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OPTIMIZATION OF USABLE LEFTOVER CUTTING STOCK PROBLEMS USING FUZZY APPROACH

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Abstract. The objectives of this paper are to analyze different linear optimization models for the One-dimensional Cutting Stock Problem and, by using a fuzzy classification approach, determine the reutilization of the leftovers from the cutting process. The optimal solutions of the proposed linear models are obtained from Simplex Method. The fuzzy classification system is a collaborative decision-making tool, which analyzes uncertain parameters in the manufacturing process in order to determine, according to the given objective, the most appropriate cutting pattern, also classifying the results provided by different linear optimization models. The comparison of the results allows to infer the most appropriate model to use according to the specifications of the problem to be solved. Results are obtained from a case study, in a packaging company, located in the state of Paraná, Brazil, which aims to select the best cutting pattern for two different scenarios: one with concentration of leftovers on standardized objects, and the other on non-standardized objects.

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1 Introduction

In industries, it is common to research for ways to reduce the disposal of raw materials because, in addition to the economic context, there is also the ecological character, since the reduction of waste results in a decrease of the volume of material loss generated by companies. In this sense, the optimization of cutting problems contributes to the reduction of raw material losses during the cutting process for manufacturing items, constituting an important area of Operational Research studies and assisting in decision making process [5].

From the knowledge of the problem to be solved, describing the real problem as a solvable mathematical problem is required. For this reason, there are different methods of mathematical modeling. One of them is the Linear Programming, which proposes to solve linear optimization problems that, in turn, aims to mathematically describe problems aimed to maximize or minimize a given objective function, subject to a set of constraints, whose variables are linearly related [17].

Among the problems that can be solved by Linear Programming, one may highlight cutting stock problems, which consist of cutting a set of objects available in stock to produce the desired items in specified quantities and sizes, in order to optimize an objective function. The cutting problems can be classified into one-dimensional and two-dimensional. The One-dimensional Cutting Problem involves only one of the relevant dimensions in the cutting process, as occurs with cutting steel bars with the same cross section, paper rolls and tubes. In the Two-dimensional Cutting Problem, two dimensions (length and width) are relevant in the cutting process. In addition, if thickness is considered, there are also the Three-dimensional Cutting Problems [3, 5].

One branch of the Cutting Stock Problem is called the *Usable Leftover Cutting Stock Problems* [9], whose purpose is to determine the cutting pattern and to analyze the leftovers generated by the cutting process. If a leftover generated is large enough, it can be considered a *retail* and then can be returned to stock for future cut plans; if it does not meet this criterion, it is discarded and considered as *loss* or *waste*.

Since this class of problem addresses a very common issue in industrial and manufacturing processes, some resolution algorithms are developed in the literature [1, 7, 13, 14]. However, some mathematical models present better performance than others depending on the decision to be made. Therefore, it is necessary to establish a way of classifying the solutions provided by these models, based on the determination of an ideal solution.

For the One-dimensional Cutting Stock Problems, [5] developed heuristic procedures that consider the usable leftovers and preserve, as main objective, the minimization of the waste. For heuristic procedures that prioritize the cut of retails of the stock, beside the instances from the literature, the author simulated a situation in multiple periods in that Cutting Stock Problems are solved in successive periods. Due to the difficulty to analyze the heuristic procedures developed for the Cutting Stock Problem with Usable Leftover, [5] also present a fuzzy strategy to aid the analysis of the obtained solutions. The computational results were obtained considering the developed heuristic procedures to the One-dimensional Cutting Stock Problem with Usable Leftover and randomly generated instances.

The One-dimensional Cutting Stock Problem with Usable Leftovers is a problem frequently encountered in practical settings, but often it is not dealt with in an explicit way [7]. In this context, the goal of this paper is to develop a fuzzy system for classification of leftovers from the cutting process of raw material, based on the study developed by [5]. For this, a case study is carried out in a factory that performs one-dimensional packaging cutting, located in the interior of the state of Paraná. The fuzzy classification system proposed in this paper considers the numerical solution provided by Simplex Method [3] which is applied to three distinct linear mathematical models and returns the output classification for two distinct scenarios. The first scenario intends to select the solution that contains large concentration of leftovers on the standardized objects, while the second scenario intends to choose the solution that concentrates the leftovers on non-standardized objects. The fuzzy approach is presented in order to facilitate the analysis of the obtained solutions, since there are uncertainties in the classification process and the leftover must be classified using linguistic variable to facilitate the use of the proposed system by a non-expert user.

A motivation of the proposed methodology is to develop a framework to decide which mathematical model, among a set of predefined structures, yields the best optimal result. This choice is not absolute and depends on the industrial context, such as the maximum stock capacity and production rate. In this regard, after choosing the optimization model to be employed, there would be no need to apply the proposed technique again, unless there are changes on the industrial scenario, resulting on another model being best suited.

2 Related Works

As defined in [10], the Cutting Stock Problem consists in filling an order at minimum cost for specified numbers of lengths of material to be cut from given stock lengths of given cost. In such paper, one of the first to address the cutting problem, a technique is described for overcoming the difficulty in the linear programming formulation of the problem, which involves a large number of variables. The technique enables one to compute with a matrix which has no more columns than rows. Later, authors extended and adapted the methods for stock cutting outlined to the specific full-scale paper trim problem [11]. In this last paper, the experiments evaluate speed-up devices and explore a connection with integer programming. Other experiments define waste as a function of stock length, examine the effect of multiple stock lengths on waste, and the effect of a cutting knife limitation.

Based on [11], a new approach to the One-dimensional Cutting Stock Problem is described in [9] and is compared to the classical model for which Gilmore and Gomory developed the column-generation technique. This model, characterized by a dynamic use of simply structured cutting patterns, enables the representation of complex combinations of cuts.

A sequential heuristic procedure for optimization of roll cutting in the clothing industry is developed by [12]. The issue of roll cutting is defined as a bicriterial multidimensional knapsack problem, and a lexicographic approach is proposed. Later, authors applied the sequential heuristic for optimizing one-dimensional stock cutting when all stock lengths are different [13]. A combined method - sequential heuristic procedure and branch-and-bound - for the solution to the general One-dimensional Cutting Stock Problem is then proposed in [14].

Many production environments require economical cutting of one-dimensional items according to bills of materials from objects of several standard lengths; however, substantial trim loss may occur [16]. Typically, the Cutting Stock Problems aim for the minimization of the waste, however [4] considered that if a unused piece is "sufficiently large" to be reused in the future, it should not be considered as waste. Some desirable characteristics are then defined and modifications in classic heuristic methods are proposed, so that the cutting patterns with undesirable waste are modified. Later, [7] reviewed published studies that consider the solution of the One-dimensional Cutting Stock Problem with the possibility of using leftovers to meet future demands, if long enough. In this context, a case in which there are several stock lengths available in limited quantities is carried out in [19], in which authors focused on low demand problems. Some heuristic methods are proposed in order to obtain an integer solution and compared with others. The proposed methods presented very small objective function value gaps.

A particular One-dimensional Cutting Stock with both cutting and reuse decision variables was presented in [2]. The problem was formulated in terms of integer linear programming and then it was efficiently solved by applying standard packages within a column generation technique. A significant improvement of performance was obtained for both economic savings and product quality.

Also considering One-dimensional Cutting Problems, in [1] the One-dimensional Cutting Problem of metallic structural tubes, used in the manufacturing of agricultural light aircrafts, is developed. The problem aims to minimize material trim losses and considers the possibility of generating leftovers with enough size to reuse. The computational results, obtained from a Brazilian aeronautical company, show that the models are useful in supporting decisions in this cutting process. More recent, a heuristic algorithm for the One-dimensional Cutting Stock problem with usable leftover is also presented in [8]. The algorithm can balance the cost of the consumed bars, the profit from leftovers and the profit from shorter stocks reduction.

A case study from the wood processing industry is presented in [15]. Authors focused on a cutting process in which material from stock is cut down in order to provide the items required by the customers. In order to reduce the cost of the cutting process, a decision support tool that incorporated an integer linear programming model as a central feature was developed.

Finally, [20] is one of the most recent papers to address the use of leftovers. Authors propose to obtain integer solutions for the Usable Leftover Cutting Stock Problems. For this, a computational study of the mathematical models of the literature is described. The effectiveness of the models is analyzed by comparing the results with the results from the heuristics proposed by [5]. Computer simulations suggested that it is possible to propose a variation of the models, finding solutions better than those already found, which is also an objective to be achieved.

Analyzing the literature, the main contributions of this paper to the literature are: (i) to evaluate the performance of mathematical models presented in [1], [13] e [14], by proposing adaptations for the case study considered in this paper; (ii) to develop a fuzzy classification system for the use of leftover from the cutting process; (iii) to assist in the decision making process; (iv) to provide waste reduction of raw material and, consequently, reduce the environmental impact.

3 Case Study

In this paper, a case study is developed from data collected in a paper packaging and labeling industry located in the interior of the state of Paraná, Brazil. In this place, one-dimensional cutting of paper rolls and bobbins is performed to make labels and packages, which are then distributed to other institutions. Table 1 describes the demand which must be met.

Table 1: Demand Specifications.

Item	**Length [m]**	**Demand [un]**
1	0.07	50
2	0.09	100
3	0.10	30
4	0.12	40

As defined in [5], the called *standardized objects* refer to the raw material of the stock that has not been used in previous cutting patterns and *non-standardized objects* refer to the raw material that has already been used in other cutting pattern and returned to stock. To meet the demand presented in Table 1, the stock of the industry in study contains two types of paper rolls: one roll measures 30m (standardized objects) and the other measures 20m (non-standardized objects). In addition, *retail* and *loss* are also defined. If a leftover can be returned to stock for future cut plans, it is considered a retail; if it does not meet this criterion and can be discarded, it is considered a loss.

Using the data of the problem to be optimized, describing the real problem as a solvable mathematical problem is required. From the presented problem, the mathematical formulation of the one-dimensional cutting problem is required, using Linear Programming, by considering the use of leftovers. For this case, a leftover can then be used (retail) if its size is larger than 0.5m. The mathematical models are described in the next section, as well as the necessary adaptations to the case study. Then, a *Fuzzy System* is developed in order to classify the optimal solutions obtained. The application is performed considering two distinct scenarios. The first intends to indicate which of the selected models prioritizes the accumulation of leftovers on standardized objects, and the second defines which of the models provided a solution that concentrates leftovers on non-standardized objects.

4 Mathematical Models

The problem under study can be modeled using linear programming, since it can be described initially as a One-dimensional Stock Cutting Problem. For the choice of these linear models, it was considered that all models must have the same objective function and the constraints should represent the situations in which the available stock must meet the demand. In order to facilitate the understanding of the models, the notations used in mathematical formulations are also described as follows [5].

- Indexes
  - i: Refers to item;
  - j: Refers to cut pattern;
  - k: Refers to object.
- Parameters
  - Y: Maximum number of item variations that can be cut from an object;
  - L_k : Length of the object type k;
  - l_i : Length of the item type i;
  - d_i : Demand of the item type i;
  - m: Number of different types of items ;
  - δ : Minimum length for an leftover be considered a retail;
  - N: Number of possible cut patterns for the cut object;
  - M: Large enough number;
  - K: Number of types of object in stock.
- Variables
  - p_{ik}: Number of items type i cut using object type k;
  - s_k: Leftover in the object k, that is $s_k = L_k - \sum_{i=1}^{m} l_i p_{ik}$;
  - z_k: Indicates whether the object k is used in cutting pattern, being:

$$z_k = \begin{cases} 0, & \text{if } p_{ik} = 0, i = 1, ..., m; \\ 1, & \text{otherwise} \end{cases} \tag{1}$$

  - y_{ik}: Indicates whether the item i is cut in the object k, being:

$$y_{ik} = \begin{cases} 0, & \text{if } p_{ik} = 0; \\ 1, & \text{otherwise} \end{cases} \tag{2}$$

  - t_k: Indicates the length of the leftover in the object k, being:

$$t_k = \begin{cases} s_k, & \text{if } z_k = 1 \text{ and } s_k \leq \delta; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

  - u_k: Indicates whether the leftover in the object k is a retail, being

$$u_k = \begin{cases} 1, & \text{if } z_k = 1 \text{ and } s_k \geq \max l_i; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

– y_k: Indicates whether object k is used in the cutting pattern, being:

$$y_k = \begin{cases} 0, & \text{if object } k \text{ is used in the cutting pattern;} \\ 1, & \text{otherwise} \end{cases} \quad (5)$$

4.1 Original Mathematical Models

In this section, the mathematical models of the literature selected for the development of this work are described. These mathematical models describe the one-dimensional cutting problems with usable leftovers that present the same objective function and are the most recent models in the literature. Therefore, considering the item-oriented approach, the models in which all items to be cut are treated explicitly, so that the way an object is cut is known with the solution of the problem, are selected and described below. The fundamental variables are the number of items of a particular type in a particular object, or type of object.

- Model 1:

  The model developed by [1] arose from a study in a Brazilian company that cuts structural pipes in its production line for the production of agricultural aircraft. For the mathematical modeling of the problem, it is assumed that the stock has objects of different sizes and in sufficient quantity to meet the demand. It is important to mention that, in the objects of the stock, the retails from previous cutting pattern are considered.

  In the authors study, two models were performed: one based on [12], making it a mixed integer problem, and another based on a simplification of the first model.

  For the present paper, the simplified model is selected for problem representation, which is presented as follows.

$$\min \sum_{k=1}^{K} t_k \quad (6)$$

subject to:

$$\sum_{i=1}^{m} l_i p_{ik} \leq L_k, \quad k = 1, ..., K \quad (7)$$

$$\sum_{k=1}^{K} p_{ik} = d_i, \quad i = 1, ..., m \quad (8)$$

$$N u_k \leq L_k z_k - \sum_{i=1}^{m} l_i p_{ik}, \quad k = 1, ..., K \quad (9)$$

$$L_k z_k + \sum_{i=1}^{m} l_i p_{ik} \leq t_k + u_k M, \quad k = 1, ..., K \tag{10}$$

$$\sum_{k=1}^{K} u_k \leq 1, \quad k = 1, ..., K \tag{11}$$

$$p_{ik} \geq 0 \, and \, integer, t_k, \geq 0, \quad i = 1, ..., m, \;\; k = 1, ..., K \tag{12}$$

$$z_k \in \{0, 1\}, u_k \in \{0, 1\}, \quad k = 1, ..., K \tag{13}$$

- Model 2:

  A heuristic method that solves the One-dimensional Stock Cutting Problem is developed by [13]. The goal is to minimize the loss of stock objects by assuming that all objects can have different lengths. In this study, two models are considered, one for each situation, which are described below.

  - Case 1: Stock meets demand
  - Case 2: Stock does not meet demand

  Since the case study of this paper presents enough objects in stock to cut the requested parts, the model formulated in Case 1 is selected. The mathematical formulation is presented as follows, according to [13].

$$\min \sum_{k=1}^{K} t_k \tag{14}$$

subject to:

$$\sum_{i=1}^{m} l_i p_{ik} + s_k = L_k, \quad k = 1, ..., K \tag{15}$$

$$\sum_{k=1}^{K} p_{ik} = d_i, \quad i = 1, ..., m \tag{16}$$

$$\sum_{k=1}^{K} u_k \leq 1 \tag{17}$$

$$\sum_{i=1}^{m} y_{ik} \leq Y \leq m, \quad k = 1, ..., K \tag{18}$$

$$p_{ik} \geq 0 \; and \; integer, \quad i = 1, ..., m, \;\; k = 1, ..., K \tag{19}$$

$$s_k \geq 0, t_k \geq 0, \quad k = 1, ..., K \tag{20}$$

$$u_k \in \{0, 1\}, \quad k = 1, ..., K \tag{21}$$

$$y_{ik} \in \{0, 1\}, \quad i = 1, ..., m, \;\; k = 1, ..., K \tag{22}$$

- Model 3:

  A characteristic of the problem described by Model 3 is that all objects that can be considered as retail will not be counted as losses, since the objective is to reduce losses in the cutting process.

  According to [14], the model joins existing methods that solve the One-dimensional Stock Cutting Problem, as well as a combination of exact methods and heuristic procedures. In order to make this study the closest to reality, the authors considered two situations that can happen in industry and then developed two different mathematical models, one for each of the two cases presented in Model 2. The difference is that, in case 2, it was considered that the distribution of items to be cut is not relevant.

  Following the same criteria of choice, the mathematical model of Case 1 was selected, which is presented as follows.

$$\min \sum_{k=1}^{K} t_k \tag{23}$$

subject to:

$$\sum_{i=1}^{m} l_i p_{ik} + s_k = L_k(1 - y_k), \quad k = 1, ..., K \tag{24}$$

$$\sum_{k=1}^{K} p_{ik} = d_i, \quad i = 1, ..., m \tag{25}$$

$$s_k - \delta u_k \geq 0, \quad k = 1, ..., K \tag{26}$$

$$\sum_{k=1}^{K} u_k \leq 1 \tag{27}$$

$$p_{ik} \geq 0 \, and \, integer, \quad i = 1, ..., m, \;\; k = 1, ..., K \tag{28}$$

$$s_k \geq 0, t_k \geq 0, \quad k = 1, ..., K \tag{29}$$

$$u_k \in \{0, 1\}, y_k \in \{0, 1\}, \quad k = 1, ..., K \tag{30}$$

4.2 Adapted Mathematical Models

The mathematical models presented in the previous section were adapted to more accurately represent the problem addressed in this paper. Thus, there were two common changes in the models, which are described below:

Restrictions limiting the number of retails have been removed: the minimum leftover size to be considered retail is much smaller than the size of the objects to be cut. Therefore, the probability of the leftover return to the stock is very high, which makes the use of this restriction unfeasible. Moreover, for the purposes of this application, there is no need to limit the number of retails generated.

Variable Substitution: The indicator variable t_k was replaced by the real variable s_k, directly indicating the size of the leftovers.

The models are described below, with their justified adaptations.

- Model 1:
  In addition to removing the restriction (11), the existence of the variable u_k was disregarded, which caused changes in the restrictions (9) and (10). This change caused modification in constraint (7) so that it was possible to calculate the leftovers in the model. Also, the meaning of variable z_k has been changed to:

$$z_k = \begin{cases} 1, & \text{if } p_{ik} = 0, i = 1, ..., m; \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

  The adapted model is described below.

$$\min \sum_{k=1}^{K} s_k \tag{32}$$

  subject to:

$$\sum_{i=1}^{m} l_i p_{ik} + s_k = L_k, \quad k = 1, ..., K \tag{33}$$

$$\sum_{k=1}^{K} p_{ik} = d_i, \quad i = 1, ..., m \tag{34}$$

$$0 \leq L_k z_k - \sum_{i=1}^{m} l_i p_{ik}, \quad k = 1, ..., K \tag{35}$$

$$L_k z_k + \sum_{i=1}^{m} l_i p_{ik} \leq s_k, \quad k = 1, ..., K \tag{36}$$

$$p_{ik} \geq 0 \; and \; integer, s_k, \geq 0, \quad i = 1, ..., m, \;\; k = 1, ..., K \tag{37}$$

$$z_k \in \{0, 1\}, \quad k = 1, ..., K \tag{38}$$

- Model 2:
  In addition to the changes made in all models, constraint (18) was removed, since the binary variable y_{ik} is unrelated to other constraints or variables, indicating that, in order to obtain the value of this variable, a heuristic method must be implemented. The adapted model is described as follows.

$$\min \sum_{k=1}^{K} s_k \tag{39}$$

subject to:

$$\sum_{i=1}^{m} l_i p_{ik} + s_k = L_k, \quad k = 1, ..., K \tag{40}$$

$$\sum_{k=1}^{K} p_{ik} = d_i, \quad i = 1, ..., m \tag{41}$$

$$p_{ik} \geq 0 \; and \; integer, \quad i = 1, ..., m, \;\; k = 1, ..., K \tag{42}$$

$$s_k \geq 0, \quad k = 1, ..., K \tag{43}$$

- Model 3:
  This model had only the common changes of all models. The adapted model is presented below.

$$\min \sum_{k=1}^{K} s_k \tag{44}$$

subject to:

$$\sum_{i=1}^{m} l_i p_{ik} + s_k = L_k(1 - y_k), \quad k = 1, ..., K \tag{45}$$

$$\sum_{k=1}^{K} p_{ik} = d_i, \quad i = 1, ..., m \tag{46}$$

$$s_k - \delta u_k \geq 0, \quad k = 1, ..., K \tag{47}$$

$$p_{ik} \geq 0 \; and \; integer, \quad i = 1, ..., m, \;\; k = 1, ..., K \tag{48}$$

$$s_k \geq 0 \quad k = 1, ..., K \tag{49}$$

$$u_k \in \{0, 1\}, y_k \in \{0, 1\}, \quad k = 1, ..., K \tag{50}$$

4.3 Simplex Method Application Results

Considering properly modified models, the Simplex method [3] is employed to obtain the optimal solutions. For this task, the software LINDO (www.lindo.com) is used, which has the ability to solve linear optimization problems. Using this tool, optimal solutions are then obtained. It is worth mentioning that all models provided the same numerical value of the objective function: 29.7 m. In Tables 2, 3 and 4 are presented, respectively, the optimal solutions obtained from Models 1, 2 and 3.

From the results provided by the Simplex method in the described models, the Fuzzy Inference process, described in the following section, begins.

Table 2: Optimal solution obtained from Model 1

Variable	**Numerical value**
p_{11}	0
p_{21}	100
p_{31}	30
p_{41}	0
p_{12}	50
p_{22}	0
p_{32}	0
p_{42}	40
z_1	0
z_2	0
s_1	18
s_2	11.7

Table 3: Optimal solution obtained from Model 2

Variable	**Numerical value**
p_{11}	0
p_{21}	100
p_{31}	30
p_{41}	40
p_{12}	50
p_{22}	0
p_{32}	0
p_{42}	0
s_1	13.2
s_2	16.5

5 Fuzzy Classification System

Fuzzy logic [18] aims to analyze data that have inaccurately defined limits, opening the dichotomous concept, so that the classification values range from 0 to 1, providing a more realistic interpretation of the information. Thus, association values indicate how well an object is compatible with a given characteristic [18]. In the case study presented in this paper, a fuzzy approach is presented to facilitate the analysis of the obtained solutions. The fuzzy system is used in order to classify the solutions of one-dimensional cutting problems with usable leftovers, since there are uncertainties in the classification process and, using linguistic variables, the use of the proposed system can be made by a non-expert user. Then, the Fuzzy classifier aims to represent the uncertainties present in the packing company reality for two distinct scenarios:

- Scenario 1: The company wants the leftovers to be concentrated on standardized objects;

Table 4: Optimal solution obtained from Model 3

Variable	**Numerical value**
p_{11}	50
p_{21}	100
p_{31}	30
p_{41}	40
p_{12}	50
p_{22}	0
p_{32}	0
p_{42}	0
y_1	0
y_2	1
u_1	1
u_2	0
s_2	9.7
s_2	20

- Scenario 2: The company wants the leftovers to be concentrated on non-standardized objects.

According to [18], the inference process involves five steps, which are explained below. The proposed methods and parameters used in each step are also described, defining the proposed Fuzzy Classification System for the problem addressed in this paper.

5.1 Input Fuzzification

In the fuzzification process, the degree to which the input variables (or attributes) belong to each of the appropriate fuzzy sets must be determined, using the called *membership functions*. Input is always a numerical value limited to the discourse universe of the variable and the output must be a diffuse degree of association (the range between 0 and 1).

For the two scenarios analyzed, the following entries are determined:

- Input 1: Percentage of retail generated from standardized objects, obtained by Equation (51):

$$Input1 = \frac{retail\ length\ from\ standardized\ objects}{total\ length\ of\ leftover} \tag{51}$$

- Input 2: Percentage of retail generated from non-standardized objects, obtained by Equation (52):

$$Input2 = \frac{retail\ length\ from\ non-standardized\ objects}{total\ length\ of\ leftover} \tag{52}$$

- Input 3: Percentage of leftover loss, obtained by Equation (53):

$$Input3 = \frac{losses\ length}{total\ length\ of\ leftover} \tag{53}$$

These input variables determine how much of the leftovers have been lost and whether the retails are concentrated more on standardized or nonstandard objects. For each scenario, inputs are discretized at the linguistic ranges described in Table 5.

Table 5: Linguistic Ranges of Fuzzified Inputs for Scenario 1

Input	**Linguistic Ranges**		
Input 1	Small(S) [0.0 0.1]	Large(L) [0.15 0.64]	Too Large(TL) [0.6 1.0]
Input 2	Small(S) [0.0 0.2]	Large(L) [0.20 0.64]	Too Large(TL) [0.6 1.0]
Input 3	Small(S) [0.0 0.35]	Large(L) [0.30 0.65]	Too Large(TL) [0.6 1.0]

Based on empirical tests, the shape of the membership functions for Scenario 1 that best fit the problem can be seen in Figures 1, 2 and 3. The membership functions are performed by considering the ranges described in Table 5, the points where the function has peaks (triangular shape) or stability intervals (trapezoidal shape). By this way, the membership functions are created in two stages. In the first stage, the numeric ranges for each linguistic variable are defined, based on the analysis of the packaging industry data. For each scenario different analyzes are performed. In the second stage, the shape of the membership functions is performed applying empirical tests and the best results are selected.

Figure 1: Input 1 Membership Functions for Scenario 1

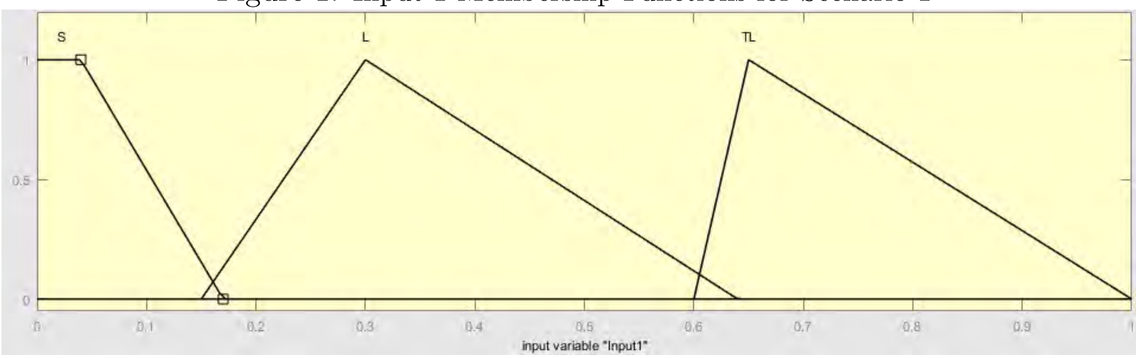


Table 6: Linguistic Ranges of Fuzzified Inputs for Scenario 2

Input	**Linguistic Ranges**		
Input 1	Small(S) [0.0 0.25]	Large(L) [0.20 0.64]	Too Large(TL) [0.6 1.0]
Input 2	Small(S) [0.0 0.17]	Large(L) [0.15 0.64]	Too Large(TL) [0.6 1.0]
Input 3	Small(S) [0.0 0.35]	Large(L) [0.30 0.65]	Too Large(TL) [0.6 1.0]

Figure 2: Input 2 Membership Functions for Scenario 1

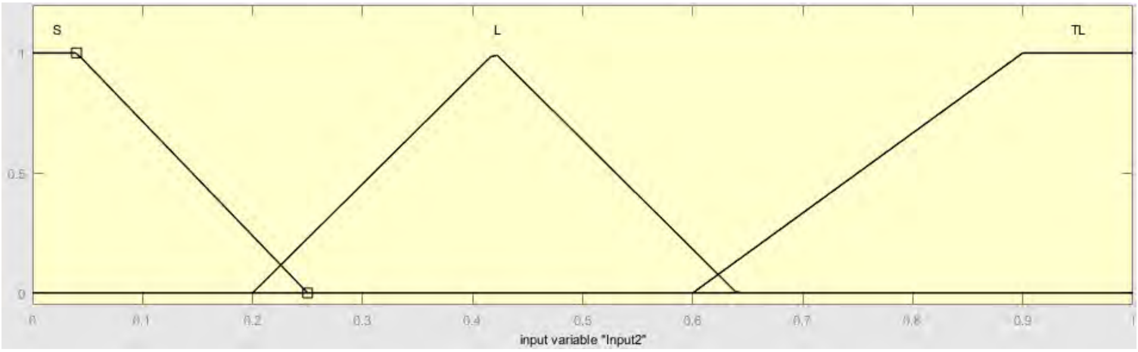
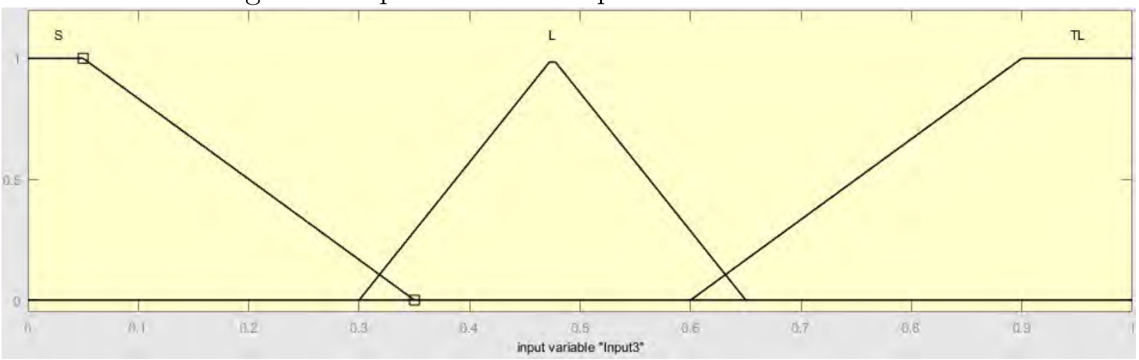


Figure 3: Input 3 Membership Functions for Scenario 1



As made to Scenario 1, the membership functions for the input of Scenario 2 are also empirically tested to define the shape of the functions that best represent the problem studied. The membership functions are performed by considering the ranges described in Table 6, the points where the function has peaks (triangular shape) or stability intervals (trapezoidal shape). Figures 4, 5 and 6 show the membership functions for the inputs of Scenario 2.

Figure 4: Input 1 Membership Functions for Scenario 2

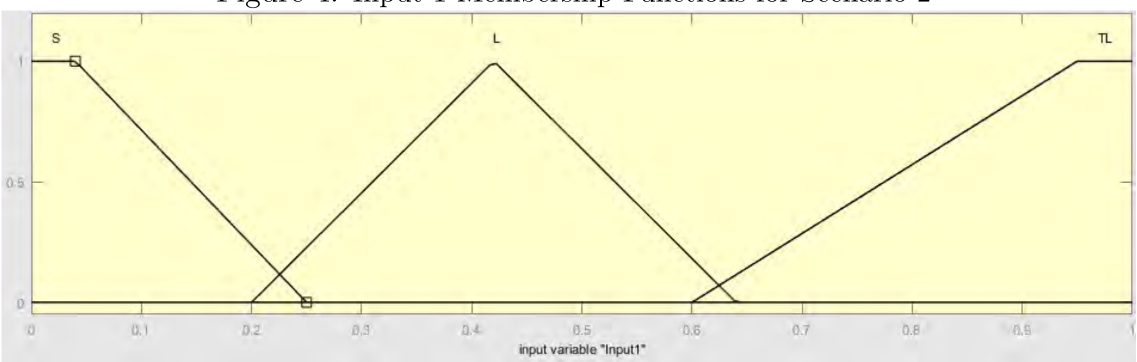


Figure 5: Input 2 Membership Functions for Scenario 2

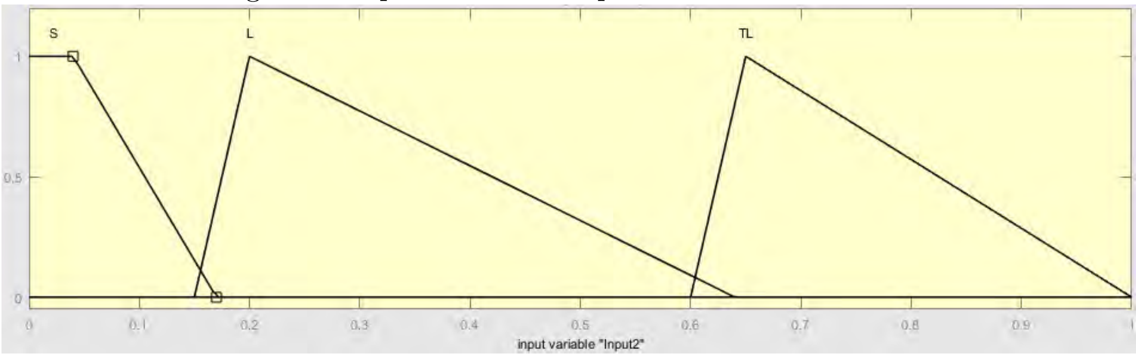
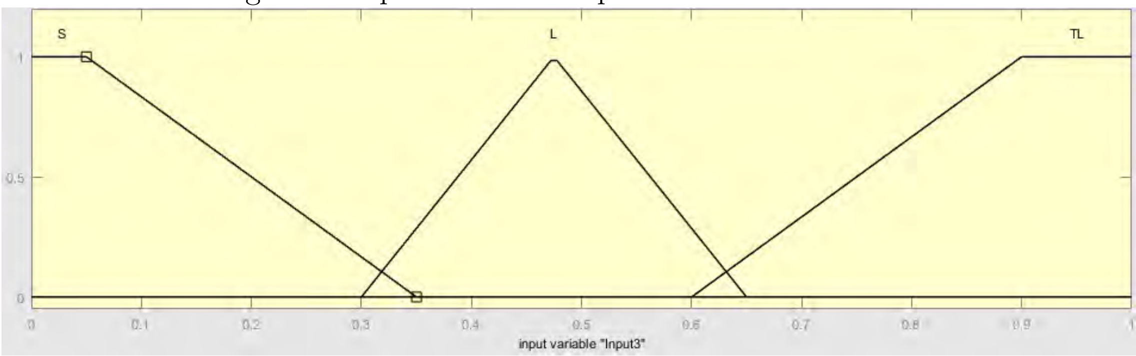


Figure 6: Input 3 Membership Functions for Scenario 2



5.2 Fuzzy Operator

If the antecedent of a given linguistic rule has more than one condition, a fuzzy operator must be used to obtain a number that represents the result of that rule's antecedent. This number will then be applied to the output function. The input to the fuzzy operator is two or more membership values from fuzzified input variables. The output is a single truth value. Basically, AND (minimum) and OR (maximum) methods.

For each scenario, different fuzzy rules are established, as the objective differs for each situation. Since the definition of these rules is related to user preferences, it is important to emphasize that as the purpose of fuzzy inference changes, the rules also change. Tables 7 and 8 describe, respectively, the fuzzy rule base for Scenarios 1 and 2. For example, according to Table 7, in Scenario 1, If the retail percentage of standardized objects (Input 1) is "too large", the retail percentage of non-standard objects (Input 2) is "small" and the generated loss (Input 3) is "small", then the solution provided by the mathematical model is a "Ideal Solution". In Table 8, considering the same inputs, the output shows that the model provides a "Undesirable Solution," according to Scenario 2.

5.3 Implication Method

It is defined as the shaping of the consequent (a fuzzy set) based on the antecedent (a single number). The input for the implication process is a single number given by the antecedent, and the output is a fuzzy set. Implication occurs for each rule.

Table 7: Fuzzy Rule Base for Scenario 1

	Input 3		
Input 1/ Input 2.	**Small**	**Large**	**Too Large**
Small/Small	Acceptable	Undesirable	Undesirable
Small/Large	Acceptable	Undesirable	Undesirable
Small/Too Large	Undesirable	Undesirable	Unacceptable
Large/Small	Desirable	Acceptable	Undesirable
Large/Large	Acceptable	Acceptable	Undesirable
Large/Too Large	Acceptable	Undesirable	Undesirable
Too Large/Small	Ideal	Desirable	Acceptable
Too Large/Large	Desirable	Acceptable	Undesirable
Too Large/Too Large	Acceptable	Acceptable	Undesirable

Table 8: Fuzzy Rule Base for Scenario 2

	Input 3		
Input 1/ Input 2	**Small**	**Large**	**Too Large**
Small/Small	Acceptable	Undesirable	Undesirable
Small/Large	Desirable	Acceptable	Undesirable
Small/Too Large	Ideal	Desirable	Acceptable
Large/Small	Acceptable	Acceptable	Undesirable
Large/Large	Acceptable	Undesirable	Undesirable
Large/Too Large	Desirable	Acceptable	Undesirable
Too Large/Small	Undesirable	Undesirable	Unacceptable
Too Large/Large	Undesirable	Undesirable	Undesirable
Too Large/Too Large	Undesirable	Undesirable	Undesirable

The implication method selected for this study is Mandami [18], which is based on the max-min inference composition rule. Each fuzzy rule is modeled by the minimum when conditioned by the logical operator “ and ”, or by the maximum when conditioned by the logical operator “ or ”.

5.4 Aggregation Method

It is the process by which the fuzzy sets that represent the outputs of each rule are combined into a single fuzzy set. Aggregation only occurs once for each output variable. The output of this process is one fuzzy set for each output variable. The method selected for the inference of this study is the maximum method [18].

5.5 Defuzzification

The input for the defuzzification process is a fuzzy set (the aggregate output fuzzy set) and the output is a single number. For the case study presented in this paper, the method it smallest of the maximums is selected for the defuzzification, since it presents the best result in relation to the centroid and bisector methods [18].

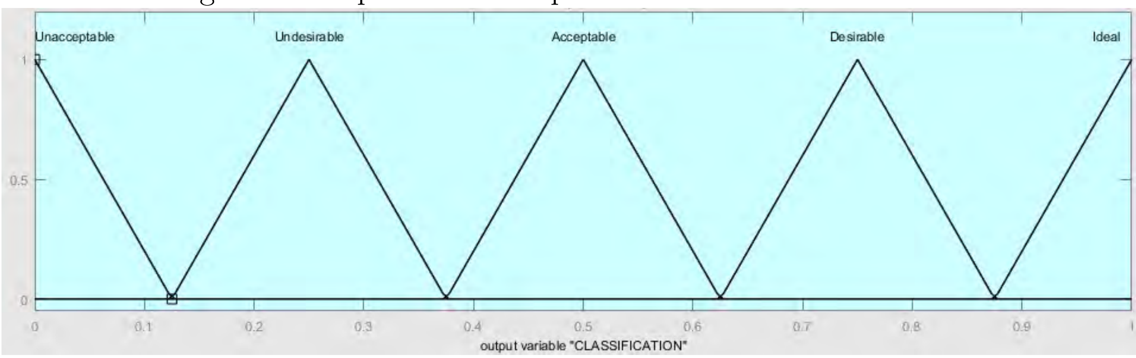
According to [5], there are 5 possible results of fuzzy inference outputs, which, for both scenarios, are represented by the linguistic ranges described in Table 9.

Table 9: Fuzzy Inference Outputs

Outputs	**Linguistic Ranges**
Ideal Solution	[0.875 1.000]
Desirable Solution	[0.625 0.875]
Acceptable Solution	[0.375 0.625]
Undesirable Solution	[0.125 0.375]
Unacceptable Solution	[0.000 0.125]

The output membership functions are illustrated in Figure 7.

Figure 7: Output Membership Functions for Scenarios 1 and 2



6 Results

After defining the methods used in the fuzzy inference process, described in Section 5, the inference results are then obtained for Scenarios 1 and 2 and are described in Sections 6.1 and 6.2. For both scenarios, the inputs are as described in Table 10 for each of the 3 mathematical models.

Table 10: Numerical inputs for each of the 3 mathematical models.

Model	**Input 1**	**Input 2**	**Input 3**
Model 1	0.606	0.394	0.000
Model 2	0.444	0.556	0.000
Model 3	0.327	0.673	0.000

The results of the proposed fuzzy inference for each scenario are presented below.

6.1 Results from Scenario 1

The inputs of each mathematical model are then entered into the proposed fuzzy inference system and the outputs shown in Table 11 are obtained.

Table 11: Fuzzy classification output for Scenario 1

Model	**Output**
Model 1	Desirable Solution (0.65)
Model 2	Acceptable Solution (0.43)
Model 3	Acceptable Solution (0.41)

Observing Table 11, we can conclude that Model 1 best corresponds to the objective of Scenario 1, since it presents a Desirable Solution, (but not an Ideal Solution). While models 2 and 3 presented an Acceptable Solution.

6.2 Results from Scenario 2

For Scenario 2, the same inputs used in Scenario 1 are entered. After performing the proposed fuzzy inference for each one of the three mathematical models, the results of the fuzzy classification are obtained for each model as described in Table 12.

Table 12: Fuzzy classification output for Scenario 2

Model	**Output**
Model 1	Acceptable Solution (0.40)
Model 2	Acceptable Solution (0.40)
Model 3	Desirable Solution (0.70)

Unlike the result presented in Scenario 1, the model that best corresponds to the objective of Scenario 2 is model 3, which presents a Desirable Solution (but not an Ideal Solution). Models 1 and 2 present the same membership degree to Acceptable Solution.

7 Conclusion

In this paper, linear optimization models for the One-dimensional Cutting Stock Problem are studied. The optimal solutions of the proposed linear models are obtained from Simplex Method. In addition, a fuzzy classification system is proposed in order to reuse the leftovers from the cutting process of raw material. The comparison of the results allows to infer the most appropriate model to use according to the objectives of the problem. A case study in a packaging company is developed, using the proposed methods.

From results provided by Simplex Method, a fuzzy inference process is proposed in order to classify the obtained optimal solutions considering the two scenarios described in this paper: one with concentration of leftovers on standardized objects, and the other on non-standardized objects. A fuzzy classification system is performed for each scenario, in order to choose the best optimization model, helping the user in the decision process.

Analyzing the results, we can conclude that it is not possible to determine the quality of the mathematical model absolutely, since the case study presented in this paper demonstrated that, according to the optimization objective, the model can present better or worse performance. While in Scenario 1, the most appropriate model to use is Model 1, for Scenario 2, the best model is Model 3. Therefore, for different optimization objectives, the most suitable model may not be the same.

In addition, it is important to emphasize that the fuzzy system proposed in this paper aims to make use of leftovers, since the problem studied is formulated mathematically as a One-dimensional Cutting Stock Problem with Usable Leftovers, However, using the same mathematical models, it is possible to make an inference with distinct input variables. For example, if the user wishes that the cutting pattern to perform the cut of equal items in the same roll of paper, it is necessary to use the variables that indicate where each item was cut off, not being necessary to use variables that inform the size of leftovers.

Thus, the fuzzy inference can be used in the optimization process in order to represent the uncertainties that may arise in the cutting process. The fuzzy is then used in order to classify the solutions of one-dimensional cutting problems with usable leftovers, since there are uncertainties in the classification process. Finally, it is noteworthy that the application of the fuzzy system described in this paper is not restricted to the Cutting Stock Problem with Usable Leftovers, and can be used in other optimization problems.

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