore $f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} = \tilde{f}_{B_r(\mathbf{x})}$ if $\mathbf{x} \in \partial\mathbb{R}^n_+$ and hence

$$\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} = 0.$$

By continuing the calculation we can deduce that

$$(*1) = \frac{2}{|B_r(\mathbf{x}) \cap \mathbb{R}^n_+|}\int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y}) - f_{B_r(\mathbf{x})\cap\mathbb{R}^n_+}|\,\mathrm{d}\mathbf{y} \le 2\cdot[f]_{ba^\infty(\mathbb{R}^n_+)}.$$

Thus if $\mathbf{x} \in \partial\mathbb{R}^n_+$, then for any $r > 0$ we have that

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \le 2\cdot[f]_{ba^\infty(\mathbb{R}^n_+)}.$$

(2).If $B_r(\mathbf{x}) \cap \partial\mathbb{R}^n_+ \neq \varnothing$ and $\mathbf{x} \notin \partial\mathbb{R}^n_+$, then $\exists\, \mathbf{x}^* \in B_r(\mathbf{x}) \cap \partial\mathbb{R}^n_+$ and $B_r(\mathbf{x}) \subset B_{2r}(\mathbf{x}^*)$. Notice that

$$\begin{aligned}\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_{2r}(\mathbf{x}^*)}|\,\mathrm{d}\mathbf{y} &\le \frac{|B_{2r}(\mathbf{x}^*)|}{|B_r(\mathbf{x})|}\cdot\frac{1}{|B_{2r}(\mathbf{x}^*)|}\cdot\int_{B_{2r}(\mathbf{x}^*)} |\tilde{f}(\mathbf{y}) - \tilde{f}_{B_{2r}(\mathbf{x}^*)}|\,\mathrm{d}\mathbf{y}\\ &\le \frac{|B_{2r}(\mathbf{x}^*)|}{|B_r(\mathbf{x})|}\cdot 2\cdot[f]_{ba^\infty(\mathbb{R}^n_+)}\\ &= 2^{n+1}\cdot[f]_{ba^\infty(\mathbb{R}^n_+)}.\end{aligned}$$

The second inequality here holds because of (1). Notice that

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})}|\tilde{f}(\mathbf{y})-\tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \le \Big(\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})}|\tilde{f}(\mathbf{y})-\tilde{f}_{B_{2r}(\mathbf{x}^*)}|\,\mathrm{d}\mathbf{y} \\ + \frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})}|\tilde{f}_{B_{2r}(\mathbf{x}^*)}-\tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y}\Big)\cdots\cdots(*2).$$

and

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})}|\tilde{f}_{B_{2r}(\mathbf{x}^*)}-\tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \le \frac{1}{|B_r(\mathbf{x})|}\cdot\int_{B_r(\mathbf{x})}|\tilde{f}(\mathbf{y})-\tilde{f}_{B_{2r}(\mathbf{x}^*)}|\,\mathrm{d}\mathbf{y}.$$

Therefore

$$(*2) \le \frac{2}{|B_r(\mathbf{x})|}\cdot\int_{B_r(\mathbf{x})}|\tilde{f}(\mathbf{y})-\tilde{f}_{B_{2r}(\mathbf{x}^*)}|\,\mathrm{d}\mathbf{y} \le 2^{n+2}\cdot[f]_{ba^\infty(\mathbb{R}^n_+)}.$$

As a result, for any $\mathbf{x}\in\mathbb{R}^n_+$ and $r>0$, we have that

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})}|\tilde{f}(\mathbf{y})-\tilde{f}_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \le \big([f]_{BMO^\infty(\mathbb{R}^n_+)}+2^{n+2}\cdot[f]_{ba^\infty(\mathbb{R}^n_+)}\big) \\ = 2^{n+2}\cdot[f]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)}$$

by (1) and (2). Therefore it is true that

$$[\tilde{f}]_{BMO} \le 2^{n+2}\cdot[f]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)}.$$

□

Lemma 3.3. *Let $f\in BMO(\mathbb{R}^n,\mathbb{R})$ and f be odd with respect to x_n, i.e. $f(\mathbf{x}',x_n)=-f(\mathbf{x}',-x_n)$, then we have that $r_{\mathbb{R}^n_+}f\in BMO^{\infty,\infty}_b(\mathbb{R}^n_+)$ and there exists a universal constant C such that*

$$||r_{\mathbb{R}^n_+}f||_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)} \le C\cdot[f]_{BMO}.$$

Proof. (1). Notice that

$$[r_{\mathbb{R}^n_+}f]_{BMO^\infty(\mathbb{R}^n_+)} \le \sup_{\substack{\mathbf{x}\in\mathbb{R}^n\\ r>0}}\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})}|f(\mathbf{y})-f_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} = [f]_{BMO}.$$

(2). Let $\mathbf{x}\in\partial\mathbb{R}^n_+$ and $r>0$. Let $B_r^+(\mathbf{x}):=B_r(\mathbf{x})\cap\mathbb{R}^n_+$ and $B_r^-(\mathbf{x}):=B_r(\mathbf{x})\cap(\mathbb{R}^n_+)^c$. We have that

$$f_{B_r(\mathbf{x})} = \frac{1}{|B_r(\mathbf{x})|}\Big(\int_{B_r^+(\mathbf{x})}f(\mathbf{y})\,\mathrm{d}\mathbf{y}+\int_{B_r^-(\mathbf{x})}f(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big).$$

Notice that by change of variables we can easily deduce that

$$\int_{B_r^-(\mathbf{x})}f(\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_{B_r^+(\mathbf{x})}f(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

Hence

$$f_{B_r(\mathbf{x})} = \frac{1}{|B_r(\mathbf{x})|} \cdot \big(\int_{B_r^+(\mathbf{x})} f(\mathbf{y})\,\mathrm{d}\mathbf{y} - \int_{B_r^+(\mathbf{x})} f(\mathbf{y})\,\mathrm{d}\mathbf{y} \big) = 0.$$

Therefore in this case, we have that

$$\frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} |f(\mathbf{y}) - f_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} = \frac{1}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} |f(\mathbf{y})|\,\mathrm{d}\mathbf{y}.$$

By taking the supremum, we can deduce that

$$\begin{aligned} \sup_{\substack{r>0 \\ \mathbf{x}\in\partial\mathbb{R}^n_+}} r^{-n} \int_{B_r(\mathbf{x})\cap\mathbb{R}^n_+} |f(\mathbf{y})|\,\mathrm{d}\mathbf{y} &\leq \sup_{\substack{r>0 \\ \mathbf{x}\in\partial\mathbb{R}^n_+}} \frac{C}{|B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} |f(\mathbf{y}) - f_{B_r(\mathbf{x})}|\,\mathrm{d}\mathbf{y} \\ &\leq C \cdot [f]_{BMO}. \end{aligned}$$

Thus

$$||r_{\mathbb{R}^n_+} f||_{b^\infty(\mathbb{R}^n_+)} \leq C \cdot [f]_{BMO}.$$

Therefore by (1) and (2), we have that

$$||r_{\mathbb{R}^n_+} f||_{BMO_b^{\infty,\infty}(\mathbb{R}^n_+)} \leq C \cdot [f]_{BMO}.$$

□

Lemma 3.4. *Let $f \in BMO(\mathbb{R}^n_+, \mathbb{R})$, then we have that $r_{\mathbb{R}^n_+} f \in BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)$ and there exists a universal constant C such that*

$$[r_{\mathbb{R}^n_+} f]_{BMO_{ba}^{\infty,\infty}(\mathbb{R}^n_+)} \leq C \cdot [f]_{BMO}.$$

Proof. Firstly let us recall the fact that in defining the BMO-seminorm it is equivalent to consider the supremum over all balls and all squares. Here we make use of this idea. Let $f \in BMO(\mathbb{R}^n_+, \mathbb{R})$, $\mathbf{x} \in \partial\mathbb{R}^n_+$ and $r > 0$, let $B_r^+(\mathbf{x})$ be the intersection of the ball $B_r(\mathbf{x})$ and the half space $\mathbb{R}^n_+$. Let $\tilde{Q}_c$ be the set of squares whose centers are on the boundary $\partial\mathbb{R}^n_+$ with sides parallel to the coordinate system. Notice that a simple triangle inequality would give us the fact that if for each half ball $B_r^+(\mathbf{x})$ there exists a constant $c_{B_r^+(\mathbf{x})}$ such that

$$\sup_{\substack{\mathbf{x}\in\partial\mathbb{R}^n_+ \\ r>0}} \frac{1}{|B_r^+(\mathbf{x})|} \int_{B_r^+(\mathbf{x})} |f(\mathbf{y}) - c_{B_r^+(\mathbf{x})}|\,\mathrm{d}\mathbf{y} < \infty, \tag{3.1}$$

then $[f]_{ba^\infty} < \infty$. Now we let $Q_* \in \tilde{Q}_c$ be the smallest square that contains $B_r(\mathbf{x})$, then we can easily deduce that

$$\begin{aligned} \frac{1}{|B_r^+(\mathbf{x})|} \int_{B_r^+(\mathbf{x})} |f(\mathbf{y}) - f_{Q_*^+}|\,\mathrm{d}\mathbf{y} &\leq \frac{|Q_*^+|}{|B_r^+(\mathbf{x})|} \cdot \frac{1}{|Q_*^+|} \int_{Q_*^+} |f(\mathbf{y}) - f_{Q_*^+}|\,\mathrm{d}\mathbf{y} \\ &\leq c \cdot \sup_{Q\in\tilde{Q}_c} \frac{1}{|Q^+|} \int_{Q^+} |f(\mathbf{y}) - f_{Q^+}|\,\mathrm{d}\mathbf{y} \end{aligned}$$

where c is a constant independent of the radius r and Q^+ is the intersection of Q and $\mathbb{R}^n_+$. Hence by (3.1) there exists a constant c such that

$$[f]_{ba^\infty(\mathbb{R}^n_+)} \leq c \cdot \sup_{Q \in \tilde{Q}_c} \frac{1}{|Q^+|} \int_{Q^+} |f(\mathbf{y}) - f_{Q^+}| \, \mathrm{d}\mathbf{y}.$$

For the opposite direction let $Q^* \in \tilde{Q}_c$ be the largest square that is contained in the ball $B_r(\mathbf{x})$, then we have

$$\frac{1}{|Q^{*+}|} \int_{Q^{*+}} |f(\mathbf{y}) - f_{B_r^+(\mathbf{x})}| \, \mathrm{d}\mathbf{y} \leq \frac{|B_r^+(\mathbf{x})|}{|Q^{*+}|} \cdot \frac{1}{|B_r^+(\mathbf{x})|} \int_{B_r^+(\mathbf{x})} |f(\mathbf{y}) - f_{B_r^+(\mathbf{x})}| \, \mathrm{d}\mathbf{y}.$$

By similar arguments as proving (3.1), if we take the supremum over all squares, we have that

$$\sup_{Q \in \tilde{Q}_c} \frac{1}{|Q^+|} \int_{Q^+} |f(\mathbf{y}) - f_{Q^+}| \, \mathrm{d}\mathbf{y} \leq c \cdot [f]_{ba^\infty(\mathbb{R}^n_+)}.$$

Therefore the seminorm $[f]_{ba^\infty(\mathbb{R}^n_+)}$ is equivalent to the seminorm $\sup\limits_{Q \in \tilde{Q}_c} \frac{1}{|Q^+|} \int_{Q^+} |f(\mathbf{y}) - f_{Q^+}| \, \mathrm{d}\mathbf{y}$. To prove Lemma 3.4, we only need to check that the seminorm $\sup\limits_{Q \in \tilde{Q}_c} \frac{1}{|Q^+|} \int_{Q^+} |f(\mathbf{y}) - f_{Q^+}| \, \mathrm{d}\mathbf{y}$ is less than infinity. This is indeed since we always have that

$$\begin{aligned}\frac{1}{|Q^+|} \int_{Q^+} |f(\mathbf{y}) - f_Q| \, \mathrm{d}\mathbf{y} &\leq \frac{|Q|}{|Q^+|} \cdot \frac{1}{|Q|} \cdot \int_Q |f(\mathbf{y}) - f_Q| \, \mathrm{d}\mathbf{y} \\ &= c \cdot [f]_{BMO} \\ &< \infty.\end{aligned}$$

By applying the argument of the square version of (2.1) again, we can deduce that

$$\frac{1}{|Q^+|} \int_{Q^+} |f(\mathbf{y}) - f_{Q^+}| \, \mathrm{d}\mathbf{y} \leq c \cdot [f]_{BMO} < \infty.$$

Therefore by taking the supremum, we are done. □

Now we are ready to prove the main lemma in this subsection.

Lemma 3.5. $\mathbb{P}_{\mathbb{R}^n_+} : \mathbf{X} \to \mathbf{X}$ *is a bounded linear operator.*

Proof. (1). Let $\mathbf{v} \in \mathbf{X}$, by Lemma 3.1 and Lemma 3.2, we can deduce that there exists a constant C such that

$$\begin{aligned}[E\mathbf{v}]_{BMO} &= \sum_{i=1}^{n-1} [E_{even}\, v^i]_{BMO} + [E_{odd}\, v^n]_{BMO} \\ &\leq C \cdot (\sum_{i=1}^{n-1} [v^i]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)} + ||v^n||_{BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)}) \\ &\leq C \cdot [\mathbf{v}]_{\mathbf{X}}.\end{aligned}$$

Therefore $E : \mathbf{X} \to BMO(\mathbb{R}^n, \mathbb{R}^n)$ is a bounded linear operator.

(2).Since the Riesz transform R_i is a bounded linear operator from $BMO(\mathbb{R}^n, \mathbb{R}^n)$ to $BMO(\mathbb{R}^n, \mathbb{R}^n)$ for each i, we can easily deduce that the projection $\mathbb{P} := I + R \otimes R$ is also a bounded linear operator from $BMO(\mathbb{R}^n, \mathbb{R}^n)$ to $BMO(\mathbb{R}^n, \mathbb{R}^n)$. As for the boundedness of Riesz transforms from BMO to BMO, please refer to Fefferman and Stein [3].

(3). Notice the fact that $(\mathbb{P}E\mathbf{v})^i$ is even with respect to x_n for i such that $1 \le i \le n-1$ whereas $(\mathbb{P}E\mathbf{v})^n$ is odd with respect to x_n. This fact will be proved in subsection 3.3. Then by Lemma 3.3 and Lemma 3.4, we can deduce that there exists a constant C such that

$$[\mathbb{P}_{\mathbb{R}^n_+}\mathbf{v}]_{\mathbf{X}} \le C \cdot [\mathbf{v}]_{\mathbf{X}}.$$

□

3.2 Trace problem

Let $\mathbf{u} \in \mathbf{X}$, then by Lemma 3.1 and Lemma 3.2 we know that $E\mathbf{u} \in BMO(\mathbb{R}^n, \mathbb{R}^n)$. Let $\mathbf{L}^2_{loc}(\Omega) := (L^2_{loc}(\Omega))^n$ where $\Omega \subseteq \mathbb{R}^n$.

Lemma 3.6. *Let $\mathbf{u} \in \mathbf{X}$, then we have that $\mathbf{u} \in \mathbf{L}^2_{loc}(\overline{\mathbb{R}^n_+})$.*

Proof. Let $u \in L^1_{loc}(\mathbb{R}^n_+)$ and $Eu \in L^1_{loc}(\mathbb{R}^n)$ be an extension of u.

(1). $Eu \in BMO$ implies that $Eu \in L^2_{loc}(\mathbb{R}^n)$. This is indeed true since if we let B be any open ball in $\mathbb{R}^n$, by the John-Nirenberg inequality we have that

$$\begin{aligned} ||Eu||^2_{L^2(B)} &= 2 \cdot \int_0^\infty \alpha\mu(\{\mathbf{x} \in B \mid |Eu(\mathbf{x}) - Eu_B| > \alpha\}) \,\mathrm{d}\alpha \\ &\le C_1 \cdot |B| \cdot \int_0^\infty \alpha \cdot \exp(-\frac{C_2\alpha}{[Eu]_{BMO}}) \,\mathrm{d}\alpha \\ &< \infty. \end{aligned}$$

The first equality above is due to $||f||^p_{L^p} = p\int_0^\infty \alpha^{p-1} d_f(\alpha)\,\mathrm{d}\alpha$ where $d_f(\alpha)$ is the distribution function of f, for this fact please refer to L.Grafakos [5].

(2). Let $K \subset\subset \overline{\mathbb{R}^n_+}$, it is certainly that $K \subset B_r(\mathbf{x}) \cap \mathbb{R}^n_+$ for some $\mathbf{x} \in \partial\mathbb{R}^n_+$ and $r > 0$, then we have that

$$||u||_{L^2(K)} \le ||u||_{L^2(B_r(\mathbf{x}) \cap \mathbb{R}^n_+)} \le ||Eu||_{L^2(B_r(\mathbf{x}))} < \infty.$$

Therefore $u \in L^2(K)$ for any $K \subset\subset \overline{\mathbb{R}^n_+}$, that means $u \in L^2_{loc}(\overline{\mathbb{R}^n_+})$.

For $\mathbf{u} \in \mathbf{X}$, we have that $E_{even}\, u^i \in BMO$ for $1 \le i \le n-1$ and $E_{odd}\, u^n \in BMO$, hence by (1) and (2) $u^i \in L^2_{loc}(\overline{\mathbb{R}^n_+})$ for $1 \le i \le n$. □

Since we have proved that $\mathbf{u} \in \mathbf{X}$ implies that $\mathbf{u} \in \mathbf{L}^2_{loc}(\overline{\mathbb{R}^n_+})$, we are able to make use of the theory of R.Temam [10] to define the trace.

Definition 3.7. *We define the space $E_{loc}(\overline{\mathbb{R}^n_+})$ in the following way :*

- $E_{loc}(\overline{\mathbb{R}^n_+}) := \{\mathbf{u} \in \mathbf{L}^2_{loc}(\overline{\mathbb{R}^n_+}) \mid \operatorname{div}\mathbf{u} \in L^2_{loc}(\overline{\mathbb{R}^n_+})\}$.

  Here $\operatorname{div}\mathbf{u}$ *means the divergence of* $\mathbf{u}$, *i.e.* $\operatorname{div}\mathbf{u} := \displaystyle\sum_{i=1}^{n} \partial_{x_i} u^i$.

- *Let* $\mathbf{u} \in E_{loc}(\overline{\mathbb{R}^n_+})$, *we define a family of seminorms* $||\cdot||_{E(\Omega_i)}$ *for all* $i \in \mathbb{N}$ *on* $E_{loc}(\overline{\mathbb{R}^n_+})$ *by*

$$||\mathbf{u}||^2_{E(\Omega_i)} := \textstyle\int_{\Omega_i} |\operatorname{div}\mathbf{u}|^2 + |\mathbf{u}|^2 \,\mathrm{d}\mathbf{x}$$

  where Ω_i *is an open domain in* $\mathbb{R}^n_+$ *with* C^2 *boundary* $\partial\Omega_i$ *for each* $i \in \mathbb{N}$, *moreover we require that* $B_i(0)' \subset \partial\Omega_i$ *for all* $i \in \mathbb{N}$ *where* $B_i(0)' := \{\mathbf{x} \in B_i(0) \mid x_n = 0\}$ *and* $\Omega_i \uparrow \mathbb{R}^n_+$ *as* $i \to \infty$.

Definition 3.8. *(Trace space)*

- *We denote the interior of the region* $\overline{\Omega_i} \cap \partial\mathbb{R}^n_+$ *in* $\mathbb{R}^{n-1}$ *by* Ω_i'.
- $\Gamma(\mathbb{R}^{n-1}) := \{\, T \in \mathscr{D}'(\mathbb{R}^{n-1}) \mid \mid < T, \phi > \mid \le C_i \cdot ||\,\phi\,||_{H^{\frac{1}{2}}(\Omega_i')}$ *for any* $\phi \in \mathscr{D}(\mathbb{R}^{n-1})$ *with* $\operatorname{supp}\phi \subset \Omega_i'\,\}$
- *We define a family of seminorms* $\{\, ||\cdot||_{\Omega_i'} \mid i \in \mathbb{N}\,\}$ *on* $\Gamma(\mathbb{R}^{n-1})$ *by :*

$$||\,T\,||_{\Omega_i'} := \sup_{\substack{\phi\in\mathscr{D}(\mathbb{R}^{n-1}),\\ \operatorname{supp}\phi\subset\Omega_i',\\ ||\,\phi\,||_{H^{\frac{1}{2}}(\Omega_i')}=1.}} \mid < T, \phi > \mid.$$

It is not hard to verify the fact that these two spaces $E_{loc}(\overline{\mathbb{R}^n_+})$ and $\Gamma(\mathbb{R}^{n-1})$ are indeed Frechet spaces, thus we omit the details here and proceed directly to define the trace.

Lemma 3.9. *Let* $\gamma : E_{loc}(\overline{\mathbb{R}^n_+}) \to \Gamma(\mathbb{R}^{n-1})$ *by* $\mathbf{u} \mapsto \gamma_{\mathbf{u}}$, *where for* $\phi \in \mathscr{D}(\mathbb{R}^{n-1})$ *with* $\operatorname{supp}\phi \subset \Omega_i'$ *we have the map*

$$\gamma_{\mathbf{u}}(\phi) := \int_{\Omega_i} \operatorname{div}\,\mathbf{u}\cdot\omega + \mathbf{u}\cdot\nabla\omega \,\mathrm{d}\mathbf{x}.$$

Here we choose $\omega \in H^1(\Omega_i)$ *with the trace operator* $\gamma_0 : H^1(\Omega_i) \to H^{\frac{1}{2}}(\partial\Omega_i)$ *such that the trace of* ω *is* ϕ. *Then we have that the map* γ *is a bounded linear operator.*

Proof. Here we make use of the theory of R.Temam [10]. Notice that for each $\phi \in \mathscr{D}(\mathbb{R}^{n-1})$ with $\operatorname{supp}\phi \subset \Omega_i'$, we can actually find an $\omega \in H^1(\Omega_i)$ such that its trace $\gamma_0\,\omega = \phi$. Let $\phi \in \mathscr{D}(\mathbb{R}^{n-1})$ with $\operatorname{supp}\phi \subset \Omega_i'$, notice that by definition we have that $\Omega_i' \subset \Omega_i$. We define a function g on $\partial\Omega_i$ by

$$g(\mathbf{x}) := \begin{cases} \phi(\mathbf{x}') & \text{if} \quad x_n = 0, \\ 0 & \text{else.} \end{cases}$$

Since $\phi \in \mathscr{D}(\mathbb{R}^{n-1})$, an easy check quickly tells us that this function $g \in H^{\frac{1}{2}}(\partial\Omega_i)$ and $||g||_{H^{\frac{1}{2}}(\partial\Omega_i)} = ||\phi||_{H^{\frac{1}{2}}(\Omega_i')}$. Then by R.Temam [10], there exists an $\omega \in H^1(\Omega_i)$ such that its trace $\gamma_0\,\omega = g$. Therefore by the definition of our $\gamma_{\mathbf{u}}$, we have that

$$\begin{aligned} |\,\gamma_{\mathbf{u}}(\phi)\,| &\le ||\,\mathrm{div}\,\mathbf{u}\,||_{L^2(\Omega_i)} \cdot ||\,\omega\,||_{L^2(\Omega_i)} + ||\,\mathbf{u}\,||_{\mathbf{L}^2(\Omega_i)} \cdot ||\,\nabla\omega\,||_{\mathbf{L}^2(\Omega_i)} \\ &\le C \cdot (\,||\,\mathrm{div}\,\mathbf{u}\,||_{L^2(\Omega_i)} + ||\,\mathbf{u}\,||_{\mathbf{L}^2(\Omega_i)}\,) \cdot ||\,\omega\,||_{H^1(\Omega_i)} \\ &\le C \cdot ||\,\mathbf{u}\,||_{E(\Omega_i)} \cdot ||\,\omega\,||_{H^1(\Omega_i)} \end{aligned}$$

by the triangle inequality and the Hölder inequality. Since by R.Temam [10], there exists $l_{\Omega_i} \in \mathcal{L}\,(\,H^{1/2}(\partial\Omega_i),\ H^1(\Omega_i)\,)$ where l_{Ω_i} is the lifting operator such that $l_{\Omega_i}g = \omega$, hence by above we have that

$$\begin{aligned} |\,\gamma_{\mathbf{u}}(\phi)\,| &\le C \cdot ||\,\mathbf{u}\,||_{E(\Omega_i)} \cdot ||\,l_{\Omega_i}g\,||_{H^1(\Omega_i)} \\ &\le C_i\, \cdot ||\,\mathbf{u}\,||_{E(\Omega_i)} \cdot ||\,g\,||_{H^{1/2}(\partial\Omega_i)} \\ &= C_i\, \cdot ||\,\mathbf{u}\,||_{E(\Omega_i)} \cdot ||\,\phi\,||_{H^{1/2}(\Omega_i')}. \end{aligned}$$

The last equality holds since $g(\mathbf{x}) = 0$ for $\mathbf{x} \notin \Omega_i'$. Therefore, we can deduce that

$$||\,\gamma_{\mathbf{u}}\,||_{\Omega_i'} \le C_i\, \cdot ||\,\mathbf{u}\,||_{E(\Omega_i)}$$

where C_i is simply a constant which depends on i. As a result, we see that

$$\gamma : E_{loc}(\overline{\mathbb{R}^n_+}) \to \Gamma(\mathbb{R}^{n-1})$$

is indeed a bounded linear operator in the sense of Frechet spaces. □

By Lemma 3.6 we know that $\mathbf{X} \subset \mathbf{L}^2_{loc}(\overline{\mathbb{R}^n_+})$ and by Lemma 3.9 there exists a bounded linear operator γ which maps $E_{loc}(\overline{\mathbb{R}^n_+})$ to $\Gamma(\mathbb{R}^{n-1})$. For the subspace $\{\,\mathbf{u} \in \mathbf{X} \mid \mathrm{div}\ \mathbf{u} \in L^2_{loc}(\overline{\mathbb{R}^n_+})\,\} \subset \mathbf{X}$, it is trivial to see that the map γ is also a bounded linear operator from $\{\,\mathbf{u} \in \mathbf{X} \mid \mathrm{div}\ \mathbf{u} \in L^2_{loc}(\overline{\mathbb{R}^n_+})\,\}$ to $\Gamma(\mathbb{R}^{n-1})$. This is how we take the trace for elements in $\mathbf{X}$.

3.3 Validity of $\mathbb{P}_{\mathbb{R}^n_+}$ as the Helmholtz projection

Lemma 3.10. *Let* $\mathbf{v} \in \mathbf{X}$*, then* div $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} = 0$ *in* $\mathbb{R}^n_+$ *in the sense of distributions.*

Proof. Let $\phi \in C_0^\infty(\mathbb{R}^n_+)$. By the definition of distributions, we have that

$$\int_{\mathbb{R}^n_+} \mathrm{div}\ \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \phi\,\mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^n_+} \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \nabla\phi\,\mathrm{d}\mathbf{x}.$$

Since supp $\phi \subset\subset \mathbb{R}^n_+$, we can easily deduce that supp $\partial_{x_i}\phi \subset\subset \mathbb{R}^n_+$ for any $1 \le i \le n$, therefore

$$\int_{\mathbb{R}^n_+} \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \nabla\phi\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \mathbb{P}E\mathbf{v} \cdot \nabla\phi\,\mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^n} \mathrm{div}\ (\mathbb{P}E\mathbf{v}) \cdot \phi\,\mathrm{d}\mathbf{x}.$$

Because div $(\mathbb{P}E\mathbf{v}) = 0$ in the sense of distributions, we have that

$$\int_{\mathbb{R}^n} \text{div } (\mathbb{P}E\mathbf{v}) \cdot \phi \, \mathrm{d}\mathbf{x} = 0.$$

Thus

$$\int_{\mathbb{R}^n_+} \text{div } \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \phi \, \mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^n_+} \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \nabla\phi \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \text{div } (\mathbb{P}E\mathbf{v}) \cdot \phi \, \mathrm{d}\mathbf{x} = 0.$$

Notice that the above equality holds for any $\phi \in C_0^\infty(\mathbb{R}^n_+)$, hence

$$\text{div } \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} = 0 \quad \text{in} \quad \mathbb{R}^n_+$$

in the sense of distributions. As for the reason why div $\mathbb{P}E\mathbf{v} = 0$ in the sense of distributions, by considering Fourier transforms we can quickly prove it through simple calculations. $\square$

Let us recall some facts about Riesz transforms. Notice that the j-th Riesz transform R_j is defined as

$$R_j(f)(\mathbf{x}) := \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|\,\mathbf{x} - \mathbf{y}\,|^{n+1}} \cdot f(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

By [9, p.232], we have that $R_j(f)$ is well-defined for any $f \in \mathscr{H}^1(\mathbb{R}^n)$ and $1 \le j \le n$. By [3], we have that for $f \in BMO$ and $1 \le j \le n$, $R_j(f) \in \mathscr{H}^1(\mathbb{R}^n)^*$. Hence by the fact that $BMO = \mathscr{H}^1(\mathbb{R}^n)^*$, there exists $h \in BMO$ such that $R_j(f) = h$ in the sense of bounded linear functionals on $\mathscr{H}^1(\mathbb{R}^n)$. Therefore for any $f \in BMO$ and $1 \le j \le n$, $R_j(f)$ is defined by its corresponding h. Based on these facts, we have the next lemma which proves an interesting property about Riesz transforms.

Lemma 3.11. *Let f belongs to BMO or $\mathscr{H}^1(\mathbb{R}^n)$,*
(1). If f is even with respect to x_n, then

$$\begin{cases} R_j(f) \text{ is even with respect to } x_n \text{ for } j \text{ satisfying } 1 \le j \le n-1, \\ R_n(f) \text{ is odd with respect to } x_n. \end{cases}$$

(2). If f is odd with respect to x_n, then

$$\begin{cases} R_j(f) \text{ is odd with respect to } x_n \text{ for } j \text{ satisfying } 1 \le j \le n-1, \\ R_n(f) \text{ is even with respect to } x_n. \end{cases}$$

Proof. For $f \in \mathscr{H}^1(\mathbb{R}^n)$, since $R_j(f)$ is well-defined for each $1 \le j \le n$, we can prove this lemma directly through change of variables. Let $g \in BMO$ be odd with respect to x_n and $1 \le j \le n-1$, let $w \in BMO$ such that $R_j(g) = w$. Let $\tilde{w}(\mathbf{x}', x_n) := w(\mathbf{x}', -x_n)$ and $f \in \mathscr{H}^1(\mathbb{R}^n)$, then by change of variables we have that

$$< \tilde{w}, f > = < w, \tilde{f} > = - < g, R_j(\tilde{f}) > .$$

Notice that the second equality above holds since $\tilde{f} \in \mathscr{H}^1(\mathbb{R}^n)$ if $f \in \mathscr{H}^1(\mathbb{R}^n)$. Again by change of variables, we can further deduce that

$$R_j(\tilde{f})(\mathbf{x}', x_n) = R_j(f)(\mathbf{x}', -x_n).$$

Then,

$$\begin{aligned}
- < g, R_j(\tilde{f}) > &= - \int_{\mathbb{R}^n} g \cdot R_j(\tilde{f}) \,\mathrm{d}\mathbf{x} \\
&= - \int_{\mathbb{R}^n} g(\mathbf{x}', x_n) \cdot R_j(f)(\mathbf{x}', -x_n) \,\mathrm{d}\mathbf{x} \\
&= - \int_{\mathbb{R}^n} g(\mathbf{x}', -x_n) \cdot R_j(f)(\mathbf{x}', x_n) \,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^n} g(\mathbf{x}', x_n) \cdot R_j(f)(\mathbf{x}', x_n) \,\mathrm{d}\mathbf{x} \\
&=< g, R_j(f) > \\
&= - < w, f > .
\end{aligned}$$

Hence $< \tilde{w} + w, f >= 0$ for any $f \in \mathscr{H}^1(\mathbb{R}^n)$ and thus w is odd with respect to x_n. The other three cases can be proved by similar arguments. $\square$

Lemma 3.12. *Let $\mathbf{v} \in \mathbf{X}$, then we have that*

$$\begin{cases} (\mathbb{P}E\mathbf{v})^i \text{ is even with respect to } x_n \text{ for } i \text{ satisfying } 1 \leq i \leq n-1, \\ (\mathbb{P}E\mathbf{v})^n \text{ is odd with respect to } x_n. \end{cases}$$

Proof. This is a direct application of Lemma 3.11. $\square$

Lemma 3.13. *Let $\mathbf{v} \in \mathbf{X}$, then the trace $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\mathbb{R}^n_+$ in the sense of distributions.*

Proof. Let B_R be the ball $B_R(0)$. Let $B_R^+ := B_R \cap \mathbb{R}^n_+$ and $B_R^- := B_R \cap (\mathbb{R}^n_+)^c$. Let $\mathbf{v} \in \mathbf{X}$ and let $\mathbf{u} := \mathbb{P}E\mathbf{v}$. By the above lemma we can see that u^n is odd with respect to x_n. Let

$$\mathbf{u}_1(\mathbf{x}', x_n) := \begin{cases} \mathbf{u}(\mathbf{x}', x_n) & \text{if } x_n > 0, \\ 0 & \text{if } x_n < 0. \end{cases}$$

and

$$\mathbf{u}_2(\mathbf{x}', x_n) := \begin{cases} 0 & \text{if } x_n > 0, \\ \mathbf{u}(\mathbf{x}', x_n) & \text{if } x_n < 0. \end{cases}$$

Let $\phi \in C_0^\infty(B_R)$, then we have that

$$\begin{aligned}
< \text{div } \mathbf{u}_1, \phi > &:= - < \mathbf{u}_1, \nabla\phi > \\
&= \int_{B_R^+} \text{div } \mathbf{u}_1 \cdot \phi \,\mathrm{d}\mathbf{x} + \int_{\{x_n=0\}\cap B_R} (\mathbf{u}_1 \cdot \mathbf{n}_1)\phi \,\mathrm{d}\mathscr{H}^{n-1}
\end{aligned}$$

where $\mathbf{n}_1$ is the normal vector on $\partial\mathbb{R}^n_+$ which points outward B^+_R. In the mean time, we also have that

$$\begin{aligned} < \operatorname{div} \mathbf{u}_2, \phi > &:= - < \mathbf{u}_2, \nabla\phi > \\ &= \int_{B^-_R} \operatorname{div} \mathbf{u}_2 \cdot \phi \,\mathrm{d}\mathbf{x} + \int_{\{x_n=0\}\cap B_R} (\mathbf{u}_2 \cdot \mathbf{n}_2)\phi \,\mathrm{d}\mathscr{H}^{n-1} \end{aligned}$$

where $\mathbf{n}_2$ is the normal vector on $\partial\mathbb{R}^n_+$ which points outward B^-_R. By similar arguments as in the proof of Lemma 3.10, we can see that div $\mathbf{u} = 0$ in B_R, div $\mathbf{u}_1 = 0$ in B^+_R and div $\mathbf{u}_2 = 0$ in B^-_R. Therefore

$$\begin{aligned} 0 &=< \operatorname{div} \mathbf{u}_1, \phi > + < \operatorname{div} \mathbf{u}_2, \phi > \\ &= \int_{B^+_R} \operatorname{div} \mathbf{u}_1 \cdot \phi \,\mathrm{d}\mathbf{x} + \int_{B^-_R} \operatorname{div} \mathbf{u}_2 \cdot \phi \,\mathrm{d}\mathbf{x} + \int_{\{x_n=0\}\cap B_R} \big(\mathbf{u}_1 \cdot \mathbf{n}_1 + \mathbf{u}_2 \cdot \mathbf{n}_2\big)\, \phi \,\mathrm{d}\mathscr{H}^{n-1} \\ &= \int_{\{x_n=0\}\cap B_R} \big(\mathbf{u}_1 \cdot \mathbf{n}_1 - \mathbf{u}_2 \cdot \mathbf{n}_1\big)\, \phi \,\mathrm{d}\mathscr{H}^{n-1}. \end{aligned}$$

Thus we see that on $\{x_n = 0\} \cap B_R$, $\big(\mathbf{u}_1 \cdot \mathbf{n}_1 - \mathbf{u}_2 \cdot \mathbf{n}_1\big) = 0$ in the sense of distributions. Notice that if $x_n < 0$, then

$$u^n_2\,(\mathbf{x}^{'}, x_n) = -u^n_1\,(\mathbf{x}^{'}, -x_n).$$

At $\{x_n = 0\} \cap B_R$, we have that

$$\mathbf{u}_1 \cdot \mathbf{n}_1 = u^n_1\,(\mathbf{x}^{'}, 0) \quad \text{and} \quad \mathbf{u}_2 \cdot \mathbf{n}_2 = -u^n_1\,(\mathbf{x}^{'}, 0).$$

and thus $u^n_1\,(\mathbf{x}^{'}, 0) = 0$ in the sense of distributions. Notice that

$$u^n_1\,(\mathbf{x}^{'}, 0) = \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \mathbf{n}\, |_{\{x_n=0\}\cap B_R}\, .$$

Since $\{x_n = 0\} \cap B_R \uparrow \partial\mathbb{R}^n_+$ as $R \to \infty$, we can easily deduce that the trace

$$\mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} \cdot \mathbf{n}\, |_{\partial\mathbb{R}^n_+} = 0$$

in the sense of distributions. □

Lemma 3.14. *Let* $\mathbf{v} \in \mathbf{X}$ *such that*

$$\begin{cases} \operatorname{div} \mathbf{v} = 0 & \text{in } \mathbb{R}^n_+, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Then we have that $\mathbf{v} \in \mathbb{P}_{\mathbb{R}^n_+}\mathbf{X}$*. Notice that both equalities above hold in the sense of distributions.*

Proof. Let $\mathbf{v} \in \mathbf{X}$ such that

$$\begin{cases} \operatorname{div} \mathbf{v} = 0 & \text{in } \mathbb{R}^n_+, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

in the sense of distributions and let E be our extension operator. Throughout the proof of this lemma we mean equal to 0 in the sense of distributions.

(1). Here we prove that div $E\mathbf{v} = 0$ in $\mathbb{R}^n$. Let B_R be the ball $B_R(0)$. Let $B_R^+ := B_R \cap \mathbb{R}^n_+$ and $B_R^- := B_R \cap (\mathbb{R}^n_+)^c$. If $x_n > 0$, then $E\mathbf{v}\,(\mathbf{x}', x_n) = \mathbf{v}\,(\mathbf{x}', x_n)$ and div $E\mathbf{v} =$ div $\mathbf{v} = 0$ in $\mathbb{R}^n_+$ by our assumptions. If $x_n < 0$, then $E\mathbf{v}\,(\mathbf{x}', x_n) = (\,\mathbf{v}'(\mathbf{x}', -x_n), -v^n(\mathbf{x}', -x_n)\,)$ and

$$\text{div } E\mathbf{v} = \sum_{i=1}^{n-1} \partial_{x_i} v^i(\mathbf{x}', -x_n) + \partial_{-x_n} v^n(\mathbf{x}', -x_n) = 0$$

since div $\mathbf{v} = 0$ in $\mathbb{R}^n_+$. Let $\phi \in C_0^\infty(B_R)$, then

$$\begin{aligned} < \text{div } E\mathbf{v}, \phi > &:= - < E\mathbf{v}, \nabla\phi > \\ &= \int_{B_R^+} \text{div } E\mathbf{v} \cdot \phi \, \mathrm{d}\mathbf{x} + \int_{B_R^-} \text{div } E\mathbf{v} \cdot \phi \, \mathrm{d}\mathbf{x} \\ &\quad - \int_{B_R \cap \{x_n = 0\}} \{((E\mathbf{v})_+ - (E\mathbf{v})_-) \cdot \mathbf{n}_+\} \phi \, \mathrm{d}\mathscr{H}^{n-1}. \end{aligned}$$

The first two terms in the last equality equal to 0 since div $E\mathbf{v} = 0$ in both B_R^+ and B_R^-. The third term equals to 0 since $(E\mathbf{v})_+ \cdot \mathbf{n}_+ = v^n(\mathbf{x}', 0)$, $(E\mathbf{v})_- \cdot \mathbf{n}_+ = -v^n(\mathbf{x}', 0)$ and $v^n(\mathbf{x}', 0) = 0$ by our assumptions. Hence div $E\mathbf{v} = 0$ in $\mathbb{R}^n$.

(2). Notice that by simply considering Fourier transforms it is easy to verify that $R_i \sum_j R_j u^j = 0$ for any $1 \le i \le n$ if div $\mathbf{u} = 0$ in $\mathbb{R}^n$. Therefore if div $\mathbf{u} = 0$ in $\mathbb{R}^n$, then $(\mathbb{P}\mathbf{u})^i = u^i$ for any $1 \le i \le n$.

Now let $\mathbf{u} := E\mathbf{v}$, by (1) and (2) we have that $\mathbb{P}\mathbf{u} = \mathbf{u}$. Then by applying the restriction on both sides of this equality, we get that $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} = \mathbf{v}$. $\square$

Definition 3.15. *We define the solenoidal subspace* $\mathbf{X}_\sigma$ *of* $\mathbf{X}$ *by*

$$\mathbf{X}_\sigma := \{\, \mathbf{v} \in \mathbf{X} \mid \text{div } \mathbf{v} = 0 \ \ \textit{in} \ \ \mathbb{R}^n_+ \ \ \& \ \ \mathbf{v} \cdot \mathbf{n} = 0 \ \ \textit{on} \ \ \partial\mathbb{R}^n_+ \,\}.$$

Here the two equalities hold in the sense of distributions.

By Lemma 3.10 and Lemma 3.13 we can see that $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{X} \subseteq \mathbf{X}_\sigma$. And by Lemma 3.14 we can see that $\mathbf{X}_\sigma \subseteq \mathbb{P}_{\mathbb{R}^n_+}\mathbf{X}$. Therefore $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{X} = \mathbf{X}_\sigma$. This fact justifies the validity of $\mathbb{P}_{\mathbb{R}^n_+}$ as the Helmholtz projection.

3.4 Characterisation of the subspace $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$

Lemma 3.16. *Let* $\mathbf{v} \in \mathbf{X}$*, then there exists* $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ *such that* $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{v} = \nabla p$.

Proof. We seek to make use of De Rham's theorem [4] here. In order to make use of De Rham's theorem, it is sufficient to show that

$$< (I - \mathbb{P})E\mathbf{v}, \phi >= 0 \quad \forall\, \phi \in C^\infty_{0,\sigma}(\mathbb{R}^n).$$

Let $\phi \in C^{\infty}_{0,\sigma}(\mathbb{R}^n)$ and $\mathbf{u} := E\mathbf{v}$, notice that

$$\{(I-\mathbb{P})\mathbf{u}\}^i = -R_i \sum_j R_j u^j.$$

Therefore by substitution $< (I-\mathbb{P})\mathbf{u}, \phi >= \sum_i < -R_i \sum_j R_j u^j, \phi^i >$. Let $f := \sum_j R_j u^j$, notice that

$$< -R_i(f), \phi^i > =< f, R_i(\phi^i) > .$$

Therefore

$$\begin{aligned} < (I-\mathbb{P})\mathbf{u}, \phi > &= \sum_i < \sum_j R_j u^j, R_i \phi^i > \\ &=< \sum_j R_j u^j, \sum_i R_i \phi^i > . \end{aligned}$$

By div $\phi = 0$ we can easily deduce that $\sum_i R_i \phi^i = 0$ by considering Fourier transforms. Thus

$$< (I-\mathbb{P})\mathbf{u}, \phi >= 0 \quad \forall\, \phi \in C^{\infty}_{0,\sigma}(\mathbb{R}^n).$$

Therefore by De Rham [4], there exists $p \in L^1_{loc}(\mathbb{R}^n)$ such that $(I-\mathbb{P})\mathbf{u} = \nabla p$. By applying the restriction operator we have that

$$r_{\mathbb{R}^n_+}\,(I-\mathbb{P})\,E\mathbf{v} = (I-\mathbb{P}_{\mathbb{R}^n_+})\,\mathbf{v} = r_{\mathbb{R}^n_+}\,\nabla p.$$

Notice that we can further deduce that $r_{\mathbb{R}^n_+}\nabla p = \nabla\,(r_{\mathbb{R}^n_+} p)$. Indeed since for any $\phi \in C^{\infty}_0(\mathbb{R}^n_+)$ we have that

$$\begin{aligned} < r_{\mathbb{R}^n_+}\nabla p, \phi > \;&:=\; \int_{\mathbb{R}^n_+} r_{\mathbb{R}^n_+}\nabla p \cdot \phi \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \nabla p \cdot \phi \,\mathrm{d}\mathbf{x} \\ &= -\int_{\mathbb{R}^n} p \cdot \mathrm{div}\,\phi \,\mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^n_+} p \cdot \mathrm{div}\,\phi \,\mathrm{d}\mathbf{x} \\ &= -\int_{\mathbb{R}^n} (r_{\mathbb{R}^n_+} p) \cdot \mathrm{div}\,\phi \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n_+} \nabla(r_{\mathbb{R}^n_+} p) \cdot \phi \,\mathrm{d}\mathbf{x} \\ &= < \nabla(r_{\mathbb{R}^n_+} p), \phi > . \end{aligned}$$

Therefore we have that $(I-\mathbb{P}_{\mathbb{R}^n_+})\mathbf{v} = \nabla(r_{\mathbb{R}^n_+} p)$. Since $p \in L^1_{loc}(\mathbb{R}^n)$, it is easy to deduce that $r_{\mathbb{R}^n_+} p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$. □

Lemma 3.17. *Let $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ such that $\nabla p \in \mathbf{X}$, then $\nabla p \in (I-\mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$.*

Proof. Let $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ such that $\nabla p \in \mathbf{X}$, it is sufficient to prove that $\mathbb{P}_{\mathbb{R}^n_+}\nabla p = 0$. Then by this fact we can see that

$$(I-\mathbb{P}_{\mathbb{R}^n_+})\nabla p = \nabla p - \mathbb{P}_{\mathbb{R}^n_+}\nabla p = \nabla p.$$

and thus $\nabla p \in (I-\mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$. Let q be defined as follow:

$$q(\mathbf{x}', x_n) := \begin{cases} p(\mathbf{x}', x_n) & \text{if } x_n > 0, \\ p(\mathbf{x}', -x_n) & \text{if } x_n < 0. \end{cases}$$

Since q is the even extension of p, $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ would imply $q \in L^1_{loc}(\mathbb{R}^n)$. Moreover, simple calculations would tell us $\nabla q = E\,\nabla p$. This is indeed since for $x_n < 0$ we have that

$$\frac{\partial}{\partial x_n} q(\mathbf{x}', x_n) = \frac{\partial}{\partial x_n} p(\mathbf{x}', -x_n) = -\frac{\partial}{\partial(-x_n)} p(\mathbf{x}', -x_n) = -\frac{\partial}{\partial z_n} p(\mathbf{x}', z_n)$$

where $z_n > 0$. Again by considering Fourier transforms, it is easy to verify that $(\mathbb{P}\nabla q)^i = 0$ for any $1 \le i \le n$. As a result,

$$\mathbb{P}_{\mathbb{R}^n_+} \nabla p = r_{\mathbb{R}^n_+} \mathbb{P} E \nabla p = r_{\mathbb{R}^n_+} \mathbb{P} \nabla q = 0.$$

Hence $\nabla p = (I - \mathbb{P}_{\mathbb{R}^n_+})\nabla p$ and we are done. $\square$

Definition 3.18. *We define the subspace $\mathbf{X}_\pi$ of $\mathbf{X}$ by*

$$\mathbf{X}_\pi := \{\, \nabla p \in \mathbf{X} \mid p \in L^1_{loc}(\overline{\mathbb{R}^n_+}) \,\}.$$

By Lemma 3.16 we can see that $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X} \subseteq \mathbf{X}_\pi$ and by Lemma 3.17 we can see that $\mathbf{X}_\pi \subseteq (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$. Therefore $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X} = \mathbf{X}_\pi$. This fact gives the characterisation of the subspace $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$.

3.5 Proof of Theorem 1.1

Proof. By Lemma 3.5 we see that $\mathbb{P}_{\mathbb{R}^n_+}$ is a bounded linear operator which maps $\mathbf{X}$ to $\mathbf{X}$. By this bounded linear map we can easily see that the vector field $\mathbf{X}$ admits a natural decomposition

$$\mathbf{X} = \mathbb{P}_{\mathbb{R}^n_+}\mathbf{X} \oplus (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$$

where both $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{X}$ and $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X}$ are linear subspaces of $\mathbf{X}$. Since this natural decomposition is induced by the projection $\mathbb{P}_{\mathbb{R}^n_+}$, this decomposition is certainly unique. Moreover, we have already proved that

$$\mathbb{P}_{\mathbb{R}^n_+}\mathbf{X} = \mathbf{X}_\sigma$$

and

$$(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{X} = \mathbf{X}_\pi.$$

As a result, Theorem 1.1 holds and we are done. $\square$

Remark 3.19. *Although the Helmholtz decomposition we established for $\mathbf{X}$ is true, due to the fact that $[\,\cdot\,]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)}$ is a seminorm, it is inevitable to think about the question where constant vectors are mapped to under this Helmholtz projection $\mathbb{P}_{\mathbb{R}^n_+}$. Unfortunately, this question is not answered in this paper, in order to avoid this ambiguity, we shall consider our Helmholtz decomposition not for the space $\mathbf{X}$ but for the quotient space $\mathbf{X}/(\mathbb{R}^{n-1}\times\{0\})$. From now on, without causing any ambiguity, we shall denote $\mathbf{X}/(\mathbb{R}^{n-1} \times \{0\})$ simply by $\mathbf{X}$.*

4 Partial Helmholtz decomposition of vector field in real Hardy spaces over the half space

4.1 Boundedness of projection $\mathbb{P}_{\mathbb{R}^n_+}$ from Y to Y

Let $\mathbf{v} \in \mathbf{Y}$ and $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{v} := r_{\mathbb{R}^n_+}\mathbb{P}E\mathbf{v}$.

Lemma 4.1. *Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, then we have that $E_{odd}f \in \mathscr{H}^1(\mathbb{R}^n)$ and*

$$||E_{odd}f||_{\mathscr{H}^1} = 2 \cdot ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}.$$

Proof. For simplicity we denote $E_{odd}f$ by $\bar{f}$. Let G_t be the heat kernel on $\mathbb{R}^n$ so that $(e^{t\Delta}g)(\mathbf{x}) = (G_t * g)(\mathbf{x})$ for a function g on $\mathbb{R}^n$. By Definition 2.8, we have that

$$\begin{aligned}||\bar{f}||_{\mathscr{H}^1} &= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, G_t * \bar{f}\,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n_-} \sup_{t>0} |\, G_t * \bar{f}\,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= (1) + (2).\end{aligned}$$

(1). For $\mathbf{x} \in \mathbb{R}^n_+$ and $t > 0$, we have that $(G_t * \bar{f})\,(\mathbf{x}, t) = (r_{\mathbb{R}^n_+}(G_t * \bar{f}))\,(\mathbf{x}, t)$. Since this is true for all $t > 0$, by taking the supremum over all $t > 0$, we have that

$$\sup_{t>0} |\, G_t * \bar{f}\,|\,(\mathbf{x}) = \sup_{t>0} |\, r_{\mathbb{R}^n_+}(G_t * \bar{f})\,|\,(\mathbf{x}).$$

Since the above equality holds for all $\mathbf{x} \in \mathbb{R}^n_+$, we can see that

$$\begin{aligned}(1) &= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+}(G_t * \bar{f})\,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+}\, e^{t\Delta}\, \bar{f}\,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}.\end{aligned}$$

(2). Notice that $(G_t * \bar{f})\,(\mathbf{x}, t)$ is actually odd with respect to x_n since $\bar{f}$ is odd with respect to x_n, hence

$$|\, G_t * \bar{f}\,|\,(\mathbf{x}', x_n, t) = |\, -(G_t * \bar{f})\,(\mathbf{x}', -x_n, t)\,| = |\, G_t * \bar{f}\,|\,(\mathbf{x}', -x_n, t).$$

Let $\bar{f}^+_{G_t}(\mathbf{x}) := \sup_{t>0} |\, G_t * \bar{f}\,|\,(\mathbf{x})$, $\bar{f}^+_{G_t}$ is even with respect to x_n. Hence,

$$(2) = \int_{\mathbb{R}^n_+} \bar{f}^+_{G_t}(\mathbf{z}', -z_n)\,\mathrm{d}\mathbf{z}'\,\mathrm{d}z_n = \int_{\mathbb{R}^n_+} \bar{f}^+_{G_t}(\mathbf{z}', z_n)\,\mathrm{d}\mathbf{z}'\,\mathrm{d}z_n = (1).$$

$\square$

Lemma 4.2. *Let $f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$, then we have that $E_{even}f \in \mathscr{H}^1(\mathbb{R}^n)$ and*

$$||E_{even}f||_{\mathscr{H}^1(\mathbb{R}^n)} = 2 \cdot ||f||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)}.$$

Proof. For simplicity we denote $E_{even}f$ by $\tilde{f}$. Let G_t be the heat kernel. By Definition 2.8, we have that

$$\begin{aligned}||\tilde{f}||_{\mathscr{H}^1} &= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, G_t * \tilde{f}\,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n_-} \sup_{t>0} |\, G_t * \tilde{f}\,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= (1) + (2).\end{aligned}$$

(1). For $\mathbf{x} \in \mathbb{R}^n_+$ and $t > 0$, we have that $(G_t * \tilde{f})\,(\mathbf{x}, t) = (r_{\mathbb{R}^n_+}(G_t * \tilde{f}))\,(\mathbf{x}, t)$. Since this is true for all $t > 0$, by taking the supremum over all $t > 0$, we have that

$$\sup_{t>0} |\, G_t * \tilde{f}\,|\,(\mathbf{x}) = \sup_{t>0} |\, r_{\mathbb{R}^n_+}(G_t * \tilde{f})\,|\,(\mathbf{x}).$$

Since the above equality holds for all $\mathbf{x} \in \mathbb{R}^n_+$, we can see that

$$(1) = ||f||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)}.$$

(2). Notice that $(G_t * \tilde{f})\,(\mathbf{x}, t)$ is even with respect to x_n since $\tilde{f}$ is even with respect to x_n. We have that $\tilde{f}^+_{G_t}(\mathbf{x}) := \sup_{t>0} |\, G_t * \tilde{f}\,|\,(\mathbf{x})$ is even with respect to x_n. Therefore,

$$(2) = \int_{\mathbb{R}^n_+} \tilde{f}^+_{G_t}(\mathbf{z}', -z_n)\,\mathrm{d}\mathbf{z}'\,\mathrm{d}z_n = \int_{\mathbb{R}^n_+} \tilde{f}^+_{G_t}(\mathbf{z}', z_n)\,\mathrm{d}\mathbf{z}'\,\mathrm{d}z_n = (1).$$

□

Lemma 4.3. *Let $f \in \mathscr{H}^1(\mathbb{R}^n)$ and f be odd with respect to x_n, i.e. $f(\mathbf{x}', x_n) = -f(\mathbf{x}', -x_n)$, then we have that $r_{\mathbb{R}^n_+} f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ and*

$$||r_{\mathbb{R}^n_+} f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} \leq ||f||_{\mathscr{H}^1}.$$

Proof. Let $f \in \mathscr{H}^1(\mathbb{R}^n)$ such that f is odd with respect to x_n, then

$$\begin{aligned}||r_{\mathbb{R}^n_+} f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} &:= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+}\, e^{t\Delta}\, E_{odd}\, r_{\mathbb{R}^n_+} f\,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+}\, e^{t\Delta}\, f\,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &\leq \int_{\mathbb{R}^n} \sup_{t>0} |\, e^{t\Delta}\, f\,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= ||f||_{\mathscr{H}^1(\mathbb{R}^n)}.\end{aligned}$$

□

Lemma 4.4. *Let $f \in \mathscr{H}^1(\mathbb{R}^n)$ and f be even with respect to x_n, i.e. $f(\mathbf{x}', x_n) = f(\mathbf{x}', -x_n)$, then we have that $r_{\mathbb{R}^n_+} f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$ and*

$$||r_{\mathbb{R}^n_+} f||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)} \leq ||f||_{\mathscr{H}^1}.$$

Proof. Let $f \in \mathscr{H}^1(\mathbb{R}^n)$ such that f is even with respect to x_n, then

$$\begin{aligned}
||r_{\mathbb{R}^n_+} f||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)} &:= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+}\, e^{t\Delta}\, E_{even}\, r_{\mathbb{R}^n_+} f \,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^n_+} \sup_{t>0} |\, r_{\mathbb{R}^n_+}\, e^{t\Delta}\, f \,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&\le \int_{\mathbb{R}^n} \sup_{t>0} |\, e^{t\Delta}\, f \,|\,(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&= ||f||_{\mathscr{H}^1(\mathbb{R}^n)}.
\end{aligned}$$

□

Lemma 4.5. $\mathbb{P}_{\mathbb{R}^n_+} : \mathbf{Y} \to \mathbf{Y}$ *is a bounded linear operator.*

Proof. The proof is basically identical to the proof of Lemma 3.5. □

4.2 Properties of projection $\mathbb{P}_{\mathbb{R}^n_+}$

Except some places due to the fact that we cannot take the trace properly, the theory in this subsection is completely identical to the theory in subsection 3.3. This is due to the fact that all properties hold not because of the space where $\mathbf{v}$ belongs to, but the properties of projection $\mathbb{P}$ itself has.

Lemma 4.6. *Let* $\mathbf{v} \in \mathbf{Y}$*, then* $\operatorname{div} \mathbb{P}_{\mathbb{R}^n_+} \mathbf{v} = 0$ *in* $\mathbb{R}^n_+$ *in the sense of distributions.*

Proof. The proof is completely identical to the proof of Lemma 3.10. □

Lemma 4.7. *Let* $\mathbf{v} \in \mathbf{Y}$ *such that*

$$\begin{cases} \operatorname{div} \mathbf{v} = 0 & \text{in } \mathbb{R}^n_+, \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Then we have that $\mathbf{v} \in \mathbb{P}_{\mathbb{R}^n_+} \mathbf{Y}$*. Notice that both equalities above hold in the sense of distributions.*

Proof. The proof is completely identical to the proof of Lemma 3.14. □

Definition 4.8. *We define the subspace* $\mathbf{Y}_\sigma$ *of* $\mathbf{Y}$ *by*

$$\mathbf{Y}_\sigma := \{\, \mathbf{v} \in \mathbf{Y} \mid \operatorname{div} \mathbf{v} = 0 \;\; \text{in } \mathbb{R}^n_+ \;\; \& \;\; \mathbf{v} \cdot \mathbf{n} = 0 \;\; \text{on } \partial\mathbb{R}^n_+ \,\}.$$

Lemma 4.9. *In the case for the space* $\mathbf{Y}$*, we have that*

$$\mathbf{Y}_\sigma \subset \mathbb{P}_{\mathbb{R}^n_+} \mathbf{Y} \subset \{\, \mathbf{v} \in \mathbf{Y} \mid \operatorname{div} \mathbf{v} = 0 \;\; \text{in } \mathbb{R}^n_+ \,\}.$$

Proof. By Lemma 4.6 and Lemma 4.7, we are done. □

4.3 Characterisation of the subspace $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$

Due to the fact that the theory in this section depends only on the properties of projection $\mathbb{P}_{\mathbb{R}^n_+}$ and the trace problem which we do not know how to deal with is not involved in any sense, it is completely identical to the theory in subsection 3.4.

Lemma 4.10. *Let $\mathbf{v} \in \mathbf{Y}$, then there exists $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ such that $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{v} = \nabla p$.*

Proof. The proof is completely identical to the proof of Lemma 3.16. □

Lemma 4.11. *Let $p \in L^1_{loc}(\overline{\mathbb{R}^n_+})$ such that $\nabla p \in \mathbf{Y}$, then $\nabla p \in (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$.*

Proof. The proof is completely identical to the proof of Lemma 3.17. □

Definition 4.12. *We define the subspace $\mathbf{Y}_\pi$ of $\mathbf{Y}$ by*

$$\mathbf{Y}_\pi := \{\, \nabla p \in \mathbf{Y} \mid p \in L^1_{loc}(\overline{\mathbb{R}^n_+}) \,\}.$$

Lemma 4.13. $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y} = \mathbf{Y}_\pi$.

Proof. By Lemma 4.10 and Lemma 4.11, we are done. □

4.4 Proof of Theorem 1.2

Proof. By Lemma 4.5 we see that $\mathbb{P}_{\mathbb{R}^n_+}$ is a bounded linear operator which maps $\mathbf{Y}$ to $\mathbf{Y}$. By this bounded linear map we can easily see that the vector field $\mathbf{Y}$ admits a natural decomposition

$$\mathbf{Y} = \mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} \oplus (I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$$

where both $\mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y}$ and $(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y}$ are linear subspaces of $\mathbf{Y}$. Since this natural decomposition is induced by the projection $\mathbb{P}_{\mathbb{R}^n_+}$, this decomposition is certainly unique. Moreover, we have already proved that

$$\mathbf{Y}_\sigma \subset \mathbb{P}_{\mathbb{R}^n_+}\mathbf{Y} \subset \{\, \mathbf{v} \in \mathbf{Y} \mid \operatorname{div} \mathbf{v} = 0 \ \text{ in } \ \mathbb{R}^n_+ \,\}$$

and

$$(I - \mathbb{P}_{\mathbb{R}^n_+})\mathbf{Y} = \mathbf{Y}_\pi.$$

As a result, Theorem 1.2 holds and we are done. □

5 Duality theorem

Before we start this section we would like to recall the definition that a function $h \in \mathscr{H}^1(\mathbb{R}^n)$ is called a 2-atom if $\operatorname{supp} h \subset B$, $||h||_{L^2(\mathbb{R}^n)} \leq |B|^{-1/2}$ and $\int_B h \, \mathrm{d}\mathbf{x} = 0$. Here $B \subset \mathbb{R}^n$ is an open ball.

5.1 Duality theorem for the case of odd extension

Throughout this subsection, we denote the odd extension operator E_{odd} by E.

Definition 5.1. *We define the set of symmetric 2-atoms by the set*

$$\{Er_{\mathbb{R}^n_+}\alpha \mid \alpha \text{ is a 2-atom s.t. } \operatorname{supp}\alpha \subset B \text{ and } B \cap \partial\mathbb{R}^n_+ \neq \varnothing\}$$
$$\bigcup \quad \{Er_{\mathbb{R}^n_+}\beta \mid \beta \text{ is a 2-atom s.t. } \operatorname{supp}\beta \subset B \subset \mathbb{R}^n_+\}.$$

Let $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+) := \{E\mathbf{v} \mid \mathbf{v} \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)\}$. Then $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+) \subset \mathscr{H}^1(\mathbb{R}^n)$ is a linear subspace.

Lemma 5.2. *The norm*

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \textit{all symmetric 2-atomic decompositions}\}$$

is equivalent to the norm $||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$ on the subspace $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.

Proof. Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, then $Ef \in \mathscr{H}^1(\mathbb{R}^n)$.

(1). By the atomic decompositions of functions of the real Hardy space $\mathscr{H}^1(\mathbb{R}^n)$, we see that Ef admits 2-atomic decompositions. Let

$$Ef = \sum_i \lambda_i\alpha_i + \sum_j \mu_j\beta_j$$

be a 2-atomic decomposition of Ef. Apply $r_{\mathbb{R}^n_+}$ firstly and then E secondly on both sides of this 2-atomic decomposition, we can deduce that

$$Ef = Er_{\mathbb{R}^n_+}Ef = \sum_i \lambda_i Er_{\mathbb{R}^n_+}\alpha_i + \sum_j \mu_j Er_{\mathbb{R}^n_+}\beta_j.$$

This is a symmetric 2-atomic decomposition of Ef with exactly the same coefficients just as the original 2-atomic decomposition. Hence we see that every 2-atomic decomposition of Ef gives rise to a symmetric 2-atomic decomposition of Ef with exactly the same coefficients. Therefore,

$$\begin{aligned}||Ef||_{\mathscr{H}^1(\mathbb{R}^n)} &= \inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all 2-atomic decompositions}\}\\ &\geq \inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all symmetric 2-atomic decompositions}\}.\end{aligned}$$

(2). Let $Ef = \sum_i \lambda_i Er_{\mathbb{R}^n_+}\alpha_i + \sum_j \mu_j Er_{\mathbb{R}^n_+}\beta_j$ be a symmetric 2-atomic decomposition. Pick an i, suppose that $\operatorname{supp}\alpha_i \subset B_i$ where B_i is a ball in $\mathbb{R}^n$ such that $B_i \cap \partial\mathbb{R}^n_+ \neq \varnothing$. Then there exists $\mathbf{x}^* \in B_i \cap \partial\mathbb{R}^n_+$ such that $\operatorname{supp} Er_{\mathbb{R}^n_+}\alpha_i \subset B_{2i}(\mathbf{x}^*)$. Moreover, we have that

$$||Er_{\mathbb{R}^n_+}\alpha_i||_{L^2(\mathbb{R}^n)} \leq 2 \cdot ||\alpha_i||_{L^2(\mathbb{R}^n)} = 2^{\frac{n}{2}+1} \cdot |B_{2i}(\mathbf{x}^*)|^{-1/2}.$$

Since E is the odd extension, we certainly have that

$$\int_{B_{2i}(\mathbf{x}^*)} Er_{\mathbb{R}^n_+}\alpha_i \,\mathrm{d}\mathbf{x} = 0.$$

Therefore, $\frac{1}{2^{\frac{n}{2}+1}} \cdot Er_{\mathbb{R}^n_+}\alpha_i$ is a 2-atom in $\mathscr{H}^1(\mathbb{R}^n)$ for any i. In addition, since $\operatorname{supp}\beta_j \subset B_j \subset \mathbb{R}^n_+$ for some ball B_j, for any j we can decompose $Er_{\mathbb{R}^n_+}\beta_j$ into the form $\beta_j + \beta_j^-$ where β_j^- is a 2-atom which is contained in $(\mathbb{R}^n_+)^c$. Hence we can rewrite the symmetric 2-atomic decomposition in the following way:

$$Ef = \sum_i (\lambda_i 2^{\frac{n}{2}+1}) \cdot (\frac{1}{2^{\frac{n}{2}+1}} Er_{\mathbb{R}^n_+}\alpha_i) + \sum_j \mu_j \cdot \beta_j + \sum_j \mu_j \cdot \beta_j^-.$$

Here $(\frac{1}{2^{\frac{n}{2}+1}} Er_{\mathbb{R}^n_+}\alpha_i)$, β_j and β_j^- are all 2-atoms for any i, j. Therefore we can get a 2-atomic decomposition of Ef from each symmetric 2-atomic decomposition of Ef with coefficients $\{\lambda_i'\}_{i=1}^\infty$ and $\{\mu_j'\}_{j=1}^\infty$ where $\lambda_i' = \lambda_i \cdot 2^{\frac{n}{2}+1}$ for all i and $\mu_j' = 2 \cdot \mu_j$ for all j. Notice that

$$\begin{aligned}\sum_i |\lambda_i| + \sum_j |\mu_j| &\geq \frac{1}{2^{\frac{n}{2}+1}} \cdot \big(\sum_i (|\lambda_i| \cdot 2^{\frac{n}{2}+1}) + \sum_j 2 \cdot |\mu_j|\big) \\ &= \frac{1}{2^{\frac{n}{2}+1}} \cdot \big(\sum_i |\lambda_i'| + \sum_j |\mu_j'|\big).\end{aligned}$$

Therefore we have that

$$\begin{aligned} &\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all symmetric 2-atomic decompositions}\} \\ \geq\ & \frac{1}{2^{\frac{n}{2}+1}} \cdot \inf\{\sum_i |\lambda_i'| + \sum_j |\mu_j'| \mid \text{all 2-atomic decompositions}\}.\end{aligned}$$

Since the norm $\inf\{\sum_i |\lambda_i'| + \sum_j |\mu_j'| \mid \text{all 2-atomic decompositions}\}$ is equivalent to the norm $||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$ by the standard theory of real Hardy spaces, we can deduce that

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all symmetric 2-atomic decompositions}\} \geq C||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$$

for some constant C. □

By making use of Lemma 5.2 we can deduce the half space atomic decomposition for elements of $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.

Theorem 5.3. *Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, then there exists sequences of non-negative numbers $\{\lambda_i\}_{i=1}^\infty$ & $\{\mu_j\}_{j=1}^\infty$, a sequence of 2-atoms $\{\alpha_i\}_{i=1}^\infty$ where for each i $\operatorname{supp}\alpha_i \subset B_i$ for some ball B_i and $B_i \cap \partial\mathbb{R}^n_+ \neq \varnothing$ and a sequence of 2-atoms $\{\beta_j\}_{j=1}^\infty$ where for each j $\operatorname{supp}\beta_j \subset B_j \subset \mathbb{R}^n_+$ for some ball B_j such that*

$$f = \sum_i \lambda_i \cdot \alpha_i \,|_{r_{\mathbb{R}^n_+}} + \sum_j \mu_j \cdot \beta_j.$$

We refer such a decomposition of f as a half space atomic decomposition of f and moreover, the norm

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \textit{all half space atomic decompositions}\}$$

is equivalent to the norm $||\cdot||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}$ *on* $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.

Proof. By Lemma 5.2, we have that

$$\begin{aligned} f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+). &\implies Ef \in \mathscr{H}^1(\mathbb{R}^n). \\ &\implies Ef \text{ admits 2-atomic decompositions.} \\ &\implies Ef \text{ admits symmetric 2-atomic decompositions.} \\ &\implies f \text{ admits half space atomic decompositions by taking} \\ &\qquad \text{restrictions of symmetric 2-atomic decompositions.} \end{aligned}$$

By Lemma 4.1 and Lemma 4.3, there exists constants C_1 and C_2 such that

$$C_1 \cdot ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} \leq ||Ef||_{\mathscr{H}^1(\mathbb{R}^n)} \leq C_2 \cdot ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}.$$

Moreover, the norm $||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$ is equivalent to the norm

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all symmetric 2-atomic decompositions}\}$$

on $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ by Lemma 5.2. Since each of the half space atomic decomposition of f gives rise naturally to a symmetric 2-atomic decomposition of Ef with exactly the same coefficients by odd extension, we have that

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all half space atomic decompositions}\} \approx ||\cdot||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}$$

on $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. $\square$

Definition 5.4. *We denote the set of all finite linear combinations of symmetric 2-atoms by* $\mathscr{H}^1_{0,s}(\mathbb{R}^n)$.

Notice that $\mathscr{H}^1_{0,s}(\mathbb{R}^n) \subset \mathscr{H}^1_0(\mathbb{R}^n) \cap E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ where $\mathscr{H}^1_0(\mathbb{R}^n)$ is the set of all finite linear combinations of 2-atoms.

Lemma 5.5. $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ *is a closed subspace of* $\mathscr{H}^1(\mathbb{R}^n)$.

Proof. Let $F \in \overline{E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}^{||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}} \setminus E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, then there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ such that $Eu_n \to F$ in $||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$ as $n \to \infty$. Since $\mathscr{H}^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, we have that

$$||Eu_n - F||_{L^1(\mathbb{R}^n)} \leq ||Eu_n - F||_{\mathscr{H}^1(\mathbb{R}^n)} \to 0.$$

This means that $Eu_n(\mathbf{x}) \to F(\mathbf{x})$ *a.e.*. Notice that for $\mathbf{x} \in \mathbb{R}^n$,

$$F(\mathbf{x}', x_n) \leftarrow Eu_n(\mathbf{x}', x_n) = -Eu_n(\mathbf{x}', -x_n) \to -F(\mathbf{x}', -x_n).$$

Therefore, F is odd with respect to x_n *a.e.* and $F \in E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. $\square$

Lemma 5.6. *$\mathscr{H}^1_{0,s}(\mathbb{R}^n)$ is dense in $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.*

Proof. Through the proof of Lemma 5.2 we know that every element of $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ admits symmetric 2-atomic decompositions and by Lemma 5.5 we see that $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ is closed in $\mathscr{H}^1(\mathbb{R}^n)$. We are done. $\square$

Theorem 5.7. *Suppose $g \in BMO^{\infty,\infty}_b(\mathbb{R}^n_+)$. Then the linear functional l defined on $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ by*

$$l(f) = \int_{\mathbb{R}^n_+} f \cdot g \, \mathrm{d}\mathbf{x}$$

for $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ is a bounded linear functional which satisfies $||l|| \le c \cdot ||g||_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)}$ with some constant c. Conversely, every bounded linear functional l on $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ can be written in the form of

$$l(f) = \int_{\mathbb{R}^n_+} f \cdot g \, \mathrm{d}\mathbf{x} \quad \text{for all } f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$$

with $g \in BMO^{\infty,\infty}_b(\mathbb{R}^n_+)$ and $||g||_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)} \le c \cdot ||l||$ with some constant c. Here $||l||$ means the norm of l as a bounded linear functional on $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.

Proof. (1). Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ and $g \in BMO^{\infty,\infty}_b(\mathbb{R}^n_+)$. Then we have the estimates

$$\begin{aligned} |\int_{\mathbb{R}^n_+} f \cdot g \, \mathrm{d}\mathbf{x}| &= \frac{1}{2} \cdot |\int_{\mathbb{R}^n} Ef \cdot Eg \, \mathrm{d}\mathbf{x}| \\ &\le \frac{1}{2} \cdot ||Ef||_{\mathscr{H}^1(\mathbb{R}^n)} \cdot ||Eg||_{BMO} \\ &\le c \cdot ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} \cdot ||g||_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)}. \end{aligned}$$

Therefore, $l : f \mapsto \displaystyle\int_{\mathbb{R}^n_+} f \cdot g \, \mathrm{d}\mathbf{x} \;\; \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)^*$ and the above inequalities imply that $||l|| \le c \cdot ||g||_{BMO^{\infty,\infty}_b(\mathbb{R}^n_+)}$ with some constant c.

(2). Let $l \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)^*$. We define $\tilde{l}(Ef) := 2 \cdot l(f)$ for all $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. Fix a ball $B \subset \mathbb{R}^n_+$, let $L^2_0(B)$ be the subspace $\{f \in L^2(B) \mid \int_B f \, \mathrm{d}\mathbf{x} = 0\}$, notice that $L^2_0(B) \subset \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. Let $u \in L^2_0(B)$ be a 2-atom, i.e. we require that $\operatorname{supp} u \subset B \subset \mathbb{R}^n_+$ for some ball B, $\int_B u \, \mathrm{d}\mathbf{x} = 0$ and $||u||_{L^2(B)} \le |B|^{-1/2}$. We then have that

$$\begin{aligned} |\tilde{l}(Eu)| &:= 2 \cdot |l(u)| \le c \cdot ||u||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} \\ &\le c \cdot ||Eu||_{\mathscr{H}^1(\mathbb{R}^n)} = c \cdot ||u^+ + u^-||_{\mathscr{H}^1(\mathbb{R}^n)} \\ &\le c \cdot (||u^+||_{\mathscr{H}^1(\mathbb{R}^n)} + ||u^-||_{\mathscr{H}^1(\mathbb{R}^n)}) \le c \cdot |B|^{1/2} \cdot ||u||_{L^2_0(B)} \\ &\le c \cdot |B|^{1/2} \cdot ||Eu||_{EL^2_0(B)}. \end{aligned}$$

Here $||\cdot||_{L^2_0(B)} := (\int_B |\cdot|^2 \, \mathrm{d}\mathbf{x})^{\frac{1}{2}}$ and $||\cdot||_{EL^2_0(B)} := (\int_{B \cup B^-} |\cdot|^2 \, \mathrm{d}\mathbf{x})^{\frac{1}{2}}$ with $B^- := \{(\mathbf{x}', -x_n) \mid (\mathbf{x}', x_n) \in B\}$. For general $w \in L^2_0(B)$, we have that $w = \lambda \cdot u$ where $u \in L^2_0(B)$ is a 2-atom, then

$$|\tilde{l}(Ew)| := 2 \cdot |l(w)| = 2 \cdot |\lambda| \cdot |l(u)| \le c \cdot |B|^{1/2} \cdot ||Ew||_{EL^2_0(B)}.$$

Thus $\tilde{l}\,|_{EL_0^2(B)}$ is a bounded linear functional on $EL_0^2(B)$.

Claim 1 : $EL_0^2(B)^* = EL_0^2(B)$.

Proof of *Claim* 1 : Let $\tilde{T} \in EL_0^2(B)^*$, by definition we have that $|\tilde{T}(Eu)| \leq c \cdot ||Eu||_{EL_0^2(B)}$. Let's define $T(u)$ for each $u \in L_0^2(B)$ by $T(u) = \frac{1}{2} \cdot \tilde{T}(Eu)$, thus

$$|T(u)| = \frac{1}{2} \cdot |\tilde{T}(Eu)| \leq c \cdot ||Eu||_{EL_0^2(B)} \leq c \cdot ||u||_{L_0^2(B)}.$$

Hence $T \in L_0^2(B)^*$. By the Riesz representation theorem for the Hilbert space $L_0^2(B)$, we deduce that there exists $g^B \in L_0^2(B)$ such that

$$T(u) = \int_B u \cdot g^B \,\mathrm{d}\mathbf{x} \text{ for all } u \in L_0^2(B).$$

Notice that

$$\tilde{T}(Eu) = 2 \cdot T(u) = 2 \cdot \int_B u \cdot g^B \,\mathrm{d}\mathbf{x} = \int_{B\cup B^-} Eu \cdot Eg^B \,\mathrm{d}\mathbf{x}$$

and $Eg^B \in EL_0^2(B)$, hence $EL_0^2(B)^* = EL_0^2(B)$ and the proof of *Claim* 1 is finished.

By *Claim* 1, $\tilde{l}\,|_{EL_0^2(B)} \in EL_0^2(B)^* = EL_0^2(B)$ implies that there exists $g^B \in L_0^2(B)$ such that $\tilde{l}\,|_{EL_0^2(B)} = Eg^B$ as a bounded linear functional on $EL_0^2(B)$, i.e.

$$\tilde{l}(Eu) = \int_{B\cup B^-} Eu \cdot Eg^B \,\mathrm{d}\mathbf{x} \text{ for all } Eu \in EL_0^2(B).$$

Since B is any ball in $\mathbb{R}^n_+$, we can find Eg^B for any $B \subset \mathbb{R}^n_+$. If $B_1 \subset B_2 \subset \mathbb{R}^n_+$, then we can easily see that $Eg^{B_2} - Eg^{B_1}$ is a constant on $B_1 \cup B_1^-$.

Consider the ball $B_r(\mathbf{x})$ where $\mathbf{x} \in \partial\mathbb{R}^n_+$ and $r > 0$. Let $B_r^+(\mathbf{x}) := B_r(\mathbf{x}) \cap \mathbb{R}^n_+$. For simplicity, we denote $B_r(\mathbf{x})$ by B_r. Let $u \in B_r^+$, notice that $Eu \in L^2(B_r)$ and $\int_{B_r} Eu \,\mathrm{d}\mathbf{x} = 0$ as E is the odd extension. Since $Eu \in EL_0^2(B_r)$ and Eu is odd with respect to x_n, we have that $L^2(B_r^+) \subset \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. By similar arguments as above, we see that $\tilde{l}\,|_{EL^2(B_r^+)}$ is a bounded linear functional on $EL^2(B_r^+)$. By the same proof of *Claim* 1, we have that $EL^2(B_r^+)^* = EL^2(B_r^+)$. Hence $\tilde{l}\,|_{EL^2(B_r^+)} \in EL^2(B_r^+)^* = EL^2(B_r^+)$ implies that $\tilde{l}\,|_{EL^2(B_r^+)} = Eg^{B_r^+} \in EL^2(B_r^+)$ as a bounded linear functional on $EL^2(B_r^+)$ for some $g^{B_r^+} \in L^2(B_r^+)$. For any ball $B_r(\mathbf{x})$ where $\mathbf{x} \in \partial\mathbb{R}^n_+$, we can find $Eg^{B_r^+}$. If $B_{r_1} \subset B_{r_2}$, then $Eg^{B_{r_2}^+} - Eg^{B_{r_1}^+}$ is a constant on B_{r_1}.

Now we seek to find a uniform $Eg(\mathbf{x})$ defined on $\mathbb{R}^n$. We define that

$$Eg(\mathbf{x}) := Eg^{B_r^+(0)} - \frac{1}{|B_1(0)|} \cdot \int_{B_1(0)} Eg^{B_r^+(0)} \,\mathrm{d}\mathbf{x} = Eg^{B_r^+(0)}.$$

The last equality holds as $\underset{B_1(0)}{Avg}\, Eg^{B_r^+(0)} = 0$. For $B \subset \mathbb{R}^n_+$, we have $Eg^B(\mathbf{x})$ defined on B, then there exists $B_R(0)$ for some R large enough such that $B \subset B_R^+(0)$. Hence

$$\begin{aligned} Eg^B(\mathbf{x}) &= Eg^B(\mathbf{x}) - Eg^{B_R^+(0)}(\mathbf{x}) + Eg^{B_R^+(0)}(\mathbf{x}) \\ &= c_B + Eg(\mathbf{x}) \end{aligned}$$

where $c_B := Eg^B(\mathbf{x}) - Eg^{B^+_R(0)}(\mathbf{x})$ is a constant which depends on B.

Next we prove that the function $g(\mathbf{x})$ defined by $g(\mathbf{x}) := r_{\mathbb{R}^n_+} Eg(\mathbf{x})$ belongs to the space $BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$.

1^*. If $B \subset \mathbb{R}^n_+$, we have that

$$\begin{aligned}\frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} &= \frac{1}{|B|}\int_B |Eg^B(\mathbf{x})|\,\mathrm{d}\mathbf{x}\\ &\leq \frac{1}{|B|}\Big(\int_B |Eg^B|^2\,\mathrm{d}\mathbf{x}\Big)^{\frac{1}{2}} \cdot |B|^{\frac{1}{2}}\\ &= |B|^{-\frac{1}{2}} \cdot ||Eg^B||_{EL^2_0(B)}\end{aligned}$$

where the second inequality above is by the Hölder inequality. Since

$$|\int_{B\cup B^-} Eg^B \cdot Eu\,\mathrm{d}\mathbf{x}| = |\tilde{l}(Eu)| \leq c\cdot |B|^{\frac{1}{2}} \cdot ||Eu||_{EL^2_0(B)},$$

we can deduce that

$$||Eg^B||_{EL^2_0(B)} = ||\tilde{l}|| \leq c\cdot |B|^{\frac{1}{2}}$$

where $||\tilde{l}||$ is the operator norm of $\tilde{l}$. Therefore we have that

$$\frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} \leq |B|^{-\frac{1}{2}} \cdot c \cdot |B|^{\frac{1}{2}} = c.$$

By taking the supremum over all balls in $\mathbb{R}^n_+$, we can deduce that

$$\sup_{B\subset\mathbb{R}^n_+} \frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} \leq c.$$

Then by the triangle inequality, we can easily get that

$$[g]_{BMO^\infty(\mathbb{R}^n_+)} \leq 2\cdot \sup_{B\subset\mathbb{R}^n_+} \frac{1}{|B|}\int_B |g(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} \leq 2\cdot c.$$

2^*. For balls of the form $B_r(\mathbf{x})$ where $\mathbf{x} \in \partial\mathbb{R}^n_+$, we have that

$$Eg(\mathbf{x}) = Eg^{B_r^+(\mathbf{x})} - c_{B_r}.$$

Now we integrate this equality over the ball $B_r(\mathbf{x})$, we have that

$$\int_{B_r(\mathbf{x})} Eg(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{B_r(\mathbf{x})} Eg^{B_r^+(\mathbf{x})}\,\mathrm{d}\mathbf{y} - \int_{B_r(\mathbf{x})} c_{B_r}\,\mathrm{d}\mathbf{y}.$$

Notice that Eg and $Eg^{B_r^+(\mathbf{x})}$ are both odd with respect to x_n, we certainly have

$$\int_{B_r(\mathbf{x})} Eg(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{B_r(\mathbf{x})} Eg^{B_r^+(\mathbf{x})}\,\mathrm{d}\mathbf{y} = 0.$$

Hence c_{B_r} must equal 0. By making use of this fact and similar arguments as the previous part, we also have that

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |Eg(\mathbf{y}) - (-c_{B_r})|\,\mathrm{d}\mathbf{y} \le c.$$

Therefore,

$$\frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} |Eg|\,\mathrm{d}\mathbf{y} = \frac{1}{|B_r^+(\mathbf{x})|}\int_{B_r^+(\mathbf{x})} |Eg|\,\mathrm{d}\mathbf{y} \le c.$$

As $Eg(\mathbf{y}) = g(\mathbf{y})$ in $B_r^+(\mathbf{x})$, we have that

$$\frac{1}{|B_r^+(\mathbf{x})|}\int_{B_r^+(\mathbf{x})} |g(\mathbf{y})|\,\mathrm{d}\mathbf{y} \le c.$$

By taking the supremum over all balls centered at $\partial\mathbb{R}^n_+$, we can easily deduce that

$$||g||_{b^\infty(\mathbb{R}^n_+)} = \sup_{\substack{r>0 \\ \mathbf{x}\in\partial\mathbb{R}^n_+}} \frac{1}{|B_r^+(\mathbf{x})|}\int_{B_r^+(\mathbf{x})} |g(\mathbf{y})|\,\mathrm{d}\mathbf{y} \le c < \infty.$$

Hence by 1^* and 2^*, $g \in BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$.

Let Eu be a (2, s)-atom, we have that

$$\int_{\mathbb{R}^n_+} g\cdot u\,\mathrm{d}\mathbf{x} = \frac{1}{2}\cdot\int_{\mathbb{R}^n} Eg\cdot Eu\,\mathrm{d}\mathbf{x} = \frac{1}{2}\cdot\tilde{l}(Eu) = l(u).$$

Since this representation has been established for the subspace $\mathscr{H}^1_{0,s}(\mathbb{R}^n)$ and $\mathscr{H}^1_{0,s}(\mathbb{R}^n)$ is dense in $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, therefore $Eg = \tilde{l} \in E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)^*$ and thus $g = l \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)^*$. $\square$

Notice that in the proof of Theorem 5.7, there is a step where we proved that for $B \subset \mathbb{R}^n_+$ and $u \in L^2_0(B)$ we have that

$$|\tilde{l}(Eu)| \le c\cdot|B|^{\frac{1}{2}}\cdot||Eu||_{EL^2_0(B)}.$$

For the ball $B_r(\mathbf{x})$ with $\mathbf{x}\in\partial\mathbb{R}^n_+$ we also have the same estimates. By L.Grafakos [6], the constant c depends only on the dimension n and it is independent of the ball B or $B_r(\mathbf{x})$, hence the later arguments in the proof are valid.

5.2 Duality theorem for the case of even extension

Throughout this subsection, we denote the even extension operator E_{even} by E.

Definition 5.8. *We define the set of symmetric 2-atoms by*

$$\begin{aligned}&\{Er_{\mathbb{R}^n_+}\alpha \mid \alpha \textit{ is a 2-atom such that } \operatorname{supp}\alpha \subset B \ \&\ B\cap\partial\mathbb{R}^n_+ \neq \varnothing \\ &\&\ \int_{\mathbb{R}^n_+}\alpha\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n_-}\alpha\,\mathrm{d}\mathbf{x} = 0\} \\ &\cup\{Er_{\mathbb{R}^n_+}\beta \mid \beta \textit{ is a 2-atom such that } \operatorname{supp}\beta \subset B \subset \mathbb{R}^n_+\}.\end{aligned}$$

Let $E\mathscr{H}^1_{even}(\mathbb{R}^n_+) := \{E\mathbf{v} \mid \mathbf{v} \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)\}$. Then $E\mathscr{H}^1_{even}(\mathbb{R}^n_+) \subset \mathscr{H}^1(\mathbb{R}^n)$ is a linear subspace.

Lemma 5.9. *The norm*

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \textit{all symmetric 2-atomic decompositions}\}$$

is equivalent to the norm $||\cdot||_{\mathscr{H}^1(\mathbb{R}^n)}$ *on the subspace* $E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$.

Proof. Let $f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$, then $Ef \in \mathscr{H}^1(\mathbb{R}^n)$.

(1). By the atomic decompositions of functions of the real Hardy space $\mathscr{H}^1(\mathbb{R}^n)$, we see that Ef admits 2-atomic decompositions. Let

$$Ef = \sum_i \lambda_i \alpha_i + \sum_j \mu_j \beta_j$$

be a 2-atomic decomposition of Ef. Notice that

$$f = r_{\mathbb{R}^n_+} Ef = \sum_i \lambda_i r_{\mathbb{R}^n_+} \alpha_i + \sum_j \mu_j r_{\mathbb{R}^n_+} \beta_j.$$

Without loss of generality, assume that $\operatorname{supp} \alpha_i \subset B_i$ for some ball B_i and $B_i \cap \partial\mathbb{R}^n_+ \neq \varnothing$, assume further that $\operatorname{supp} \beta_j \subset B_j \subset \mathbb{R}^n_+$ or $\mathbb{R}^n_-$. Therefore we have that

$$f = \sum_i \lambda_i r_{\mathbb{R}^n_+} \alpha_i + \sum_j \mu_j \beta_j.$$

Let $B_i^+ := B_i \cap \mathbb{R}^n_+$ and $B_i^- := B_i \cap \mathbb{R}^n_-$. Since α_i can be any 2-atom, we know that $\displaystyle\int_{B_i} \alpha_i \,\mathrm{d}\mathbf{x} = 0$ but $\displaystyle\int_{B_i^+} \alpha_i \,\mathrm{d}\mathbf{x}$ and $\displaystyle\int_{B_i^-} \alpha_i \,\mathrm{d}\mathbf{x}$ are not necessarily zero. Here we need to do some tricks to $\displaystyle\int_{B_i^+} \alpha_i \,\mathrm{d}\mathbf{x}$ and $\displaystyle\int_{B_i^-} \alpha_i \,\mathrm{d}\mathbf{x}$. Since E is the even extension, except

$$Ef = Er_{\mathbb{R}^n_+} Ef = \sum_i \lambda_i Er_{\mathbb{R}^n_+} \alpha_i + \sum_j \mu_j Er_{\mathbb{R}^n_+} \beta_j$$

we also have that

$$Ef = Er_{\mathbb{R}^n_-} Ef = \sum_i \lambda_i Er_{\mathbb{R}^n_-} \alpha_i + \sum_j \mu_j Er_{\mathbb{R}^n_-} \beta_j.$$

Therefore,

$$\begin{aligned} 2Ef &= Er_{\mathbb{R}^n_+} Ef + Er_{\mathbb{R}^n_-} Ef \\ &= \sum_i \lambda_i \cdot (Er_{\mathbb{R}^n_+} \alpha_i + Er_{\mathbb{R}^n_-} \alpha_i) + \sum_j \mu_j \cdot (Er_{\mathbb{R}^n_+} \beta_j + Er_{\mathbb{R}^n_-} \beta_j). \end{aligned}$$

Suppose that $\operatorname{supp} \alpha_i \subset B_i(\mathbf{x})$ and $B_i(\mathbf{x}) \cap \partial\mathbb{R}^n_+ \neq \varnothing$, there exists $\mathbf{x}^* \in B_i(\mathbf{x}) \cap \partial\mathbb{R}^n_+$ such that $\operatorname{supp} Er_{\mathbb{R}^n_+} \alpha_i \subset B_{2r_i}(\mathbf{x}^*)$ and $\operatorname{supp} Er_{\mathbb{R}^n_-} \alpha_i \subset B_{2r_i}(\mathbf{x}^*)$. Therefore we have that

$\operatorname{supp}(Er_{\mathbb{R}^n_+}\alpha_i + Er_{\mathbb{R}^n_-}\alpha_i) \subset B_{2r_i}(\mathbf{x}^*)$. Notice that $Er_{\mathbb{R}^n_+}\alpha_i + Er_{\mathbb{R}^n_-}\alpha_i$ is also even with respect to x_n. Let's consider $r_{\mathbb{R}^n_+}(Er_{\mathbb{R}^n_+}\alpha_i + Er_{\mathbb{R}^n_-}\alpha_i) = r_{\mathbb{R}^n_+}\alpha_i + r_{\mathbb{R}^n_+}Er_{\mathbb{R}^n_-}\alpha_i$. There is no doubt that $\operatorname{supp}(r_{\mathbb{R}^n_+}\alpha_i + r_{\mathbb{R}^n_+}Er_{\mathbb{R}^n_-}\alpha_i) \subset B_{2r_i}(\mathbf{x}^*) \cap \mathbb{R}^n_+$ and

$$\begin{aligned}\int_{\mathbb{R}^n_+} r_{\mathbb{R}^n_+}\alpha_i + r_{\mathbb{R}^n_+}Er_{\mathbb{R}^n_-}\alpha_i \,\mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^n_+} r_{\mathbb{R}^n_+}\alpha_i \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n_-} Er_{\mathbb{R}^n_-}\alpha_i \,\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{R}^n} \alpha_i \,\mathrm{d}\mathbf{x}\\ &= 0.\end{aligned}$$

Let $\alpha_i^* := r_{\mathbb{R}^n_+}\alpha_i + r_{\mathbb{R}^n_+}Er_{\mathbb{R}^n_-}\alpha_i$, notice that

$$\begin{aligned}||\alpha_i^*||_{L^2(\mathbb{R}^n)} &= ||r_{\mathbb{R}^n_+}\alpha_i + r_{\mathbb{R}^n_+}Er_{\mathbb{R}^n_-}\alpha_i||_{L^2(\mathbb{R}^n)}\\ &\le ||\alpha_i||_{L^2(\mathbb{R}^n)} + ||Er_{\mathbb{R}^n_-}\alpha_i||_{L^2(\mathbb{R}^n)}\\ &\le ||\alpha_i||_{L^2(\mathbb{R}^n)} + 2\cdot ||\alpha_i||_{L^2(\mathbb{R}^n)}\\ &\le 3\cdot 2^{\frac{n}{2}} \cdot |B_{2r_i}(\mathbf{x}^*)|^{-1/2}.\end{aligned}$$

Let $c_{3,2} := 3\cdot 2^{\frac{n}{2}}$. Therefore $c_{3,2}^{-1}\cdot\alpha_i^*$ is a 2-atom and more importantly, we have that

$$\int_{\mathbb{R}^n_-} c_{3,2}^{-1}\cdot\alpha_i^* \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n_+} c_{3,2}^{-1}\cdot\alpha_i^* \,\mathrm{d}\mathbf{x} = 0.$$

Hence $E(c_{3,2}^{-1}\cdot\alpha_i^*) = c_{3,2}^{-1}\cdot E\alpha_i^*$ is a symmetric 2-atom. We have that

$$2\cdot Ef = \sum_i \lambda_i^{'}\cdot(c_{3,2}^{-1}\cdot E\alpha_i^*) + \sum_j \mu_j\cdot Er_{\mathbb{R}^n_+}\beta_j + \sum_j \mu_j\cdot Er_{\mathbb{R}^n_-}\beta_j$$

where $\lambda_i^{'} := \lambda_i\cdot c_{3,2}$. Therefore from a 2-atomic decomposition of Ef we can get a symmetric 2-atomic decomposition of Ef. In addition, for a 2-atomic decomposition $Ef = \sum_i \lambda_i\alpha_i + \sum_j \mu_j\beta_j$ such that $\sum_i |\lambda_i| + \sum_j |\mu_j| < \infty$, the corresponding symmetric 2-atomic decomposition of this 2-atomic decomposition is $Ef = \sum_i \frac{\lambda_i^{'}}{2}\cdot(c_{3,2}^{-1}\cdot E\alpha_i^*) + \sum_j \frac{\mu_j}{2}\cdot Er_{\mathbb{R}^n_+}\beta_j + \sum_j \frac{\mu_j}{2}\cdot Er_{\mathbb{R}^n_-}\beta_j$. In this case we have that

$$\sum_i \frac{|\lambda_i^{'}|}{2} + \sum_j \frac{|\mu_j|}{2} + \sum_j \frac{|\mu_j|}{2} \le 3\cdot 2^{\frac{n}{2}-1}\cdot(\sum_i |\lambda_i| + \sum_j |\mu_j|) < \infty.$$

Therefore,

$$\sum_i |\lambda_i| + \sum_j |\mu_j| \ge \frac{1}{3\cdot 2^{\frac{n}{2}-1}}\cdot\big(\sum_i |\lambda_i^{''}| + \sum_j |\mu_j^{''}|\big)$$

where $\lambda_i^{''} := \frac{\lambda_i^{'}}{2}$ for all i and $\mu_j^{''} := \frac{\mu_j}{2}$ for all j. $\lambda_i^{''}$ and $\mu_j^{''}$ are the coefficients of the corresponding symmetric 2-atomic decomposition induced by the original 2-atomic decomposition. As a result, we have that

$$\begin{aligned} &\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all 2-atomic decompositions}\} \\ \geq \quad & C_1 \cdot \inf\{\sum_i |\lambda_i^{''}| + \sum_j |\mu_j^{''}| \mid \text{all symmetric 2-atomic decompositions}\} \end{aligned}$$

where $C_1 := \frac{1}{3\cdot 2^{\frac{n}{2}-1}}$.

(2). Let $Ef = \sum_i \lambda_i \cdot Er_{\mathbb{R}^n_+}\alpha_i + \sum_j \mu_j \cdot Er_{\mathbb{R}^n_+}\beta_j$ be a symmetric 2-atomic decomposition. Since α_i is a 2-atom, we have that

$$||Er_{\mathbb{R}^n_+}\alpha_i||_{L^2(\mathbb{R}^n)} \leq 2^{\frac{n}{2}+1} \cdot |B_{2r_i}(\mathbf{x}^*)|^{-1/2}.$$

Therefore

$$Ef = \sum_i (\lambda_i \cdot 2^{\frac{n}{2}+1}) \cdot (\frac{1}{2^{\frac{n}{2}+1}} \cdot Er_{\mathbb{R}^n_+}\alpha_i) + \sum_j \mu_j \beta_j^+ + \sum_j \mu_j \beta_j^-$$

is a 2-atomic decomposition of Ef. Thus every symmetric 2-atomic decomposition of Ef gives rise to a 2-atomic decomposition. For this symmetric 2-atomic decomposition of Ef where $\sum_i |\lambda_i| + \sum_j |\mu_j| < \infty$, the coefficients of the corresponding 2-atomic decomposition of Ef satisfies

$$\sum_i (|\lambda_i| \cdot 2^{\frac{n}{2}+1}) + \sum_j 2 \cdot |\mu_j| \leq 2^{\frac{n}{2}+1} \cdot (\sum_i |\lambda_i| + \sum_j |\mu_j|).$$

Therefore,

$$\begin{aligned} &\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \text{all symmetric 2-atomic decompositions}\} \\ \geq \quad & C_2 \cdot \inf\{\sum_i |\lambda_i^{'}| + \sum_j |\mu_j^{'}| \mid \text{all 2-atomic decompositions}\} \end{aligned}$$

where $C_2 := \frac{1}{2^{\frac{n}{2}+1}}$. □

Theorem 5.10. *Let $f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$, then there exists sequences of non-negative numbers $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\mu_j\}_{j=1}^{\infty}$, a sequence of 2-atoms $\{\alpha_i\}_{i=1}^{\infty}$ where for each i $\operatorname{supp}\alpha_i \subset B_i$ & $B_i \cap \partial\mathbb{R}^n_+ \neq \varnothing$ & $\int_{\mathbb{R}^n_+} \alpha_i \, \mathrm{d}\mathbf{x} = 0$ for some ball B_i and a sequence of 2-atoms $\{\beta_j\}_{j=1}^{\infty}$ where for each j $\operatorname{supp}\beta_j \subset B_j \subset \mathbb{R}^n_+$ for some ball B_j such that*

$$f = \sum_i \lambda_i \cdot \alpha_i \mid_{\mathbb{R}^n_+} + \sum_j \mu_j \cdot \beta_j.$$

We refer such a decomposition of f as a half space atomic decomposition of f and moreover, the norm

$$\inf\{\sum_i |\lambda_i| + \sum_j |\mu_j| \mid \textit{all half space atomic decompositions}\}$$

is equivalent to the norm $||\cdot||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)}$ on $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$.

Proof. By Lemma 5.9 we are done. □

Definition 5.11. *We denote the set of all finite linear combinations of symmetric 2-atoms by $\mathscr{H}^1_{0,s}(\mathbb{R}^n)$.*

By similar arguments as in the previous subsection, we can easily deduce that $\mathscr{H}^1_{0,s}(\mathbb{R}^n) \subset \mathscr{H}^1_0(\mathbb{R}^n) \cap E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$, $E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ is a closed subspace of $\mathscr{H}^1(\mathbb{R}^n)$ and $\mathscr{H}^1_{0,s}(\mathbb{R}^n)$ is dense in $E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$. Then by making use of these facts, we can prove our duality theorem for the case of even extension.

Theorem 5.12. *Suppose $g \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$. Then the linear functional l defined on $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ by*

$$l(f) = \int_{\mathbb{R}^n_+} f \cdot g \,\mathrm{d}\mathbf{x}$$

for $f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$ is a bounded linear functional which satisfies $||l|| \le c \cdot [g]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)}$ with some constant c. Conversely, every bounded linear functional l on $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ can be written in the form of

$$l(f) = \int_{\mathbb{R}^n_+} f \cdot g \,\mathrm{d}\mathbf{x} \quad \textit{for all } f \in \mathscr{H}^1_{even}(\mathbb{R}^n_+)$$

with $g \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ and $[g]_{BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)} \le c \cdot ||l||$ with some constant c. Here $||l||$ means the norm of l as a bounded linear functional on $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$.

Proof. The only difference from the proof of Theorem 5.7 is the last part where here we prove that the unified function $g(\mathbf{x}) \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ instead of $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$. For the rest of the details, please refer to the proof of Theorem 5.7.

We define the unified function $Eg(\mathbf{x})$ on $\mathbb{R}^n$ by

$$\begin{aligned} Eg(\mathbf{x}) &:= Eg^{B^+_r(0)} - \frac{1}{|B_1(0)|}\int_{B_1(0)} Eg^{B^+_r(0)} \,\mathrm{d}\mathbf{x} \\ &= Eg^{B^+_r(0)} - \underset{B_1(0)}{Avg} Eg^{B^+_r(0)}. \end{aligned}$$

For $B \subset \mathbb{R}^n_+$ we have $Eg^B(\mathbf{x})$ defined on the ball B, then there exists $B_r(0)$ for some r large enough such that $B \subset B_r(0)$. We can rewrite $Eg^B(\mathbf{x})$ as

$$Eg^B(\mathbf{x}) = Eg^B(\mathbf{x}) - Eg^{B^+_r(0)}(\mathbf{x}) + Eg^{B^+_r(0)}(\mathbf{x}) - \underset{B_1(0)}{Avg} Eg^{B^+_r(0)} + \underset{B_1(0)}{Avg} Eg^{B^+_r(0)}.$$

Notice that $Eg^B(\mathbf{x}) - Eg^{B_r^+(0)}(\mathbf{x})$ and $\underset{B_1(0)}{Avg} Eg^{B_r^+(0)}$ are both constants which depend on B, hence let $c_B := Eg^B(\mathbf{x}) - Eg^{B_r^+(0)}(\mathbf{x}) + \underset{B_1(0)}{Avg} Eg^{B_r^+(0)}$, we have that $Eg^B(\mathbf{x}) = c_B + Eg(\mathbf{x})$. Next we prove that the function $g(\mathbf{x})$ defined by $g(\mathbf{x}) := r_{\mathbb{R}^n_+} Eg(\mathbf{x}) \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$.

$*1$. If $B \subset \mathbb{R}^n_+$, we have that

$$\frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} \leq c \cdot |B|^{-1/2} \cdot ||Eg^B||_{EL^2_0(B)}$$

by the Hölder inequality. Since

$$|\int_{B\cup B^-} Eg^B \cdot Eu \,\mathrm{d}\mathbf{x} = |\tilde{l}(Eu)| \leq c \cdot |B|^{-1/2} \cdot ||Eu||_{EL^2_0(B)},$$

we have that

$$||Eg^B||_{EL^2_0(B)} = ||\tilde{l}|| \leq c \cdot |B|^{1/2}.$$

Therefore we can deduce that

$$\frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} \leq c.$$

Notice that the c here is just a number which is independent of B. Therefore by taking the supremum over all balls contained in $\mathbb{R}^n_+$, we can see that

$$\sup_{B\subset\mathbb{R}^n_+} \frac{1}{|B|}\int_B |Eg(\mathbf{x}) - (-c_B)|\,\mathrm{d}\mathbf{x} \leq c.$$

and thus,

$$[r_{\mathbb{R}^n_+} Eg]_{BMO^\infty(\mathbb{R}^n_+)} = [g]_{BMO^\infty(\mathbb{R}^n_+)} \leq 2 \cdot c.$$

$*2$. If $B_r(\mathbf{x})$ is a ball where $\mathbf{x} \in \partial\mathbb{R}^n_+$ and $r > 0$, we have that $Eg(\mathbf{x}) = Eg^{B_r^+}(\mathbf{x}) - c_{B_r}$. Therefore we have the following calculations:

$$\begin{aligned} 2 \cdot \int_{B_r^+} g(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{B_r} Eg(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \int_{B_r} Eg^{B_r^+}(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{B_r} c_{B_r}\,\mathrm{d}\mathbf{x} \\ &= 0 - c_{B_r} \cdot |B_r|. \end{aligned}$$

Hence $c_{B_r} = -g_{B_r^+}$ and we have that

$$\begin{aligned} \frac{1}{|B_r|}\int_{B_r} |Eg(\mathbf{x}) - (-c_{B_r})|\,\mathrm{d}\mathbf{x} &= \frac{1}{|B_r|}\int_{B_r} |Eg(\mathbf{x}) - g_{B_r^+}|\,\mathrm{d}\mathbf{x} \\ &= \frac{1}{|B_r^+|}\int_{B_r^+} |g(\mathbf{x}) - g_{B_r^+}|\,\mathrm{d}\mathbf{x} \leq c. \end{aligned}$$

Take the supremum over all balls centered on $\mathbb{R}^n_+$, we have that

$$[g]_{ba^\infty(\mathbb{R}^n_+)} = \sup_{\substack{r>0 \\ \mathbf{x}\in\partial\mathbb{R}^n_+}} \frac{1}{|B_r^+|}\int_{B_r^+} |g(\mathbf{x}) - g_{B_r^+}|\,\mathrm{d}\mathbf{x} \leq c$$

and hence $g \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$. $\square$

5.3 Proof of Theorem 1.3

Proof. By Theorem 5.7 and Theorem 5.12, we are done. □

5.4 Comments

Remark 5.13. *If we look at the proof of Lemma* 4.1 *and Lemma* 4.2*, we can see that it is completely all right for us to replace the heat kernel* $e^{t\Delta}$ *in the definition of* $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ *and* $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ *by any radial symmetric function* $\varphi \in \mathscr{S}(\mathbb{R}^n)$ *such that* $\int_{\mathbb{R}^n} \varphi \, \mathrm{d}\mathbf{x} = 1$. *Therefore, the definitions of the norms* $|| \cdot ||_{\mathscr{H}^1_{even}(\mathbb{R}^n_+)}$ *and* $|| \cdot ||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)}$ *are independent of the choice of* φ *if* φ *is radial symmetric with integral over* $\mathbb{R}^n$ *equals* 1.

Remark 5.14. *When we established the half space atomic decompositions for* $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ *and* $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$*, we made use of the 2-atomic decomposition of* $\mathscr{H}^1(\mathbb{R}^n)$ *in order to carry out the arguments of Fefferman and Stein [3] to prove the duality theorem. However, if we carry out the arguments using the p-atomic decomposition of* $\mathscr{H}^1(\mathbb{R}^n)$ *instead where* $p \geq 1$*, then we get the half space atomic decompositions for* $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ *and* $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ *in the form of symmetric p-atomic decompositions.*

In [1], it is proved that $BMO_M(\mathbb{R}^n_+)$ and $BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$ are actually the same space. Since $BMO_M(\mathbb{R}^n_+)$ is the dual space of $\mathscr{H}^1_M(\mathbb{R}^n_+)$ and $BMO_b^{\infty,\infty}(\mathbb{R}^n_+)$ is the dual space of $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, it is natural to ask the question about the relation between $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_M(\mathbb{R}^n_+)$. Here we give an answer to this question.

Lemma 5.15. $\mathscr{H}^1_{odd}(\mathbb{R}^n_+) = \mathscr{H}^1_M(\mathbb{R}^n_+)$.

Proof. (1). By the theory of Miyachi [7], $f \in \mathscr{H}^1_M(\mathbb{R}^n_+)$ implies that f admits the half space atomic decomposition of the form

$$f = \sum_i \lambda_i \alpha_i + \sum_j \mu_j \beta_j$$

where $\{\beta_j\}_{j=1}^{\infty}$ is a sequence of 1-atom such that β_j is supported on some ball B_j with $2B_j \subset \mathbb{R}^n_+$ for each j and $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence of $(1, \mathbb{R}^n_+)$-atom such that α_i is supported on some ball B_i with $2B_i \subset \mathbb{R}^n_+$ but $5B_i \cap (\mathbb{R}^n_+)^c \neq \varnothing$ for each i. Let $B_i = B_r(\mathbf{x}_i)$ and $\mathbf{x}^* := (\mathbf{x}_i^{'}, 0)$. Since $2B_i \subset \mathbb{R}^n_+$ but $5B_i \cap (\mathbb{R}^n_+)^c \neq \varnothing$, we can easily deduce that $B_i \subset B_{6r}(\mathbf{x}^*)$. Notice that $\alpha_i = r_{\mathbb{R}^n_+} E_{odd}\alpha_i$ and $\int_{B_{6r}(\mathbf{x}^*)} E_{odd}\alpha_i \, \mathrm{d}\mathbf{x} = 0$, therefore we have that

$$E_{odd} f = \sum_i (\lambda_i \cdot 6^n) \cdot (\frac{1}{6^n} \cdot E_{odd}\alpha_i) + \sum_j \mu_j E_{odd}\beta_j. \tag{5.1}$$

Here $\frac{1}{6^n} \cdot E_{odd}\alpha_i$ is a 1-atom for any i, hence by (5.1) we see that $E_{odd} f \in \mathscr{H}^1(\mathbb{R}^n)$ and thus by Remark 5.14 $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$.

(2). Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, let η be the standard mollifier. For $\mathbf{x} \in \mathbb{R}^n_+$ and $0 < t < dist(\mathbf{x}, \partial\mathbb{R}^n_+)$, we have that $(\eta_t * f)(\mathbf{x}) = (\eta_t * E_{odd} f)(\mathbf{x})$ since $\operatorname{supp} \eta_t \subset B_t(0)$. Hence for

$\mathbf{x} \in \mathbb{R}^n_+$,

$$\sup_{0<t<dist(\mathbf{x},\partial\mathbb{R}^n_+)} |\eta_t * f|(\mathbf{x}) = \sup_{0<t<dist(\mathbf{x},\partial\mathbb{R}^n_+)} |\eta_t * E_{odd}f|(\mathbf{x})$$
$$\leq \sup_{t>0} |\eta_t * E_{odd}f|(\mathbf{x}).$$

Thus

$$\begin{aligned} ||f||_{\mathscr{H}^1_M(\mathbb{R}^n_+)} &:= \int_{\mathbb{R}^n_+} \sup_{0<t<dist(\mathbf{x},\partial\mathbb{R}^n_+)} |\eta_t * f|(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &\leq \int_{\mathbb{R}^n_+} \sup_{t>0} |\eta_t * E_{odd}f|(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= ||f||_{\mathscr{H}^1_{odd}(\mathbb{R}^n_+)} \end{aligned}$$

and therefore $f \in \mathscr{H}^1_M(\mathbb{R}^n_+)$. $\square$

Remark 5.16. *Let us consider a function $f \in L^2(B^+_r(0))$ with integral over $B^+_r(0)$ not equals to 0. Notice that although $\int_{B^+_r(0)} f\,\mathrm{d}\mathbf{x} \neq 0$, the odd extension $E_{odd}f$ has integral zero over the ball $B_r(0)$. Hence we have that $E_{odd}f \in L^2(B_r(0))$, $\int_{B_r(0)} E_{odd}f\,\mathrm{d}\mathbf{x} = 0$ and thus $E_{odd}f \in \mathscr{H}^1(\mathbb{R}^n)$. Then $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. However, $\int_{B^+_r(0)} f\,\mathrm{d}\mathbf{x} \neq 0$ implies that $\int_{B_r(0)} E_{even}f\,\mathrm{d}\mathbf{x} \neq 0$ and thus $E_{even}f \notin \mathscr{H}^1(\mathbb{R}^n)$. Hence $f \notin \mathscr{H}^1_{even}(\mathbb{R}^n_+)$. Therefore $\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ and $\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ are two different spaces.*

Remark 5.17. *Let us consider the function $log|\mathbf{x}|$, by the standard theory of BMO spaces we see that $log|\mathbf{x}| \in BMO$. Then $log|\mathbf{x}|\ |_{\mathbb{R}^n_+} \in BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$. However, $log|\mathbf{x}|\ |_{\mathbb{R}^n_+} \notin BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$ since the integral*

$$\frac{1}{B^+_r(0)} \int_{B^+_r(0)} |\,log|\mathbf{x}|\,|\,\mathrm{d}\mathbf{x} \to \infty \quad as\ \ r \to \infty.$$

Therefore $BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+)$ and $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+)$ are also two different spaces.

Remark 5.18. *Notice that by Theorem 5.3 we can easily see that $\mathscr{H}^1_{odd}(\mathbb{R}^n_+) = \mathscr{H}^1(\mathbb{R}^n_+)$ where $\mathscr{H}^1(\mathbb{R}^n_+) := \{r_{\mathbb{R}^n_+}f \,|\, f \in \mathscr{H}^1(\mathbb{R}^n)\}$. Moreover, by Lemma 3.2 and Lemma 3.4, we can also see that $BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+) = BMO(\mathbb{R}^n_+)$ where $BMO(\mathbb{R}^n_+) := \{r_{\mathbb{R}^n_+}f \,|\, f \in BMO(\mathbb{R}^n)\}$. As a result, we can clarify the relationship between various function spaces in this paper as follow:*

$$\begin{array}{ccccc} BMO(\mathbb{R}^n_+) & = & BMO^{\infty,\infty}_{ba}(\mathbb{R}^n_+) & =^* & \mathscr{H}^1_{even}(\mathbb{R}^n_+) \\ & & \cup & & \cap \\ & & BMO^{\infty,\infty}_{b}(\mathbb{R}^n_+) & =^* & \mathscr{H}^1_{odd}(\mathbb{R}^n_+) = \mathscr{H}^1(\mathbb{R}^n_+) \\ & & \| & & \| \\ & & BMO_M(\mathbb{R}^n_+) & =^* & \mathscr{H}^1_M(\mathbb{R}^n_+). \end{array}$$

Here $A =^* B$ means that A is the dual space of B.

6 Dual operator of the Helmholtz projection

6.1 Dual operators of E_{odd} and $r_{\mathbb{R}^n_+}$

In this subsection, for simplicity, we shall denote the odd extension operator E_{odd} by E. Since $E : \mathscr{H}^1_{odd}(\mathbb{R}^n_+) \to E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, we have that $E^* : E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)^* \to \mathscr{H}^1_{odd}(\mathbb{R}^n_+)^*$. By the theories in section 5 we have that $E^* : EBMO^{\infty,\infty}_b(\mathbb{R}^n_+) \to BMO^{\infty,\infty}_b(\mathbb{R}^n_+)$.

Lemma 6.1. *The dual operator of E is indeed $2 \cdot r_{\mathbb{R}^n_+}$, i.e. $E^* = 2 \cdot r_{\mathbb{R}^n_+}$.*

Proof. Let $f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$ and $g \in BMO^{\infty,\infty}_b(\mathbb{R}^n_+)$, by the definition of dual operator, we can deduce that

$$< E^*Eg, f > := < Eg, Ef > = 2 < g, f > .$$

Therefore, we have that

$$< E^*Eg - 2g, f > = 0 \text{ for all } f \in \mathscr{H}^1_{odd}(\mathbb{R}^n_+).$$

Let $B_r(0)$ be the ball centered at 0 with radius r and $B^+_r(0) := B_r(0) \cap \mathbb{R}^n_+$. For simplicity, we denote $B^+_r(0)$ by B^+_r. Notice that from the previous chapter, we see that $L^2(B^+_r) \subset \mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. Hence fix $r > 0$, we have that

$$< E^*Eg - 2g, f > = 0 \text{ for all } f \in L^2(B^+_r).$$

Since $C^\infty_0(B^+_r) \subset L^2(B^+_r)$, by the fundamental lemma of variational calculus, we see that

$$E^*Eg - 2g = 0 \quad \text{a.e. in } B^+_r.$$

This means $E^* = 2 \cdot r_{\mathbb{R}^n_+}$ and we are done. □

By similar arguments as above, we can also deduce that ${r_{\mathbb{R}^n_+}}^* : BMO^{\infty,\infty}_b(\mathbb{R}^n_+) \to EBMO^{\infty,\infty}_b(\mathbb{R}^n_+)$ and the dual operator of $r_{\mathbb{R}^n_+}$, where $r_{\mathbb{R}^n_+}$ corresponds to the restriction of $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$, is indeed $\frac{1}{2} \cdot E$.

6.2 Dual operators of E_{even} and $r_{\mathbb{R}^n_+}$

We denote the even extension operator E_{even} by E. By similar arguments as in the previous subsection, we have that the dual operator of E is indeed $2 \cdot r_{\mathbb{R}^n_+}$ and the dual operator of $r_{\mathbb{R}^n_+}$, which corresponds to the restriction of $E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$, is indeed $\frac{1}{2} \cdot E$.

6.3 Proof of Theorem 1.4

Proof. Since $\mathbb{P}_{\mathbb{R}^n_+}$ is a bounded linear operator from $\mathbf{Y}$ to $\mathbf{Y}$ and $\mathbf{X}$ is the dual space of $\mathbf{Y}$, we have that

$${\mathbb{P}_{\mathbb{R}^n_+}}^* : \mathbf{X} \to \mathbf{X}.$$

Then let $\mathbf{v} \in \mathbf{X}$ and $\mathbf{u} \in \mathbf{Y}$, we have that

$$< \mathbb{P}_{\mathbb{R}^n_+}{}^* \mathbf{v}, \mathbf{u} > = \sum_{i=1}^{n-1} < v^i, r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{u})^i > + < v^n, r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{u})^n > .$$

Notice that $(\mathbb{P}E\mathbf{u})^i$ is even with respect to x_n for $1 \leq i \leq n-1$ and $(\mathbb{P}E\mathbf{u})^n$ is odd with respect to x_n. Hence for $1 \leq i \leq n-1$, the $r_{\mathbb{R}^n_+}$ in $r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{u})^i$ corresponds to the restriction of $E\mathscr{H}^1_{even}(\mathbb{R}^n_+)$ whereas for $i = n$, the $r_{\mathbb{R}^n_+}$ in $r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{u})^n$ corresponds to the restriction of $E\mathscr{H}^1_{odd}(\mathbb{R}^n_+)$. Therefore,

$$< \mathbb{P}_{\mathbb{R}^n_+}{}^* \mathbf{v}, \mathbf{u} > = \frac{1}{2} < E\mathbf{v}, \mathbb{P}E\mathbf{u} > \cdots\cdots (*).$$

By [8], we see that the dual operator of $\mathbb{P} : \mathscr{H}^1(\mathbb{R}^n) \to \mathscr{H}^1(\mathbb{R}^n)$ is itself as a map from BMO to BMO. Therefore

$$\begin{aligned}
(*) &= \frac{1}{2} < \mathbb{P}E\mathbf{v}, E\mathbf{u} > \\
&= \frac{1}{2}\Big(\sum_{i=1}^{n-1} < (\mathbb{P}E\mathbf{v})^i, E_{even}u^i > + < (\mathbb{P}E\mathbf{v})^n, E_{odd}u^n > \Big) \\
&= \frac{1}{2}\Big(\sum_{i=1}^{n-1} < 2r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{v})^i, u^i > + < 2r_{\mathbb{R}^n_+}(\mathbb{P}E\mathbf{v})^n, u^n > \Big) \\
&= < \mathbb{P}_{\mathbb{R}^n_+}\mathbf{v}, \mathbf{u} > .
\end{aligned}$$

□

Remark 6.2. *When we are considering the dual operator of $\mathbb{P}_{\mathbb{R}^n_+}$, notice that the space* $\mathbf{X}$ *must be viewed as* $\mathbf{X}/(\mathbb{R}^{n-1} \times \{0\})$*!*

6.4 Proof of Corollary 1.5

Proof. By [2, Th 2.19] and Theorem 1.4 in this paper, we are done. □

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