

SOME ITERATION AND PARSEVAL-GOLDSTEIN TYPE IDENTITIES WITH THEIR APPLICATIONS

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Abstract. In the present paper, identities involving the generalized Laplace transform, the Stieltjes transform, the generalized Stieltjes transform, the exponential transform and the upper incomplete gamma transform are given. Using these identities, some new Parseval-Goldstein type identities are obtained for these integral transforms and the other well-known. As applications of the identities and theorems, some generalized integrals and illustrative examples are also given.

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1 Introduction

The well-known Laplace transform is defined as

$$\mathcal{L}\{f(x); y\} = \int_0^{\infty} \exp(-xy) f(x) dx. \quad (1.1)$$

In this paper, we introduce the generalized Laplace transform as follow

$$\mathcal{L}_{\lambda}\{f(x); y\} = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} x^{\lambda-1} \exp(-xy) f(x) dx. \quad (1.2)$$

The classical Stieltjes transform is defined as

$$\mathcal{S}\{f(x); y\} = \int_0^{\infty} \frac{f(x)}{x+y} dx. \quad (1.3)$$

The generalized Stieltjes transform of a real-valued function $f(x)$ is defined as

$$\mathcal{S}_{\mu}\{f(x); y\} = \int_0^{\infty} \frac{f(x)}{(x+y)^{\mu}} dx. \quad (1.4)$$

Some Parseval-Goldstein type identities were given in (for example) [7, 8, 9, 10] for the Stieltjes transform and the other transforms. There are many similar results in the literature on various integral transforms (see, for instance [8, 9, 11, 12]). Some of the results from Yürekli [9, 10] are applied to generalized functions by Adawi and Alawneh [5].

Brown et al. [6] introduced the exponential integral transform

$$\mathcal{E}_1\{f(x); y\} = \int_0^{\infty} \exp(xy) E_1(xy) f(x) dx \quad (1.5)$$

where $E_1(x)$ is the exponential integral function defined as

$$E_1(x) = -\text{Ei}(-x) = \int_x^{\infty} \frac{e^{-t}}{t} dt. \quad (1.6)$$

They showed that the three times iteration of the classical Laplace transform is (1.5) the exponential integral transform .

In this paper we introduce the upper incomplete gamma transform:

$$\Gamma_{\rho}\{f(x); y\} = \int_0^{\infty} \exp(xy) \Gamma(\rho, xy) f(x) dx \quad (1.7)$$

where the upper incomplete gamma function is defined as [4, p. 463, 45:3:2]:

$$\Gamma(\rho, x) = \int_x^{\infty} t^{\rho-1} e^{-t} dt. \quad (1.8)$$

Since

$$\Gamma(0, x) = E_1(x), \quad (1.9)$$

the upper incomplete gamma transform (1.8) is a generalization of the exponential integral transform.

In this paper, identities involving the generalized Laplace transform, the exponential integral transform, the Stieltjes transform, the generalized Stieltjes transform and the upper incomplete gamma transforms are given. Using these identities, a number of new Parseval-Goldstein type identities are obtained for these and many other well-known integral transforms. As applications of the identities and theorems, some illustrative examples are also given.

2 The Main Theorem

In the following lemmas and theorems, we give useful identities involving the generalized Laplace transform, the generalized Stieltjes transform and the Stieltjes transform.

Lemma 1. *The following iteration identity hold true:*

$$\mathcal{L}_\mu \left\{ \mathcal{L}_\lambda \{ f(x); y \}; t \right\} = \frac{1}{\Gamma(\lambda)} \mathcal{S}_\mu \{ x^{\lambda-1} f(x); t \} \quad (2.1)$$

provided that the integrals involved converge absolutely.

Proof. We begin with the definition of (1.2) the generalized Laplace transform, we have

$$\mathcal{L}_\mu \left\{ \mathcal{L}_\lambda \{ f(x); y \}; t \right\} = \frac{1}{\Gamma(\mu)} \int_0^\infty y^{\mu-1} e^{-ty} \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} e^{-yx} f(x) dx \right) dy. \quad (2.2)$$

Changing the order of integration and using the known result [1, p.137, Eq.4:3:1] for the inner integral we find that

$$\begin{aligned} \mathcal{L}_\mu \left\{ \mathcal{L}_\lambda \{ f(x); y \}; t \right\} &= \frac{1}{\Gamma(\lambda)\Gamma(\mu)} \int_0^\infty x^{\lambda-1} f(x) \left(\int_0^\infty y^{\mu-1} e^{-y(t+x)} dy \right) dx \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \frac{x^{\lambda-1} f(x)}{(t+x)^\mu} dx. \end{aligned}$$

From the definition of (1.4) the generalized Stieltjes transform, we obtain (2.1). $\square$

Corollary 1. *If we take $\lambda = \mu$ in Lemma 1 then*

$$\mathcal{L}_\mu \left\{ \mathcal{L}_\mu \{ f(x); y \}; t \right\} = \frac{1}{\Gamma(\mu)} \mathcal{S}_\mu \{ x^{\mu-1} f(x); t \}. \quad (2.3)$$

Remark 1. *If $\lambda = \mu = 1$ then we have the known equality that,*

$$\mathcal{L} \left\{ \mathcal{L} \{ f(x); y \}; t \right\} = \mathcal{S} \{ f(x); t \}.$$

Theorem 1. *Under the conditions stated in Lemma 1, the following Parseval-Goldstein type relations*

$$\int_0^\infty x^{\alpha-1} \mathcal{L}_\mu \{f(t); x\} \mathcal{L}_\lambda \{g(y); x\} dx = \frac{\Gamma(\alpha)}{\Gamma(\lambda)\Gamma(\mu)} \int_0^\infty y^{\lambda-1} g(y) \mathcal{S}_\alpha \{t^{\mu-1} f(t); y\} dy, \quad (2.4)$$

$$\int_0^\infty x^{\alpha-1} \mathcal{L}_\mu \{f(t); x\} \mathcal{L}_\lambda \{g(y); x\} dx = \frac{\Gamma(\alpha)}{\Gamma(\mu)\Gamma(\lambda)} \int_0^\infty t^{\mu-1} f(t) \mathcal{S}_\alpha \{y^{\lambda-1} g(y); t\} dt \quad (2.5)$$

hold true.

Proof. We only give the proof of (2.4), as the proof of (2.5) is similar. So by the definition of (1.2) the generalized Laplace transform of $g(y)$, we have

$$\int_0^\infty x^{\alpha-1} \mathcal{L}_\mu \{f(t); x\} \mathcal{L}_\lambda \{g(y); x\} dx = \frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\alpha-1} \mathcal{L}_\mu \{f(t); x\} \left(\int_0^\infty y^{\lambda-1} e^{-xy} g(y) dy \right) dx.$$

Changing the order of integration and using the definition of (1.2) the generalized Laplace transform, we obtain

$$\begin{aligned} \int_0^\infty x^{\alpha-1} \mathcal{L}_\mu \{f(t); x\} \mathcal{L}_\lambda \{g(y); x\} dx &= \frac{1}{\Gamma(\lambda)} \int_0^\infty y^{\lambda-1} g(y) \left(\int_0^\infty x^{\alpha-1} e^{-xy} \mathcal{L}_\mu \{f(t); x\} dx \right) dy \\ &= \frac{\Gamma(\alpha)}{\Gamma(\lambda)} \int_0^\infty y^{\lambda-1} g(y) \mathcal{L}_\alpha \{ \mathcal{L}_\mu \{f(t); x\}; y \} dy. \end{aligned}$$

From (2.1), we find that

$$\int_0^\infty x^{\alpha-1} \mathcal{L}_\mu \{f(t); x\} \mathcal{L}_\lambda \{g(y); x\} dx = \frac{\Gamma(\alpha)}{\Gamma(\lambda)\Gamma(\mu)} \int_0^\infty y^{\lambda-1} g(y) \mathcal{S}_\alpha \{t^{\mu-1} f(t); y\} dy.$$

□

Corollary 2. *We have for $\operatorname{Re}(\alpha) > 0$*

$$\int_0^\infty y^{\lambda-1} g(y) \mathcal{S}_\alpha \{t^{\mu-1} f(t); y\} dy = \int_0^\infty t^{\mu-1} f(t) \mathcal{S}_\alpha \{y^{\lambda-1} g(y); t\} dt. \quad (2.6)$$

Proof. From the equality of the left hand side of (2.4) and (2.5) we get (2.6). □

Corollary 3. *We have for $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\mu) > 0$ and $\operatorname{Re}(\lambda) > 0$*

$$\mathcal{S}_\mu \{x^{\lambda-1} \mathcal{L}_\lambda \{g(y); x\}; y\} = \frac{\Gamma(\alpha)}{\Gamma(\lambda)} \mathcal{L}_\mu \{ \mathcal{S}_\alpha \{y^{\lambda-1} g(y); t\}; y \}. \quad (2.7)$$

Proof. If we take $f(t) = e^{-yt}$ in (2.5), we obtain (2.7). □

Lemma 2. *The following identity hold true:*

$$\mathcal{L}_\mu \{y^\lambda \mathcal{L}_\lambda \{f(x); y\}; t\} = \frac{1}{\beta(\mu, \lambda)} \mathcal{S}_{\mu+\lambda} \{x^{\lambda-1} f(x); t\} \quad (2.8)$$

provided that the integrals involved converge absolutely.

Proof. To prove (2.8) we start with the definition of (1.2) the generalized Laplace transform then

$$\mathcal{L}_\mu \{y^\lambda \mathcal{L}_\lambda \{f(x); y\}; t\} = \frac{1}{\Gamma(\mu)} \int_0^\infty y^{\mu+\lambda-1} e^{-ty} \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} e^{-yx} f(x) dx \right) dy.$$

Interchange of the order of integration and using the known result [1, p.137, Eq.4.3:1] for $\text{Re}(\mu + \lambda) > 0$, we obtain

$$\begin{aligned} \mathcal{L}_\mu \{y^\lambda \mathcal{L}_\lambda \{f(x); y\}; t\} &= \frac{1}{\Gamma(\mu)} \int_0^\infty x^{\lambda-1} f(x) \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty y^{\mu+\lambda-1} e^{-y(x+t)} dy \right) dx \\ &= \frac{1}{\Gamma(\mu)\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} f(x) (\Gamma(\mu + \lambda)(x+t)^{-(\mu+\lambda)}) dx \\ &= \frac{\Gamma(\mu + \lambda)}{\Gamma(\mu)\Gamma(\lambda)} \int_0^\infty \frac{x^{\lambda-1} f(x)}{(x+t)^{\mu+\lambda}} dx. \end{aligned}$$

From the definition (1.3) of Stieltjes transform, we get

$$\mathcal{L}_\mu \{y^\lambda \mathcal{L}_\lambda \{f(x); y\}; t\} = \frac{1}{\beta(\mu, \lambda)} \mathcal{S}_{\mu+\lambda} \{x^{\lambda-1} f(x); t\}.$$

□

Theorem 2. *If the conditions in Lemma 2 are satisfied, then the Parseval-Goldstein type relation*

$$\int_0^\infty y^{\lambda+\mu-1} \mathcal{L}_\lambda \{f(x); y\} \mathcal{L}_\mu \{g(t); y\} dy = \frac{1}{\beta(\mu, \lambda)} \int_0^\infty t^{\mu-1} g(t) \mathcal{S}_{\mu+\lambda} \{x^{\lambda-1} f(x); t\} dt \quad (2.9)$$

hold true.

Proof. We start with the proof of (2.9). By the definition of (1.2) the generalized Laplace transform of $g(t)$ function and then changing the order of integration, we obtain

$$\begin{aligned} &\int_0^\infty y^{\lambda+\mu-1} \mathcal{L}_\lambda \{f(x); y\} \mathcal{L}_\mu \{g(t); y\} dy \\ &= \int_0^\infty y^{\lambda+\mu-1} \mathcal{L}_\lambda \{f(x); y\} \left(\frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} e^{-yt} g(t) dt \right) dy \\ &= \int_0^\infty t^{\mu-1} g(t) \left(\frac{1}{\Gamma(\mu)} \int_0^\infty y^{\mu-1} e^{-ty} [y^\lambda \mathcal{L}_\lambda \{f(x); y\}] dy \right) dt \\ &= \int_0^\infty t^{\mu-1} g(t) \mathcal{L}_\mu \{y^\lambda \mathcal{L}_\lambda \{f(x); y\}; t\} dt. \end{aligned} \quad (2.10)$$

Using (2.8) for (2.10), we find that

$$\int_0^\infty y^{\lambda+\mu-1} \mathcal{L}_\lambda \{f(x); y\} \mathcal{L}_\mu \{g(t); y\} dy = \frac{1}{\beta(\mu, \lambda)} \int_0^\infty t^{\mu-1} g(t) \mathcal{S}_{\mu+\lambda} \{x^{\lambda-1} f(x); t\} dt.$$

□

Lemma 3. *The following identities hold true:*

$$\mathcal{L}_\lambda \{y^{1-\lambda} \mathcal{S} \{f(x); y\}; t\} = \frac{1}{\Gamma(\lambda)} \mathcal{E}_1 \{f(x); t\} \quad (2.11)$$

$$\mathcal{L}_\lambda \{\mathcal{S} \{f(x); y\}; t\} = \Gamma_{1-\lambda} \{x^{\lambda-1} f(x); t\} \quad (2.12)$$

$$\mathcal{S} \{\mathcal{L}_\lambda \{f(x); y\}; t\} = \frac{1}{\Gamma(\lambda)} \mathcal{E}_1 \{x^{\lambda-1} f(x); t\} \quad (2.13)$$

$$\mathcal{S} \{y^{\lambda-1} \mathcal{L}_\lambda \{f(x); y\}; t\} = t^{\lambda-1} \Gamma_{1-\lambda} \{x^{\lambda-1} f(x); t\} \quad (2.14)$$

provided that the integrals involved converge absolutely.

Proof. We first prove (2.11). By the definitions of (1.2) the generalized Laplace transform and (1.3) the Stieltjes transform, we have

$$\mathcal{L}_\lambda \{y^{1-\lambda} \mathcal{S} \{f(x); y\}; t\} = \frac{1}{\Gamma(\lambda)} \int_0^\infty y^{\lambda-1} e^{-ty} \left(\int_0^\infty y^{1-\lambda} \frac{f(x)}{x+y} dx \right) dy.$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, we obtain

$$\mathcal{L}_\lambda \{y^{1-\lambda} \mathcal{S} \{f(x); y\}; t\} = \frac{1}{\Gamma(\lambda)} \int_0^\infty f(x) \left(\int_0^\infty \frac{y^{\lambda-1} e^{-ty}}{y^{\lambda-1}(x+y)} dy \right) dx. \quad (2.15)$$

Using the formula [2, p.217, Eq.14:2:11] for the inner integral given in (2.15) and using the definition of (1.5) the exponential integral transform, we find

$$\begin{aligned} \mathcal{L}_\lambda \{y^{1-\lambda} \mathcal{S} \{f(x); y\}; t\} &= \frac{1}{\Gamma(\lambda)} \int_0^\infty f(x) (-\exp(tx)) \text{Ei}(-tx) dx \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \exp(tx) E_1(tx) f(x) dx \\ &= \frac{1}{\Gamma(\lambda)} \mathcal{E}_1 \{f(x); t\}. \end{aligned}$$

For the proof of (2.12), we use the definitions of (1.2) the generalized Laplace transform and (1.3) the Stieltjes transform, we have

$$\mathcal{L}_\lambda \{\mathcal{S} \{f(x); y\}; t\} = \frac{1}{\Gamma(\lambda)} \int_0^\infty y^{\lambda-1} e^{-ty} \left(\int_0^\infty \frac{f(x)}{x+y} dx \right) dy.$$

By the permission we change the order of integration, then we find

$$\mathcal{L}_\lambda \{\mathcal{S} \{f(x); y\}; t\} = \frac{1}{\Gamma(\lambda)} \int_0^\infty f(x) \left(\int_0^\infty \frac{y^{\lambda-1} e^{-ty}}{x+y} dy \right) dx \quad (2.16)$$

and using the formula [2, p.217, Eq.14:2:17] for the inner integral given in (2.16), we obtain

$$\begin{aligned} \mathcal{L}_\lambda \{\mathcal{S} \{f(x); y\}; t\} &= \frac{1}{\Gamma(\lambda)} \int_0^\infty f(x) (x^{\lambda-1} e^{tx} \Gamma(\lambda) \Gamma(1-\lambda, tx)) dx \\ &= \int_0^\infty e^{tx} \Gamma(1-\lambda, tx) x^{\lambda-1} f(x) dx \\ &= \Gamma_{1-\lambda} \{x^{\lambda-1} f(x); t\}. \end{aligned}$$

The proofs of (2.13) and (2.14) are similar. □

Remark 2. If we take $f(x) = x^{\lambda-1}f(x)$ in (2.11) of Lemma 3 then

$$\mathcal{L}_\lambda \{y^{1-\lambda} \mathcal{S} \{x^{\lambda-1} f(x); y\}; t\} = \mathcal{S} \{\mathcal{L}_\lambda \{f(x); y\}; t\}.$$

Remark 3. From the right sides of (2.12) and (2.14), we have

$$t^{\lambda-1} \mathcal{L}_\lambda \{\mathcal{S} \{f(x); y\}; t\} = \mathcal{S} \{y^{\lambda-1} \mathcal{L}_\lambda \{f(x); y\}; t\}.$$

Theorem 3. Under the conditions stated in Lemma 3, the following Parseval-Goldstein type relation

$$\int_0^\infty x^{\lambda-1} f(x) \mathcal{L}_\lambda \{g(y); x\} dx = \int_0^\infty y^{\lambda-1} g(y) \mathcal{L}_\lambda \{f(x); y\} dy \quad (2.17)$$

holds true.

Proof. We begin with the definition of (1.2) the generalized Laplace transform, we have

$$\int_0^\infty x^{\lambda-1} f(x) \mathcal{L}_\lambda \{g(y); x\} dx = \int_0^\infty x^{\lambda-1} f(x) \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty y^{\lambda-1} e^{-xy} g(y) dy \right) dx.$$

Interchange of the order of integration and again the definition of (1.2) the generalized Laplace transform, we obtain

$$\begin{aligned} \int_0^\infty x^{\lambda-1} f(x) \mathcal{L}_\lambda \{g(y); x\} dx &= \int_0^\infty y^{\lambda-1} g(y) \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} e^{-yx} f(x) dx \right) dy \\ &= \int_0^\infty y^{\lambda-1} g(y) \mathcal{L}_\lambda \{f(x); y\} dy. \end{aligned}$$

□

Theorem 4. If the hypothesis in Lemma 3 are satisfied, then

$$\int_0^\infty \mathcal{L}_\lambda \{f(t); x\} \mathcal{S} \{g(y); x\} dx = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} f(t) \mathcal{E}_1 \{g(y); t\} dt, \quad (2.18)$$

$$\int_0^\infty \mathcal{L}_\lambda \{f(t); x\} \mathcal{S} \{g(y); x\} dx = \frac{1}{\Gamma(\lambda)} \int_0^\infty g(y) \mathcal{E}_1 \{t^{\lambda-1} f(t); y\} dy. \quad (2.19)$$

Proof. To prove (2.18), we start with the definition of (1.2) the generalized Laplace transform, we have

$$\int_0^\infty \mathcal{L}_\lambda \{f(t); x\} \mathcal{S} \{g(y); x\} dx = \int_0^\infty \mathcal{S} \{g(y); x\} \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-xt} f(t) dt \right) dx.$$

Changing the order of integration, expanding with $x^{\lambda-1}$ and using the (2.11) of Lemma 3, we obtain

$$\begin{aligned} \int_0^\infty \mathcal{L}_\lambda \{f(t); x\} \mathcal{S} \{g(y); x\} dx &= \int_0^\infty t^{\lambda-1} f(t) \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} e^{-xt} (x^{1-\lambda} \mathcal{S} [g(y); x]) dx \right) dy \\ &= \int_0^\infty t^{\lambda-1} f(t) \mathcal{L}_\lambda \{x^{1-\lambda} \mathcal{S} \{g(y); x\}; t\} dt \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} f(t) \mathcal{E}_1 \{g(y); t\} dt. \end{aligned}$$

To prove (2.19), from the definition of (1.3) the Stieltjes transform, we have

$$\int_0^\infty \mathcal{L}_\lambda \{f(t); x\} \mathcal{S} \{g(y); x\} dx = \int_0^\infty \mathcal{L}_\lambda \{f(t); x\} \left(\int_0^\infty \frac{g(y)}{y+x} dy \right) dx.$$

Changing the order of integration and using the (2.13) of Lemma 3, we find that

$$\begin{aligned} \int_0^\infty \mathcal{L}_\lambda \{f(t); x\} \mathcal{S} \{g(y); x\} dx &= \int_0^\infty g(y) \left(\int_0^\infty \frac{\mathcal{L}_\lambda \{f(t); x\}}{y+x} dx \right) dy \\ &= \int_0^\infty g(y) \mathcal{S} \{ \mathcal{L}_\lambda \{f(t); x\}; y \} dy \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty g(y) \mathcal{E}_1 \{ t^{\lambda-1} f(t); y \} dy. \end{aligned}$$

□

Remark 4. From (2.18) and (2.19) of Theorem 4, we obtain

$$\int_0^\infty t^{\lambda-1} f(t) \mathcal{E}_1 \{g(y); t\} dt = \int_0^\infty g(y) \mathcal{E}_1 \{t^{\lambda-1} f(t); y\} dy.$$

Theorem 5. The following Parseval-Goldstein type relations hold true:

$$\int_0^\infty x^{\lambda-1} \mathcal{L}_\lambda \{f(t); x\} \mathcal{S} \{g(y); x\} dx = \int_0^\infty t^{\lambda-1} f(t) \Gamma_{1-\lambda} \{y^{\lambda-1} g(y); t\} dt, \quad (2.20)$$

$$\int_0^\infty x^{\lambda-1} \mathcal{L}_\lambda \{t^{1-\lambda} f(t); x\} \mathcal{S} \{g(y); x\} dx = \int_0^\infty f(t) \Gamma_{1-\lambda} \{y^{\lambda-1} g(y); t\} dt. \quad (2.21)$$

Proof. To prove (2.20), we start with the definition of (1.2) the generalized Laplace transform,

$$\int_0^\infty x^{\lambda-1} \mathcal{L}_\lambda \{f(t); x\} \mathcal{S} \{g(y); x\} dx = \int_0^\infty x^{\lambda-1} \mathcal{S} \{g(y); x\} \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-xt} f(t) dt \right) dx.$$

Changing the order of integration and using the (2.12) of Lemma 3, we get

$$\begin{aligned} \int_0^\infty x^{\lambda-1} \mathcal{L}_\lambda \{f(t); x\} \mathcal{S} \{g(y); x\} dx &= \int_0^\infty t^{\lambda-1} f(t) \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} e^{-xt} \mathcal{S} \{g(y); x\} dx \right) dt \\ &= \int_0^\infty t^{\lambda-1} f(t) \mathcal{L}_\lambda \{ \mathcal{S} \{g(y); x\}; t \} dt \\ &= \int_0^\infty t^{\lambda-1} f(t) \Gamma_{1-\lambda} \{y^{\lambda-1} g(y); t\} dt. \end{aligned}$$

The proof of (2.21) is similar. □

Remark 5. We have

$$\int_0^\infty y^{\lambda-1} g(y) \Gamma_{1-\lambda} \{t^{\lambda-1} f(t); y\} dy = \int_0^\infty t^{\lambda-1} f(t) \Gamma_{1-\lambda} \{y^{\lambda-1} g(y); t\} dt.$$

3 Examples

Example 1. *We show*

$$\int_0^\infty \frac{x^{\lambda-1} [\psi(\lambda) - \ln x]}{(x+u)^\mu} dx = \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)} \frac{1}{u^{\mu-\lambda}} [\psi(\mu-\lambda) - \ln u], \quad (3.1)$$

where $\operatorname{Re}(\lambda) > 0$.

We put

$$f(x) = \psi(\lambda) - \ln x$$

in (2.1). Using Eq.4:369:1 in [3, p.578], we obtain

$$\begin{aligned} & \mathcal{L}_\mu \{ \mathcal{L}_\lambda \{ [\psi(\lambda) - \ln(x)]; y \}; u \} \\ &= \frac{1}{\Gamma(\mu)} \int_0^\infty y^{\mu-1} e^{-uy} \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} e^{-yx} [\psi(\lambda) - \ln x] dx \right) dy \\ &= \frac{1}{\Gamma(\mu)} \int_0^\infty y^{\mu-\lambda-1} e^{-uy} \ln y dy. \end{aligned}$$

Using the result [3, p.573, Eq.4:352:1] for the integral giving above, we get

$$\mathcal{L}_\mu \{ \mathcal{L}_\lambda \{ [\psi(\lambda) - \ln x]; y \}; u \} = \frac{\Gamma(\mu-\lambda)}{\Gamma(\mu)} \frac{[\psi(\mu-\lambda) - \ln u]}{u^{\mu-\lambda}}. \quad (3.2)$$

From the equality (2.1) of Lemma 1 and (3.2), we find

$$\int_0^\infty \frac{x^{\lambda-1} [\psi(\lambda) - \ln x]}{(x+u)^\mu} dx = \frac{\Gamma(\mu-\lambda)}{\Gamma(\mu)} \frac{[\psi(\mu-\lambda) - \ln u]}{u^{\mu-\lambda}}. \quad (3.3)$$

Example 2. *We show*

$$\begin{aligned} & \int_0^\infty \exp(ay) \Gamma(1-\alpha, ay) (y^{\lambda-1} (\psi(\lambda) - \ln y)) dy \\ &= -\frac{\Gamma(\lambda)}{\Gamma(\alpha)} \pi a^{-\lambda} \csc((\alpha-\lambda-1)\pi) [\ln a - \pi \cot((\alpha-\lambda-1)\pi)] \end{aligned} \quad (3.4)$$

where $\operatorname{Re}(a) > 0$ and $0 < \operatorname{Re}(\mu-\lambda) < 1$.

We put

$$g(y) = \psi(\lambda) - \ln y$$

in (2.4) and then using the known result [1, p.149, Eq.4:6:12] we get,

$$\begin{aligned} & \int_0^\infty x^{\alpha-1} \mathcal{L}_\mu \{ f(t); x \} \mathcal{L}_\lambda \{ \psi(\lambda) - \ln y; x \} dx = \\ & \frac{\Gamma(\alpha)}{\Gamma(\lambda)\Gamma(\mu)} \int_0^\infty y^{\lambda-1} (\psi(\lambda) - \ln y) \mathcal{S}_\alpha \{ t^{\mu-1} f(t); y \} dy \end{aligned}$$

$$\int_0^\infty x^{\alpha-\lambda-1} \ln x \mathcal{L}_\mu \{f(t); x\} dx = \frac{\Gamma(\alpha)}{\Gamma(\lambda)\Gamma(\mu)} \int_0^\infty y^{\lambda-1} (\psi(\lambda) - \ln y) \mathcal{S}_\alpha \{t^{\mu-1} f(t); y\} dy. \quad (3.5)$$

If we take

$$f(t) = t^{1-\mu} e^{-at}$$

in (3.5) and using the known results [1, p.143, Eq.4:5:1] and [2, p.233, Eq.14:4:10] for $\text{Re}(a) > 0$ and from the definition of (1.7) the upper incomplete gamma integral transform, we obtain

$$\begin{aligned} \int_0^\infty \frac{x^{\alpha-\lambda-1} \ln x}{x+a} dx &= \frac{a^{\alpha-1} \Gamma(\alpha)}{\Gamma(\lambda)} \int_0^\infty y^{\lambda-1} (\psi(\lambda) - \ln y) e^{ay} \Gamma(1-\alpha, ay) dy \\ \mathcal{S} \{x^{\alpha-\lambda-1} \ln x; a\} &= \frac{a^{\alpha-1} \Gamma(\alpha)}{\Gamma(\lambda)} \Gamma_{1-\alpha} \{y^{\lambda-1} (\psi(\lambda) - \ln y); a\}. \end{aligned}$$

Using the known result [2, p.218, Eq.14:2:28] for $0 < \text{Re}(\alpha - \lambda) < 1$, we find

$$\Gamma_{1-\alpha} \{y^{\lambda-1} (\psi(\lambda) - \ln y); a\} = -\frac{\Gamma(\lambda)}{\Gamma(\alpha)} \pi a^{-\lambda} \csc((\alpha - \lambda - 1)\pi) [\ln a - \pi \cot((\alpha - \lambda - 1)\pi)].$$

Example 3. We show

$$\mathcal{E}_1 \{x^\nu \log x; y\} = \frac{-\pi \csc(\nu\pi)}{y^{\nu+1}} \Gamma(\nu+1) [\psi(\nu+1) - \log y - \pi \cot(\nu\pi)],$$

where $-1 < \text{Re}(\nu) < 0$.

We put

$$f(x) = x^\nu \log x$$

in (2.11). Using Eq.14:2:28 in [2, p.218], we obtain

$$\mathcal{S} \{x^\nu \log x; t\} = -\pi t^\nu \csc(\nu\pi) [\log t - \pi \cot(\nu\pi)]. \quad (3.6)$$

Substituting the results (3.6) into identity (2.11), we find

$$\begin{aligned} \frac{1}{\Gamma(\lambda)} \mathcal{E}_1 \{x^\nu \log x; y\} &= \mathcal{L}_\lambda \{t^{1-\lambda} (-\pi t^\nu \csc(\nu\pi) [\log t - \pi \cot(\nu\pi)]) ; y\} \\ \mathcal{E}_1 \{x^\nu \log x; y\} &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-yt} (-\pi t^{1-\lambda+\nu} \csc(\nu\pi) [\log t - \pi \cot(\nu\pi)]) dt \\ &= \int_0^\infty e^{-yt} [-\pi t^\nu \csc(\nu\pi) \log t + \pi^2 t^\nu \csc(\nu\pi) \cot(\nu\pi)] dt \\ &= -\pi \csc(\nu\pi) \left[\int_0^\infty e^{-yt} t^\nu \log(t) dt - \pi \cot(\nu\pi) \int_0^\infty e^{-yt} t^\nu dt \right]. \end{aligned}$$

For the integrals giving above we use the results [1, p.148, Eq.4:6:11] and [1, p.137, Eq.4:3:1], then we get

$$\mathcal{E}_1 \{x^\nu \log x; y\} = \frac{-\pi \csc(\nu\pi)}{y^{\nu+1}} \Gamma(\nu+1) [\psi(\nu+1) - \log y - \pi \cot(\nu\pi)].$$

Example 4. *We show*

$$\mathcal{E}_1\{x^{\lambda-1}e^{-ax}; u\} = \frac{\Gamma(\lambda)^2}{\Gamma(\lambda+1)a^\lambda} {}_2F_1(\lambda, 1; \lambda+1; 1-u/a), \quad (3.7)$$

where $|\arg a| < \pi$ and $\operatorname{Re}(\lambda) > -1$.

We put

$$f(x) = e^{-ax}$$

in (2.13) of Lemma 3 and we get,

$$\begin{aligned} \frac{1}{\Gamma(\lambda)} \mathcal{E}_1\{x^{\lambda-1}e^{-ax}; u\} &= \mathcal{S}\{\mathcal{L}_\lambda\{e^{-ax}; y\}; u\} \\ &= \int_0^\infty \frac{1}{y+u} \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} e^{-x(y+a)} dx \right) dy \end{aligned}$$

Using the known result [1, p.144, Eq.4:5:3] for the inner integral giving above, we find that

$$\begin{aligned} \frac{1}{\Gamma(\lambda)} \mathcal{E}_1\{x^{\lambda-1}e^{-ax}; u\} &= \int_0^\infty \frac{dy}{(y+u)(y+a)^\lambda} \\ \mathcal{E}_1\{x^{\lambda-1}e^{-ax}; u\} &= \Gamma(\lambda) \mathcal{S}\{(y+a)^{-\lambda}; u\}. \end{aligned}$$

From the known result [2, p.217, Eq.14:2:9] for (1.3) the Stieltjes transform, we obtain

$$\mathcal{E}_1\{x^{\lambda-1}e^{-ax}; u\} = \frac{\Gamma(\lambda)^2}{\Gamma(\lambda+1)a^\lambda} {}_2F_1(\lambda, 1; \lambda+1; 1-u/a).$$

Example 5. *We show*

$$\Gamma_{1-\lambda}\{x^{\lambda-1}e^{-ax}; t\} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+1)a^\lambda} {}_2F_1(\lambda, \lambda; \lambda+1; 1-t/a)$$

where $|\arg a| < \pi$ and $\operatorname{Re}(\lambda) > -1$.

We put

$$f(x) = e^{-ax}$$

in (2.14) of Lemma 3 and we get,

$$\begin{aligned} t^{\lambda-1} \Gamma_{1-\lambda}\{x^{\lambda-1}e^{-ax}; t\} &= \mathcal{S}\{y^{\lambda-1} \mathcal{L}_\lambda\{e^{-ax}; y\}; t\} \\ &= \int_0^\infty \frac{y^{\lambda-1}}{y+t} \left(\frac{1}{\Gamma(\lambda)} \int_0^\infty x^{\lambda-1} e^{-x(y+a)} dx \right) dy \end{aligned}$$

Using the known result [1, p.144, Eq.4:5:3] we obtain

$$\begin{aligned} t^{\lambda-1} \Gamma_{1-\lambda}\{x^{\lambda-1}e^{-ax}; t\} &= \int_0^\infty \frac{y^{\lambda-1}}{(y+t)(y+a)^\lambda} dy \\ &= \mathcal{S}\{y^{\lambda-1}(y+a)^{-\lambda}; t\}. \end{aligned}$$

From the known result [2, p.217, Eq.14:2:9] for (1.3) the Stieltjes transform, we find

$$\Gamma_{1-\lambda}\{x^{\lambda-1}e^{-ax}; t\} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+1)a^\lambda} {}_2F_1(\lambda, \lambda; \lambda+1; 1-t/a).$$

4 Conclusion

In this work, we obtained some new properties of the generalized Laplace transform. Several relationships between the generalized Laplace transform and the generalized Stieltjes transform are studied. Moreover, we have shown that many early known identities obtained as a special cases of parameters in these relationships. In the last section, we also show that how these identities work while calculating improper integrals.

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