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A CLASS OF APPROXIMATE OPTIMAL CONTROL PROBLEMS FOR 1-D PHASE-FIELD SYSTEM WITH SINGULARITY AND ITS NUMERICAL ALGORITHM

Dedicated to Professor Masahiro Kubo on the Occasion of his 60th Birthday

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Abstract. We study an optimal control problem for a one dimensional phase-filed system associated with the total variation energy, from the view-point of numerical analysis. Our state system consists of two parabolic PDEs: a heat equation and a singular diffusion equation of an order parameter. In this paper, we give a class of approximate optimal control problems for our original phase-filed system with singularity. Then, we show the necessary condition of the optimal pair by using the control problem of the approximate state system. In addition, by means of necessary conditions for the approximate control problem, we propose the numerical scheme to find the stationary point of the cost functional to the approximate control problem, and show the convergence of our numerical algorithm. Furthermore, we perform the simple numerical experiments.

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1 Introduction

In this paper, we consider a class of approximate problems for the following one dimensional phase-filed system with singularity:

Problem (P;$f, h, \ell)^0$.

$$[u + w]_t - u_{xx} = a_0 f(t, x) \quad \text{in } Q := (0, T) \times (0, L), \tag{1.1}$$

$$w_t - \kappa \left(\frac{w_x}{|w_x|} \right)_x + \partial I_{[-1,1]}(w) + g(w) \ni u \quad \text{in } Q, \tag{1.2}$$

$$- u_x(t, 0) + n_0(u(t, 0) - b_1) = a_1 h(t), \quad t \in (0, T), \tag{1.3}$$

$$u_x(t, L) + n_0(u(t, L) - b_2) = a_2 \ell(t), \quad t \in (0, T), \tag{1.4}$$

$$w_x(t, 0) = w_x(t, L) = 0, \quad t \in (0, T), \tag{1.5}$$

$$u(0, x) = u_0(x), \quad w(0, x) = w_0(x), \quad x \in (0, L), \tag{1.6}$$

where $0 < T < \infty$ and $0 < L < \infty$ are fixed positive constants, a_0, a_1, a_2 are given nonnegative constants, $\kappa > 0$, $n_0 > 0$, b_1, b_2 are given constants, g is a given continuous function on $\mathbb{R}$, $[f, h, \ell]$ is a triplet of given functions, and u_0, w_0 are given initial data. In addition, $\partial I_{[-1,1]}(\cdot)$ is the subdifferential of an indicator function $I_{[-1,1]}(\cdot)$ on the closed interval $[-1, 1]$, that is defined as:

$$I_{[-1,1]}(z) := \begin{cases} 0, & \text{if } z \in [-1, 1], \\ +\infty, & \text{otherwise}. \end{cases} \tag{1.7}$$

The system (P;$f, h, \ell)^0$ is based on the modeling method of Visintin [48] as a possible mathematical model of solid-liquid phase transitions in a mesoscopic length scale. In the physical context, the unknown function $u = u(t, x)$ is the relative temperature, and $w = w(t, x)$ is the nonconserved order parameter that indicates the physical phase of material: $w = 1$ (resp. $w = -1$) corresponds to pure liquid (resp. solid), for instance. Note that the equation (1.2) is derived as the L^2-gradient flow of the free energy functional as follows:

$$\mathscr{F}_u(w) := \kappa \int_0^L |Dw| + \int_0^L \Big\{ I_{[-1,1]}(w) + \widehat{g}(w) - wu \Big\} dx, \quad w \in L^2(0, L),$$

where $\int_0^L |Dw|$ is the total variation of a function $w \in L^2(0, L)$ and $\widehat{g}$ is a non-negative primitive of g. Therefore, we can regard (1.2) as one kind of mathematical formulation of Gibbs-Thomson law.

Many mathematicians studied the singular diffusion equation (1.2) with or without constraint $\partial I_{[-1,1]}(w)$ from the various point of view (cf. [3, 4, 5, 14, 16, 17, 18, 19, 20, 25, 26, 28, 29, 33, 35, 36, 37, 38, 41, 44, 48]). For instance, Kenmochi–Shirakawa studied in [25] the precise structure of steady-state solution, and characterized in [26] the asymptotic stability of steady-states, by means of an original concept named "local stability". Furthermore, the line of results [25, 26] was enhanced by Shirakawa–Kimura [44], under the higher dimensional setting of spatial domain.

In addition, Ohtsuka–Shirakawa–Yamazaki [36, 37, 38] considered the optimal control problem of (1.2) with respect to the temperature control u in the case when $g(w) = -w$.

The system $(\mathrm{P};f,h,\ell)^0$ was considered by Kenmochi–Shirakawa [27] and Shirakawa [42, 43]. In particular, Kenmochi–Shirakawa [27] discussed the large-time behavior of solutions to $(\mathrm{P};0,0,0)^0$ on the basis of the previous work [25, 26] of stability analysis. In addition, Shirakawa–Yamazaki [45] considered the optimal control problem and its optimality condition for $(\mathrm{P};f,h,\ell)^0$ with $g(w) = \nu w^3 - w$ for some small constant $\nu \geq 0$ via the limiting observation of approximate problems : in such approximate problems, the singular diffusion term $\left(\frac{w_x}{|w_x|}\right)_x$ and the constraint $\partial I_{[-1,1]}(w)$ as in (1.2) were approximated by

$$\left(\frac{w_x^\varepsilon}{\sqrt{|w_x^\varepsilon|^2+\varepsilon^2}} + \varepsilon w_x^\varepsilon\right)_x \qquad \text{and} \qquad K^\varepsilon(w^\varepsilon), \tag{1.8}$$

respectively, for given small parameter $\varepsilon \in (0,1]$. Here, K^ε is a nondecreasing function on $\mathbb{R}$ defined by

$$K^\varepsilon(r) := \mathrm{sign}(r)\int_0^{|r|} \min\left\{\frac{1}{\varepsilon}, \frac{[s-1]^+}{\varepsilon^2}\right\} ds \qquad \text{for } r \in \mathbb{R}, \tag{1.9}$$

where $[\,\cdot\,]^+$ denotes the positive part of a function and $\mathrm{sign}(\cdot)$ is a signum function so that $\mathrm{sign}(0) = 0$.

In this present paper, we consider a class of approximate functions for singular diffusion term $\left(\frac{w_x}{|w_x|}\right)_x$ in $(\mathrm{P};f,h,\ell)^0$. Then we investigate the following approximate problems, denoted by $(\mathrm{P};f,h,\ell)^\varepsilon$, with small parameter $\varepsilon \in (0,1]$:

Problem (P;f,h,ℓ)$^\varepsilon$.

$$[u^\varepsilon + w^\varepsilon]_t - u^\varepsilon_{xx} = a_0 f(t,x) \qquad \text{in } Q, \tag{1.10}$$

$$w^\varepsilon_t - \kappa\left(a^\varepsilon(w^\varepsilon_x) + \varepsilon w^\varepsilon_x\right)_x + K^\varepsilon(w^\varepsilon) + g(w^\varepsilon) = u^\varepsilon \;\; \text{in } Q, \tag{1.11}$$

$$-u^\varepsilon_x(t,0) + n_0(u^\varepsilon(t,0) - b_1) = a_1 h(t), \quad t \in (0,T), \tag{1.12}$$

$$u^\varepsilon_x(t,L) + n_0(u^\varepsilon(t,L) - b_2) = a_2 \ell(t), \quad t \in (0,T), \tag{1.13}$$

$$w^\varepsilon_x(t,0) = w^\varepsilon_x(t,L) = 0, \quad t \in (0,T), \tag{1.14}$$

$$u^\varepsilon(0,x) = u_0(x), \quad w^\varepsilon(0,x) = w_0(x), \quad x \in (0,L), \tag{1.15}$$

where a^ε is a given function on $\mathbb{R}$ with $a^\varepsilon(r) \to a^0(r) := \frac{r}{|r|}$ in an appropriate sense as $\varepsilon \to 0$. The typical example is $a^\varepsilon(r) = \frac{r}{\sqrt{r^2+\varepsilon^2}}$ (cf. (1.8)). Then, we clarify the class of approximate functions a^ε so that $(\mathrm{P};f,h,\ell)^\varepsilon$ is the approximate problem for $(\mathrm{P};f,h,\ell)^0$ as $\varepsilon \to 0$.

In addition, we consider a class of approximate optimal control problems, denoted by $(\mathrm{OP})^\varepsilon$, as follows:

Problem (OP)$^\varepsilon$: Find a triplet of control functions $[f^\varepsilon_*, h^\varepsilon_*, \ell^\varepsilon_*] \in \mathcal{U}$, call *optimal control*, such that

$$J^\varepsilon(f^\varepsilon_*, h^\varepsilon_*, \ell^\varepsilon_*) = \inf_{[f,h,\ell]\in\mathcal{U}} J^\varepsilon(f,h,\ell).$$

Here, we set $\mathcal{U} := L^2(0,T;L^2(0,L)) \times L^2(0,T) \times L^2(0,T)$ as a control space, and $J^\varepsilon(f,h,\ell)$ is the cost functional defined by

$$\begin{aligned} J^\varepsilon(f,h,\ell) \ &:= \frac{c_0}{2}\int_0^T |(u^\varepsilon - u_d)(t)|^2_{L^2(0,L)}\,dt + \frac{c_1}{2}\int_0^T |(w^\varepsilon - w_d)(t)|^2_{L^2(0,L)}\,dt \\ &\quad + \frac{m_0}{2}\int_0^T a_0^2|f(t)|^2_{L^2(0,L)}\,dt + \frac{m_1}{2}\int_0^T a_1^2|h(t)|^2\,dt + \frac{m_2}{2}\int_0^T a_2^2|\ell(t)|^2\,dt, \end{aligned} \tag{1.16}$$

where $|\cdot|_{L^2(0,L)}$ is a standard norm of $L^2(0,L)$, c_0, c_1, m_0, m_1, m_2 are given nonnegative constants, and u_d, w_d are the given desired target profiles in $L^2(0,T;L^2(0,L))$. In addition, a couple of functions $[u^\varepsilon, w^\varepsilon]$ is a unique solution to the initial-boundary value state problem (P;f,h,ℓ)$^\varepsilon$ with the control parameter $[f,h,\ell] \in \mathcal{U}$.

Note that (OP)$^\varepsilon$ can be regarded as an optimal control problem in solid-liquid phase transition phenomena. Indeed, if the constant a_0 is equal to 0, then (OP)$^\varepsilon$ is a boundary control problem. Similarly, if $a_1 = a_2 = 0$, then (OP)$^\varepsilon$ reduces to a distributed control problem with the heat source as control. Note that b_1 (resp. b_2) denotes the outside temperature at $x = 0$ (resp. $x = L$). There is a vast amount of literature on optimal control of phase transitions problems. In particular, we refer to the contributions [1, 10, 13, 21, 34, 35, 36, 37, 38, 39, 40, 45, 46, 47].

In addition, note that (OP)^0 is an optimal control problem for our original phase-filed system (P;f,h,ℓ)^0 with singularity. Therefore, in this present paper, we show the relationship between (OP)$^\varepsilon$ and its limiting problem (OP)^0 as $\varepsilon \to 0$. Furthermore, by using necessary conditions for (OP)$^\varepsilon$, we propose the numerical scheme to find the stationary point of the cost functional $J^\varepsilon(\cdot,\cdot,\cdot)$ to (OP)$^\varepsilon$, and show the convergence of our numerical algorithm. Moreover, we give some numerical experiments for (OP)$^\varepsilon$ under the simple situations.

The main novelties found in this paper are the following:

(i) to prove the existence-uniqueness of solutions to (P;f,h,ℓ)$^\varepsilon$ for any $\varepsilon \ge 0$;

(ii) to prove continuous dependence of solutions to the systems (P;f,h,ℓ)$^\varepsilon$ with respect to $\varepsilon \to 0$;

(iii) to prove the existence of an optimal control (optimal pair) to (OP)$^\varepsilon$ for any $\varepsilon \ge 0$;

(iv) to show the relationship between the limits (ω-limit points) of sequences of approximate optimal pairs and the optimal pairs of the limiting problem (OP)^0;

(v) to show the necessary conditions to the approximate optimal control problems (OP)$^\varepsilon$ for any $\varepsilon > 0$;

(vi) to derive a weak formula of the necessary conditions to the original problem (OP)^0 through the limiting observation of approximate situations, as $\varepsilon \downarrow 0$;

(vii) to propose the numerical scheme to find the stationary point of the cost functional to the approximate control problem (OP)$^\varepsilon$;

(viii) to show the convergence of our numerical algorithm;

(ix) to perform numerical experiments of $(\mathrm{OP})^\varepsilon$ for the sufficient small $\varepsilon > 0$ under the simple situations.

Consequently, an effective class of approximate optimal pairs of our control problem $(\mathrm{OP})^0$ will be presented as a further conclusion derived from the main results. In addition, it is worthy of considering approximate optimal control problems $(\mathrm{OP})^\varepsilon$ from the view-point of numerical analysis.

The plan of this paper is as follows. In Section 2, we recall the fundamentals of the theory of functions of bounded variation, including the exact definition of the total variation functional. In Section 3, we prove Theorem 3.1 concerned with the item (i) listed in the above. In Section 4, we discuss the continuous dependence of solutions to the systems $(\mathrm{P};f,h,\ell)^\varepsilon$ with respect to $\varepsilon \to 0$, corresponding to the item (ii) listed in the above. In Sections 5, 6, and 7, we consider optimal control problems $(\mathrm{OP})^\varepsilon$ for any $\varepsilon \geq 0$, which correspond to the items (iii), (iv), (v), and (vi) listed in the above. In Section 8, we mention and prove the main theorem, concerned with the items (vii) and (viii) listed in the above. In the final Section 9, we show the item (ix). Indeed, we give three numerical experiments to $(\mathrm{OP})^\varepsilon$ for the sufficient small $\varepsilon > 0$.

1.1 Notations and basic assumptions

First, we mention the notations that are used throughout this paper.

For each dimension $n \in \mathbb{N}$, we denote by $\mathscr{L}^n$ the n-dimensional Lebesgue measure, and we use this measure unless otherwise specified.

For any reflexive Banach space B, we denote by $|\cdot|_B$ the norm of B, and denote by B' the dual space of B. Additionally, we denote by $\langle \cdot, \cdot \rangle_{B',B}$ the duality pairing between B' and B.

In particular, we put $H := L^2(0,L)$ with the usual real Hilbert structure, and denote by $(\cdot,\cdot)_H$ the inner product in H, for simplicity.

Also, let X be the Sobolev space $H^1(0,L)$ with the norm

$$|z|_X := \left\{ |z_x|_H^2 + n_0 \left(|z(0)|^2 + |z(L)|^2 \right) \right\}^{1/2} \qquad \text{for any } z \in X,$$

which is equivalent to the standard norm of $H^1(0,L)$. We denote by X' the dual space of X. Also, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X' and X. By identifying Hilbert spaces with their duals, we suppose that

$$X \subset H = H' \subset X' \tag{1.17}$$

with dense and compact embeddings, and then we have $\langle v, z \rangle = (v,z)_H$ for $v \in H$ and $z \in X$. Furthermore, let $F : X \to X'$ be the duality mapping defined by

$$\langle Fv, z \rangle := (v_x, z_x)_H + n_0 \left(v(0)z(0) + v(L)z(L) \right) \qquad \text{for all } v, z \in X. \tag{1.18}$$

Also, for given $f \in H$, $h \in \mathbb{R}$, $\ell \in \mathbb{R}$, $a_0 \in \mathbb{R}$, $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}$, $b_1 \in \mathbb{R}$, $b_2 \in \mathbb{R}$, and $n_0 \in \mathbb{R}$, an element $\widetilde{f} \in X'$ is uniquely determined by

$$\langle \widetilde{f}, z \rangle := (a_0 f, z)_H + (a_1 h + n_0 b_1) z(0) + (a_2 \ell + n_0 b_2) z(L) \qquad \text{for all } z \in X.$$

For this $\widetilde{f}$, it is easy to check that $Fv = \widetilde{f}$ is formally equivalent to

$$\begin{cases} -v_{xx} = a_0 f \quad \text{in } (0,L), \\ -v_x(0) + n_0(v(0) - b_1) = a_1 h, \quad v_x(L) + n_0(v(L) - b_2) = a_2 \ell. \end{cases} \tag{1.19}$$

Note that X' becomes a Hilbert space with inner product $(\cdot,\cdot)_{X'}$ given by

$$(v,z)_{X'} := \langle v, F^{-1}z \rangle \quad \text{for all } v, z \in X'.$$

We next list some notation and definitions of subdifferentials of convex functions. For a proper (i.e., not identically equal to infinity), l.s.c. (lower semi-continuous), and convex function $\psi : H \to \mathbb{R} \cup \{\infty\}$, the effective domain $D(\psi)$ of ψ is defined by $D(\psi) := \{z \in H;\ \psi(z) < \infty\}$. We denote by $\partial\psi$ the subdifferential of ψ in the topology of H. In general, the subdifferential is a possibly multi-valued operator from H into itself, and for any $z \in H$, the value $\partial\psi(z)$ is defined as:

$$\partial\psi(z) := \{z^* \in H\ ;\ (z^*, y - z)_H \le \psi(y) - \psi(z) \quad \text{for all } y \in H\}.$$

Then, a set $D(\partial\psi) := \{z \in H\ ;\ \partial\psi(z) \ne \emptyset\}$ is called the domain of $\partial\psi$. For various properties and related notions of a proper, l.s.c., convex function ψ and its subdifferential $\partial\psi$, we refer to the monograph by Brézis [11]. In particular, for those in Banach spaces, we quote the books by Barbu [8, 9].

We also recall a notion of convergence for convex functions, developed by Mosco [32].

Definition 1.1 (cf. [32])**.** Let ψ, ψ_n $(n \in \mathbb{N})$ be proper, l.s.c., and convex functions on H. Then, we say that ψ_n converges to ψ on H in the sense of Mosco [32] as $n \to \infty$ if the following two conditions are satisfied:

(i) for any subsequence $\{\psi_{n_k}\}_{k\in\mathbb{N}} \subset \{\psi_n\}_{n\in\mathbb{N}}$, if $z_k \to z$ weakly in H as $k \to \infty$, then

$$\liminf_{k\to\infty} \psi_{n_k}(z_k) \ge \psi(z);$$

(ii) for any $z \in D(\psi)$, there is a sequence $\{z_n\}_{n\in\mathbb{N}}$ in H such that

$$z_n \to z \text{ in } H \text{ as } n \to \infty \quad \text{and} \quad \lim_{n\to\infty} \psi_n(z_n) = \psi(z).$$

As well as, if the sequence of convex functions $\{\psi_\varepsilon\}_{\varepsilon\in\Xi}$ is labeled by a continuous argument $\varepsilon \in \Xi$ with a infinite set $\Xi \subset \mathbb{R}$, then for any $\varepsilon_0 \in \Xi$, the Mosco-convergence of $\{\psi_\varepsilon\}_{\varepsilon\in\Xi}$, as $\varepsilon \to \varepsilon_0$, is defined by those of subsequences $\{\psi_{\varepsilon_n}\}_{n\in\mathbb{N}}$, for all sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \Xi$, satisfying $\varepsilon_n \to \varepsilon_0$ as $n \to \infty$.

Finally, throughout this paper, N_i, $i = 1, 2, 3, \cdots$, denotes positive (or nonnegative) constants depending only on their argument(s).

2 Preliminaries

In this section, we recall the fundamentals concerned with the total variation and functions of bounded variation. These notions are rigorously defined as follows.

Definition 2.1. **(I)** Let $z \in L^1(0,L)$. Then, z is called a function of bounded variation, or simply a BV-function, on $(0,L)$, if and only if:

$$V_0(z) := \sup \left\{ \int_0^L z\varpi_x dx; \begin{array}{l} \varpi \in C^1[0,L] \text{ with a compact support on } (0,L), \\ |\varpi| \leq 1 \text{ on } [0,L] \end{array} \right\} < \infty.$$

Here, we call $V_0(z)$ the total variation of z.
(II) We denote by $BV(0,L)$ the space of all BV-functions on $(0,L)$.

Here are listed usual properties of BV-functions and the space $BV(0,L)$, in forms of some propositions and remarks.

Proposition 2.1 (cf. [15, Chapter 5]). *Let $z \in BV(0,L)$. Then, there exists a Radon measure $|Dz|$ on $(0,L)$, and $|Dz|$-measurable function $\sigma_z : (0,L) \to \mathbb{R}$ such that*

(i) $V_0(z) = \int_0^L |Dz|$, *and* $|\sigma_z| = 1$, $|Dz|$*-a.e. on* $(0,L)$*;*

(ii) $\int_0^L z\varpi_x dx = -\int_0^L \varpi\ \sigma_z |Dz|$ *for any* $\varpi \in C^1[0,L]$ *with a compact support on* $(0,L)$*.*

Remark 2.1. If z belongs to the Sobolev space $W^{1,1}(0,L)$, then $|Dz|$ is absolutely continuous with respect to the Lebesgue measure, and it follows that:

$$\int_U |Dz| = \int_U |z_x(x)| dx \qquad \text{for all Borel subsets } U \subset (0,L)$$

and

$$\sigma_z(x) = \begin{cases} \dfrac{z_x(x)}{|z_x(x)|}, & \text{if } z_x(x) \neq 0, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{a.a. } x \in (0,L).$$

Proposition 2.2 (cf. [7, Chapter 10], [15, Chapter 5]). **(I)** *The functional* $z \in L^1(0,L) \mapsto V_0(z)$ *forms a proper, l.s.c., and convex function on* $L^1(0,L)$*.*
(II) *The space* $BV(0,L)$ *is a Banach space with the norm:*

$$|z|_{BV(0,L)} := |z|_{L^1(0,L)} + V_0(z) \qquad \text{for all } z \in BV(0,L).$$

Proposition 2.3 (cf. [2, Corollary 3.49], [7, Chapter 10]). *$BV(0,L)$ is continuously embedded in $L^\infty(0,L)$, and compactly embedded in $L^p(0,L)$ for any $1 \leq p < \infty$.*

Next, let us set a proper, l.s.c., and convex functional $\mathcal{I}_{[-1,1]}$ on H, by putting:

$$\mathcal{I}_{[-1,1]}(z) := \int_0^L I_{[-1,1]}(z(x))dx \quad \text{for all } z \in H;$$

to define the following total variation functional V^0 with a constraint by the indicator function $I_{[-1,1]}$:

$$V^0(z) = V_0(z) + \frac{1}{\kappa}\mathcal{I}_{[-1,1]}(z) \quad \text{for all } z \in H. \tag{2.1}$$

Clearly, V^0 is proper, l.s.c., and convex on H, and its effective domain is formulated by:

$$D(V^0) = \{z \in BV(0,L)\ ;\ |z| \leq 1, \text{ a.e. on } (0,L)\}.$$

Finally, we recall the decomposition result of the subdifferential ∂V^0 of V^0. For the detailed proof, we refer to [44, Theorem 3.1].

Proposition 2.4 (cf. [44, Theorem 3.1])**.** *The subdifferential ∂V^0 of V^0 is decomposed into the following form:*

$$\partial V^0(z) = \partial\left(V_0|_H\right)(z) + \frac{1}{\kappa}\partial\mathcal{I}_{[-1,1]}(z) \text{ in } H \quad \text{for all } z \in H,$$

where $V_0|_H$ denotes the restriction of V_0 onto H.

3 Solvability of (P;f, h, ℓ)$^\varepsilon$

In this section, we discuss the existence-uniqueness of solutions to (P;f, h, ℓ)$^\varepsilon$ for any $\varepsilon \geq 0$.

We begin with giving some assumptions on data. Throughout this paper, we assume the following conditions (A1)–(A4).

(A1) $\widehat{a}^0$ is an absolute value function on $\mathbb{R}$, i.e., $\widehat{a}^0(r) := |r|$ for all $r \in \mathbb{R}$. In addition, let $\{\widehat{a}^\varepsilon\}_{\varepsilon\in(0,1]} \subset C^1(\mathbb{R})$ be a sequence of convex functions and C^1-regularizations for $\widehat{a}^0(\cdot) := |\cdot|$, such that:

$$\widehat{a}^\varepsilon(r) \geq 0 \ \text{ for any } r \in \mathbb{R} \text{ and any } \varepsilon \in (0,1],$$

$$\left\{\begin{array}{ll} \widehat{a}^\varepsilon(r) \to \widehat{a}^0(r) \ \text{ for any } r \in \mathbb{R}, \\ \widehat{a}^\varepsilon(\cdot) \to \widehat{a}^0(\cdot) \ \text{ on } \mathbb{R}, \ \text{ in the sense of Mosco,} \end{array}\right. \text{ as } \varepsilon \downarrow 0,$$

and there exists a constant $\delta_0 > 0$, independent of $\varepsilon \in (0,1]$, satisfying:

$$|a^\varepsilon(r)| \leq \delta_0(|r|+1) \ \text{ for any } r \in \mathbb{R} \text{ and any } \varepsilon \in (0,1],$$

where $a^\varepsilon := (\widehat{a}^\varepsilon)'$ is the derivative of $\widehat{a}^\varepsilon$. Furthermore, there exist bounded functions $\delta_1 : (0,1] \to (0,1]$ and $\delta_2 : (0,1] \to [0,\infty)$ such that

$$\delta_1(\varepsilon) \to 1, \quad \delta_2(\varepsilon) \to 0 \ \text{ as } \varepsilon \downarrow 0,$$

and

$$\widehat{a}^\varepsilon(r) \geq \delta_1(\varepsilon)\widehat{a}^0(r) - \delta_2(\varepsilon) \ \text{ for any } r \in \mathbb{R} \text{ and any } \varepsilon \in (0,1].$$

(A2) g is a continuous and semi-monotone function on $\mathbb{R}$, i.e., there is a constant $C_g > 0$ such that $g(r) + C_g r$ is monotone in $r \in \mathbb{R}$. In addition, the function $g(r)$ has a non-negative potential function $\widehat{g}(r)$, that is,

$$\widehat{g}(r) \geq 0 \quad \text{and} \quad (\widehat{g})'(r) = g(r) \quad \text{for any } r \in \mathbb{R}.$$

(A3) $T > 0$, $L > 0$, $\kappa > 0$, $n_0 > 0$, $c_0 \geq 0$, $c_1 \geq 0$, $m_0 \geq 0$, $m_1 \geq 0$, $m_2 \geq 0$ are fixed constants. Also, a_0, a_1, a_2, b_1, b_2 are fixed real numbers.

(A4) u_d and w_d are the given desired target profiles in $L^2(0,T;H)$.

Remark 3.1 (cf. [14, 31]). The assumption (A1) was introduced in [14, (A4)]. The similar assumption was found in [31, Definition 3.1]. In addition, the typical examples of $\widehat{a}^\varepsilon$ are the followings:

- (Hyperbola type) $\widehat{a}^\varepsilon(r) = \sqrt{r^2 + \varepsilon^2}$ for any $r \in \mathbb{R}$ and any $\varepsilon \in (0,1]$.
- (Hyperbolic-tangent type) $\widehat{a}^\varepsilon(r) = \varepsilon \log\left(\cosh\left(\frac{r}{\varepsilon}\right)\right)$ for any $r \in \mathbb{R}$ and any $\varepsilon \in (0,1]$.
- (Arctangent type) $\widehat{a}^\varepsilon(r) = \frac{2\varepsilon}{\pi}\left[\frac{r}{\varepsilon}\tan^{-1}\left(\frac{r}{\varepsilon}\right) - \frac{1}{2}\log\left(1 + \left(\frac{r}{\varepsilon}\right)^2\right)\right]$ for any $r \in \mathbb{R}$ and any $\varepsilon \in (0,1]$.
- (p-growth type) $\widehat{a}^\varepsilon(r) = \frac{1}{1+\varepsilon^2}|r^2 + \varepsilon^2|^{\frac{1+\varepsilon^2}{2}}$ for any $r \in \mathbb{R}$ and any $\varepsilon \in (0,1]$.

Clearly, such functions satisfy (A1).

We now give the notion of solutions to (P;f,h,ℓ)$^\varepsilon$. To this end, for given $f \in L^2(0,T;H)$, $h \in L^2(0,T)$, and $\ell \in L^2(0,T)$, we define $\widetilde{f} \in L^2(0,T;X')$ by putting

$$\begin{gathered}\langle \widetilde{f}(t), z\rangle := (a_0 f(t), z)_H + (a_1 h(t) + n_0 b_1) z(0) + (a_2 \ell(t) + n_0 b_2) z(L) \\ \text{for all } z \in X \text{ and a.a. } t \in (0,T).\end{gathered} \tag{3.1}$$

In addition, let K^ε be a function on $\mathbb{R}$ defined by (1.9). Clearly, K^ε is a C^1-function with derivative $(K^\varepsilon)' \in W^{1,\infty}(\mathbb{R})$. We fix a primitive $\widehat{K}^\varepsilon \in C^2(\mathbb{R}) \cap W^{3,\infty}_{loc}(\mathbb{R})$ of K^ε such that

$$\widehat{K}^\varepsilon(0) = 0 \quad \text{and} \quad \widehat{K}^\varepsilon(r) \geq 0 \quad \text{for all } r \in \mathbb{R}. \tag{3.2}$$

Then, for any $\varepsilon \in (0,1]$, let us set:

$$V^\varepsilon(z) := \begin{cases} \displaystyle\int_0^L \widehat{a}^\varepsilon(z_x(x))dx + \frac{\varepsilon}{2}\int_0^L |z_x(x)|^2 dx + \frac{1}{\kappa}\int_0^L \widehat{K}^\varepsilon(z(x))\,dx, & \text{if } z \in X, \\ \infty, & \text{otherwise.}\end{cases} \tag{3.3}$$

Clearly, each functional V^ε ($\varepsilon \in (0,1]$) forms a proper, l.s.c., and convex functional on H.

Based on functionals V^ε ($\varepsilon \in (0,1]$) and V^0 (cf. (2.1)), the solutions to (P;f,h,ℓ)$^\varepsilon$, for $\varepsilon \geq 0$, are defined as follows.

Definition 3.1. Let $\varepsilon \in [0,1]$, $u_0 \in X'$, and $w_0 \in H$. Then, a couple of functions $[u^\varepsilon, w^\varepsilon]$ is called a solution to (P;$f,h,\ell)^\varepsilon$, or (P;$u_0, w_0, f, h, \ell)^\varepsilon$ when the initial data are specified, on $[0,T]$, if the following conditions are satisfied:

(S1) $u^\varepsilon \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H)$.

(S2) $w^\varepsilon \in W^{1,2}(0,T;H)$ with $V^\varepsilon(w^\varepsilon) \in L^\infty(0,T)$.

(S3) For all $z \in X$ and a.a. $t \in (0,T)$,

$$\langle (u^\varepsilon)'(t), z\rangle + ((w^\varepsilon)'(t), z)_H + \langle Fu^\varepsilon(t), z\rangle = \langle \widetilde{f}(t), z\rangle.$$

(S4) There is a function $(w^\varepsilon)^* \in L^2(0,T;H)$ such that $(w^\varepsilon)^*(t) \in \partial V^\varepsilon(w^\varepsilon(t))$ in H and

$$(w^\varepsilon)'(t) + \kappa(w^\varepsilon)^*(t) + g(w^\varepsilon(t)) = u^\varepsilon(t) \text{ in } H, \text{ a.a. } t \in (0,T).$$

(S5) $u^\varepsilon(0) = u_0$ in X' and $w^\varepsilon(0) = w_0$ in H.

Remark 3.2. By the definition of subdifferentials, we observe that the evolution equation in (S4) of Definition 3.1 is equivalent to the following variational inequality:

$$\begin{gathered}((w^\varepsilon)'(t) + g(w^\varepsilon(t)) - u^\varepsilon(t), w^\varepsilon(t) - z)_H + \kappa V^\varepsilon(w^\varepsilon(t)) - \kappa V^\varepsilon(z) \le 0 \\ \text{for any } z \in D(V^\varepsilon) \text{ and a.a. } t \in (0,T).\end{gathered} \tag{3.4}$$

Note that (3.4) corresponds to a weak formulation of the second equation of (P;$f,h,\ell)^\varepsilon$, for any $\varepsilon \ge 0$.

Remark 3.3. Let $\varepsilon \in (0,1]$. Then, note that the subdifferential operator ∂V^ε is single-valued. In addition, we observe from the definition of subdifferential that $w^* = \partial V^\varepsilon(w^\varepsilon)$ if and only if

$$(w^*, z)_H = (a^\varepsilon(w^\varepsilon_x) + \varepsilon w^\varepsilon_x, z_x)_H + \frac{1}{\kappa}\left(K^\varepsilon(w^\varepsilon), z\right)_H, \quad \forall z \in D(V^\varepsilon).$$

The expression of ∂V^ε is obtained by computing the first variations of the convex function V^ε, and the variational inequality (3.4) implicitly includes the homogeneous Neumann type boundary condition.

Remark 3.4 (cf. [45, Remarks 3.1, 3.2, and 3.3])**.** Let $\varepsilon = 0$. By Proposition 2.4, the condition (S4) of Definition 3.1 is equivalent to the following condition (S4)$'$:

(S4)$'$ There is a function $(w_0^\varepsilon)^* \in L^2(0,T;H)$ and a function $\xi^\varepsilon \in L^2(0,T;H)$ such that

$$(w_0^\varepsilon)^*(t) \in \partial\left(V_0|_H\right)(w^\varepsilon(t)) \text{ in } H, \quad \xi^\varepsilon(t) \in \partial \mathcal{I}_{[-1,1]}(w^\varepsilon(t)) \text{ in } H,$$

$$(w^\varepsilon)'(t) + \kappa(w_0^\varepsilon)^*(t) + \xi^\varepsilon(t) + g(w^\varepsilon(t)) = u^\varepsilon(t) \text{ in } H$$

for a.a. $t \in (0,T)$.

Note that the function $(w_0^\varepsilon)^* \in L^2(0,T;H)$ as in (S4)' somehow links to the first variation of the total variation functional $V_0|_H$. In addition, as is well-known (cf. [11, Proposition 2.16]),

$$\partial \mathcal{I}_{[-1,1]}(z) = \left\{ \xi \in H;\ \xi \in \partial I_{[-1,1]}(z), \text{ a.e. on } (0,L) \right\} \text{ for any } z \in D(\mathcal{I}_{[-1,1]}).$$

Hence, the subdifferential ∂V^0 corresponds to the rigorous expression of the singular term $-\left(\frac{w_x}{|w_x|}\right)_x + \frac{1}{\kappa}\partial I_{[-1,1]}(w)$ as in (1.2), and the variational inequality (3.4) implicitly includes the homogeneous Neumann type boundary condition.

We now mention the first main result in this paper, which is concerned with the existence-uniqueness of solutions to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ for each $\varepsilon \in [0,1]$.

Theorem 3.1 (cf. [45, Propositions 3.1 and 3.2])**.** *Assume* (A1)*,* (A2)*,* (A3)*, and $\varepsilon \in [0,1]$. Let $[f,h,\ell]$ be arbitrary triplet of functions in $\mathcal{U}$. Then, for each $u_0 \in H$ and $w_0 \in D(V^\varepsilon)$, there is a unique solution $[u^\varepsilon, w^\varepsilon]$ to* (P;$u_0, w_0, f, h, \ell)^\varepsilon$ *on $[0,T]$. In addition, there is a positive constant N_1, dependent only on T and n_0, and independent of $\varepsilon \in [0,1]$, such that the following bounded estimate holds:*

$$\begin{aligned}
&|(u^\varepsilon)'|^2_{L^2(0,T;X')} + |u^\varepsilon|^2_{L^\infty(0,T;H)} + |u^\varepsilon|^2_{L^2(0,T;X)} + |(w^\varepsilon)'|^2_{L^2(0,T;H)} + |w^\varepsilon|^2_{L^\infty(0,T;H)} \\
&\qquad + \kappa \sup_{0\le t\le T} V^\varepsilon(w^\varepsilon(t)) + \sup_{0\le t\le T} \int_0^L \widehat{g}(w^\varepsilon(t,x))\,dx \\
&\le \ N_1 \Biggl(|u_0|^2_H + |w_0|^2_H + \kappa V^\varepsilon(w_0) + \int_0^L \widehat{g}(w_0(x))\,dx + a_0^2 |f|^2_{L^2(0,T;H)} \\
&\qquad + a_1^2 |h|^2_{L^2(0,T)} + a_2^2 |\ell|^2_{L^2(0,T)} + b_1^2 + b_2^2 \Biggr).
\end{aligned} \tag{3.5}$$

Proof. By a similar argument to [24, Theorem 2.1], we get the unique solution $[u^\varepsilon, w^\varepsilon]$ to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ on $[0,T]$. In fact, let $[u_i^\varepsilon, w_i^\varepsilon]$ $(i=1,2)$ be two solutions to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ on $[0,T]$. Then, note that the following variational identity holds:

$$\begin{aligned}
&\langle ((u_1^\varepsilon)' - (u_2^\varepsilon)')(\tau), z\rangle + (((w_1^\varepsilon)' - (w_2^\varepsilon)')(\tau), z)_H + \langle (Fu_1^\varepsilon - Fu_2^\varepsilon)(\tau), z\rangle = 0 \\
&\qquad \text{for all } z \in X \text{ and a.a. } \tau \in (0,T).
\end{aligned} \tag{3.6}$$

By integrating (3.6) in time, we obtain that

$$\begin{aligned}
&((u_1^\varepsilon - u_2^\varepsilon)(t), z)_H + ((w_1^\varepsilon - w_2^\varepsilon)(t), z)_H + \left((\textstyle\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau)d\tau)_x, z_x \right)_H \\
&+ n_0 \left\{ \left(\textstyle\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,0)d\tau \right) z(0) + \left(\textstyle\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,L)d\tau \right) z(L) \right\} = 0 \\
&\qquad \text{for all } z \in X \text{ and all } t \in [0,T].
\end{aligned} \tag{3.7}$$

Taking $z = (u_1^\varepsilon - u_2^\varepsilon)(t)$ in (3.7), we get that

$$\begin{aligned}
&|(u_1^\varepsilon - u_2^\varepsilon)(t)|^2_H + ((w_1^\varepsilon - w_2^\varepsilon)(t), (u_1^\varepsilon - u_2^\varepsilon)(t))_H + \frac{1}{2}\frac{d}{dt}\left| \left(\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau)d\tau \right)_x \right|^2_H \\
&+ \frac{n_0}{2}\frac{d}{dt}\left\{ \left| \int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,0)d\tau \right|^2 + \left| \int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,L)d\tau \right|^2 \right\} = 0 \\
&\qquad \text{for all } t \in [0,T].
\end{aligned} \tag{3.8}$$

By using the Schwarz inequality in (3.8), and integrating in time, we obtain:

$$\begin{aligned}
&\frac{1}{2}\int_0^t |(u_1^\varepsilon - u_2^\varepsilon)(\tau)|_H^2 d\tau + \frac{1}{2}\left|\left(\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau)d\tau\right)_x\right|_H^2 \\
&+\frac{n_0}{2}\left\{\left|\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,0)d\tau\right|^2 + \left|\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,L)d\tau\right|^2\right\} \\
\leq\ &\frac{1}{2}\int_0^t |(w_1^\varepsilon - w_2^\varepsilon)(\tau)|_H^2 d\tau \quad \text{for all } t \in [0,T].
\end{aligned} \tag{3.9}$$

Note from the monotonicity of ∂V^ε that

$$(((w_1^\varepsilon)^* - (w_2^\varepsilon)^*)(\tau), (w_1^\varepsilon - w_2^\varepsilon)(\tau))_H \geq 0, \qquad \text{a.a. } \tau \in (0,T),$$

where $(w_i^\varepsilon)^*(\tau) \in \partial V^\varepsilon(w_i^\varepsilon(\tau))$ in H for a.a. $\tau \in (0,T)$ $(i = 1,2)$. Therefore, it follows from (S4) of Definition 3.1 and (A2), i.e., the monotonicity of $g(r) + C_g r$, that

$$\begin{aligned}
\frac{1}{2}\frac{d}{d\tau}|(w_1^\varepsilon - w_2^\varepsilon)(\tau)|_H^2 \leq ((u_1^\varepsilon - u_2^\varepsilon)(\tau), (w_1^\varepsilon - w_2^\varepsilon)(\tau))_H + C_g|(w_1^\varepsilon - w_2^\varepsilon)(\tau)|_H^2 \\
\text{for a.a. } \tau \in (0,T).
\end{aligned} \tag{3.10}$$

By using the Schwarz inequality in (3.10), and integrating in time, we obtain:

$$\begin{aligned}
\frac{1}{2}|(w_1^\varepsilon - w_2^\varepsilon)(t)|_H^2 \leq \left(\frac{1}{2} + C_g\right)\int_0^t |(w_1^\varepsilon - w_2^\varepsilon)(\tau)|_H^2 d\tau + \frac{1}{2}\int_0^t |(u_1^\varepsilon - u_2^\varepsilon)(\tau)|_H^2 d\tau \\
\text{for all } t \in [0,T].
\end{aligned} \tag{3.11}$$

Hence, we infer from (3.9) and (3.11) that

$$\frac{1}{2}|(w_1^\varepsilon - w_2^\varepsilon)(t)|_H^2 \leq (1 + C_g)\int_0^t |(w_1^\varepsilon - w_2^\varepsilon)(\tau)|_H^2 d\tau \quad \text{for all } t \in [0,T]. \tag{3.12}$$

Thus, applying the Gronwall inequality to (3.12), we observe that

$$w_1^\varepsilon(t) = w_2^\varepsilon(t) \text{ in } H \text{ for all } t \in [0,T]. \tag{3.13}$$

By the quite standard arguments, we conclude from (3.6) with (3.13) that

$$u_1^\varepsilon(t) = u_2^\varepsilon(t) \text{ in } H \text{ for all } t \in [0,T]. \tag{3.14}$$

Thus, the solutions to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ on $[0,T]$ is unique.

Now, we show the existence of solutions to (P;$u_0, w_0, f, h, \ell)^\varepsilon$. Note that (P;$u_0, w_0, f, h, \ell)^\varepsilon$ can be reformulated to abstract evolution equations of the form:

$$(u^\varepsilon)'(t) + (w^\varepsilon)'(t) + \partial\varphi(u^\varepsilon(t)) \ni \widetilde{f}(t) \ \text{ in } X', \ \text{ for } t \in (0,T), \tag{3.15}$$

$$(w^\varepsilon)'(t) + \kappa\partial V^\varepsilon(w^\varepsilon(t)) + g(w^\varepsilon(t)) \ni u^\varepsilon(t) \ \text{ in } H, \ \text{ for } t \in (0,T), \tag{3.16}$$

$$u^\varepsilon(0) = u_0 \text{ in } X' \text{ and } \ w^\varepsilon(0) = w_0 \text{ in } H, \tag{3.17}$$

where $\partial\varphi(\cdot)$ is the subdifferential of a convex function $\varphi(\cdot)$ on X' defined by

$$\varphi(z) := \begin{cases} \frac{1}{2}|z|_H^2, & \text{if } z \in H, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.18}$$

Also, $\partial V^\varepsilon(\cdot)$ is the subdifferential of the convex functional V^ε on H defined in (3.3).

We first show the existence of a local (in time) solution to (3.15)–(3.17) by employing the fixed point argument for continuous operators in compact convex sets. To this end, for $T > 0$ and $M > 0$, we define a (non-empty) compact convex subset $E(T, M)$ of $L^2(0, T; H)$ by

$$E(T, M) := \left\{ u \in L^2(0,T;H) \;\middle|\; \begin{array}{c} u \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H), \\ |u'|^2_{L^2(0,T;X')} + |u|^2_{L^2(0,T;X)} + \sup\limits_{0 \le t \le T} |u(t)|_H^2 \le M \end{array} \right\}.$$

Now, for each $\overline{u} \in E(T, M)$ we consider the following problem, denoted by $(\mathrm{P2})_{\overline{u}}$, with a given function $\overline{u} \in E(T, M)$. For a moment, we often omit the superscript $\varepsilon \in [0, 1]$.

Problem $(\mathbf{P2})_{\overline{u}}$. Find a function $w : [0, T] \to H$ which fulfills the following equation:

$$w'(t) + \kappa\partial V^\varepsilon(w(t)) + g(w(t)) \ni \overline{u}(t) \ \text{ in } H, \quad \text{for } t \in (0, T), \tag{3.19}$$

$$w(0) = w_0 \ \text{ in } H. \tag{3.20}$$

Taking account of (2.1) with Propositions 2.2–2.3 and (3.3) with (A1), we observe that for each $\varepsilon \in [0, 1]$, V^ε is proper, l.s.c., and convex on H such that the level set of V^ε is compact in H, i.e.,

$$\{z \in H \ ; \ V^\varepsilon(z) \le r\} \text{ is compact in } H \text{ for any } r > 0. \tag{3.21}$$

Therefore, by using the abstract theory established by Brézis [11] and the perturbation theory (cf. [12, 22]), we observe that $(\mathrm{P2})_{\overline{u}}$ has a unique solution $w \in W^{1,2}(0, T; H)$ with $V^\varepsilon(w) \in L^\infty(0, T)$ for each $w_0 \in D(V^\varepsilon)$ and $\overline{u} \in E(T, M)$. Indeed, (3.19) is equivalent to the following equation:

$$w'(t) + \kappa\partial V^\varepsilon(w(t)) + g(w(t)) + C_g w(t) - C_g w(t) \ni \overline{u}(t) \ \text{ in } H, \quad \text{for } t \in (0, T),$$

where C_g is the positive constant in (A2). Therefore, by applying the general theory of evolution equations with monotone and Lipschitz linear perturbations, we can get the unique solution to $(\mathrm{P2})_{\overline{u}}$ on $[0, T]$.

Moreover, by the standard calculation (cf. (3.29) below), we can obtain the following inequality:

$$\begin{aligned} &\int_0^t |w'(\tau)|_H^2 d\tau + 2\kappa V^\varepsilon(w(t)) + 2\int_0^L \widehat{g}(w(t,x))dx \\ \le\ & 2\kappa V^\varepsilon(w_0) + 2\int_0^L \widehat{g}(w_0(x))dx + \int_0^T |\overline{u}(\tau)|_H^2 d\tau, \quad \forall t \in (0, T). \end{aligned} \tag{3.22}$$

Next, for the function w constructed above, we consider the following problem, denoted by $(\mathrm{P1})_w$.

Problem $(\mathbf{P1})_w$. Find a function $u : [0,T] \to X'$ which fulfills the following equation:

$$u'(t) + \partial\varphi(u(t)) \ni \widetilde{f}(t) - w'(t) \quad \text{in } X', \quad \text{for } t \in (0,T), \tag{3.23}$$

$$u(0) = u_0 \quad \text{in } X'. \tag{3.24}$$

We observe from (1.17) and (3.18) that φ is proper, l.s.c., and convex on X' such that the level set of φ is compact in X'. Since $\widetilde{f} - w' \in L^2(0,T;X')$, we can apply the abstract theory established by Brézis [11]. Thus, we observe that $(\mathrm{P1})_w$ has a unique solution $u \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H)$ for each $u_0 \in H$ and solution w to $(\mathrm{P2})_{\overline{u}}$. Moreover, by the standard calculation (cf. (3.28) below), we can obtain the following inequality:

$$\begin{aligned}
&|u(t)|_H^2 + 2\int_0^t |u_x(\tau)|_H^2 d\tau + n_0\left(\int_0^t |u(\tau,0)|^2 d\tau + \int_0^t |u(\tau,L)|^2 d\tau\right) \\
\le\ & e^{2T}\left(|u_0|_H^2 + a_0^2\int_0^T |f(\tau)|_H^2 d\tau + \frac{2a_1^2}{n_0}\int_0^T |h(\tau)|^2 d\tau + \frac{2a_2^2}{n_0}\int_0^T |\ell(\tau)|^2 d\tau\right) \\
&+2n_0Te^{2T}(b_1^2 + b_2^2) + e^{2T}\int_0^T |w'(\tau)|_H^2 d\tau, \quad \forall t \in (0,T).
\end{aligned} \tag{3.25}$$

Note that the solution u to $(\mathrm{P1})_w$ satisfies the following identity (cf. (S3) in Definition 3.1):

$$\begin{gathered}
\int_0^T \langle u'(t), \zeta(t)\rangle dt + \int_0^T \langle Fu(t), \zeta(t)\rangle dt = \int_0^T \langle \widetilde{f}(t), \zeta(t)\rangle dt - \int_0^T (w'(t), \zeta(t))_H dt \\
\text{for all } \zeta \in L^2(0,T;X).
\end{gathered} \tag{3.26}$$

From (1.18), (3.25), and (3.26), we infer that

$$\begin{aligned}
|u'|_{L^2(0,T;X')}^2 \ \le N_2 \Big(& |u_x|_{L^2(0,T;H)}^2 + n_0|u(\cdot,0)|_{L^2(0,T)}^2 + n_0|u(\cdot,L)|_{L^2(0,T)}^2 \\
& + a_0^2|f|_{L^2(0,T;H)}^2 + \frac{a_1^2}{n_0}|h|_{L^2(0,T)}^2 + \frac{a_2^2}{n_0}|\ell|_{L^2(0,T)}^2 \\
& + |w'|_{L^2(0,T;H)}^2 + n_0T(b_1^2 + b_2^2)\Big)
\end{aligned} \tag{3.27}$$

for some constant $N_2 > 0$ independent of the given function $\overline{u} \in E(T,M)$.

Here, we define an operator $S : E(T,M) \longrightarrow L^2(0,T;H)$ as follows. For each $\overline{u} \in E(T,M)$, we denote by w a unique solution to $(\mathrm{P2})_{\overline{u}}$, and subsequently, we denote by u a unique solution to $(\mathrm{P1})_w$. On that basis, for any given $\overline{u} \in E(T,M)$, we put $S\overline{u} = u$ via the solution w.

Now, we show that S is a self-mapping on $E(T_0,M_0)$ for some positive constants T_0 and M_0, i.e., $S\overline{u}(= u) \in E(T_0,M_0)$ for any $\overline{u} \in E(T_0,M_0)$.

Here, we take $M_0 > 0$ so large such that

$$(4N_2 + 8)\left(|u_0|_H^2 + a_0^2|f|_{L^2(0,T;H)}^2 + \frac{a_1^2}{n_0}|h|_{L^2(0,T)}^2 + \frac{a_2^2}{n_0}|\ell|_{L^2(0,T)}^2\right)$$

$$+N_2\left(a_0^2|f|^2_{L^2(0,T;H)}+\frac{a_1^2}{n_0}|h|^2_{L^2(0,T)}+\frac{a_2^2}{n_0}|\ell|^2_{L^2(0,T)}\right)$$

$$+n_0(b_1^2+b_2^2)+(6N_2+8)\left(\kappa V^\varepsilon(w_0)+\int_0^L \widehat{g}(w_0(x))dx+1\right)\le M_0,$$

and then, choose $T_0\in(0,T]$ so small such that

$$e^{2T_0}\le 2,\quad M_0T_0\le 1,\quad N_2T_0+4T_0e^{2T_0}+2N_2T_0e^{2T_0}\le 1.$$

Then, estimates (3.22), (3.25), (3.27) implies that $S\overline{u}(=u)$ belongs to the set $E(T_0,M_0)$ for $\overline{u}\in E(T_0,M_0)$. Thus, the mapping S maps the set $E(T_0,M_0)$ into itself for T_0 and M_0 chosen as above.

Moreover, on account of the convergence theory as in [6], we observe that S is continuous in $E(T_0,M_0)$ with respect to the topology of $L^2(0,T;H)$ (cf. Corollary 4.2 below). Therefore, the Schauder fixed point theorem guarantees that S has at least one fixed point u in $E(T_0,M_0)$. The pair of functions $[u,w]$, consisting of the fixed point u of S and the solution w to $(\mathrm{P})_{\overline{u}}$ when $\overline{u}=u$, is a solution to $(\mathrm{P};u_0,w_0,f,h,\ell)^\varepsilon$ on the time interval $[0,T_0]$. Thus, we have shown that the problem $(\mathrm{P};u_0,w_0,f,h,\ell)^\varepsilon$ has a local (in time) solution $[u,w]$ on $[0,T_0]$.

We now give the energy estimate of the local (in time) solution $[u,w]$ to $(\mathrm{P};u_0,w_0,f,h,\ell)^\varepsilon$ on $[0,T_0]$. To this end, take $z=u(t)$ in (S3) of Definition 3.1. Then, by using the Schwarz inequality, we have:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u(t)|_H^2+(w'(t),u(t))_H+|u_x(t)|_H^2+\frac{n_0}{2}|u(t,0)|^2+\frac{n_0}{2}|u(t,L)|^2\\
\le\ &\frac{1}{2}|u(t)|_H^2+\frac{a_0^2}{2}|f(t)|_H^2+\frac{a_1^2}{n_0}|h(t)|^2+\frac{a_2^2}{n_0}|\ell(t)|^2+n_0(b_1^2+b_2^2)\\
&\qquad\text{for a.a. } t\in(0,T_0).
\end{aligned}\tag{3.28}$$

Next, multiplying (3.16) by $w'(t)$ (cf. (S4) of Definition 3.1), we get:

$$\begin{aligned}
&|w'(t)|_H^2+\kappa\frac{d}{dt}V^\varepsilon(w(t))+\frac{d}{dt}\int_0^L\widehat{g}(w(t,x))dx=(u(t),w'(t))_H\\
&\qquad\text{for a.a. } t\in(0,T_0).
\end{aligned}\tag{3.29}$$

Adding (3.29) to (3.28), and applying the Gronwall inequality to the resultant, we have:

$$\begin{aligned}
&\frac{1}{2}|u(t)|_H^2+\kappa V^\varepsilon(w(t))+\int_0^L\widehat{g}(w(t,x))dx+\\
&\qquad+\int_0^t\left\{|u_x(\tau)|_H^2+\frac{n_0}{2}|u(\tau,0)|^2+\frac{n_0}{2}|u(\tau,L)|^2+|w'(\tau)|_H^2\right\}d\tau\\
\le\ &e^T\left(\frac{1}{2}|u_0|_H^2+\kappa V^\varepsilon(w_0)+\int_0^L\widehat{g}(w_0(x))dx+\frac{a_0^2}{2}|f|^2_{L^2(0,T;H)}\right.\\
&\qquad\left.+\frac{a_1^2}{n_0}|h|^2_{L^2(0,T)}+\frac{a_2^2}{n_0}|\ell|^2_{L^2(0,T)}+n_0T(b_1^2+b_2^2)\right),\quad\forall t\in[0,T_0].
\end{aligned}\tag{3.30}$$

In addition, it follows from (3.26) and (3.30) that

$$\begin{aligned} |u'|^2_{L^2(0,T_0;X')} \leq\ & N_3 \left(|u_0|^2_H + \kappa V^\varepsilon(w_0) + \int_0^L \widehat{g}(w_0(x))dx + a_0^2|f|^2_{L^2(0,T;H)} \right. \\ & \left. + \frac{a_1^2}{n_0}|h|^2_{L^2(0,T)} + \frac{a_2^2}{n_0}|\ell|^2_{L^2(0,T)} + n_0(b_1^2 + b_2^2) \right), \end{aligned} \tag{3.31}$$

where N_3 is a positive constant independent of T_0 and given data u_0, w_0, f, h, and ℓ.

Multiplying (3.16) by $w(t)$ (cf. (S4) of Definition 3.1), we observe from (A2) that:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|w(t)|^2_H + \kappa V^\varepsilon(w(t)) \leq C_g|w(t)|^2_H + (u(t), w(t))_H \\ \text{for a.a. } t \in (0, T_0). \end{aligned} \tag{3.32}$$

By using the Schwarz inequality in (3.32), applying the Gronwall inequality to the resultant, we have:

$$|w(t)|^2_H + 2\kappa \int_0^t V^\varepsilon(w(\tau))d\tau \leq e^{(1+2C_g)T}\left(|w_0|^2_H + |u|^2_{L^2(0,T_0;H)}\right), \quad \forall t \in [0, T_0]. \tag{3.33}$$

Therefore, by (3.30), (3.31), and (3.33), we can find a positive constant N_4, independent of T_0, such that the following bounded estimate holds:

$$\begin{aligned} & |u'|^2_{L^2(0,T_0;X')} + |u|^2_{L^\infty(0,T_0;H)} + |u|^2_{L^2(0,T_0;X)} + |w'|^2_{L^2(0,T_0;H)} + |w|^2_{L^\infty(0,T_0;H)} \\ & \quad + \kappa \sup_{0\leq t\leq T_0} V^\varepsilon(w(t)) + \sup_{0\leq t\leq T_0} \int_0^L \widehat{g}(w(t,x))\,dx \\ \leq\ & N_4 \left(|u_0|^2_H + |w_0|^2_H + \kappa V^\varepsilon(w_0) + \int_0^L \widehat{g}(w_0(x))\,dx + a_0^2|f|^2_{L^2(0,T;H)} \right. \\ & \quad \left. + a_1^2|h|^2_{L^2(0,T)} + a_2^2|\ell|^2_{L^2(0,T)} + b_1^2 + b_2^2 \right). \end{aligned} \tag{3.34}$$

Hence, by (3.34), we can extend the solution to $(\mathrm{P};u_0, w_0, f, h, \ell)^\varepsilon$ beyond the time interval $[0, T_0]$. Namely, we get the existence of a solution to $(\mathrm{P};u_0, w_0, f, h, \ell)^\varepsilon$on $[0, T]$.

The a priori estimate (3.5) can be obtained by calculations similar to (3.34).

Thus, the proof of Theorem 3.1 has been completed. □

4 Continuous dependence of solutions to (P;$f, h, \ell)^\varepsilon$

In this section, we discuss the continuous dependence of solutions to systems $(\mathrm{P};u_0, w_0, f, h, \ell)^\varepsilon$ with respect to $\varepsilon \to 0$.

We begin with proving the Mosco convergence of V^ε on H as $\varepsilon \to 0$.

Lemma 4.1 (cf. [31, Theorem 4.1], [41, Lemma 3.1])**.** *Let V^0 and V^ε ($\varepsilon \in (0, 1]$) be convex functions given in* (2.1) *and* (3.3)*, respectively. Then:*

$$V^\varepsilon(\cdot) \longrightarrow V^0(\cdot) \quad \textit{on } H \textit{ in the sense of Mosco [32] as } \varepsilon \to 0. \tag{4.1}$$

Proof. The proof of this lemma is merely a slight modification of that as in [31, Theorem 4.1] and [41, Lemma 3.1].

Indeed, note that:

$$\int_0^L \widehat{K}^\varepsilon(\cdot)dx \longrightarrow \int_0^L I_{[-1,1]}(\cdot)dx \ \text{ on } H \text{ in the sense of Mosco [32] as } \varepsilon \to 0; \tag{4.2}$$

we easily show (4.2), therefore, we omit the detailed proof of (4.2).

Now, we show (i) of Definition 1.1 by using (A1), Proposition 2.2(I), and (4.2). To this end, assume $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,1]$, $\{z_k\}_{k\in\mathbb{N}} \subset H$, and $z \in H$ so that

$$\varepsilon_k \to 0 \text{ and } z_k \to z \text{ weakly in } H \text{ as } k \to \infty.$$

Note that we may suppose $\liminf_{k\to\infty} V^{\varepsilon_k}(z_k) < \infty$, because the other case is trivial. Then, from (A1), Proposition 2.2(I), and (4.2), we infer that:

$$\begin{aligned}
\infty > \ & \liminf_{k\to\infty} V^{\varepsilon_k}(z_k) \\
= \ & \liminf_{k\to\infty} \left[\int_0^L \widehat{a}^{\varepsilon_k}((z_k)_x(x))dx + \frac{\varepsilon_k}{2} \int_0^L |(z_k)_x(x)|^2 dx + \frac{1}{\kappa} \int_0^L \widehat{K}^{\varepsilon_k}(z_k(x))\,dx \right] \\
\geq \ & \liminf_{k\to\infty} \left[\int_0^L \left\{ \delta_1(\varepsilon_k)\widehat{a}^0((z_k)_x(x)) - \delta_2(\varepsilon_k) \right\} dx + \frac{1}{\kappa} \int_0^L \widehat{K}^{\varepsilon_k}(z_k(x))\,dx \right] \\
\geq \ & \lim_{k\to\infty} \delta_1(\varepsilon_k) \liminf_{k\to\infty} \int_0^L \widehat{a}^0((z_k)_x(x))dx - \lim_{k\to\infty} \delta_2(\varepsilon_k)L \\
& \qquad + \frac{1}{\kappa} \liminf_{k\to\infty} \int_0^L \widehat{K}^{\varepsilon_k}(z_k(x))\,dx \\
\geq \ & \liminf_{k\to\infty} V_0(z_k) + \frac{1}{\kappa} \int_0^L I_{[-1,1]}(z(x))dx \\
\geq \ & V_0(z) + \frac{1}{\kappa} \int_0^L I_{[-1,1]}(z(x))dx = V^0(z),
\end{aligned}$$

which implies that (i) of Definition 1.1 holds.

Next, we show (ii) of Definition 1.1. To this end, Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1]$ be any sequence such that $\varepsilon_n \to 0$ as $n \to \infty$, and let z be any element of $D(V^0)$. According to the result in [2, Theorem 3.9] and [15, Chapter 5], there is a sequence $\{\tilde{z}_i\}_{i\in\mathbb{N}\cup\{0\}} \subset C^\infty(0,L) \cap D(V^0)$ such that

$$|\tilde{z}_i - z|_H < \frac{1}{2^{i+1}} \ \text{ and } \ |V_0(\tilde{z}_i) - V_0(z)| < \frac{1}{2^{i+1}} \ \text{ for all } i \in \mathbb{N} \cup \{0\}. \tag{4.3}$$

By (A1), we can find a sequence $\{n_i\}_{i\in\mathbb{N}}$ such that

$$n_0 = 1,\ n_i \geq i,\ n_{i+1} > n_i,$$

and for any $i \in \mathbb{N} \cup \{0\}$,

$$\begin{cases} \sup_{n \geq n_i} \left| \int_0^L \widehat{a}^{\varepsilon_n}((\tilde{z}_i)_x(x))dx - \int_0^L \widehat{a}^0((\tilde{z}_i)_x(x))dx \right| < \frac{1}{2^{i+1}}, \\ \sup_{n \geq n_i} \frac{\varepsilon_n}{2} |(\tilde{z}_i)_x|_H^2 < \frac{1}{2^{i+1}}. \end{cases} \tag{4.4}$$

Based on these, let us define:

$$z_n := \tilde{z}_i \ \text{ if } n_i \leq n < n_{i+1} \ \text{ for some } i \in \mathbb{N} \cup \{0\}.$$

Then, we infer from (4.3) and (4.4) that

$$|z_n - z|_H = |\tilde{z}_i - z|_H < \frac{1}{2^{i+1}}$$

and

$$\begin{aligned} & |V^{\varepsilon_n}(z_n) - V^0(z)| \\ \leq & \left| \int_0^L \widehat{a}^{\varepsilon_n}((z_n)_x(x))dx - \int_0^L \widehat{a}^0((z_n)_x(x))dx \right| \\ & \qquad + \left| \int_0^L \widehat{a}^0((z_n)_x(x))dx - V_0(z) \right| + \frac{\varepsilon_n}{2} |(z_n)_x|_H^2 \\ \leq & \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} < \frac{1}{2^{i-1}} \\ & \qquad \text{for any } i \in \mathbb{N} \cup \{0\} \ \text{ and } \ n \geq n_i, \end{aligned}$$

which implies that (ii) of Definition 1.1 holds.

Thus, the proof of Lemma 4.1 is complete. □

Taking account of (4.1), we get the following Corollary 4.1. For the detailed proof, we refer to [6] or [18, Appendix], for instance.

Corollary 4.1 (cf. [6], [18, Appendix]). *Let V^0 and V^ε ($\varepsilon \in (0,1]$) be convex functions given in* (2.1) *and* (3.3), *respectively. Define*

$$\widehat{V}^0(z) := \int_0^T V^0(z(t))dt \quad \textit{and} \quad \widehat{V}^\varepsilon(z) := \int_0^T V^\varepsilon(z(t))dt, \quad \forall z \in L^2(0,T;H).$$

Then:

$$\widehat{V}^\varepsilon(\cdot) \longrightarrow \widehat{V}^0(\cdot) \ \ \textit{on } L^2(0,T;H) \textit{ in the sense of Mosco [32] as } \varepsilon \to 0.$$

Now, we mention the main theorem concerning the continuous dependence of solutions to systems $(\mathrm{P};u_0, w_0, f, h, \ell)^\varepsilon$ with respect to $\varepsilon \to 0$.

Theorem 4.1. *Assume* (A1)*,* (A2)*, and* (A3)*. Let* $[f,h,\ell] \in \mathcal{U}$*,* $u_0 \in H$*, and* $w_0 \in D(V^0)$*. Also, let* $\varepsilon \in (0,1]$*,* $\{[f^\varepsilon, h^\varepsilon, \ell^\varepsilon]\}_{\varepsilon\in(0,1]} \subset \mathcal{U}$*,* $\{u_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset H$*, and* $\{w_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset D(V^\varepsilon)$*. Furthermore, suppose that*

$$f^\varepsilon \to f \text{ weakly in } L^2(0,T;H), \tag{4.5}$$

$$h^\varepsilon \to h \text{ weakly in } L^2(0,T), \tag{4.6}$$

$$\ell^\varepsilon \to \ell \text{ weakly in } L^2(0,T), \tag{4.7}$$

$$u_0^\varepsilon \to u_0 \text{ in } X', \quad w_0^\varepsilon \to w_0 \text{ in } H, \quad \text{and} \quad V^\varepsilon(w_0^\varepsilon) \to V^0(w_0) \tag{4.8}$$

as $\varepsilon \to 0$*. Then, the unique solution* $[u^\varepsilon, w^\varepsilon]$ *to* $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f^\varepsilon, h^\varepsilon, \ell^\varepsilon)^\varepsilon$ *converges to the solution* $[u,w]$ *to* $(\mathrm{P};u_0, w_0, f, h, \ell)^0$ *in the following sense:*

$$[u^\varepsilon, w^\varepsilon] \longrightarrow [u,w] \quad \text{in } L^2(0,T;H) \times C([0,T];H) \text{ as } \varepsilon \to 0. \tag{4.9}$$

Proof. Note from the Mosco convergence (4.1) in Lemma 4.1 that for each $w_0 \in D(V^0)$, we can always find a sequence $\{w_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset D(V^\varepsilon)$ satisfying (4.8).

From (4.8), we infer that

$$V^\varepsilon(w_0^\varepsilon) \text{ is bounded uniformly in } \varepsilon \in (0,1]. \tag{4.10}$$

Therefore, we observe from (A1), Proposition 2.1(i), Remark 2.1, and the definitions of $V_0(\cdot)$ and $V^\varepsilon(\cdot)$ that

$$V_0(w_0^\varepsilon) \text{ is bounded uniformly in } \varepsilon \in (0,1]. \tag{4.11}$$

Therefore, it follows from Propositions 2.2–2.3, (4.8), and (4.11) that $|w_0^\varepsilon|_{BV(0,L)}$ is bounded uniformly in $\varepsilon \in (0,1]$, hence,

$$|w_0^\varepsilon|_{L^\infty(0,L)} \text{ is bounded uniformly in } \varepsilon \in (0,1]. \tag{4.12}$$

Thus, we observe from (A2) and (4.12) that

$$\int_0^L \widehat{g}(w_0^\varepsilon(x))\,dx \text{ is bounded uniformly in } \varepsilon \in (0,1]. \tag{4.13}$$

Now, let $[u^\varepsilon, w^\varepsilon]$ be the unique solution to $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f^\varepsilon, h^\varepsilon, \ell^\varepsilon)^\varepsilon$ on $[0,T]$. Then, from (4.5)–(4.8) and the stability estimate (3.5) with (4.10)–(4.13), we observe that

$$u^\varepsilon \text{ is bounded in } W^{1,2}(0,T;X') \cap L^2(0,T;X) \cap L^\infty(0,T;H), \tag{4.14}$$

$$w^\varepsilon \text{ is bounded in } W^{1,2}(0,T;H), \tag{4.15}$$

and

$$\sup_{0\le t\le T} V^\varepsilon(w^\varepsilon(t)) \text{ is bounded} \tag{4.16}$$

uniformly in $\varepsilon \in (0,1]$.

Additionally, from similar arguments as above (cf. (4.10)–(4.12)), we infer that

$$\sup_{0\le t\le T} V_0(w^\varepsilon(t)) \ \text{ is bounded uniformly in } \varepsilon\in(0,1], \tag{4.17}$$

therefore,

$$\sup_{0\le t\le T} |w^\varepsilon(t)|_{BV(0,L)} \ \text{ is bounded uniformly in } \varepsilon\in(0,1].$$

Hence, we have

$$\sup_{0\le t\le T} |w^\varepsilon(t)|_{L^\infty(0,L)} \ \text{ is bounded uniformly in } \varepsilon\in(0,1] \tag{4.18}$$

and

$$\sup_{0\le t\le T} \int_0^L \widehat{g}(w^\varepsilon(t,x))dx \ \text{ is bounded uniformly in } \varepsilon\in(0,1]. \tag{4.19}$$

Thus, by (4.14)–(4.19), there is a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ of $\{\varepsilon\}_{\varepsilon\in(0,1]}$ and functions $u \in W^{1,2}(0,T;X')\cap L^2(0,T;X)\cap L^\infty(0,T;H)$ and $w\in W^{1,2}(0,T;H)\cap L^\infty(Q)$ with $V_0(w)\in L^\infty(0,T)$ such that $\varepsilon_k\to 0$,

$$\left.\begin{array}{ll} u^{\varepsilon_k}\to u & \text{in } L^2(0,T;H),\\ & \text{in } C([0,T];X'),\\ & \text{weakly in } W^{1,2}(0,T;X'),\\ & \text{weakly in } L^2(0,T;X),\\ & \text{weakly-}* \text{ in } L^\infty(0,T;H), \end{array}\right\} \tag{4.20}$$

$$\left.\begin{array}{ll} w^{\varepsilon_k}\to w & \text{in } C([0,T];H),\\ & \text{weakly in } W^{1,2}(0,T;H),\\ & \text{weakly-}* \text{ in } L^\infty(Q), \end{array}\right\} \tag{4.21}$$

and

$$w^{\varepsilon_k}(t)\to w(t) \ \text{ weakly-}* \text{ in } BV(0,L), \quad \text{for any } t\in[0,T]$$

as $k\to\infty$.

We now show that the pair of functions $[u,w]$ is the solution to (P;$u_0,w_0,f,h,\ell)^0$ on $[0,T]$. To this end, we recall Corollary 4.1. Let z be any element in $D(\widehat{V}^0)$. Then, by the Mosco convergence of $\hat{V}^\varepsilon(\cdot)$, we can find a sequence $\{z_k\}_{k\in\mathbb{N}}\subset L^2(0,T;H)$ such that

$$z_k\to z \ \text{ in } L^2(0,T;H) \ \text{ and } \ \widehat{V}^{\varepsilon_k}(z_k)\to\widehat{V}^0(z) \tag{4.22}$$

as $k\to\infty$.

Since $[u^{\varepsilon_k},w^{\varepsilon_k}]$ is the unique solution to (P;$u_0^{\varepsilon_k},w_0^{\varepsilon_k},f^{\varepsilon_k},h^{\varepsilon_k},\ell^{\varepsilon_k})^{\varepsilon_k}$ on $[0,T]$, we easily observe that:

$$\begin{aligned} &\int_0^T\langle (u^{\varepsilon_k})'(t),\varpi(t)\rangle dt+\int_0^T((w^{\varepsilon_k})'(t),\varpi(t))_H dt+\int_0^T\langle Fu^{\varepsilon_k}(t),\varpi(t)\rangle dt\\ =\ &\int_0^T(a_0f^{\varepsilon_k}(t),\varpi(t))_H dt+\int_0^T(a_1h^{\varepsilon_k}(t)+n_0b_1)\varpi(t,0)dt\\ &\quad+\int_0^T(a_2\ell^{\varepsilon_k}(t)+n_0b_2)\varpi(t,L)dt \quad \text{ for any } \varpi\in L^2(0,T;X), \end{aligned} \tag{4.23}$$

$$\begin{aligned}&\int_0^T ((w^{\varepsilon_k})'(t) + g(w^{\varepsilon_k}(t)) - u^{\varepsilon_k}(t), w^{\varepsilon_k}(t) - z_k(t))_H dt \\ &\qquad +\kappa \int_0^T V^{\varepsilon_k}(w^{\varepsilon_k}(t))dt - \kappa \int_0^T V^{\varepsilon_k}(z_k(t))dt \leq 0,\end{aligned} \tag{4.24}$$

and

$$u^{\varepsilon_k}(0) = u_0^{\varepsilon_k} \text{ in } X' \text{ and } \ w^{\varepsilon_k}(0) = w_0^{\varepsilon_k} \text{ in } H. \tag{4.25}$$

Therefore, from (4.5)–(4.8), (4.20)–(4.25), the Mosco convergence of $\hat{V}^\varepsilon(\cdot)$, and the Lebesgue dominated convergence theorem, we observe that:

$$\begin{aligned}&\int_0^T \langle u'(t), \varpi(t)\rangle dt + \int_0^T (w'(t), \varpi(t))_H dt + \int_0^T \langle Fu(t), \varpi(t)\rangle dt \\ = &\int_0^T (a_0 f(t), \varpi(t))_H dt + \int_0^T (a_1 h(t) + n_0 b_1)\varpi(t,0)dt \\ &\quad + \int_0^T (a_2\ell(t) + n_0 b_2)\varpi(t,L)dt \qquad \text{for any } \varpi \in L^2(0,T;X),\end{aligned} \tag{4.26}$$

$$\begin{aligned}&\int_0^T (w'(t) + g(w(t)) - u(t), w(t) - z(t))_H dt \\ &\qquad +\kappa \int_0^T V^0(w(t))dt - \kappa \int_0^T V^0(z(t))dt \leq 0 \\ &\text{for any } z \in L^2(0,T;H) \text{ with } V^0(z) \in L^1(0,T),\end{aligned} \tag{4.27}$$

and

$$u(0) = u_0 \text{ in } X' \text{ and } \ w(0) = w_0 \text{ in } H. \tag{4.28}$$

Thus, we conclude from (4.20), (4.21), and (4.26)–(4.28) that $[u, w]$ is a unique solution to (P;$u_0, w_0, f, h, \ell)^0$ on $[0, T]$, whence (4.9) holds without extracting any subsequence from $\{\varepsilon\}_{\varepsilon\in(0,1]}$. Thus, the proof of Theorem 4.1 has been completed. □

By the slight modification of the proof of Theorem 4.1, we have the following convergence result of solutions to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ on $[0, T]$ for the fixed parameter $\varepsilon \in [0, 1]$.

Corollary 4.2 (cf. [6], [18, Appendix]). *Assume* (A1), (A2), *and* (A3). *Let $\varepsilon \in [0, 1]$ be a fixed parameter, and let $[f, h, \ell] \in \mathcal{U}$, $u_0 \in H$, and $w_0 \in D(V^\varepsilon)$. Also, let $\{[f_n, h_n, \ell_n]\}_{n\in\mathbb{N}} \subset \mathcal{U}$, $\{u_{0n}\}_{n\in\mathbb{N}} \subset H$, and $\{w_{0n}\}_{n\in\mathbb{N}} \subset D(V^\varepsilon)$. Furthermore, suppose that*

$$f_n \to f \text{ weakly in } L^2(0,T;H),$$

$$h_n \to h \text{ weakly in } L^2(0,T),$$

$$\ell_n \to \ell \text{ weakly in } L^2(0,T),$$

$$u_{0n} \to u_0 \text{ in } X', \quad w_{0n} \to w_0 \text{ in } H, \quad \text{and} \quad V^\varepsilon(w_{0n}) \to V^\varepsilon(w_0)$$

as $n \to \infty$. Then, the sequence of solutions $[u_n, w_n]$ to (P;$u_{0n}, w_{0n}, f_n, h_n, \ell_n)^\varepsilon$ *converges to the solution $[u, w]$ to* (P;$u_0, w_0, f, h, \ell)^\varepsilon$ *in the following sense:*

$$[u_n, w_n] \longrightarrow [u, w] \quad \text{in } L^2(0,T;H) \times C([0,T];H) \text{ as } n \to \infty.$$

5 Optimal control to $(\mathrm{OP})^{\varepsilon}$

In this section, we consider a class of approximate optimal control problems $(\mathrm{OP})^{\varepsilon}$. Indeed, we prove the following main result, which is concerned with the existence of an optimal control to $(\mathrm{OP})^{\varepsilon}$ for each $\varepsilon \in [0,1]$ and the relationship between the limits (ω-limit points) of sequences of approximate optimal pairs and the optimal pairs of the limiting problem $(\mathrm{OP})^{0}$.

Theorem 5.1. *Suppose* (A1)–(A4). *Then, the following two statements hold.*

(I) *Let $\varepsilon \in [0,1]$, $u_0^{\varepsilon} \in H$, and $w_0^{\varepsilon} \in D(V^{\varepsilon})$. Then, the problem $(\mathrm{OP})^{\varepsilon}$ has at least one optimal control $[f_*^{\varepsilon}, h_*^{\varepsilon}, \ell_*^{\varepsilon}] \in \mathcal{U}$, so that:*

$$J^{\varepsilon}(f_*^{\varepsilon}, h_*^{\varepsilon}, \ell_*^{\varepsilon}) = \inf_{[f,h,\ell]\in\mathcal{U}} J^{\varepsilon}(f,h,\ell).$$

(II) *Assume $u_0 \in H$, $\{u_0^{\varepsilon}\}_{\varepsilon\in(0,1]} \subset H$, $w_0 \in D(V^0)$, $\{w_0^{\varepsilon}\}_{\varepsilon\in(0,1]} \subset D(V^{\varepsilon})$,*

$$u_0^{\varepsilon} \to u_0 \text{ in } X',\ w_0^{\varepsilon} \to w_0 \text{ in } H, \text{ and } V^{\varepsilon}(w_0^{\varepsilon}) \longrightarrow V^0(w_0) \text{ as } \varepsilon \to 0. \tag{5.1}$$

Let $[f_*^{\varepsilon}, h_*^{\varepsilon}, \ell_*^{\varepsilon}] \in \mathcal{U}$ be the optimal control of $(\mathrm{OP})^{\varepsilon}$ obtained in (I). *In addition, let $[u_*^{\varepsilon}, w_*^{\varepsilon}]$ be the unique solution to $(\mathrm{P};u_0^{\varepsilon}, w_0^{\varepsilon}, f_*^{\varepsilon}, h_*^{\varepsilon}, \ell_*^{\varepsilon})^{\varepsilon}$ on $[0,T]$. Then, there exist a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$, the triplet of functions $[f_{**}, h_{**}, \ell_{**}] \in \mathcal{U}$, and the unique solution $[u_{**}, w_{**}]$ to $(\mathrm{P};u_0, w_0, f_{**}, h_{**}, \ell_{**})^0$ on $[0,T]$ such that $[f_{**}, h_{**}, \ell_{**}]$ is the optimal control of $(\mathrm{OP})^0$, $\varepsilon_k \to 0$,*

$$f_*^{\varepsilon_k} \to f_{**} \quad \text{weakly in } L^2(0,T;H), \tag{5.2}$$
$$h_*^{\varepsilon_k} \to h_{**} \quad \text{weakly in } L^2(0,T), \tag{5.3}$$
$$\ell_*^{\varepsilon_k} \to \ell_{**} \quad \text{weakly in } L^2(0,T), \tag{5.4}$$

and

$$[u_*^{\varepsilon_k}, w_*^{\varepsilon_k}] \longrightarrow [u_{**}, w_{**}] \quad \text{in } L^2(0,T;H) \times C([0,T];H) \tag{5.5}$$

as $k \to \infty$.

Proof. By Corollary 4.2 and taking a minimizing sequence $\{[f_n, h_n, \ell_n]\}_{n\in\mathbb{N}} \subset \mathcal{U}$ so that

$$\lim_{n\to\infty} J^{\varepsilon}(f_n, h_n, \ell_n) = \inf_{[f,h,\ell]\in\mathcal{U}} J^{\varepsilon}(f,h,\ell),$$

we can prove (I). Such an argument is quite standard, thus, we omit the detailed proof of (I).

Next, let us prove (II), which is concerned with the relationship between the optimal control problems $(\mathrm{OP})^{\varepsilon}$ and $(\mathrm{OP})^0$. To this end, let us fix any sequence $\{[f_*^{\varepsilon}, h_*^{\varepsilon}, \ell_*^{\varepsilon}]\}_{\varepsilon\in(0,1]} \subset \mathcal{U}$ of the optimal controls $[f_*^{\varepsilon}, h_*^{\varepsilon}, \ell_*^{\varepsilon}]$ to $(\mathrm{OP})^{\varepsilon}$ for $\varepsilon \in (0,1]$. Let $[f,h,\ell]$ be any function in $\mathcal{U}$. In addition, let $[u^{\varepsilon}, w^{\varepsilon}]$ be a unique solution to $(\mathrm{P};u_0^{\varepsilon}, w_0^{\varepsilon}, f, h, \ell)^{\varepsilon}$ on $[0,T]$, and let $[u,w]$ be a unique solution to $(\mathrm{P};u_0, w_0, f, h, \ell)^0$ on $[0,T]$. Then, we observe from Theorem 4.1 with (5.1) that

$$[u^{\varepsilon}, w^{\varepsilon}] \longrightarrow [u,w] \text{ in } L^2(0,T;H) \times C([0,T];H) \text{ as } \varepsilon \to 0. \tag{5.6}$$

Since $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]$ is the optimal control to $(\mathrm{OP})^\varepsilon$, we observe that

$$\begin{aligned} J^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon) &\leq J^\varepsilon(f, h, \ell) \\ &= \frac{c_0}{2}\int_0^T |(u^\varepsilon - u_d)(t)|_H^2\, dt + \frac{c_1}{2}\int_0^T |(w^\varepsilon - w_d)(t)|_H^2\, dt \\ &\quad + \frac{m_0}{2}\int_0^T a_0^2 |f(t)|_H^2\, dt \\ &\quad + \frac{m_1}{2}\int_0^T a_1^2 |h(t)|^2\, dt + \frac{m_2}{2}\int_0^T a_2^2 |\ell(t)|^2\, dt. \end{aligned} \tag{5.7}$$

Clearly, it follows from (1.16), (5.6), and (5.7) that $\{[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]\}_{\varepsilon \in (0,1]}$ is bounded in $\mathcal{U}$ with respect to $\varepsilon \in (0,1]$. Therefore, there are a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$ and the triplet of functions $[f_{**}, h_{**}, \ell_{**}] \in \mathcal{U}$ such that $\varepsilon_k \to 0$,

$$f_*^{\varepsilon_k} \to f_{**} \quad \text{weakly in } L^2(0,T;H), \tag{5.8}$$

$$h_*^{\varepsilon_k} \to h_{**} \quad \text{weakly in } L^2(0,T), \tag{5.9}$$

$$\ell_*^{\varepsilon_k} \to \ell_{**} \quad \text{weakly in } L^2(0,T) \tag{5.10}$$

as $k \to \infty$.

Let $[u_*^{\varepsilon_k}, w_*^{\varepsilon_k}]$ be a unique solution to $(\mathrm{P}; u_0^{\varepsilon_k}, w_0^{\varepsilon_k}, f_*^{\varepsilon_k}, h_*^{\varepsilon_k}, \ell_*^{\varepsilon_k})^{\varepsilon_k}$ on $[0,T]$. Then, from Theorem 4.1 with (5.1) and (5.8)–(5.10), we infer that $[u_*^{\varepsilon_k}, w_*^{\varepsilon_k}]$ converges to the unique solution $[u_{**}, w_{**}]$ to $(\mathrm{P}; u_0, w_0, f_{**}, h_{**}, \ell_{**})^0$ on $[0,T]$ in the sense that

$$[u_*^{\varepsilon_k}, w_*^{\varepsilon_k}] \longrightarrow [u_{**}, w_{**}] \ \text{ in } L^2(0,T;H) \times C([0,T];H) \text{ as } k \to \infty, \tag{5.11}$$

hence, the convergence (5.5) holds.

Now, by using (5.6)–(5.11) and the weak lower semicontinuity of L^2-norm, we see that

$$J^0(f_{**}, h_{**}, \ell_{**}) \leq \liminf_{k\to\infty} J^{\varepsilon_k}(f_*^{\varepsilon_k}, h_*^{\varepsilon_k}, \ell_*^{\varepsilon_k}) \leq \lim_{k\to\infty} J^{\varepsilon_k}(f, h, \ell) = J^0(f, h, \ell).$$

Since $[f, h, \ell]$ is any function in $\mathcal{U}$, we infer from the above inequality that $[f_{**}, h_{**}, \ell_{**}]$ is the optimal control to $(\mathrm{OP})^0$. Hence, the assertion (II) of Theorem 5.1 holds. Thus, the proof of Theorem 5.1 has been completed. □

Remark 5.1. Unfortunately, Theorem 5.1 does not cover the uniqueness of optimal controls. Although Hoffmann–Jiang [21] reported the uniqueness of optimal controls for a regular Fix–Caginalp system, their technique is not applicable to our problem $(\mathrm{OP})^\varepsilon$ because of the nonlinear terms $a^\varepsilon(w_x)$ and $K^\varepsilon(w)$. Therefore, the uniqueness question of optimal controls to $(\mathrm{OP})^\varepsilon$ is still open.

Remark 5.2. Theorem 5.1(II) shows that the weak limit function of optimal control of $(\mathrm{OP})^\varepsilon$ is an optimal control for $(\mathrm{OP})^0$. Note that we can approximate any optimal control of $(\mathrm{OP})^0$ by considering the following approximate control problems:

($\star$) Let $\alpha > 0$ be a fixed constant. In addition, let $[f_*, h_*, \ell_*] \in \mathcal{U}$ be any optimal control of $(\mathrm{OP})^0$ obtained in Theorem 5.1(I). Then, for each $\varepsilon \in (0,1]$, we consider the following approximate optimal control problem:

Problem $(\mathrm{OP})^\varepsilon_\alpha$ **.** Find a triplet of control functions $[f^\varepsilon_*, h^\varepsilon_*, \ell^\varepsilon_*] \in \mathcal{U}$, called optimal control, such that

$$J^\varepsilon_\alpha(f^\varepsilon_*, h^\varepsilon_*, \ell^\varepsilon_*) = \inf_{[f,h,\ell]\in\mathcal{U}} J^\varepsilon_\alpha(f,h,\ell).$$

Here, $J^\varepsilon_\alpha(f,h,\ell)$ is the cost functional defined by

$$\begin{aligned} J^\varepsilon_\alpha(f,h,\ell) \quad &:= \frac{c_0}{2}\int_0^T |(u^\varepsilon - u_d)(t)|^2_H\,dt + \frac{c_1}{2}\int_0^T |(w^\varepsilon - w_d)(t)|^2_H\,dt \\ &\quad + \frac{m_0}{2}\int_0^T a_0^2|f(t)|^2_H\,dt + \frac{m_1}{2}\int_0^T a_1^2|h(t)|^2\,dt \\ &\quad + \frac{m_2}{2}\int_0^T a_2^2|\ell(t)|^2\,dt + \frac{\alpha}{2}\int_0^T |(f-f_*)(t)|^2_H\,dt \\ &\quad + \frac{\alpha}{2}\int_0^T |(h-h_*)(t)|^2\,dt + \frac{\alpha}{2}\int_0^T |(\ell-\ell_*)(t)|^2\,dt, \end{aligned} \tag{5.12}$$

where $[f,h,\ell] \in \mathcal{U}$ is the control, and a couple of functions $[u^\varepsilon, w^\varepsilon]$ is a unique solution to the state problem $(\mathrm{P};u^\varepsilon_0, w^\varepsilon_0, f, h, \ell)^\varepsilon$.

Then, by arguments similar to those in [45, Theorem 3.3(II)], we can prove that there is a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$ such that $\varepsilon_k \to 0$,

$$f^{\varepsilon_k}_* \to f_* \ \text{ in } L^2(0,T;H), \quad h^{\varepsilon_k}_* \to h_* \ \text{ in } L^2(0,T), \quad \ell^{\varepsilon_k}_* \to \ell_* \ \text{ in } L^2(0,T),$$

and

$$[u^{\varepsilon_k}_*, w^{\varepsilon_k}_*] \longrightarrow [u_*, w_*] \ \text{ in } L^2(0,T;H) \times C([0,T];H)$$

as $k \to \infty$, where $[u^{\varepsilon_k}_*, w^{\varepsilon_k}_*]$ is a unique solution to $(\mathrm{P};u^{\varepsilon_k}_0, w^{\varepsilon_k}_0, f^{\varepsilon_k}_*, h^{\varepsilon_k}_*, \ell^{\varepsilon_k}_*)^{\varepsilon_k}$ and $[u_*, w_*]$ is a unique solution to $(\mathrm{P};u_0, w_0, f_*, h_*, \ell_*)^0$ on $[0,T]$.

6 Optimality condition for $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$

In this section we show the necessary condition of an optimal pair $[u^\varepsilon_*, w^\varepsilon_*, f^\varepsilon_*, h^\varepsilon_*, \ell^\varepsilon_*]$ to $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$, where $[u^\varepsilon_*, w^\varepsilon_*]$ is the unique solution to $(\mathrm{P};u^\varepsilon_0, w^\varepsilon_0, f^\varepsilon_*, h^\varepsilon_*, \ell^\varepsilon_*)^\varepsilon$, and $[f^\varepsilon_*, h^\varepsilon_*, \ell^\varepsilon_*] \in \mathcal{U}$ is the optimal control to $(\mathrm{OP})^\varepsilon$ obtained in Theorem 5.1(I).

Theorem 6.1. *Suppose the same conditions as in Theorem* 5.1. *Additionally, assume*

(A5) $\{\widehat{a}^\varepsilon\}_{\varepsilon\in(0,1]} \subset C^2(\mathbb{R})$ *is a sequence of convex functions and C^2-regularizations for* $\widehat{a}^0(\cdot) := |\cdot|$. *Moreover, there exists a positive constant δ_3, independent of $\varepsilon \in (0,1]$, such that*

$$0 \le (\widehat{a}^\varepsilon)''(r) \le \frac{\delta_3}{\varepsilon} \qquad \textit{for any } r \in \mathbb{R}.$$

(A6) *g is a C^1 function on $\mathbb{R}$.*

For the fixed number $\varepsilon \in (0,1]$, let $u_0^\varepsilon \in H$, $w_0^\varepsilon \in D(V^\varepsilon)$, and let $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon] \in \mathcal{U}$ be the optimal control to $(\mathrm{OP})^\varepsilon$ obtained in Theorem 5.1(I). In addition, let $[u_*^\varepsilon, w_*^\varepsilon]$ be the unique solution to $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon)^\varepsilon$ on $[0,T]$. Then, there exists a unique solution $[p^\varepsilon, q^\varepsilon]$ to the adjoint equation on $[0,T]$ as follows:

$$p^\varepsilon \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X), \tag{6.1}$$

$$q^\varepsilon \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H), \tag{6.2}$$

$$-(p^\varepsilon)' - p_{xx}^\varepsilon - q^\varepsilon = c_0(u_*^\varepsilon - u_d) \quad \text{in } Q, \tag{6.3}$$

$$\begin{aligned} &\int_0^T (-(p^\varepsilon)'(\tau), \zeta(\tau))_H d\tau + \int_0^T \langle -(q^\varepsilon)'(\tau), \zeta(\tau)\rangle d\tau \\ &\quad + \kappa \int_0^T \left(((a^\varepsilon)'((w_*^\varepsilon)_x(\tau)) + \varepsilon) q_x^\varepsilon(\tau), \zeta_x(\tau)\right)_H d\tau \\ &\quad + \int_0^T \left((K^\varepsilon)'(w_*^\varepsilon(\tau)) q^\varepsilon(\tau), \zeta(\tau)\right)_H d\tau + \int_0^T \left(g'(w_*^\varepsilon(\tau)) q^\varepsilon(\tau), \zeta(\tau)\right)_H d\tau \\ = \;& c_1 \int_0^T \left((w_*^\varepsilon - w_d)(\tau), \zeta(\tau)\right)_H d\tau \quad \text{for all } \zeta \in L^2(0,T;X), \end{aligned} \tag{6.4}$$

$$-p_x^\varepsilon(t,0) + n_0 p^\varepsilon(t,0) = p_x^\varepsilon(t,L) + n_0 p^\varepsilon(t,L) = 0, \quad t \in (0,T), \tag{6.5}$$

$$p^\varepsilon(T,x) = q^\varepsilon(T,x) = 0, \quad x \in (0,L), \tag{6.6}$$

where $(a^\varepsilon)'(\cdot)$ and $g'(\cdot)$ are the derivatives of $a^\varepsilon(\cdot)$ and $g(\cdot)$, respectively. Moreover, p^ε satisfies the following equations:

$$a_0(p^\varepsilon + m_0 a_0 f_*^\varepsilon) = 0 \quad \text{in } L^2(0,T;H), \tag{6.7}$$

$$a_1(p^\varepsilon(\cdot,0) + m_1 a_1 h_*^\varepsilon) = 0 \quad \text{in } L^2(0,T), \tag{6.8}$$

$$a_2(p^\varepsilon(\cdot,L) + m_2 a_2 \ell_*^\varepsilon) = 0 \quad \text{in } L^2(0,T). \tag{6.9}$$

We prove Theorem 6.1 by showing the result of Gâteaux differentiability of the cost functional $J^\varepsilon(\cdot,\cdot,\cdot)$. To this end, we fix $\varepsilon \in (0,1]$ and the initial data $[u_0^\varepsilon, w_0^\varepsilon] \in H \times D(V^\varepsilon)$. Then, we define the solution operator Λ^ε to $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$ as follows.

Definition 6.1. (I) *We denote by $\Lambda^\varepsilon : \mathcal{U} \to L^2(0,T;H) \times L^2(0,T;H)$* a solution operator to $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$ that assigns to any control $[f,h,\ell] \in \mathcal{U}$ the unique solution $[u^\varepsilon, w^\varepsilon] := \Lambda^\varepsilon(f,h,\ell)$ to the state system $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$.
(II) Let $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon] \in \mathcal{U}$ be the optimal control to $(\mathrm{OP})^\varepsilon$. Then, $[u_*^\varepsilon, w_*^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon] = [\Lambda^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon), f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]$ is called the optimal pair to the optimal control problem $(\mathrm{OP})^\varepsilon$.

For a moment, we often omit the superscript $\varepsilon \in (0,1]$.

At first, we show the Gâteaux differentiability of Λ^ε and J^ε. For any $\lambda \in [-1,1] \setminus \{0\}$, any $[f,h,\ell] \in \mathcal{U}$, and any $[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}$, we put $[u_\lambda, w_\lambda] := \Lambda^\varepsilon(f + \lambda\check{f}, h + \lambda\check{h}, \ell + \lambda\check{\ell})$, $[u,w] := \Lambda^\varepsilon(f,h,\ell)$, $\theta_\lambda := \frac{u_\lambda - u}{\lambda}$, and $\chi_\lambda := \frac{w_\lambda - w}{\lambda}$.

Note that the pair of functions $[\theta_\lambda, \chi_\lambda]$ satisfies the following system:

$$\begin{aligned}\langle \theta'_\lambda(t), z\rangle + \langle \chi'_\lambda(t), z\rangle + ((\theta_\lambda)_x(t), z_x)_H + n_0\theta_\lambda(t,0)z(0) + n_0\theta_\lambda(t,L)z(L) \\ = (a_0\check{f}(t), z)_H + a_1\check{h}(t)z(0) + a_2\check{\ell}(t)z(L), \\ \text{a.a. } t \in (0,T), \ \text{ for all } z \in X;\end{aligned} \tag{6.10}$$

$$\begin{aligned}\langle \chi'_\lambda(t), z\rangle + \kappa(\overline{a}^\varepsilon_\lambda(t)(\chi_\lambda)_x(t), z_x)_H + \left(\overline{K}^\varepsilon_\lambda(t)\chi_\lambda(t), z\right)_H + (\overline{g}_\lambda(t)\chi_\lambda(t), z)_H = (\theta_\lambda(t), z)_H, \\ \text{a.a. } t \in (0,T), \ \text{ for all } z \in X;\end{aligned} \tag{6.11}$$

$$\theta_\lambda(0,x) = \chi_\lambda(0,x) = 0, \qquad \text{a.a. } x \in (0,L), \tag{6.12}$$

where notations $\overline{a}^\varepsilon_\lambda$, $\overline{K}^\varepsilon_\lambda$, and $\overline{g}_\lambda$ are functions on Q, given as:

$$\begin{aligned}\overline{a}^\varepsilon_\lambda(t,x) &= \int_0^1 (a^\varepsilon)'(w_x(t,x) + s((w_\lambda)_x(t,x) - w_x(t,x)))ds + \varepsilon; \\ \overline{K}^\varepsilon_\lambda(t,x) &= \int_0^1 (K^\varepsilon)'(w(t,x) + s(w_\lambda(t,x) - w(t,x)))ds; \\ \overline{g}_\lambda(t,x) &= \int_0^1 g'\left(w(t,x) + s(w_\lambda(t,x) - w(t,x))\right)ds;\end{aligned}$$

for $(t,x) \in Q$, with use of the derivatives $(a^\varepsilon)'$, $(K^\varepsilon)'$, and g' of the single-valued functions.

Now, we give the uniform estimate of solutions $[\theta_\lambda, \chi_\lambda]$ to (6.10)–(6.12) with respect to $\lambda \in [-1,1] \setminus \{0\}$.

Lemma 6.1. *Suppose all the same conditions in Theorem 6.1. Then, there is a positive number $N_5 > 0$, dependent on $\varepsilon, T, \kappa, n_0$ and independent of λ, such that*

$$\begin{aligned}&\sup_{0\le t\le T} |\theta_\lambda(t)|^2_H + \int_0^T |\theta'_\lambda(t)|^2_{X'}dt + \int_0^T |\theta_\lambda(t)|^2_X dt \\ &\quad + \sup_{0\le t\le T} |\chi_\lambda(t)|^2_H + \int_0^T |\chi'_\lambda(t)|^2_{X'}dt + \int_0^T |\chi_\lambda(t)|^2_X dt \\ &\le N_5\left(a_0^2|\check{f}|^2_{L^2(0,T;H)} + a_1^2|\check{h}|^2_{L^2(0,T)} + a_2^2|\check{\ell}|^2_{L^2(0,T)}\right)\end{aligned} \tag{6.13}$$

for any $[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}$.

Proof. Clearly, we observe from (A1) and (A5) that $(\widehat{a}^\varepsilon)'(\cdot) = a^\varepsilon(\cdot) \in C^1(\mathbb{R})$ and

$$0 \le (a^\varepsilon)'(r) \le \frac{\delta_3}{\varepsilon} \quad \text{for any } r \in \mathbb{R}. \tag{6.14}$$

In addition, from the definitions of K^ε in (1.9) we infer that

$$0 \le \overline{K}^\varepsilon_\lambda(t,x) \le \frac{1}{\varepsilon}, \ \ \text{a.a. } (t,x) \in Q. \tag{6.15}$$

Here, from the boundedness (3.5) of solutions to (P;$f,h,\ell)^\varepsilon$, we note that

$$
\begin{aligned}
&\sup_{0\le t\le T}|w(t)|_H^2+\sup_{0\le t\le T}|w_\lambda(t)|_H^2+\kappa\sup_{0\le t\le T}V^\varepsilon(w(t))+\kappa\sup_{0\le t\le T}V^\varepsilon(w_\lambda(t))\\
\le\ & N_6\biggl(|u_0^\varepsilon|_H^2+|w_0^\varepsilon|_H^2+\kappa V^\varepsilon(w_0^\varepsilon)+\int_0^L\widehat{g}(w_0^\varepsilon(x))dx\\
&\quad+a_0^2|f|^2_{L^2(0,T;H)}+a_1^2|h|^2_{L^2(0,T)}+a_2^2|\ell|^2_{L^2(0,T)}\\
&\quad+a_0^2|\check{f}|^2_{L^2(0,T;H)}+a_1^2|\check{h}|^2_{L^2(0,T)}+a_2^2|\check{\ell}|^2_{L^2(0,T)}+b_1^2+b_2^2\biggr),
\end{aligned}\tag{6.16}
$$

where $N_6>0$ is a positive constant independent of $\lambda\in[-1,1]\setminus\{0\}$. Since the embedding $BV(0,L)\hookrightarrow L^\infty(0,L)$ is continuous (cf. Proposition 2.3), we infer from (6.16) that

$$
\begin{aligned}
&\sup_{0\le t\le T}|w(t)|^2_{L^\infty(0,L)}+\sup_{0\le t\le T}|w_\lambda(t)|^2_{L^\infty(0,L)}\\
\le\ & N_6'\biggl(|u_0^\varepsilon|_H^2+|w_0^\varepsilon|_H^2+\kappa V^\varepsilon(w_0^\varepsilon)+\int_0^L\widehat{g}(w_0^\varepsilon(x))dx\\
&\quad+a_0^2|f|^2_{L^2(0,T;H)}+a_1^2|h|^2_{L^2(0,T)}+a_2^2|\ell|^2_{L^2(0,T)}\\
&\quad+a_0^2|\check{f}|^2_{L^2(0,T;H)}+a_1^2|\check{h}|^2_{L^2(0,T)}+a_2^2|\check{\ell}|^2_{L^2(0,T)}+b_1^2+b_2^2\biggr),
\end{aligned}\tag{6.17}
$$

for some positive constant N_6' independent of $\lambda\in[-1,1]\setminus\{0\}$ (cf. (4.18)). Thus, by (6.17), we find a positive constant N_7, independent of $\lambda\in[-1,1]\setminus\{0\}$, such that

$$
\sup_{0\le t\le T}|\overline{g}_\lambda(t)|_{L^\infty(0,L)}\le N_7,\quad\text{for all }\lambda\in[-1,1]\setminus\{0\}.\tag{6.18}
$$

Now, we show a priori estimate (6.13). Taking account of (6.14)–(6.18), we can get the following estimate:

$$
\begin{aligned}
&\sup_{0\le t\le T}|\theta_\lambda(t)|_H^2+\int_0^T|\theta_\lambda(t)|_X^2dt+\sup_{0\le t\le T}|\chi_\lambda(t)|_H^2+\int_0^T|\chi_\lambda(t)|_X^2dt\\
\le\ & N_8\left(a_0^2|\check{f}|^2_{L^2(0,T;H)}+a_1^2|\check{h}|^2_{L^2(0,T)}+a_2^2|\check{\ell}|^2_{L^2(0,T)}\right),
\end{aligned}\tag{6.19}
$$

where $N_8>0$ is some positive constant, dependent on ε,T,κ,n_0 and independent of $\lambda\in[-1,1]\setminus\{0\}$. In fact, taking the sum of (6.10) with $z=\theta_\lambda$, (6.11) with $z=\theta_\lambda$, and (6.11) with $z=\frac{\kappa}{\varepsilon}(\frac{\delta_3}{\varepsilon}+\varepsilon)^2\chi_\lambda$, and applying the Gronwall-type inequality (e.g., [23, Proposition 0.4.1]), we get (6.19). Such calculations are standard one, so we omit the detailed arguments (cf. (8.34) in Lemma 8.1).

By using (6.14), (6.15), and (6.18), we infer from (6.11) that

$$
\begin{aligned}
\left|\int_0^T\langle\chi_\lambda'(t),\zeta(t)\rangle dt\right|\le N_9\left(|\chi_\lambda|_{L^2(0,T;X)}+|\theta_\lambda|_{L^2(0,T;H)}\right)|\zeta|_{L^2(0,T;X)}&\\
\text{for any }\zeta\in L^2(0,T;X),&
\end{aligned}\tag{6.20}
$$

where $N_9>0$ is some positive constant, dependent on ε,κ and independent of $\lambda\in[-1,1]\setminus\{0\}$. Hence, we infer from (6.19) and (6.20) that

$$
|\chi_\lambda'|_{L^2(0,T;X')}\le N_9'\left(|a_0||\check{f}|_{L^2(0,T;H)}+|a_1||\check{h}|_{L^2(0,T)}+|a_2||\check{\ell}|_{L^2(0,T)}\right)\tag{6.21}
$$

for some positive constant $N_9' > 0$, dependent on $\varepsilon, T, \kappa, n_0$ and independent of $\lambda \in [-1,1] \setminus \{0\}$.

Similarly, we infer from (6.10), (6.19), and (6.21) that

$$|\theta_\lambda'|_{L^2(0,T;X')} \le N_{10} \left(|a_0||\check{f}|_{L^2(0,T;H)} + |a_1||\check{h}|_{L^2(0,T)} + |a_2||\check{\ell}|_{L^2(0,T)} \right) \tag{6.22}$$

for some positive constant $N_{10} > 0$, dependent on $\varepsilon, T, \kappa, n_0$ and independent of $\lambda \in [-1,1] \setminus \{0\}$.

By (6.19), (6.21), and (6.22), we get the boundedness (6.13). Thus, the proof of Lemma 6.1 has been completed. □

Now, let us mention the result of the Gâteaux differentiability of Λ^ε and J^ε.

Proposition 6.1. *Assume the same conditions in Theorem 6.1. Then, the following two statements hold.*

(I) *The solution operator Λ^ε admits the Gâteaux derivative at any $[f, h, \ell] \in \mathcal{U}$. More precisely, for arbitrary $[f, h, \ell] \in \mathcal{U}$, there exists a pair of functions $[\theta, \chi] \in L^2(0,T;H) \times L^2(0,T;H)$ such that:*

$$D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f,h,\ell) := \lim_{\lambda \to 0} \frac{\Lambda^\varepsilon(f + \lambda \check{f}, h + \lambda \check{h}, \ell + \lambda \check{\ell}) - \Lambda^\varepsilon(f,h,\ell)}{\lambda} = [\theta, \chi] \quad \text{for all direction } [\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}, \tag{6.23}$$

$$\theta \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H), \tag{6.24}$$

$$\chi \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H), \tag{6.25}$$

and $[\theta, \chi]$ solves the following linear system:

$$\begin{aligned} &\langle \theta'(t), z \rangle + \langle \chi'(t), z \rangle + (\theta_x(t), z_x)_H + n_0 \left(\theta(t,0) z(0) + \theta(t,L) z(L) \right) \\ &\qquad = (a_0 \check{f}(t), z)_H + a_1 \check{h}(t) z(0) + a_2 \check{\ell}(t) z(L), \\ &\qquad \text{a.a. } t \in (0,T), \text{ for all } z \in X; \end{aligned} \tag{6.26}$$

$$\begin{aligned} &\langle \chi'(t), z \rangle + \kappa \left(((a^\varepsilon)'(w_x(t)) + \varepsilon) \chi_x(t), z_x \right)_H + ((K^\varepsilon)'(w(t)) \chi(t), z)_H \\ &\qquad + (g'(w(t)) \chi(t), z)_H = (\theta(t), z)_H, \\ &\qquad \text{a.a. } t \in (0,T), \text{ for all } z \in X; \end{aligned} \tag{6.27}$$

$$\theta(0,x) = \chi(0,x) = 0, \qquad \text{a.a. } x \in (0,L). \tag{6.28}$$

(II) *The cost function J^ε admits the Gâteaux derivative at any $[f, h, \ell] \in \mathcal{U}$. More* *precisely,*

$$
\begin{aligned}
D_{[\check f,\check h,\check \ell]}J^\varepsilon(f,h,\ell) &:= \lim_{\lambda\to 0}\frac{J^\varepsilon(f+\lambda\check f,h+\lambda\check h,\ell+\lambda\check\ell)-J^\varepsilon(f,h,\ell)}{\lambda}\\
&= c_0\int_0^T((u-u_d)(t),\theta(t))_H dt + c_1\int_0^T((w-w_d)(t),\chi(t))_H dt\\
&\quad +m_0a_0^2\int_0^T(f(t),\check f(t))_H dt\\
&\quad +m_1a_1^2\int_0^T h(t)\check h(t)dt + m_2a_2^2\int_0^T \ell(t)\check\ell(t)dt
\end{aligned}
\tag{6.29}
$$

for any $[f,h,\ell]\in\mathcal U$ and any direction $[\check f,\check h,\check\ell]\in\mathcal U$, where $[u,w]=\Lambda^\varepsilon(f,h,\ell)$ is the solution to (P;$u_0^\varepsilon,w_0^\varepsilon,f,h,\ell)^\varepsilon$, *$u_d$ and w_d are the given target profiles in $L^2(0,T;H)$, and $[\theta,\chi](=D_{[\check f,\check h,\check\ell]}\Lambda^\varepsilon(f,h,\ell))$ is the pair of functions obtained in the assertion* (I).

Proof. At first, we show (I). To this end, we put $[u_\lambda,w_\lambda]:=\Lambda^\varepsilon(f+\lambda\check f,h+\lambda\check h,\ell+\lambda\check\ell)$, $[u,w]:=\Lambda^\varepsilon(f,h,\ell)$, $\theta_\lambda:=\frac{u_\lambda-u}{\lambda}$, and $\chi_\lambda:=\frac{w_\lambda-w}{\lambda}$ for all $[f,h,\ell]\in\mathcal U$, $[\check f,\check h,\check\ell]\in\mathcal U$, and $\lambda\in[-1,1]\setminus\{0\}$. Then, by the uniform estimate (6.13) of $[\theta_\lambda,\chi_\lambda]$, there is a subsequence $\{\lambda_n\}_{n\in\mathbb N}\subset\{\lambda\}_{\lambda\in[-1,1]\setminus\{0\}}$ and the functions $\theta,\chi\in W^{1,2}(0,T;X')\cap L^2(0,T;X)\subset C([0,T];H)$ such that $\lambda_n\to 0$,

$$
\theta_{\lambda_n}\to\theta \quad \left.\begin{array}{l}
\text{in } C([0,T];X'),\\
\text{in } L^2(0,T;H),\\
\text{weakly in } W^{1,2}(0,T;X'),\\
\text{weakly in } L^2(0,T;X),\\
\text{weakly-}* \text{ in } L^\infty(0,T;H),
\end{array}\right\}
\tag{6.30}
$$

$$
\chi_{\lambda_n}\to\chi \quad \left.\begin{array}{l}
\text{in } C([0,T];X'),\\
\text{in } L^2(0,T;H),\\
\text{weakly in } W^{1,2}(0,T;X'),\\
\text{weakly in } L^2(0,T;X),\\
\text{weakly-}* \text{ in } L^\infty(0,T;H),
\end{array}\right\}
\tag{6.31}
$$

as $n\to\infty$, and

$$
\begin{aligned}
&\sup_{0\le t\le T}|\theta(t)|_H^2+\int_0^T|\theta'(t)|_{X'}^2dt+\int_0^T|\theta(t)|_X^2dt\\
&\quad+\sup_{0\le t\le T}|\chi(t)|_H^2+\int_0^T|\chi'(t)|_{X'}^2dt+\int_0^T|\chi(t)|_X^2dt\\
&\le N_5\left(a_0^2|\check f|^2_{L^2(0,T;H)}+a_1^2|\check h|^2_{L^2(0,T)}+a_2^2|\check\ell|^2_{L^2(0,T)}\right),
\end{aligned}
\tag{6.32}
$$

where N_5 is the same constant as in Lemma 6.1.

Now, let us show a pair of the limit functions $[\theta,\chi]$ of $[\theta_{\lambda_n},\chi_{\lambda_n}]$ satisfies (6.26)–(6.28). To this end, note from (6.13) that

$$
\begin{aligned}
|w_\lambda-w|_{L^2(0,T;X)} &= \lambda|\chi_\lambda|_{L^2(0,T;X)}\\
&\le \lambda N_5^{\frac12}\left(|a_0||\check f|_{L^2(0,T;H)}+|a_1||\check h|_{L^2(0,T)}+|a_2||\check\ell|_{L^2(0,T)}\right)\\
&\to 0 \ \text{ as } \lambda\to 0.
\end{aligned}
\tag{6.33}
$$

So, taking a subsequence if necessary, we see from the definition of functions $\overline{a}^{\varepsilon}_{\lambda}$, $\overline{K}^{\varepsilon}_{\lambda}$, $\overline{g}_{\lambda}$ ($\lambda \in [-1,1] \setminus \{0\}$) and continuity of functions $(a^{\varepsilon})'$, $(K^{\varepsilon})'$, and $g'(\cdot)$ that

$$\begin{cases} \overline{a}^{\varepsilon}_{\lambda_n}(t,x) \to (a^{\varepsilon})'(w_x(t,x)) + \varepsilon, \\ \overline{K}^{\varepsilon}_{\lambda_n}(t,x) \to (K^{\varepsilon})'(w(t,x)), \\ \overline{g}_{\lambda_n}(t,x) \to g'(w(t,x)), \end{cases} \quad \text{a.a. } (t,x) \in Q, \text{ in the pointwise sense, as } n \to \infty.$$

Here, let us fix arbitrary $0 \leq t_0 < t_1 \leq T$. Since functions $\overline{a}^{\varepsilon}_{\lambda}$, $\overline{K}^{\varepsilon}_{\lambda}$ and $\overline{g}_{\lambda}$ ($\lambda \in [-1,1] \setminus \{0\}$) are respectively bounded in senses of (6.14), (6.15), and (6.18), we can apply the Lebesgue dominated convergence theorem to show that

$$\begin{cases} \overline{a}^{\varepsilon}_{\lambda_n} \to (a^{\varepsilon})'(w_x) + \varepsilon, \\ \overline{K}^{\varepsilon}_{\lambda_n} \to (K^{\varepsilon})'(w), \\ \overline{g}_{\lambda_n} \to g'(w), \end{cases} \quad \text{in } L^2(t_0,t_1;H), \text{ as } n \to \infty. \tag{6.34}$$

Combining (6.30), (6.31), (6.32), and (6.34), it is deduced that:

$$\theta_{\lambda_n} \to \theta \text{ weakly in } L^2(t_0,t_1;X), \tag{6.35}$$

$$\theta'_{\lambda_n} \to \theta' \text{ weakly in } L^2(t_0,t_1;X'), \tag{6.36}$$

$$\chi_{\lambda_n} \to \chi \text{ weakly in } L^2(t_0,t_1;X), \tag{6.37}$$

$$\chi'_{\lambda_n} \to \chi' \text{ weakly in } L^2(t_0,t_1;X'), \tag{6.38}$$

and

$$\begin{cases} \overline{a}^{\varepsilon}_{\lambda_n}(\chi_{\lambda_n})_x \to ((a^{\varepsilon})'(w_x) + \varepsilon)\chi_x, \\ \overline{K}^{\varepsilon}_{\lambda_n}\chi_{\lambda_n} \to (K^{\varepsilon})'(w)\chi, \\ \overline{g}_{\lambda_n}\chi_{\lambda_n} \to g'(w)\chi, \end{cases} \quad \text{weakly in } L^2(t_0,t_1;H) \tag{6.39}$$

as $n \to \infty$.

Here, note from (6.10) and (6.11) that

$$\begin{aligned} &\int_{t_0}^{t_1} \langle \theta'_{\lambda_n}(t), z \rangle dt + \int_{t_0}^{t_1} \langle \chi'_{\lambda_n}(t), z \rangle dt + \int_{t_0}^{t_1} ((\theta_{\lambda_n})_x(t), z_x)_H \, dt \\ &\qquad + n_0 \int_{t_0}^{t_1} \theta_{\lambda_n}(t,0) z(0) dt + n_0 \int_{t_0}^{t_1} \theta_{\lambda_n}(t,L) z(L) dt \\ &= \int_{t_0}^{t_1} (a_0 \check{f}(t), z)_H dt + \int_{t_0}^{t_1} a_1 \check{h}(t) z(0) dt + \int_{t_0}^{t_1} a_2 \check{\ell}(t) z(L) dt \\ &\qquad \text{for all } z \in X \text{ and } n = 1,2,3,\cdots \end{aligned} \tag{6.40}$$

and

$$\begin{aligned} &\int_{t_0}^{t_1} \langle \chi'_{\lambda_n}(t), z \rangle dt + \kappa \int_{t_0}^{t_1} (\overline{a}^{\varepsilon}_{\lambda_n}(t)(\chi_{\lambda_n})_x(t), z_x)_H dt + \int_{t_0}^{t_1} (\overline{K}^{\varepsilon}_{\lambda_n}(t)\chi_{\lambda_n}(t), z)_H dt \\ &\qquad + \int_{t_0}^{t_1} (\overline{g}_{\lambda_n}(t)\chi_{\lambda_n}(t), z)_H dt = \int_{t_0}^{t_1} (\theta_{\lambda_n}(t), z)_H dt \\ &\qquad \text{for all } z \in X \text{ and } n = 1,2,3,\cdots. \end{aligned} \tag{6.41}$$

On account of (6.35)–(6.39), we obtain the variational form (6.26) (resp. (6.27)) by taking the limits in (6.40) (resp. (6.41)) as $n \to \infty$.

On the other hand, by (6.30) and (6.31),

$$\theta(0,\cdot) = \lim_{n\to\infty} \theta_{\lambda_n}(0,\cdot) = 0 \ (\in H) \ \text{ in } X',$$
$$\chi(0,\cdot) = \lim_{n\to\infty} \chi_{\lambda_n}(0,\cdot) = 0 \ (\in H) \ \text{ in } X',$$

which implies (6.28).

Furthermore, by the usual method with helps from the facts that $(a^\varepsilon)' \geq 0$ (on $\mathbb{R}$), $(K^\varepsilon)' \geq 0$ (on $\mathbb{R}$), and $g'(w) + C_g \geq 0$, a.e. in Q, we easily prove that the solutions to the Cauchy problem $\{(6.26)–(6.28)\}$ are uniquely determined within (6.24)–(6.25). Hence, the uniqueness of solution to $\{(6.26)–(6.28)\}$ guarantees that of cluster points of the sequence $[\theta_\lambda, \chi_\lambda]$ as $\lambda \to 0$. Namely:

(∗) $[\theta_\lambda, \chi_\lambda]$ originally converges to the unique solution $[\theta, \chi]$ to $\{(6.26)–(6.28)\}$ in the senses as in (6.30)–(6.31), as $\lambda \to 0$, and hence the operator $\mathcal{X}_{[f,h,\ell]} : \mathcal{U} \to L^2(0,T;H) \times L^2(0,T;H)$, defined by $\mathcal{X}_{[f,h,\ell]}(\check{f}, \check{h}, \check{\ell}) := D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f,h,\ell)$ for all direction $[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}$, is well-defined.

Now, on account of the linearity inherent in (6.26)–(6.27) and the estimate (6.32), we observe that each operator $\mathcal{X}_{[f,h,\ell]}$ ($[f,h,\ell] \in \mathcal{U}$) is a bounded and linear operator from $\mathcal{U}$ into $L^2(0,T;H) \times L^2(0,T;H)$, and hence, the solution operator Λ^ε admits the Gâteaux derivative at any $[f,h,\ell] \in \mathcal{U}$. Thus, we conclude the assertion (I) of this proposition.

Next, we show (II). The Gâteaux differentiability of the cost function J^ε will be a direct consequence of the assertion (I). In fact, note from (6.13) that

$$\begin{aligned} |u_\lambda - u|_{L^2(0,T;X)} &= \lambda|\theta_\lambda|_{L^2(0,T;X)} \\ &\leq \lambda N_5^{\frac{1}{2}} \left(|a_0||\check{f}|_{L^2(0,T;H)} + |a_1||\check{h}|_{L^2(0,T)} + |a_2||\check{\ell}|_{L^2(0,T)}\right) \\ &\to 0 \ \text{ as } \lambda \to 0. \end{aligned} \tag{6.42}$$

Then, by virtue of (6.33), (6.42), and (∗), it is computed that

$$\begin{aligned} & D_{[\check{f},\check{h},\check{\ell}]} J^\varepsilon(f,h,\ell) \\ := {} & \lim_{\lambda\to 0} \frac{J^\varepsilon(f+\lambda\check{f}, h+\lambda\check{h}, \ell+\lambda\check{\ell}) - J^\varepsilon(f,h,\ell)}{\lambda} \\ = {} & \lim_{\lambda\to 0} \left\{ \frac{c_0}{2}\int_0^T ((u_\lambda + u - 2u_d)(t), \theta_\lambda(t))_H dt + \frac{c_1}{2}\int_0^T ((w_\lambda + w - 2w_d)(t), \chi_\lambda(t))_H dt \right. \\ & \qquad + \frac{m_0 a_0^2}{2}\int_0^T ((2f + \lambda\check{f})(t), \check{f}(t))_H dt \\ & \qquad \left. + \frac{m_1 a_1^2}{2}\int_0^T (2h + \lambda\check{h})(t)\check{h}(t) dt + \frac{m_2 a_2^2}{2}\int_0^T (2\ell + \lambda\check{\ell})(t)\check{\ell}(t) dt \right\} \\ = {} & c_0 \int_0^T ((u - u_d)(t), \theta(t))_H dt + c_1 \int_0^T ((w - w_d)(t), \chi(t))_H dt \\ & + m_0 a_0^2 \int_0^T (f(t), \check{f}(t))_H dt + m_1 a_1^2 \int_0^T h(t)\check{h}(t) dt + m_2 a_2^2 \int_0^T \ell(t)\check{\ell}(t) dt \end{aligned}$$

for any $[f,h,\ell]\in\mathcal{U}$ and any direction $[\check{f},\check{h},\check{\ell}]\in\mathcal{U}$.

Clearly, we infer from (6.32) and $(*)$ that for any $[f,h,\ell]\in\mathcal{U}$, the functional:

$$[\check{f},\check{h},\check{\ell}]\in\mathcal{U}\mapsto D_{[\check{f},\check{h},\check{\ell}]}J^\varepsilon(f,h,\ell)$$

will form a bounded linear functional on $\mathcal{U}$. Hence, the cost functional J^ε admits the Gâteaux derivative at any $[f,h,\ell]\in\mathcal{U}$ with the directional derivative as in (6.29).

Thus, the proof of Proposition 6.1 has been completed. $\square$

By taking account of Proposition 6.1, we can prove Theorem 6.1 concerning the necessary condition of an optimal pair $[u_*^\varepsilon,w_*^\varepsilon,f_*^\varepsilon,h_*^\varepsilon,\ell_*^\varepsilon]=[\Lambda^\varepsilon(f_*^\varepsilon,h_*^\varepsilon,\ell_*^\varepsilon),f_*^\varepsilon,h_*^\varepsilon,\ell_*^\varepsilon]$ to $(\mathrm{OP})^\varepsilon$ with $\varepsilon>0$.

Proof of Theorem 6.1. By using the Schauder fixed point theorem and the general results by Ladyženskaja–Solonnikov–Ural'ceva [30, Chapter 3], we can get the unique solution $[p^\varepsilon,q^\varepsilon]$ to the adjoint equations (6.1)–(6.6).

Now, let $[u_*^\varepsilon,w_*^\varepsilon,f_*^\varepsilon,h_*^\varepsilon,\ell_*^\varepsilon]=[\Lambda^\varepsilon(f_*^\varepsilon,h_*^\varepsilon,\ell_*^\varepsilon),f_*^\varepsilon,h_*^\varepsilon,\ell_*^\varepsilon]$ be the optimal pair to the problem $(\mathrm{OP})^\varepsilon$ with $\varepsilon>0$. Let $[\theta_*^\varepsilon,\chi_*^\varepsilon]$ be the limit of $\frac{\Lambda^\varepsilon(f_*^\varepsilon+\lambda\check{f},h_*^\varepsilon+\lambda\check{h},\ell_*^\varepsilon+\lambda\check{\ell})-\Lambda^\varepsilon(f_*^\varepsilon,h_*^\varepsilon,\ell_*^\varepsilon)}{\lambda}$ as $\lambda\to0$ in the sense of (6.23).

Since $[f_*^\varepsilon,h_*^\varepsilon,\ell_*^\varepsilon]$ is a minimizer for $J^\varepsilon(\cdot,\cdot,\cdot)$, we have

$$\begin{aligned}
0 &\le \lim_{\lambda\to0}\frac{J^\varepsilon(f_*^\varepsilon+\lambda\check{f},h_*^\varepsilon+\lambda\check{h},\ell_*^\varepsilon+\lambda\check{\ell})-J^\varepsilon(f_*^\varepsilon,h_*^\varepsilon,\ell_*^\varepsilon)}{\lambda}\\
&= c_0\int_0^T((u_*^\varepsilon-u_d)(t),\theta_*^\varepsilon(t))_H dt+c_1\int_0^T((w_*^\varepsilon-w_d)(t),\chi_*^\varepsilon(t))_H dt\\
&\qquad+m_0a_0^2\int_0^T(f_*^\varepsilon(t),\check{f}(t))_H dt+m_1a_1^2\int_0^T h_*^\varepsilon(t)\check{h}(t)dt+m_2a_2^2\int_0^T\ell_*^\varepsilon(t)\check{\ell}(t)dt\\
&= \int_0^T\langle-(p^\varepsilon)'(t),\theta_*^\varepsilon(t)\rangle dt+\int_0^T(p_x^\varepsilon(t),(\theta_*^\varepsilon)_x(t))_H dt+n_0\int_0^T p^\varepsilon(t,0)\theta_*^\varepsilon(t,0)dt\\
&\qquad+n_0\int_0^T p^\varepsilon(t,L)\theta_*^\varepsilon(t,L)dt-\int_0^T(q^\varepsilon(t),\theta_*^\varepsilon(t))_H dt\\
&\qquad+\int_0^T(-(p^\varepsilon)'(t),\chi_*^\varepsilon(t))_H dt+\int_0^T\langle-(q^\varepsilon)'(t),\chi_*^\varepsilon(t)\rangle dt\\
&\quad+\kappa\int_0^T\left(((a^\varepsilon)'\left((w_*^\varepsilon)_x(t)\right)+\varepsilon)q_x^\varepsilon(t),(\chi_*^\varepsilon)_x(t)\right)_H dt+\int_0^T\left((K^\varepsilon)'(w_*^\varepsilon(t))q^\varepsilon(t),\chi_*^\varepsilon(t)\right)_H dt\\
&\qquad+\int_0^T\left(g'(w_*^\varepsilon(t))q^\varepsilon(t),\chi_*^\varepsilon(t)\right)_H dt\\
&\qquad+m_0a_0^2\int_0^T(f_*^\varepsilon(t),\check{f}(t))_H dt+m_1a_1^2\int_0^T h_*^\varepsilon(t)\check{h}(t)dt+m_2a_2^2\int_0^T\ell_*^\varepsilon(t)\check{\ell}(t)dt\\
&= \int_0^T\langle(\theta_*^\varepsilon)'(t),p^\varepsilon(t)\rangle dt+\int_0^T((\theta_*^\varepsilon)_x(t),p_x^\varepsilon(t))_H dt+n_0\int_0^T\theta_*^\varepsilon(t,0)p^\varepsilon(t,0)dt
\end{aligned}$$

$$+n_0\int_0^T \theta_*^\varepsilon(t,L)p^\varepsilon(t,L)dt-\int_0^T (q^\varepsilon(t),\theta_*^\varepsilon(t))_H dt$$

$$+\int_0^T \langle(\chi_*^\varepsilon)'(t),p^\varepsilon(t)\rangle dt+\int_0^T \langle(\chi_*^\varepsilon)'(t),q^\varepsilon(t)\rangle dt$$

$$+\kappa\int_0^T (((a^\varepsilon)'((w_*^\varepsilon)_x(t))+\varepsilon)(\chi_*^\varepsilon)_x(t),q_x^\varepsilon(t))_H\,dt+\int_0^T ((K^\varepsilon)'(w_*^\varepsilon(t))\chi_*^\varepsilon(t),q^\varepsilon(t))_H\,dt$$

$$+\int_0^T (g'(w_*^\varepsilon(t))\chi_*^\varepsilon(t),q^\varepsilon(t))_H\,dt$$

$$+m_0a_0^2\int_0^T (f_*^\varepsilon(t),\check{f}(t))_H dt+m_1a_1^2\int_0^T h_*^\varepsilon(t)\check{h}(t)dt+m_2a_2^2\int_0^T \ell_*^\varepsilon(t)\check{\ell}(t)dt$$

$$=\int_0^T (a_0p^\varepsilon(t)+m_0a_0^2f_*^\varepsilon(t),\check{f}(t))_H dt+\int_0^T (a_1p^\varepsilon(t,0)+m_1a_1^2h_*^\varepsilon(t))\check{h}(t)dt$$

$$+\int_0^T (a_2p^\varepsilon(t,L)+m_2a_2^2\ell_*^\varepsilon(t))\check{\ell}(t)dt$$

for any $[\check{f},\check{h},\check{\ell}]\in\mathcal{U}$. Here, we use the equations (6.3)–(6.6) and (6.26)–(6.28) for $[p^\varepsilon,q^\varepsilon]$ and $[\theta_*^\varepsilon,\chi_*^\varepsilon]$, respectively. Since $[\check{f},\check{h},\check{\ell}]\in\mathcal{U}$ is arbitrary, we infer from the inequality as above that the equations in (6.7)–(6.9) hold. Thus, the proof of Theorem 6.1 has been completed. $\square$

7 Optimality condition for $(\mathrm{OP})^0$

In previous Section 6, we proved Theorem 6.1, which is concerned with the optimality condition to the approximate problem $(\mathrm{OP})^\varepsilon$ with $\varepsilon>0$. But, in general, it is difficult to show the necessary condition of the optimal control to $(\mathrm{OP})^0$, i.e., $\varepsilon=0$, since (1.2) is the singular diffusion equation with constraint $\partial I_{[-1,1]}(\cdot)$. Therefore, by using Theorem 6.1, more precisely, by the limiting observation of $(\mathrm{OP})^\varepsilon$ as $\varepsilon\to 0$, we derive the optimality condition to $(\mathrm{OP})^0$.

Now, we mention the main result in this paper, which is concerned with the necessary condition of the optimal control to $(\mathrm{OP})^0$

Theorem 7.1. *Suppose that all the assumptions of Theorem 6.1 are fulfilled. Let* $u_0\in H$, $w_0\in D(V^0)$, *and let* $[f_{**},h_{**},\ell_{**}]$ *be the optimal control to* $(\mathrm{OP})^0$ *obtained in Theorem 5.1(II). Let* $[u_{**},w_{**}]$ *be the unique solution to* $(\mathrm{P};u_0,w_0,f_{**},h_{**},\ell_{**})^0$ *on* $[0,T]$. *Additionally, let us set:*

$$W:=\{z\in H^1(Q)\ ;\ z(0,x)=0,\ a.a.\ x\in(0,L)\}.$$

Then, there are the functions $p\in W^{1,2}(0,T;H)\cap L^\infty(0,T;X)$, $q\in L^\infty(0,T;H)$, *and an element* $\mu\in W'$ *satisfying the following:*

$$-p'-p_{xx}-q=c_0(u_{**}-u_d)\quad in\ Q, \tag{7.1}$$

$$\int_0^T (-p'(\tau),z(\tau))_H d\tau+\int_0^T (q(\tau),z'(\tau))_H d\tau+\langle\mu,z\rangle_{W',W}+\int_0^T (g'(w_{**}(\tau))q(\tau),z(\tau))_H d\tau$$

$$= c_1 \int_0^T ((w_{**} - w_d)(\tau), z(\tau))_H d\tau \quad \textit{for all } z \in W. \tag{7.2}$$

$$- p_x(t,0) + n_0 p(t,0) = p_x(t,L) + n_0 p(t,L) = 0, \quad t \in (0,T), \tag{7.3}$$

$$p(T,x) = 0, \quad x \in (0,L). \tag{7.4}$$

Moreover, p satisfies the following equations:

$$a_0(p + m_0 a_0 f_{**}) = 0 \quad \textit{in } L^2(0,T;H), \tag{7.5}$$

$$a_1(p(\cdot,0) + m_1 a_1 h_{**}) = 0 \quad \textit{in } L^2(0,T), \tag{7.6}$$

$$a_2(p(\cdot,L) + m_2 a_2 \ell_{**}) = 0 \quad \textit{in } L^2(0,T). \tag{7.7}$$

Proof. Let $u_0 \in H$ and $w_0 \in D(V^0)$. Then, note from (1.17) and Lemma 4.1 that we find sequences $\{u_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset H$ and $\{w_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset D(V^\varepsilon)$ satisfying

$$u_0^\varepsilon \to u_0 \text{ in } X', \ w_0^\varepsilon \to w_0 \text{ in } H, \text{ and } V^\varepsilon(w_0^\varepsilon) \longrightarrow V^0(w_0) \text{ as } \varepsilon \to 0. \tag{7.8}$$

Now, let $[f_{**}, h_{**}, \ell_{**}]$ be the optimal control to $(\mathrm{OP})^0$ obtained in Theorem 5.1(II). Namely, there exists a subsequence of $\varepsilon \in (0,1]$ (which we also denote ε for simplicity) such that $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]$ is the optimal control to $(\mathrm{OP})^\varepsilon$ and

$$f_*^\varepsilon \to f_{**} \quad \text{weakly in } L^2(0,T;H), \tag{7.9}$$

$$h_*^\varepsilon \to h_{**} \quad \text{weakly in } L^2(0,T), \tag{7.10}$$

$$\ell_*^\varepsilon \to \ell_{**} \quad \text{weakly in } L^2(0,T), \tag{7.11}$$

and

$$[u_*^\varepsilon, w_*^\varepsilon] \longrightarrow [u_{**}, w_{**}] \text{ in } L^2(0,T;H) \times C([0,T];H) \tag{7.12}$$

as $\varepsilon \to 0$, where $[u_*^\varepsilon, w_*^\varepsilon]$ is the unique solution to $(\mathrm{P}; u_0^\varepsilon, w_0^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon)^\varepsilon$ on $[0,T]$, and $[u_{**}, w_{**}]$ is the unique solution to $(\mathrm{P}; u_0, w_0, f_{**}, h_{**}, \ell_{**})^0$ on $[0,T]$.

Now, by taking the limit with respect to ε, we prove Theorem 7.1. To this end, we give a priori estimate of the solution $[p^\varepsilon, q^\varepsilon]$ to the adjoint equations (6.3)–(6.6).

Now, we multiply (6.3) by p^ε. Then, by applying the Schwarz inequality, we have

$$\begin{aligned} &-\frac{1}{2}\frac{d}{d\tau}|p^\varepsilon(\tau)|_H^2 + |p_x^\varepsilon(\tau)|_H^2 + n_0|p^\varepsilon(\tau,0)|^2 + n_0|p^\varepsilon(\tau,L)|^2 \\ \le\ & |p^\varepsilon(\tau)|_H^2 + \frac{1}{2}|q^\varepsilon(\tau)|_H^2 + \frac{c_0^2}{2}|(u_*^\varepsilon - u_d)(\tau)|_H^2, \quad \text{a.a. } \tau \in (0,T). \end{aligned} \tag{7.13}$$

By integrating (7.13) in τ over $[T-t,T]$ ($t \in [0,T]$), we have

$$\begin{aligned} &\frac{1}{2}|p^\varepsilon(T-t)|_H^2 + \int_{T-t}^T |p_x^\varepsilon(\tau)|_H^2 d\tau \\ &\qquad + n_0\int_{T-t}^T |p^\varepsilon(\tau,0)|^2 d\tau + n_0 \int_{T-t}^T |p^\varepsilon(\tau,L)|^2 d\tau \\ \le\ & \int_{T-t}^T |p^\varepsilon(\tau)|_H^2 d\tau + \frac{1}{2}\int_{T-t}^T |q^\varepsilon(\tau)|_H^2 d\tau + \frac{c_0^2}{2}\int_{T-t}^T |(u_*^\varepsilon - u_d)(\tau)|_H^2 d\tau \end{aligned} \tag{7.14}$$

for all $t \in [0,T]$.

Next, multiply (6.3) by $-(p^\varepsilon)'$. Then, by applying the Schwarz inequality, we get

$$\begin{aligned} &\frac{1}{2}|(p^\varepsilon)'(\tau)|_H^2 - \frac{1}{2}\frac{d}{d\tau}\left\{|p_x^\varepsilon(\tau)|_H^2 + n_0|p^\varepsilon(\tau,0)|^2 + n_0|p^\varepsilon(\tau,L)|^2\right\} \\ \leq\ & |q^\varepsilon(\tau)|_H^2 + c_0^2|(u_*^\varepsilon - u_d)(\tau)|_H^2, \quad \text{a.a. } \tau \in (0,T). \end{aligned} \tag{7.15}$$

By integrating (7.15) in τ over $[T-t,T]$ ($t \in [0,T]$), we have

$$\begin{aligned} &\frac{1}{2}\int_{T-t}^T |(p^\varepsilon)'(\tau)|_H^2 d\tau + \frac{1}{2}\left\{|p_x^\varepsilon(T-t)|_H^2 + n_0|p^\varepsilon(T-t,0)|^2 + n_0|p^\varepsilon(T-t,L)|^2\right\} \\ \leq & \int_{T-t}^T |q^\varepsilon(\tau)|_H^2 d\tau + c_0^2\int_{T-t}^T |(u_*^\varepsilon - u_d)(\tau)|_H^2 d\tau, \quad \forall t \in [0,T]. \end{aligned} \tag{7.16}$$

Here, note that the pair of functions $[p^\varepsilon, q^\varepsilon]$ satisfies the following variational identity (cf. (6.4)):

$$\begin{aligned} &\int_{T-t}^T (-(p^\varepsilon)'(\tau), \zeta(\tau))_H d\tau + \int_{T-t}^T \langle -(q^\varepsilon)'(\tau), \zeta(\tau)\rangle d\tau \\ &+ \kappa\int_{T-t}^T (((a^\varepsilon)'((w_*^\varepsilon)_x(\tau)) + \varepsilon)q_x^\varepsilon(\tau), \zeta_x(\tau))_H\, d\tau + \int_{T-t}^T ((K^\varepsilon)'(w_*^\varepsilon(\tau))q^\varepsilon(\tau), \zeta(\tau))_H\, d\tau \\ &+ \int_{T-t}^T (g'(w_*^\varepsilon(\tau))q^\varepsilon(\tau), \zeta(\tau))_H\, d\tau \\ =& c_1\int_{T-t}^T ((w_*^\varepsilon - w_d)(\tau), \zeta(\tau))_H\, d\tau \quad \text{ for all } t \in [0,T] \text{ and all } \zeta \in L^2(T-t,T;X). \end{aligned} \tag{7.17}$$

Therefore, let us assign q^ε to the test function ζ as in (7.17). Then, by applying the Schwarz inequality, we see that

$$\begin{aligned} \frac{1}{2}|q^\varepsilon(T-t)|_H^2 \ \leq\ & (2+C_g)\int_{T-t}^T |q^\varepsilon(\tau)|_H^2 d\tau + \frac{1}{4}\int_{T-t}^T |(p^\varepsilon)'(\tau)|_H^2 d\tau \\ & + \frac{c_1^2}{4}\int_{T-t}^T |(w_*^\varepsilon - w_d)(\tau)|_H^2 d\tau, \quad \forall t \in [0,T], \end{aligned} \tag{7.18}$$

since a^ε and K^ε are nondecreasing on $\mathbb{R}$ (cf. (6.14), (6.15)), and $g'(w_*^\varepsilon) + C_g \geq 0$, a.e. in Q. Adding (7.14), (7.16), and (7.18), we have

$$\begin{aligned} &\frac{1}{2}\left\{|p^\varepsilon(T-t)|_H^2 + |q^\varepsilon(T-t)|_H^2 + |p_x^\varepsilon(T-t)|_H^2 + n_0|p^\varepsilon(T-t,0)|^2 + n_0|p^\varepsilon(T-t,L)|^2\right\} \\ &\quad + \frac{1}{4}\int_{T-t}^T |(p^\varepsilon)'(\tau)|_H^2 d\tau + \int_{T-t}^T |p_x^\varepsilon(\tau)|_H^2 d\tau + n_0\int_{T-t}^T |p^\varepsilon(\tau,0)|^2 d\tau + n_0\int_{T-t}^T |p^\varepsilon(\tau,L)|^2 d\tau \\ \leq & \int_{T-t}^T |p^\varepsilon(\tau)|_H^2 d\tau + \left(\frac{7}{2} + C_g\right)\int_{T-t}^T |q^\varepsilon(\tau)|_H^2 d\tau + \frac{3c_0^2}{2}\int_{T-t}^T |(u_*^\varepsilon - u_d)(\tau)|_H^2 d\tau \\ &\quad + \frac{c_1^2}{4}\int_{T-t}^T |(w_*^\varepsilon - w_d)(\tau)|_H^2 d\tau, \quad \forall t \in [0,T]. \end{aligned} \tag{7.19}$$

Thus, by (7.12) and applying the Gronwall-type inequality (e.g., [23, Proposition 0.4.1]) to (7.19), we have

$$\begin{aligned}&\int_0^T \left\{|p^\varepsilon(t)|_H^2+|q^\varepsilon(t)|_H^2+|p_x^\varepsilon(t)|_H^2+n_0|p^\varepsilon(t,0)|^2+n_0|p^\varepsilon(t,L)|^2\right\}dt\\ \le& N_{11}\left(\int_0^T|(u_{**}-u_d)(t)|_H^2dt+\int_0^T|(w_{**}-w_d)(t)|_H^2dt+1\right)\end{aligned}\tag{7.20}$$

for some constant $N_{11}>0$, independent of $\varepsilon\in(0,1]$ and dependent on T. Hence, it follows from (7.19) and (7.20) that

$$\begin{aligned}&\sup_{0\le t\le T}\left\{|p^\varepsilon(t)|_H^2+|q^\varepsilon(t)|_H^2+|p_x^\varepsilon(t)|_H^2+n_0|p^\varepsilon(t,0)|^2+n_0|p^\varepsilon(t,L)|^2\right\}\\ &\quad+\int_0^T|(p^\varepsilon)'(t)|_H^2dt+\int_0^T|p_x^\varepsilon(t)|_H^2dt+n_0\int_0^T|p^\varepsilon(t,0)|^2dt+n_0\int_0^T|p^\varepsilon(t,L)|^2dt\\ \le& N_{12}\left(\int_0^T|(u_{**}-u_d)(t)|_H^2dt+\int_0^T|(w_{**}-w_d)(t)|_H^2dt+1\right)\end{aligned}\tag{7.21}$$

for some constant $N_{12}>0$, independent of $\varepsilon\in(0,1]$ and dependent on T.

Now, for any $\varepsilon\in(0,1]$, let us define a bounded and linear functional $\mu^\varepsilon\in W'$ on W, by putting, for all $\zeta\in W$,

$$\langle\mu^\varepsilon,\zeta\rangle_{W',W}:=\int_0^T\{(\kappa((a^\varepsilon)'((w_*^\varepsilon)_x(t))+\varepsilon)q_x^\varepsilon(t),\zeta_x(t))_H+((K^\varepsilon)'(w_*^\varepsilon(t))q^\varepsilon(t),\zeta(t))_H\}\,dt.$$

Here, note from (3.5) and (7.8) that (cf. (4.18), (6.17)):

$$\{w_*^\varepsilon\}\text{ is bounded in }L^\infty(Q)\text{ uniformly in }\varepsilon\in(0,1].\tag{7.22}$$

In addition, we infer from (A6) snd (7.22) that (cf. (4.19), (6.18)):

$$\{g'(w_*^\varepsilon)\}\text{ is bounded in }L^\infty(Q)\text{ uniformly in }\varepsilon\in(0,1].\tag{7.23}$$

Then, on account of (6.6), (7.12), (7.17), and (7.21)–(7.23), there exists a positive constant N_{13}, independent of $\varepsilon\in(0,1]$, such that

$$\begin{aligned}|\langle\mu^\varepsilon,\zeta\rangle_{W',W}|\le&\left|\int_0^T((p^\varepsilon)'(t),\zeta(t))_H\,dt\right|+\left|\int_0^T\langle(q^\varepsilon)'(t),\zeta(t)\rangle dt\right|\\ &+\left|\int_0^T(g'(w_*^\varepsilon(t))q^\varepsilon(t),\zeta(t))_H\,dt\right|+\left|c_1\int_0^T((w_*^\varepsilon-w_d)(t),\zeta(t))_H\,dt\right|\\ =&\left|\int_0^T((p^\varepsilon)'(t),\zeta(t))_H\,dt\right|+\left|\int_0^T(-q^\varepsilon(t),\zeta'(t))_H\,dt\right|\\ &+\left|\int_0^T(g'(w_*^\varepsilon(t))q^\varepsilon(t),\zeta(t))_H\,dt\right|+\left|c_1\int_0^T((w_*^\varepsilon-w_d)(t),\zeta(t))_H\,dt\right|\\ \le& N_{13}\left(|u_{**}-u_d|_{L^2(0,T;H)}+|w_{**}-w_d|_{L^2(0,T;H)}+1\right)|\zeta|_W\\ &\text{for any }\zeta\in W:=\{z\in H^1(Q)\,;\,z(0,x)=0,\ \text{a.a. }x\in(0,L)\}.\end{aligned}$$

Therefore, we get

$$|\mu^\varepsilon|_{W'} \le N_{13}\left(|u_{**}-u_d|_{L^2(0,T;H)} + |w_{**}-w_d|_{L^2(0,T;H)} + 1\right) \tag{7.24}$$

for all $\varepsilon \in (0,1]$.

By the boundedness estimates (7.21) and (7.24), there are the functions $p \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$, $q \in L^\infty(0,T;H)$ and an element $\mu \in W'$ such that

$$\left.\begin{array}{ll} p^\varepsilon \to p & \text{in } C([0,T];H), \\ & \text{weakly in } W^{1,2}(0,T;H), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;X), \end{array}\right\} \tag{7.25}$$

$$q^\varepsilon \to q \quad \text{weakly-}* \text{ in } L^\infty(0,T;H), \tag{7.26}$$

$$\mu^\varepsilon \to \mu \quad \text{weakly in } W' \tag{7.27}$$

as $\varepsilon \to 0$, by taking a subsequence if necessary.

Taking account of the convergence (7.9)–(7.12) and (7.25)–(7.27), we observe that the equations (7.1)–(7.7) hold. In fact, the system $\{(6.3)–(6.6)\}$ is equivalent to the following variational identities:

$$\begin{aligned} & \int_0^T (-(p^\varepsilon)'(t), \zeta(t))_H dt + \int_0^T (p_x^\varepsilon(t), \zeta_x(t))_H dt + n_0 \int_0^T p^\varepsilon(t,0)\zeta(t,0)dt \\ & \quad + n_0 \int_0^T p^\varepsilon(t,L)\zeta(t,L)dt - \int_0^T (q^\varepsilon(t), \zeta(t))_H dt \\ = & \int_0^T c_0((u_*^\varepsilon - u_d)(t), \zeta(t))_H dt \qquad \text{for all } \zeta \in L^2(0,T;X) \end{aligned} \tag{7.28}$$

and

$$\begin{aligned} & \int_0^T (-(p^\varepsilon)'(t), z(t))_H dt + \int_0^T (q^\varepsilon(t), z'(t))_H dt + \langle \mu^\varepsilon, z\rangle_{W',W} \\ & \quad + \int_0^T \left(g'(w_*^\varepsilon(t))q^\varepsilon(t), z(t)\right)_H dt \\ = & \; c_1 \int_0^T \left((w_*^\varepsilon - w_d)(t), z(t)\right)_H dt \qquad \text{for all } z \in W. \end{aligned} \tag{7.29}$$

Thus, we easily see from (7.9)–(7.12), (A6) with (7.23) (cf. (6.39)), and (7.25)–(7.29) that the equations (7.1)–(7.4) hold. Moreover, we easily see from (6.7)–(6.9), (7.9)–(7.11), and (7.25) that (7.5)–(7.7) hold. Thus, the proof of Theorem 7.1 has been completed. □

Remark 7.1. Theorem 7.1 is to be proved through the limiting observation of the approximate situations shown in Theorem 6.1. In addition, the identities (6.4) and (7.2) can be regarded as some variational forms of the equations:

$$-p_t^\varepsilon - q_t^\varepsilon - \kappa(((a^\varepsilon)'((w_*^\varepsilon)_x) + \varepsilon)q_x^\varepsilon)_x + (K^\varepsilon)'(w_*^\varepsilon)q^\varepsilon + g'(w_*^\varepsilon)q^\varepsilon = c_1(w_*^\varepsilon - w_d)$$

and

$$-p_t - q_t + \mu + g'(w_{**})q = c_1(w_{**} - w_d)$$

in the distribution sense, respectively.

Remark 7.2. In Remark 5.2 we mentioned that any optimal control of $(\mathrm{OP})^0$ can be approximated by the control problem $(\mathrm{OP})^\varepsilon_\alpha$ with $\alpha > 0$. Then, by arguments similar to those in Theorem 7.1 and [45, Theorem 3.5], we can show the necessary conditions for any optimal control of $(\mathrm{OP})^0$, which are the same ones (7.5)–(7.7) as in Theorem 7.1.

Remark 7.3. In Remark 5.2 we mentioned that for each optimal control $[f_*, h_*, \ell_*]$ of $(\mathrm{OP})^0$, we can find the sequence of optimal controls of $(\mathrm{OP})^\varepsilon_\alpha$ which converges to $[f_*, h_*, \ell_*]$ strongly in $\mathcal{U}$. However, it is very difficult to give the numerical experiments of $(\mathrm{OP})^\varepsilon_\alpha$, since the cost function J^ε_α defined by (5.12) depends on the unknown optimal control $[f_*, h_*, \ell_*]$ of $(\mathrm{OP})^0$. Therefore, in the numerical analysis, we are forced to adopt $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$ as the approximate problem of $(\mathrm{OP})^0$, since the cost function J^ε defined by (1.16) is independent of optimal controls of $(\mathrm{OP})^0$. Thus, from the viewpoint of applications, the main results for $(\mathrm{OP})^\varepsilon$ would be more useful than those for $(\mathrm{OP})^\varepsilon_\alpha$ with $\alpha > 0$.

8 Numerical Scheme for $(\mathrm{OP})^\varepsilon$

Note from the singularity and nonlinearity in (1.2) that the numerical consideration of $(\mathrm{OP})^0$ is very difficult (cf. Theorem 7.1 and Remark 7.1). In Section 5, we proved the relationship between the limits (ω-limit points) of sequences of approximate optimal pairs of $(\mathrm{OP})^\varepsilon$ as $\varepsilon \to 0$ and the optimal pairs of the limiting problem $(\mathrm{OP})^0$ (cf. Theorem 5.1(II)). Therefore, it is worth considering the approximate optimal control problem $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$ from the viewpoint of numerical analysis.

In this section, we propose the numerical scheme to find the stationary point of the cost functional J^ε to $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$, and show the convergence of our numerical algorithm. To this end, we fix the small parameter $\varepsilon \in (0, 1]$ and the pair of initial data $[u^\varepsilon_0, w^\varepsilon_0] \in H \times D(V^\varepsilon)$. Then, we define the solution operator Λ^ε_{ad} of the adjoint system $\{(6.3)–(6.6)\}$:

Definition 8.1. We denote by $\Lambda^\varepsilon_{ad} : \mathcal{U} \to L^2(0, T; H) \times L^2(0, T; H)$ the solution operator that assigns to any control $[f, h, \ell] \in \mathcal{U}$ the unique solution $[p^\varepsilon, q^\varepsilon] := \Lambda^\varepsilon_{ad}(f, h, \ell)$ to the adjoint system $\{(6.3)–(6.6)\}$ on $[0, T]$.

For a moment, we often omit the superscript $\varepsilon \in (0, 1]$.

Now, by the similar idea used in [1, 34, 37, 38, 39, 46], namely, by using the necessary conditions (6.7), (6.8), and (6.9) of $(\mathrm{OP})^\varepsilon$ obtained in Theorem 6.1, we propose the following numerical algorithm, denoted by (NA), to find the stationary point of the cost functional J^ε with $\varepsilon > 0$.

Numerical Algorithm (NA) of $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$

(Step 0) Give the stop parameter μ;

(Step 1) Choose the triplet of initial functions $[f, h, \ell] \in \mathcal{U}$, and put $[f_n, h_n, \ell_n] := [f, h, \ell]$;

(Step 2) Solve the approximate state system $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f_n, h_n, \ell_n)^\varepsilon$ for n, and let $[u_n, w_n] := \Lambda^\varepsilon(f_n, h_n, \ell_n)$, where Λ^ε is the solution operator to $(\mathrm{P};u_0^\varepsilon,\ w_0^\varepsilon, f_n, h_n, \ell_n)^\varepsilon$ defined in Definition 6.1(I);

(Step 3) Solve the adjoint system $\{(6.3)\text{–}(6.6)\}$ for n, and let $[p_n, q_n] := \Lambda_{ad}^\varepsilon(f_n, h_n, \ell_n)$;

(Step 4) Put

$$d_{0n} := a_0(p_n+m_0a_0f_n),\ d_{1n} := a_1(p_n(\cdot,0)+m_1a_1h_n),\ \text{and}\ \ d_{2n} := a_2(p_n(\cdot,L)+m_2a_2\ell_n).$$

Test: If

$$|[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}} < \mu,$$

then, STOP; Otherwise, go to (Step 5); note here that $\mathcal{U}$ is the product Hilbert space endowed with the usual norm

$$|[f,h,\ell]|_{\mathcal{U}}^2 := |f|_{L^2(0,T;H)}^2 + |h|_{L^2(0,T)}^2 + |\ell|_{L^2(0,T)}^2, \quad \forall [f,h,\ell] \in \mathcal{U}; \qquad (8.1)$$

(Step 5) Put

$$f_{n+1} := f_n - \rho_n d_{0n},\ \ h_{n+1} := h_n - \rho_n d_{1n},\ \text{and}\ \ \ell_{n+1} := \ell_n - \rho_n d_{2n},$$

where ρ_n is some appropriate constant found by using a line search. More precisely, let $\beta \in (0,1)$. Then, find the minimal constant $\varsigma_n \in \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} & J^\varepsilon\left(f_n - \beta^{\varsigma_n} d_{0n}, h_n - \beta^{\varsigma_n} d_{1n}, \ell_n - \beta^{\varsigma_n} d_{2n}\right) - J^\varepsilon(f_n, h_n, \ell_n) \\ \leq\ & -\mu\beta^{\varsigma_n}\, |[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}}\,, \end{aligned}$$

and put the constant $\rho_n := \beta^{\varsigma_n}$;

(Step 6) Set $n = n+1$, and go to (Step 2).

Remark 8.1. In (Step 5), we need to find the constant ρ_n (cf. the so-called "the learning rate" in neural networks) for each step n, because of the nonlinear term $(a^\varepsilon(w_x^\varepsilon))_x$ in (1.11) (cf. Remark 3.1). If the main diffusion term in (1.11) is just only linear (i.e., w_{xx}^ε), we can take the constant $\rho \equiv \rho_n$ independent of n. Indeed, Aiki et al. [1] considered the optimal control problem for phase-field equations of a regular Fix–Caginalp type with dynamic boundary conditions, and proved the existence of a constant ρ, independent of n, in the descend method. For the detailed statement, we refer to [1, Section 4].

Now, we mention our final theoretical result in this paper, which is concerned with the convergence of the numerical algorithm (NA).

Theorem 8.1. *Suppose that all the assumptions of Theorem 6.1 are fulfilled. Let* $\varepsilon \in (0,1]$ *and* $[u_0^\varepsilon, w_0^\varepsilon] \in H \times D(V^\varepsilon)$. *Let* $\{[f_n, h_n, \ell_n]\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{U}$ *defined by the numerical algorithm* (NA). *In addition, let* $[p_n, q_n] = \Lambda_{ad}^\varepsilon(f_n, h_n, \ell_n)$. *Then:*

(I) $\lim_{n\to\infty} J^\varepsilon(f_n, h_n, \ell_n)$ *exists.*

(II)

$$\lim_{n\to\infty} a_0(p_n + m_0 a_0 f_n) = 0 \quad in\ L^2(0,T;H), \tag{8.2}$$

$$\lim_{n\to\infty} a_1(p_n(\cdot,0) + m_1 a_1 h_n) = 0 \quad in\ L^2(0,T), \tag{8.3}$$

$$\lim_{n\to\infty} a_2(p_n(\cdot,L) + m_2 a_2 \ell_n) = 0 \quad in\ L^2(0,T). \tag{8.4}$$

(III) *There are the triplet of functions* $[f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**}] \in \mathcal{U}$*, the pair of functions* $[p^\varepsilon_{**}, q^\varepsilon_{**}] \in L^2(0,T;H) \times L^2(0,T;H)$*, and a subsequence* $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ *such that* $p^\varepsilon_{**} \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$*,* $q^\varepsilon_{**} \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \cap L^\infty(0,T;H)$*,* $[p^\varepsilon_{**}, q^\varepsilon_{**}]$ *is a unique solution of the adjoint system* $\{(6.3)–(6.6)\}$ *for* $(\mathrm{P}; u^\varepsilon_0, w^\varepsilon_0, f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**})^\varepsilon$*, i.e.,* $[p^\varepsilon_{**}, q^\varepsilon_{**}] = \Lambda^\varepsilon_{ad}(f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**})$*,*

$$f_{n_k} \longrightarrow f^\varepsilon_{**} \quad in\ L^2(0,T;H), \tag{8.5}$$

$$h_{n_k} \longrightarrow h^\varepsilon_{**} \quad in\ L^2(0,T), \tag{8.6}$$

$$\ell_{n_k} \longrightarrow \ell^\varepsilon_{**} \quad in\ L^2(0,T), \tag{8.7}$$

$$p_{n_k} \longrightarrow p^\varepsilon_{**} \quad \left.\begin{array}{l} in\ C([0,T];H), \\ in\ L^2(0,T;X), \end{array}\right\} \tag{8.8}$$

$$q_{n_k} \longrightarrow q^\varepsilon_{**} \quad in\ L^2(0,T;H) \tag{8.9}$$

as $k \to \infty$*, and*

$$a_0(p^\varepsilon_{**} + m_0 a_0 f^\varepsilon_{**}) = 0 \quad in\ L^2(0,T;H), \tag{8.10}$$

$$a_1(p^\varepsilon_{**}(\cdot,0) + m_1 a_1 h^\varepsilon_{**}) = 0 \quad in\ L^2(0,T), \tag{8.11}$$

$$a_2(p^\varepsilon_{**}(\cdot,L) + m_2 a_2 \ell^\varepsilon_{**}) = 0 \quad in\ L^2(0,T). \tag{8.12}$$

Hence,

$$\begin{aligned} &D_{[\check f,\check h,\check \ell]} J^\varepsilon(f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**}) \\ :=\ &\lim_{\lambda\to 0} \frac{J^\varepsilon(f^\varepsilon_{**} + \lambda \check f, h^\varepsilon_{**} + \lambda \check h, \ell^\varepsilon_{**} + \lambda \check \ell) - J^\varepsilon(f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**})}{\lambda} = 0 \end{aligned} \tag{8.13}$$

for all direction $[\check f, \check h, \check \ell] \in \mathcal{U}$*;*

thus, $[f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**}] \in \mathcal{U}$ *is the stationary point of the cost functional* J^ε *with* $\varepsilon \in (0,1]$*.*

To prove Theorem 8.1, we need some lemmas.

Note from Corollary 4.2 that we have the result of continuous dependence of solutions to the approximate state system $(\mathrm{P}; u^\varepsilon_0, w^\varepsilon_0, f, h, \ell)^\varepsilon$. In addition, note from Proposition 6.1(I) that the solution operator Λ^ε admits the Gâteaux derivative at any $[f,h,l] \in \mathcal{U}$.

Now we show the continuity of Gâteaux derivative of Λ^ε, which is the key to proving Theorem 8.1.

Lemma 8.1. *Assume the same conditions as in Theorem 8.1. Let* $\varepsilon \in (0,1]$, $\xi \in [-1,1] \setminus \{0\}$*, and fix the pair of initial data* $[u_0^\varepsilon, w_0^\varepsilon] \in H \times D(V^\varepsilon)$*. Then, the Gâteaux derivative of the control-to-state mapping* Λ^ε *is continuous in the following sense:*

$$\begin{aligned} & [\theta_\xi, \chi_\xi] &&:= D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3) \\ \longrightarrow\quad & [\theta, \chi] &&= D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f,h,\ell) \quad \text{in } L^2(0,T;H) \times L^2(0,T;H) \\ & \text{for all } [f,h,\ell] \in \mathcal{U}, && [\varpi_1,\varpi_2,\varpi_3] \in \mathcal{U}, \text{ and all direction } [\check{f},\check{h},\check{\ell}] \in \mathcal{U} \end{aligned} \tag{8.14}$$

as $\xi \to 0$.

Proof. For any $[f,h,\ell] \in \mathcal{U}$, $[\varpi_1,\varpi_2,\varpi_3] \in \mathcal{U}$, and $\xi \in [-1,1] \setminus \{0\}$, we put $[u_\xi, w_\xi] := \Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ and $[u,w] := \Lambda^\varepsilon(f,h,\ell)$. Then, we observe from Corollary 4.2 that

$$[u_\xi, w_\xi] \longrightarrow [u,w] \ \text{ in } L^2(0,T;H) \times C([0,T];H) \text{ as } \xi \to 0. \tag{8.15}$$

In addition, we have:

$$w_\xi \longrightarrow w \text{ in } L^2(0,T;X) \text{ as } \xi \to 0. \tag{8.16}$$

Indeed, subtract (1.11) for (P;$u_0^\varepsilon, w_0^\varepsilon, f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)^\varepsilon$ from the one for (P;$u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$, and multiply it by $w_\xi - w$. Then, from the monotonicity of $a^\varepsilon(w_x^\varepsilon)$, $K^\varepsilon(w^\varepsilon)$, and $g(w^\varepsilon) + C_g w^\varepsilon$ (cf. (A1), (1.9), and (A2)), and the Schwarz inequality, we observe that

$$\frac{1}{2}\frac{d}{dt}|(w_\xi - w)(t)|_H^2 + \varepsilon\kappa|(w_\xi - w)_x(t)|_H^2 \le \left(\frac{1}{2} + C_g\right)|(w_\xi - w)(t)|_H^2 + \frac{1}{2}|(u_\xi - u)(t)|_H^2 \tag{8.17}$$

for a.a. $t \in (0,T)$. Hence, applying the Gronwall inequality to (8.17), we conclude that

$$\frac{1}{2}\sup_{t\in[0,T]}|(w_\xi - w)(t)|_H^2 + \varepsilon\kappa\int_0^T |(w_\xi - w)_x(t)|_H^2 dt \le \frac{1}{2}e^{(1+2C_g)T}|u_\xi - u|_{L^2(0,T;H)}^2. \tag{8.18}$$

Thus, we infer from (8.15) and (8.18) that the convergence (8.16) holds.

Now, we show (8.14) by using the convergences (8.15) and (8.16). Note from Proposition 6.1(I) that $[\theta_\xi, \chi_\xi] = D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ satisfies the following variational identities:

$$\begin{gathered} \langle \theta_\xi'(t), z\rangle + \langle \chi_\xi'(t), z\rangle + ((\theta_\xi)_x(t), z_x)_H + n_0\left(\theta_\xi(t,0)z(0) + \theta_\xi(t,L)z(L)\right) \\ = (a_0\check{f}(t), z)_H + a_1\check{h}(t)z(0) + a_2\check{\ell}(t)z(L), \\ \text{a.a. } t \in (0,T), \ \text{ for all } z \in X; \end{gathered} \tag{8.19}$$

$$\begin{gathered} \langle \chi_\xi'(t), z\rangle + \kappa\left(((a^\varepsilon)'((w_\xi)_x(t)) + \varepsilon)(\chi_\xi)_x(t), z_x\right)_H + ((K^\varepsilon)'(w_\xi(t))\chi_\xi(t), z)_H \\ + (g'(w_\xi(t))\chi_\xi(t), z)_H = (\theta_\xi(t), z)_H, \\ \text{a.a. } t \in (0,T), \ \text{ for all } z \in X; \end{gathered} \tag{8.20}$$

$$\theta_\xi(0,x) = \chi_\xi(0,x) = 0, \quad \text{a.a. } x \in (0,L). \tag{8.21}$$

Then, by arguments similar to Lemma 6.1, we can obtain the uniform estimate of functions θ_ξ and χ_ξ with respect to $\xi \in [-1,1] \setminus \{0\}$. Indeed, taking $z = \theta_\xi$ in (8.19), using the Schwarz inequality, and integrating in time, we obtain:

$$\begin{aligned} &\frac{1}{2}|\theta_\xi(t)|_H^2 + \int_0^t \langle \chi_\xi'(s), \theta_\xi(s)\rangle ds + \int_0^t |(\theta_\xi)_x(s)|_H^2 ds \\ &\quad + \frac{n_0}{2}\int_0^t |\theta_\xi(s,0)|^2 ds + \frac{n_0}{2}\int_0^t |\theta_\xi(s,L)|^2 ds \\ &\le \frac{1}{2}\int_0^t |\theta_\xi(s)|_H^2 ds + \frac{a_0^2}{2}\int_0^t |\check{f}(s)|_H^2 ds + \frac{a_1^2}{2n_0}\int_0^t |\check{h}(s)|^2 ds + \frac{a_2^2}{2n_0}\int_0^t |\check{\ell}(s)|^2 ds \\ &\qquad \text{for all } t \in [0,T]. \end{aligned} \tag{8.22}$$

Next, we note from (A1) and (A5) that $(\widehat{a}^\varepsilon)'(\cdot) = a^\varepsilon(\cdot) \in C^1(\mathbb{R})$ and

$$0 \le (a^\varepsilon)'(r) \le \frac{\delta_3}{\varepsilon} \quad \text{for any } r \in \mathbb{R}. \tag{8.23}$$

In addition, note from (1.9) that $K^\varepsilon(\cdot) \in C^1(\mathbb{R})$ and

$$0 \le (K^\varepsilon)'(r) \le \frac{1}{\varepsilon} \quad \text{for any } r \in \mathbb{R}. \tag{8.24}$$

Furthermore, note from (A2) and (A6) that $g(\cdot) \in C^1(\mathbb{R})$ and

$$g'(r) + C_g \ge 0 \ \text{ for any } r \in \mathbb{R}. \tag{8.25}$$

Then, taking $z = \chi_\xi$ in (8.20), using (8.23)–(8.25) and the Schwarz inequality, and integrating in time, we obtain:

$$\begin{aligned} &\frac{1}{2}|\chi_\xi(t)|_H^2 + \varepsilon\kappa \int_0^t |(\chi_\xi)_x(s)|_H^2 ds \le \left(\frac{1}{2} + C_g\right)\int_0^t |\chi_\xi(s)|_H^2 ds + \frac{1}{2}\int_0^t |\theta_\xi(s)|_H^2 ds \\ &\qquad \text{for all } t \in [0,T]. \end{aligned} \tag{8.26}$$

Similarly, taking $z = \theta_\xi$ in (8.20), using (8.23), (8.24), and the Höder inequality, and integrating in time, we obtain:

$$\begin{aligned} \left|\int_0^t \langle \chi_\xi'(s), \theta_\xi(s)\rangle ds\right| &\le \kappa\left(\frac{\delta_3}{\varepsilon} + \varepsilon\right)\int_0^t |(\chi_\xi)_x(s)|_H |(\theta_\xi)_x(s)|_H ds \\ &\quad + \frac{1}{\varepsilon}\int_0^t |\chi_\xi(s)|_H |\theta_\xi(s)|_H ds \\ &\quad + \int_0^t |g'(w_\xi(s))\chi_\xi(s)|_H |\theta_\xi(s)|_H ds + \int_0^t |\theta_\xi(s)|_H^2 ds \\ &\quad \text{for all } t \in [0,T]. \end{aligned} \tag{8.27}$$

Note from (3.5) that we get the the following uniform estimate of solutions $[u_\xi, w_\xi] :=$

$\Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ with respect to $\xi \in [-1,1]\setminus\{0\}$:

$$
\begin{aligned}
&|u_\xi'|^2_{L^2(0,T;X')} + |u_\xi|^2_{L^\infty(0,T;H)} + |u_\xi|^2_{L^2(0,T;X)} + |w_\xi'|^2_{L^2(0,T;H)} \\
&\quad + |w_\xi|^2_{L^\infty(0,T;H)} + \kappa \sup_{0\le t\le T} V^\varepsilon(w_\xi(t)) + \sup_{0\le t\le T}\int_0^L \widehat{g}(w_\xi(t,x))\,dx \\
\le\ & N_1 \Bigg(|u_0^\varepsilon|_H^2 + |w_0^\varepsilon|_H^2 + \kappa V^\varepsilon(w_0^\varepsilon) + \int_0^L \widehat{g}(w_0^\varepsilon(x))\,dx + a_0^2|f+\xi\varpi_1|^2_{L^2(0,T;H)} \\
&\quad + a_1^2|h+\xi\varpi_2|^2_{L^2(0,T)} + a_2^2|\ell+\xi\varpi_3|^2_{L^2(0,T)} + b_1^2 + b_2^2 \Bigg),
\end{aligned}
\tag{8.28}
$$

where N_1 is the same positive constant in (3.5).

Taking account of (8.28), $\xi \in [-1,1]\setminus\{0\}$, and the continuous embedding $BV(0,L) \hookrightarrow L^\infty(0,L)$ (cf. Proposition 2.3), we see that

$$
\begin{aligned}
\sup_{t\in[0,T]} |w_\xi(t)|_{L^\infty(0,L)} \le N_{14} \Bigg(& |u_0^\varepsilon|_H^2 + |w_0^\varepsilon|_H^2 + \kappa V^\varepsilon(w_0^\varepsilon) + \int_0^L \widehat{g}(w_0^\varepsilon(x))\,dx \\
& + a_0^2|f|^2_{L^2(0,T;H)} + a_0^2|\varpi_1|^2_{L^2(0,T;H)} \\
& + a_1^2|h|^2_{L^2(0,T)} + a_1^2|\varpi_2|^2_{L^2(0,T)} \\
& + a_2^2|\ell|^2_{L^2(0,T)} + a_2^2|\varpi_3|^2_{L^2(0,T)} + b_1^2 + b_2^2 \Bigg),
\end{aligned}
\tag{8.29}
$$

hence, we observe from (A6) that

$$
\sup_{t\in[0,T]} |g'(w_\xi(t))|_{L^\infty(0,L)} \le N_{14,\varepsilon},
\tag{8.30}
$$

where N_{14} and $N_{14,\varepsilon}$ are positive constants independent of $\xi \in [-1,1]\setminus\{0\}$. Therefore, it follows from (8.27), (8.30), and the Schwarz inequality that

$$
\begin{aligned}
\left| \int_0^t \langle \chi_\xi'(s), \theta_\xi(s)\rangle ds \right| \le\ & \frac{1}{2}\int_0^t |(\theta_\xi)_x(s)|_H^2 ds + \frac{1}{2}\kappa^2\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2 \int_0^t |(\chi_\xi)_x(s)|_H^2 ds \\
& + \left(\frac{1}{2\varepsilon^2} + \frac{N_{14,\varepsilon}^2}{2}\right)\int_0^t |\chi_\xi(s)|_H^2 ds + 2\int_0^t |\theta_\xi(s)|_H^2 ds \\
& \text{for all } t \in [0,T].
\end{aligned}
\tag{8.31}
$$

Hence, we infer from (8.22) and (8.31) that

$$
\begin{aligned}
&\frac{1}{2}|\theta_\xi(t)|_H^2 + \frac{1}{2}\int_0^t |(\theta_\xi)_x(s)|_H^2 ds + \frac{n_0}{2}\int_0^t |\theta_\xi(s,0)|^2 ds + \frac{n_0}{2}\int_0^t |\theta_\xi(s,L)|^2 ds \\
\le\ & \frac{1}{2}\kappa^2\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2 \int_0^t |(\chi_\xi)_x(s)|_H^2 ds + \frac{5}{2}\int_0^t |\theta_\xi(s)|_H^2 ds \\
&\quad + \left(\frac{1}{2\varepsilon^2} + \frac{N_{14,\varepsilon}^2}{2}\right)\int_0^t |\chi_\xi(s)|_H^2 ds \\
&\quad + \frac{a_0^2}{2}\int_0^t |\check{f}(s)|_H^2 ds + \frac{a_1^2}{2n_0}\int_0^t |\check{h}(s)|^2 ds + \frac{a_2^2}{2n_0}\int_0^t |\check{\ell}(s)|^2 ds \\
&\qquad \text{for all } t \in [0,T].
\end{aligned}
\tag{8.32}
$$

Now, by adding (8.32) and (8.26)$\times\frac{\kappa}{\varepsilon}\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2$, we have

$$\begin{aligned}
&\frac{1}{2}|\theta_\xi(t)|_H^2+\frac{\kappa}{2\varepsilon}\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2|\chi_\xi(t)|_H^2+\frac{1}{2}\int_0^t|(\theta_\xi)_x(s)|_H^2ds\\
&\quad+\frac{1}{2}\kappa^2\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2\int_0^t|(\chi_\xi)_x(s)|_H^2ds\\
&\quad+\frac{n_0}{2}\int_0^t|\theta_\xi(s,0)|^2ds+\frac{n_0}{2}\int_0^t|\theta_\xi(s,L)|^2ds\\
\leq\ &\left(\frac{5}{2}+\frac{\kappa}{2\varepsilon}\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2\right)\int_0^t|\theta_\xi(s)|_H^2ds\\
&\quad+\left(\frac{1}{2\varepsilon^2}+\frac{N_{14,\varepsilon}^2}{2}+\frac{\kappa}{\varepsilon}\left(\frac{1}{2}+C_g\right)\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2\right)\int_0^t|\chi_\xi(s)|_H^2ds\\
&\quad+\frac{a_0^2}{2}\int_0^t|\check{f}(s)|_H^2ds+\frac{a_1^2}{2n_0}\int_0^t|\check{h}(s)|^2ds+\frac{a_2^2}{2n_0}\int_0^t|\check{\ell}(s)|^2ds\\
&\qquad\qquad\text{for all } t\in[0,T].
\end{aligned}\tag{8.33}$$

Thus, by applying the Gronwall inequality to (8.33) and the standard calculations, we have

$$\begin{aligned}
&|\theta_\xi(t)|_H^2+C_1(\varepsilon)|\chi_\xi(t)|_H^2+\int_0^t|(\theta_\xi)_x(s)|_H^2ds+C_2(\varepsilon)\int_0^t|(\chi_\xi)_x(s)|_H^2ds\\
&\quad+n_0\int_0^t|\theta_\xi(s,0)|^2ds+n_0\int_0^t|\theta_\xi(s,L)|^2ds\\
\leq\ &N_{15,\varepsilon}\left(a_0^2|\check{f}|_{L^2(0,T;H)}^2+\frac{a_1^2}{n_0}|\check{h}|_{L^2(0,T)}^2+\frac{a_2^2}{n_0}|\check{\ell}|_{L^2(0,T)}^2\right)\quad\text{for all } t\in[0,T],
\end{aligned}\tag{8.34}$$

where $C_1(\varepsilon)$, $C_2(\varepsilon)$, and $N_{15,\varepsilon}$ are positive constants dependent on ε and are independent of $\xi\in[-1,1]\setminus\{0\}$. In addition, by (8.20), (8.23), and (8.24), we have (cf. (8.27)):

$$\begin{aligned}
\left|\int_0^T\langle\chi_\xi'(s),z(s)\rangle ds\right|\ &\leq\kappa\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)|(\chi_\xi)_x|_{L^2(0,T;H)}|z_x|_{L^2(0,T;H)}\\
&\quad+\frac{1}{\varepsilon}|\chi_\xi|_{L^2(0,T;H)}|z|_{L^2(0,T;H)}\\
&\quad+\left|g'(w_\xi)\chi_\xi\right|_{L^2(0,T;H)}|z|_{L^2(0,T;H)}\\
&\quad+|\theta_\xi|_{L^2(0,T;H)}|z|_{L^2(0,T;H)},\qquad\forall z\in L^2(0,T;X).
\end{aligned}\tag{8.35}$$

Hence, we infer from (8.30), (8.34), and (8.35) that

$$|\chi_\xi'|_{L^2(0,T;X')}\leq N_{16,\varepsilon}\left(a_0|\check{f}|_{L^2(0,T;H)}+\frac{a_1}{\sqrt{n_0}}|\check{h}|_{L^2(0,T)}+\frac{a_2}{\sqrt{n_0}}|\check{\ell}|_{L^2(0,T)}\right),\tag{8.36}$$

where $N_{16,\varepsilon}$ is a positive constant dependent on ε and is independent of $\xi\in[-1,1]\setminus\{0\}$.

Similarly, we observe from (8.19), (8.34), and (8.36) that

$$|\theta'_\xi|_{L^2(0,T;X')} \le \tilde{N}_{16,\varepsilon}\left(a_0|\check{f}|_{L^2(0,T;H)} + \frac{a_1}{\sqrt{n_0}}|\check{h}|_{L^2(0,T)} + \frac{a_2}{\sqrt{n_0}}|\check{\ell}|_{L^2(0,T)}\right), \tag{8.37}$$

where $\tilde{N}_{16,\varepsilon}$ is a positive constant dependent on ε and is independent of $\xi \in [-1,1] \setminus \{0\}$.

By the uniform estimates (8.34), (8.36), and (8.37) of $[\theta_\xi, \chi_\xi]$, there is a subsequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \{\xi\}_{\xi\in[-1,1]\setminus\{0\}}$ and the functions $\overline{\theta}, \overline{\chi} \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \cap L^\infty(0,T;H)$ such that $\xi_n \to 0$,

$$\left.\begin{array}{ll} \theta_{\xi_n} \to \overline{\theta} & \text{in } C([0,T];X'), \\ & \text{in } L^2(0,T;H), \\ & \text{weakly in } W^{1,2}(0,T;X'), \\ & \text{weakly in } L^2(0,T;X), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;H), \end{array}\right\} \tag{8.38}$$

$$\theta_{\xi_n}(\cdot,0) \to \overline{\theta}(\cdot,0) \;\; \text{weakly in } L^2(0,T), \tag{8.39}$$

$$\theta_{\xi_n}(\cdot,L) \to \overline{\theta}(\cdot,L) \;\; \text{weakly in } L^2(0,T), \tag{8.40}$$

and

$$\left.\begin{array}{ll} \chi_{\xi_n} \to \overline{\chi} & \text{in } C([0,T];X'), \\ & \text{in } L^2(0,T;H), \\ & \text{weakly in } W^{1,2}(0,T;X'), \\ & \text{weakly in } L^2(0,T;X), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;H), \end{array}\right\} \tag{8.41}$$

as $n \to \infty$.

Here, from (8.16), (8.23), (8.24), (8.30), continuity of functions $(a^\varepsilon)'$, $(K^\varepsilon)'$, and g', and the Lebesgue dominated convergence theorem, we note that:

$$\left\{\begin{array}{l} (a^\varepsilon)'((w_{\xi_n})_x) \longrightarrow (a^\varepsilon)'(w_x), \\ (K^\varepsilon)'(w_{\xi_n}) \longrightarrow (K^\varepsilon)'(w), \\ g'(w_{\xi_n}) \longrightarrow g'(w), \end{array}\right. \quad \begin{array}{l} \text{in } L^2(0,T;H), \\ \text{weakly-}* \text{ in } L^\infty(Q), \end{array} \quad \text{as } n \to \infty. \tag{8.42}$$

Thus, taking a subsequence if necessary, we observe from (8.23), (8.24), (8.30), and (8.38)–(8.42) that:

$$\left\{\begin{array}{l} (a^\varepsilon)'((w_{\xi_n})_x)(\chi_{\xi_n})_x \rightharpoonup (a^\varepsilon)'(w_x)\overline{\chi}_x, \\ (K^\varepsilon)'(w_{\xi_n})\chi_{\xi_n} \rightharpoonup (K^\varepsilon)'(w)\overline{\chi}, \\ g'(w_{\xi_n})\chi_{\xi_n} \rightharpoonup g'(w)\overline{\chi}, \end{array}\right. \quad \text{weakly in } L^2(0,T;H), \;\; \text{as } n \to \infty. \tag{8.43}$$

Note from Proposition 6.1(I) that $[\theta_{\xi_n}, \chi_{\xi_n}] = D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f + \xi_n\varpi_1, h + \xi_n\varpi_2, \ell + \xi_n\varpi_3)$ satisfies the following variational identities:

$$\begin{aligned} &\int_0^T \langle \theta'_{\xi_n}(t), z(t)\rangle dt + \int_0^T \langle \chi'_{\xi_n}(t), z(t)\rangle dt + \int_0^T ((\theta_{\xi_n})_x(t), z_x(t))_H \, dt \\ &\qquad + n_0 \int_0^T \theta_{\xi_n}(t,0)z(t,0)dt + n_0 \int_0^T \theta_{\xi_n}(t,L)z(t,L)dt \\ &= \int_0^T (a_0\check{f}(t), z(t))_H dt + a_1 \int_0^T \check{h}(t)z(t,0)dt + a_2 \int_0^T \check{\ell}(t)z(t,L)dt \end{aligned} \tag{8.44}$$

for all $z \in L^2(0,T;X)$ and all direction $[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}$;

$$\begin{aligned}&\int_0^T \langle \chi'_{\xi_n}(t), z(t)\rangle dt + \kappa \int_0^T (((a^\varepsilon)'((w_{\xi_n})_x(t)) + \varepsilon)(\chi_{\xi_n})_x(t), z_x(t))_H \, dt \\ &+ \int_0^T ((K^\varepsilon)'(w_{\xi_n}(t))\chi_{\xi_n}(t), z(t))_H dt + \int_0^T (g'(w_{\xi_n}(t))\chi_{\xi_n}(t), z(t))_H dt \\ &\qquad = \int_0^T (\theta_{\xi_n}(t), z(t))_H dt \quad \text{ for all } z \in L^2(0,T;X);\end{aligned} \tag{8.45}$$

$$\theta_{\xi_n}(0,x) = \chi_{\xi_n}(0,x) = 0 \ (\in H) \ \text{ in } X'. \tag{8.46}$$

By (8.38)–(8.43), and by taking the limits in (8.44)–(8.46) as $n \to \infty$, we observe that $[\overline{\theta}, \overline{\chi}]$ satisfies the following system:

$$\begin{aligned}&\int_0^T \langle \overline{\theta}'(t), z(t)\rangle dt + \int_0^T \langle \overline{\chi}'(t), z(t)\rangle dt + \int_0^T \left(\overline{\theta}_x(t), z_x(t)\right)_H dt \\ &\qquad + n_0 \int_0^T \overline{\theta}(t,0) z(t,0) dt + n_0 \int_0^T \overline{\theta}(t,L) z(t,L) dt \\ &= \int_0^T (a_0 \check{f}(t), z(t))_H dt + a_1 \int_0^T \check{h}(t) z(t,0) dt + a_2 \int_0^T \check{\ell}(t) z(t,L) dt \\ &\qquad \text{for all } z \in L^2(0,T;X) \text{ and all direction } [\check{f}, \check{h}, \check{\ell}] \in \mathcal{U};\end{aligned} \tag{8.47}$$

$$\begin{aligned}&\int_0^T \langle \overline{\chi}'(t), z(t)\rangle dt + \kappa \int_0^T (((a^\varepsilon)'(w_x(t)) + \varepsilon)\overline{\chi}_x(t), z_x(t))_H \, dt \\ &\qquad + \int_0^T ((K^\varepsilon)'(w(t))\overline{\chi}(t), z(t))_H dt + \int_0^T (g'(w(t))\overline{\chi}(t), z(t))_H dt \\ &= \int_0^T (\overline{\theta}(t), z(t))_H dt \quad \text{ for all } z \in L^2(0,T;X);\end{aligned} \tag{8.48}$$

$$\overline{\theta}(0,x) = \overline{\chi}(0,x) = 0 \ (\in H) \ \text{ in } X'. \tag{8.49}$$

Since the solutions of the Cauchy problem $\{(8.47)–(8.49)\}$ are uniquely determined, we observe that $[\overline{\theta}, \overline{\chi}] = [\theta, \chi]$ and the convergence (8.14) holds without extracting any subsequence from $\{\xi\}_{\xi \in [-1,1] \setminus \{0\}}$, i.e.,

$$\begin{aligned}[\theta_\xi, \chi_\xi] &= D_{[\check{f},\check{h},\check{\ell}]} \Lambda^\varepsilon (f + \xi \varpi_1, h + \xi \varpi_2, \ell + \xi \varpi_3) \\ \longrightarrow \quad [\theta, \chi] &= D_{[\check{f},\check{h},\check{\ell}]} \Lambda^\varepsilon (f, h, \ell) \quad \text{ in } L^2(0,T;H) \times L^2(0,T;H) \\ &\text{for all } [f,h,\ell] \in \mathcal{U}, \ [\varpi_1, \varpi_2, \varpi_3] \in \mathcal{U}, \text{ and all direction } [\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}\end{aligned}$$

as $\xi \to 0$.

Thus, the proof of this lemma has been completed. □

Note from Proposition 6.1(II) that the cost functional J^ε admits the Gâteaux derivative at any $[f,h,l] \in \mathcal{U}$. Moreover, by Lemma 8.1 we can prove the continuity of Gâteaux derivative of J^ε as follows:

Corollary 8.1. *Assume the same conditions as in Theorem 8.1. Let* $\varepsilon \in (0,1]$ *and* $\xi \in [-1,1] \setminus \{0\}$. *Then, the Gâteaux derivative of the cost functional* J^ε *is continuous in the following sense:*

$$\begin{aligned} & D_{[\check{f},\check{h},\check{\ell}]} J^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3) \longrightarrow D_{[\check{f},\check{h},\check{\ell}]} J^\varepsilon(f,h,\ell) \\ & \textit{for all } [f,h,\ell] \in \mathcal{U},\ [\varpi_1,\varpi_2,\varpi_3] \in \mathcal{U},\ \textit{and all direction } [\check{f},\check{h},\check{\ell}] \in \mathcal{U} \end{aligned} \tag{8.50}$$

as $\xi \to 0$.

Proof. Note from (6.29) that

$$\begin{aligned} & D_{[\check{f},\check{h},\check{\ell}]} J^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3) \\ = \ & c_0 \int_0^T ((u_\xi - u_d)(t), \theta_\xi(t))_H dt + c_1 \int_0^T ((w_\xi - w_d)(t), \chi_\xi(t))_H dt \\ & + m_0 a_0^2 \int_0^T ((f+\xi\varpi_1)(t), \check{f}(t))_H dt \\ & + m_1 a_1^2 \int_0^T (h+\xi\varpi_2)(t)\check{h}(t) dt + m_2 a_2^2 \int_0^T (\ell+\xi\varpi_3)(t)\check{\ell}(t) dt \end{aligned}$$

for any $[f,h,\ell] \in \mathcal{U}$, $[\varpi_1,\varpi_2,\varpi_3] \in \mathcal{U}$, and any direction $[\check{f},\check{h},\check{\ell}] \in \mathcal{U}$, where $[u_\xi, w_\xi] = \Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ and $[\theta_\xi, \chi_\xi] = D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$. Thus, taking account of (8.14) and (8.15), we easily observe that the convergence (8.50) holds. □

Lemma 8.2. *Suppose the same conditions as in Theorem 8.1. Fix* $\varepsilon \in (0,1]$ *and the pair of initial data* $[u_0^\varepsilon, w_0^\varepsilon] \in H \times D(V^\varepsilon)$. *In addition, for any* $\xi \in [-1,1] \setminus \{0\}$, $[f,h,\ell] \in \mathcal{U}$, *and* $[\varpi_1,\varpi_2,\varpi_3] \in \mathcal{U}$, *let* $[p_\xi, q_\xi] = \Lambda^\varepsilon_{ad}(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$. *Then,* $[p_\xi, q_\xi] = \Lambda^\varepsilon_{ad}(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ *converges to* $[p,q] = \Lambda^\varepsilon_{ad}(f,h,\ell)$ *in* $L^2(0,T;H) \times L^2(0,T;H)$ *as* $\xi \to 0$. *Furthermore,*

$$p_\xi \to p \quad \textit{in } L^2(0,T;X) \quad \textit{as } \xi \to 0. \tag{8.51}$$

Proof. For any $\xi \in [-1,1] \setminus \{0\}$, $[f,h,\ell] \in \mathcal{U}$, and $[\varpi_1,\varpi_2,\varpi_3] \in \mathcal{U}$, let $[u_\xi, w_\xi] = \Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$. Then, note from Theorem 6.1 that $[p_\xi, q_\xi] = \Lambda^\varepsilon_{ad}(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ satisfies the following:

$$-p_\xi' - (p_\xi)_{xx} - q_\xi = c_0(u_\xi - u_d) \quad \text{in } Q; \tag{8.52}$$

$$\begin{aligned} & \int_t^T (-p_\xi'(\tau), \zeta(\tau))_H d\tau + \int_t^T \langle -q_\xi'(\tau), \zeta(\tau)\rangle d\tau \\ & \quad + \kappa \int_t^T (((a^\varepsilon)'((w_\xi)_x(\tau)) + \varepsilon)(q_\xi)_x(\tau), \zeta_x(\tau))_H \, d\tau \\ & \quad + \int_t^T ((K^\varepsilon)'(w_\xi(\tau))q_\xi(\tau), \zeta(\tau))_H \, d\tau + \int_t^T (g'(w_\xi(\tau))q_\xi(\tau), \zeta(\tau))_H \, d\tau \\ = \ & c_1 \int_t^T ((w_\xi - w_d)(\tau), \zeta(\tau))_H \, d\tau \\ & \qquad \text{for all } t \in [0,T] \text{ and } \zeta \in L^2(0,T;X); \end{aligned} \tag{8.53}$$

$$-(p_\xi)_x(t,0)+n_0p_\xi(t,0)=(p_\xi)_x(t,L)+n_0p_\xi(t,L)=0,\quad t\in(0,T), \tag{8.54}$$

$$p_\xi(T,x)=q_\xi(T,x)=0,\quad x\in(0,L). \tag{8.55}$$

Now, we give the uniform estimate of functions p_ξ and q_ξ with respect to $\xi\in[-1,1]\setminus\{0\}$.

Multiplying (8.52) by p_ξ and using the Schwarz inequality, we obtain:

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}|p_\xi(t)|_H^2+|(p_\xi)_x(t)|_H^2+n_0|p_\xi(t,0)|^2+n_0|p_\xi(t,L)|^2\\ \le\ &|p_\xi(t)|_H^2+\frac{1}{2}|q_\xi(t)|_H^2+\frac{c_0^2}{2}|(u_\xi-u_d)(t)|_H^2,\qquad \text{a.a. } t\in(0,T).\end{aligned} \tag{8.56}$$

Then, by (8.56) and the standard calculation (e.g., Gronwall inequality), we have

$$\begin{aligned}&|p_\xi(t)|_H^2+\int_t^T|(p_\xi)_x(s)|_H^2ds+n_0\int_t^T|p_\xi(s,0)|^2ds+n_0\int_t^T|p_\xi(s,L)|^2ds\\ \le\ &N_{17}\left(\int_t^T|q_\xi(s)|_H^2ds+c_0^2\int_t^T|(u_\xi-u_d)(s)|_H^2ds\right)\qquad\text{for all } t\in[0,T],\end{aligned} \tag{8.57}$$

where N_{17} is a positive constant independent of $\xi\in[-1,1]\setminus\{0\}$.

Next, multiplying (8.52) by $-p_\xi'$, using the Schwarz inequality, and integrating in time, we obtain:

$$\begin{aligned}&\frac{1}{2}\int_t^T|p_\xi'(s)|_H^2ds+\frac{1}{2}|(p_\xi)_x(t)|_H^2+\frac{n_0}{2}|p_\xi(t,0)|+\frac{n_0}{2}|p_\xi(t,L)|\\ \le\ &\int_t^T|q_\xi(s)|_H^2ds+c_0^2\int_t^T|(u_\xi-u_d)(s)|_H^2ds\qquad\text{for all } t\in[0,T].\end{aligned} \tag{8.58}$$

In addition, taking $\zeta=q_\xi$ in (8.53), using (8.23)–(8.25), (8.58), and the Schwarz inequality, and integrating in time, we obtain:

$$\begin{aligned}&\frac{1}{2}|q_\xi(t)|_H^2+\varepsilon\kappa\int_t^T|(q_\xi)_x(s)|_H^2ds\\ \le\ &(2+C_g)\int_t^T|q_\xi(s)|_H^2ds+c_0^2\int_t^T|(u_\xi-u_d)(s)|_H^2ds\\ &+\frac{c_1^2}{2}\int_t^T|(w_\xi-w_d)(s)|_H^2ds\qquad\text{for all } t\in[0,T].\end{aligned} \tag{8.59}$$

On account of (8.15), (8.59), and the Gronwall inequality, we can get the following estimate:

$$\begin{aligned}&|q_\xi(t)|_H^2+\varepsilon\kappa\int_t^T|(q_\xi)_x(s)|_H^2ds\\ \le\ &N_{18}\left(c_0^2|u-u_d|_{L^2(0,T;H)}^2+c_1^2|w-w_d|_{L^2(0,T;H)}^2+1\right)\qquad\text{for all } t\in[0,T],\end{aligned} \tag{8.60}$$

where N_{18} is a positive constant independent of $\xi \in [-1,1] \setminus \{0\}$. Consequently, we infer from (8.57), (8.58), and (8.60) that

$$\begin{aligned} & |p_\xi(t)|_H^2 + \int_t^T |(p_\xi)_x(s)|_H^2 ds + n_0 \int_t^T |p_\xi(s,0)|^2 ds + n_0 \int_t^T |p_\xi(s,L)|^2 ds \\ & \quad + \int_t^T |p_\xi'(s)|_H^2 ds + |(p_\xi)_x(t)|_H^2 + n_0 |p_\xi(t,0)| + n_0 |p_\xi(t,L)| \\ \leq \;& N_{19} \left(c_0^2 |u-u_d|^2_{L^2(0,T;H)} + c_1^2 |w - w_d|^2_{L^2(0,T;H)} + 1 \right) \quad \text{for all } t \in [0,T], \end{aligned} \tag{8.61}$$

where N_{19} is a positive constant independent of $\xi \in [-1,1] \setminus \{0\}$.

Additionally, we infer from (8.23), (8.24), and (8.53) that (cf. (8.27)):

$$\begin{aligned} & \left| \int_0^T \langle -q_\xi'(\tau), \zeta(\tau) \rangle d\tau \right| \\ \leq \;& |p_\xi'|_{L^2(0,T;H)} |\zeta|_{L^2(0,T;H)} + \kappa \left(\frac{\delta_3}{\varepsilon} + \varepsilon \right) |(q_\xi)_x|_{L^2(0,T;H)} |\zeta_x|_{L^2(0,T;H)} \\ & + \frac{1}{\varepsilon} |q_\xi|_{L^2(0,T;H)} |\zeta|_{L^2(0,T;H)} + |g'(w_\xi) q_\xi|_{L^2(0,T;H)} |\zeta|_{L^2(0,T;H)} \\ & + c_1 |w_\xi - w_d|_{L^2(0,T;H)} |\zeta|_{L^2(0,T;H)} \qquad \text{for all } \zeta \in L^2(0,T;X), \end{aligned}$$

which implies from (8.30), (8.60), and (8.61) that

$$|q_\xi'|_{L^2(0,T;X')} \leq \tilde{N}_{19,\varepsilon} \left(c_0^2 |u-u_d|^2_{L^2(0,T;H)} + c_1^2 |w-w_d|^2_{L^2(0,T;H)} + 1 \right), \tag{8.62}$$

where $\tilde{N}_{19,\varepsilon}$ is a positive constant dependent on ε and is independent of $\xi \in [-1,1] \setminus \{0\}$.

By the uniform estimates (8.60)–(8.62) of $[p_\xi, q_\xi]$, there are a subsequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \{\xi\}_{\xi \in [-1,1]\setminus\{0\}}$, and the functions $\overline{p} \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$, $\overline{q} \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \cap L^\infty(0,T;H)$ such that $\xi_n \to 0$,

$$\left. \begin{array}{ll} p_{\xi_n} \to \overline{p} & \text{in } C([0,T];H), \\ & \text{weakly in } W^{1,2}(0,T;H), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;X), \end{array} \right\} \tag{8.63}$$

$$p_{\xi_n}(\cdot,0) \to \overline{p}(\cdot,0) \;\; \text{weakly in } L^2(0,T), \tag{8.64}$$

$$p_{\xi_n}(\cdot,L) \to \overline{p}(\cdot,L) \;\; \text{weakly in } L^2(0,T), \tag{8.65}$$

and

$$\left. \begin{array}{ll} q_{\xi_n} \to \overline{q} & \text{in } C([0,T];X'), \\ & \text{in } L^2(0,T;H), \\ & \text{weakly in } W^{1,2}(0,T;X'), \\ & \text{weakly in } L^2(0,T;X), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;H), \end{array} \right\} \tag{8.66}$$

as $n \to \infty$.

Then, by (8.42), the uniqueness of the adjoint system $\{(6.3)–(6.6)\}$, and the similar argument in Lemma 8.1, we can show that

$$\begin{aligned} & [p_\xi, q_\xi] = \Lambda_{ad}^\varepsilon(f + \xi\varpi_1, h + \xi\varpi_2, \ell + \xi\varpi_3) \\ \longrightarrow \;& [\overline{p}, \overline{q}] = [p,q] = \Lambda_{ad}^\varepsilon(f,h,\ell) \quad \text{in } L^2(0,T;H) \times L^2(0,T;H) \text{ as } \xi \to 0. \end{aligned}$$

Finally, we show (8.51). To this end, subtract (6.3) for $[p_\xi, q_\xi] = \Lambda^\varepsilon_{ad}(f + \xi\varpi_1, h + \xi\varpi_2, \ell + \xi\varpi_3)$ from the one for $[p, q] = \Lambda^\varepsilon_{ad}(f, h, \ell)$, and multiply it by $p_\xi - p$. Then, using the Schwarz inequality, we obtain:

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}|(p_\xi - p)(t)|^2_H + |(p_\xi - p)_x(t)|^2_H \\ &\qquad +n_0|(p_\xi - p)(t, 0)|^2 + n_0|(p_\xi - p)(t, L)|^2 \\ &\leq |(p_\xi - p)(t)|^2_H + \frac{1}{2}|(q_\xi - q)(t)|^2_H + \frac{c_0^2}{2}|(u_\xi - u)(t)|^2_H, \\ &\qquad \text{for a.a. } t \in (0, T). \end{aligned} \tag{8.67}$$

Then, by (8.67) and the standard calculation (e.g., Gronwall inequality), we have

$$\begin{aligned} &|(p_\xi - p)(t)|^2_H + \int_t^T |(p_\xi - p)_x(s)|^2_H ds \\ &\qquad +n_0\int_t^T |(p_\xi - p)(s, 0)|^2 ds + n_0\int_t^T |(p_\xi - p)(s, L)|^2 ds \\ &\leq N_{20}\left(\int_t^T |(q_\xi - q)(s)|^2_H ds + c_0^2\int_t^T |(u_\xi - u)(s)|^2_H ds\right) \quad \text{for all } t \in [0, T], \end{aligned} \tag{8.68}$$

where N_{20} is a positive constant independent of $\xi \in [-1, 1] \setminus \{0\}$. Hence, we conclude from (8.15), (8.66), and (8.68) that (8.51) holds.

Thus, the proof of this lemma has been completed. □

Definition 8.2. We define the function $\gamma : [0, \infty) \to [0, \infty)$ by

$$\gamma(t) := \inf\left\{ |[\xi\varpi_1, \xi\varpi_2, \xi\varpi_3]|_{\mathcal{U}} \; ; \; \left| {}^{\mathrm{t}}\begin{bmatrix} \xi\varpi_1 + a_0(p_\xi - p) \\ \xi\varpi_2 + a_1(p_\xi - p)(\cdot, 0) \\ \xi\varpi_3 + a_2(p_\xi - p)(\cdot, L) \end{bmatrix} \right|_{\mathcal{U}} \geq t \right\}, \quad \text{for } t \geq 0, \tag{8.69}$$

where $[\varpi_1, \varpi_2, \varpi_3] \in \mathcal{U}$, $\xi \in \mathbb{R}$, the symbol ${}^{\mathrm{t}}\begin{bmatrix} \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{bmatrix}$ means the transposed matrix of $\begin{bmatrix} \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{bmatrix}$, namely, ${}^{\mathrm{t}}\begin{bmatrix} \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{bmatrix} = [\varpi_1, \varpi_2, \varpi_3]$, $|\cdot|_{\mathcal{U}}$ is the norm of $\mathcal{U}$ defined in (8.1), $[p_\xi, q_\xi] = \Lambda^\varepsilon_{ad}(f + \xi\varpi_1, h + \xi\varpi_2, \ell + \xi\varpi_3)$, and $[p, q] = \Lambda^\varepsilon_{ad}(f, h, \ell)$. Clearly, $\gamma(\cdot)$ is a well-defined increasing function with $\gamma(0) = 0$, because of the continuity of Λ^ε_{ad} and (8.51) (cf. Lemma 8.2).

Lemma 8.3. *Assume the same conditions as in Theorem 8.1. Let $n \in \mathbb{N}$ be a fixed number, and let $\{[f_k, h_k, \ell_k]; k = 1, 2, \cdots, n\}$ be a sequence in $\mathcal{U}$ defined by the numerical algorithm* (NA). *Let $[p_n, q_n] = \Lambda^\varepsilon_{ad}(f_n, h_n, \ell_n)$, $\beta \in (0, 1)$, and $\mu \in (0, 1)$. Put*

$$d_{0n} := a_0(p_n + m_0a_0f_n), \quad d_{1n} := a_1(p_n(\cdot, 0) + m_1a_1h_n), \quad d_{2n} := a_2(p_n(\cdot, L) + m_2a_2\ell_n).$$

Assume that at least one of the following conditions is satisfied:

$$d_{0n} \neq 0 \text{ in } L^2(0,T;H),\ d_{1n} \neq 0 \text{ in } L^2(0,T),\ \text{or } d_{2n} \neq 0 \text{ in } L^2(0,T). \tag{8.70}$$

Then, there is a minimal constant $\varsigma_n \in \mathbb{N} \cup \{0\}$ *such that*

$$\begin{aligned} & J^\varepsilon\left(f_n - \beta^{\varsigma_n} d_{0n}, h_n - \beta^{\varsigma_n} d_{1n}, \ell_n - \beta^{\varsigma_n} d_{2n}\right) - J^\varepsilon(f_n, h_n, \ell_n) \\ \leq\ & -\mu\beta^{\varsigma_n} \left|[d_{0n}, d_{1n}, d_{2n}]\right|^2_{\mathcal{U}}. \end{aligned} \tag{8.71}$$

Proof. By assumption (8.70) and by the definition of the Gâteaux derivative of $J^\varepsilon(\cdot,\cdot,\cdot)$, there is a constant $\delta_{\mu,n} > 0$ such that

$$\begin{aligned} & \left| \frac{J^\varepsilon(f_n - \lambda d_{0n}, h_n - \lambda d_{1n}, \ell_n - \lambda d_{2n}) - J^\varepsilon(f_n, h_n, \ell_n)}{\lambda} - D_{[-d_{0n}, -d_{1n}, -d_{2n}]} J^\varepsilon(f_n, h_n, \ell_n) \right| \\ & < (1-\mu)|[d_{0n}, d_{1n}, d_{2n}]|^2_{\mathcal{U}} \quad \text{for any } \lambda \in (-\delta_{\mu,n}, \delta_{\mu,n}) \setminus \{0\}. \end{aligned} \tag{8.72}$$

Put $[u_n, w_n] = \Lambda^\varepsilon[f_n, h_n, \ell_n]$ and $[\theta_n, \chi_n] = D_{[-d_{0n}, -d_{1n}, -d_{2n}]}\Lambda^\varepsilon(f_n, h_n, \ell_n)$. Then, by the proof of Theorem 6.1, we observe that

$$\begin{aligned} & D_{[-d_{0n}, -d_{1n}, -d_{2n}]} J^\varepsilon(f_n, h_n, \ell_n) \\ = \ & c_0 \int_0^T ((u_n - u_d)(t), \theta_n(t))_H dt + c_1 \int_0^T ((w_n - w_d)(t), \chi_n(t))_H dt \\ & + m_0 a_0^2 \int_0^T (f_n(t), -d_{0n}(t))_H dt + m_1 a_1^2 \int_0^T h_n(t)(-d_{1n}(t)) dt \\ & + m_2 a_2^2 \int_0^T \ell_n(t)(-d_{2n}(t)) dt \\ = \ & \int_0^T (a_0 p_n(t) + m_0 a_0^2 f_n(t), -d_{0n}(t))_H dt \\ & \quad + \int_0^T (a_1 p_n(t,0) + m_1 a_1^2 h_n(t))(-d_{1n}(t)) dt \\ & \quad + \int_0^T (a_2 p_n(t,L) + m_2 a_2^2 \ell_n(t))(-d_{2n}(t)) dt \\ = \ & -|[d_{0n}, d_{1n}, d_{2n}]|^2_{\mathcal{U}}. \end{aligned} \tag{8.73}$$

Therefore, we observe from (8.72) that

$$J^\varepsilon(f_n - \lambda d_{0n}, h_n - \lambda d_{1n}, \ell_n - \lambda d_{2n}) - J^\varepsilon(f_n, h_n, \ell_n) \leq -\lambda\mu|[d_{0n}, d_{1n}, d_{2n}]|^2_{\mathcal{U}}$$

for any $\lambda \in (0, \delta_{\mu,n})$. Therefore, we have only to take a minimal constant $\varsigma_n \in \mathbb{N} \cup \{0\}$ such that

$$0 < \beta^{\varsigma_n} < \delta_{\mu,n}.$$

Thus, the proof of this lemma has been completed. □

Lemma 8.4. *Assume the same conditions as in Theorem 8.1. Let $n \in \mathbb{N}$ be a fixed number, and let $\{[f_k, h_k, \ell_k]; k = 1, 2, \cdots, n\}$ be a sequence in $\mathcal{U}$ defined by the numerical algorithm* (NA). *Let $[p_n, q_n] = \Lambda^{\varepsilon}_{ad}(f_n, h_n, \ell_n)$, $\beta \in (0, 1)$, and $\mu \in (0, 1)$. Put*

$$d_{0n} := a_0(p_n + m_0 a_0 f_n), d_{1n} := a_1(p_n(\cdot, 0) + m_1 a_1 h_n), d_{2n} := a_2(p_n(\cdot, L) + m_2 a_2 \ell_n). \quad (8.74)$$

Assume that at least one of the following conditions is satisfied:

$$d_{0n} \neq 0 \text{ in } L^2(0, T; H),\ d_{1n} \neq 0 \text{ in } L^2(0, T),\ \text{or } d_{2n} \neq 0 \text{ in } L^2(0, T).$$

Let ς_n be the constant obtained in Lemma 8.3, and put

$$M_{max} := \max\{m_0 a_0^2,\ m_1 a_1^2,\ m_2 a_2^2\}. \quad (8.75)$$

Then, we have

$$\beta\gamma\left((1-\mu)|[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}}\right) \leq \beta^{\varsigma_n} M_{max} |[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}}, \quad (8.76)$$

where $\gamma(\cdot)$ is the function defined by (8.69) *in Definition 8.2.*

Proof. From the definition of ς_n obtained in Lemma 8.3, we observe that

$$\begin{aligned} & J^{\varepsilon}\left(f_n - \frac{\beta^{\varsigma_n}}{\beta} d_{0n}, h_n - \frac{\beta^{\varsigma_n}}{\beta} d_{1n}, \ell_n - \frac{\beta^{\varsigma_n}}{\beta} d_{2n}\right) - J^{\varepsilon}(f_n, h_n, \ell_n) \\ & > -\mu \frac{\beta^{\varsigma_n}}{\beta} |[d_{0n}, d_{1n}, d_{2n}]|^2_{\mathcal{U}}. \end{aligned} \quad (8.77)$$

Here, by (8.73), the mean-valued theorem, and the continuity of $D_{[-d_{0n}, -d_{1n}, -d_{2n}]} J^{\varepsilon}(f_n + \xi\varpi_1, h_n + \xi\varpi_2, \ell_n + \xi\varpi_3)$ with respect to ξ, there is a constant $\vartheta \in (0, 1)$ satisfying

$$\begin{aligned} & J^{\varepsilon}\left(f_n - \frac{\beta^{\varsigma_n}}{\beta} d_{0n}, h_n - \frac{\beta^{\varsigma_n}}{\beta} d_{1n}, \ell_n - \frac{\beta^{\varsigma_n}}{\beta} d_{2n}\right) - J^{\varepsilon}(f_n, h_n, \ell_n) \\ = & \int_0^{\frac{\beta^{\varsigma_n}}{\beta}} \frac{d}{d\xi} J^{\varepsilon}\left(f_n - \xi d_{0n}, h_n - \xi d_{1n}, \ell_n - \xi d_{2n}\right) d\xi \\ = & \frac{\beta^{\varsigma_n}}{\beta} D_{[-d_{0n}, -d_{1n}, -d_{2n}]} J^{\varepsilon}\left(f_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{0n}, h_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{1n}, \ell_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{2n}\right) \\ = & \frac{\beta^{\varsigma_n}}{\beta}\left[\int_0^T \left(a_0 p_{n,\vartheta}(t) + m_0 a_0^2\big(f_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{0n}(t)\big), -d_{0n}(t)\right)_H dt \right. \\ & + \int_0^T \left(a_1 p_{n,\vartheta}(t, 0) + m_1 a_1^2(h_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{1n}(t))\right)(-d_{1n}(t)) dt \\ & \left. + \int_0^T \left(a_2 p_{n,\vartheta}(t, L) + m_2 a_2^2(\ell_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{2n}(t))\right)(-d_{2n}(t)) dt \right], \end{aligned} \quad (8.78)$$

where $[p_{n,\vartheta}, q_{n,\vartheta}] = \Lambda^{\varepsilon}_{ad}\left(f_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{0n}, h_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{1n}, \ell_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{2n}\right)$.

It follows from (8.77) and (8.78) that

$$
\begin{aligned}
&(1-\mu)\,|[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}}^2 \\
\leq\ & |[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}}^2 + \int_0^T \left(a_0 p_{n,\vartheta}(t) + m_0 a_0^2\big(f_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{0n}(t)\big), -d_{0n}(t)\right)_H dt \\
&\quad + \int_0^T \Big(a_1 p_{n,\vartheta}(t,0) + m_1 a_1^2 (h_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{1n}(t))\Big)(-d_{1n}(t))dt \\
&\quad + \int_0^T \Big(a_2 p_{n,\vartheta}(t,L) + m_2 a_2^2 (\ell_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{2n}(t))\Big)(-d_{2n}(t))dt \\
=\ & \int_0^T \left(a_0 p_{n,\vartheta}(t) + m_0 a_0^2\big(f_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{0n}(t)\big) - d_{0n}(t), -d_{0n}(t)\right)_H dt \\
&\quad + \int_0^T \Big(a_1 p_{n,\vartheta}(t,0) + m_1 a_1^2 (h_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{1n}(t)) - d_{1n}(t)\Big)(-d_{1n}(t))dt \\
&\quad + \int_0^T \Big(a_2 p_{n,\vartheta}(t,L) + m_2 a_2^2 (\ell_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{2n}(t)) - d_{2n}(t)\Big)(-d_{2n}(t))dt.
\end{aligned}
$$

Hence, we infer from the Hölder inequality and (8.74) that

$$
\begin{aligned}
(1-\mu)\big|[d_{0n}, d_{1n}, d_{2n}]\big|_{\mathcal{U}} \ &\leq \left|{}^{\mathrm{t}}\begin{bmatrix} a_0 p_{n,\vartheta} + m_0 a_0^2\big(f_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{0n}\big) - d_{0n} \\ a_1 p_{n,\vartheta}(\cdot,0) + m_1 a_1^2\big(h_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{1n}\big) - d_{1n} \\ a_2 p_{n,\vartheta}(\cdot,L) + m_2 a_2^2\big(\ell_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta} d_{2n}\big) - d_{2n} \end{bmatrix}\right|_{\mathcal{U}} \\
&= \left|{}^{\mathrm{t}}\begin{bmatrix} -\vartheta\frac{\beta^{\varsigma_n}}{\beta} m_0 a_0^2 d_{0n} + a_0(p_{n,\vartheta} - p_n) \\ -\vartheta\frac{\beta^{\varsigma_n}}{\beta} m_1 a_1^2 d_{1n} + a_1(p_{n,\vartheta}(\cdot,0) - p_n(\cdot,0)) \\ -\vartheta\frac{\beta^{\varsigma_n}}{\beta} m_2 a_2^2 d_{2n} + a_2(p_{n,\vartheta}(\cdot,L) - p_n(\cdot,L)) \end{bmatrix}\right|_{\mathcal{U}}.
\end{aligned}
\tag{8.79}
$$

By the definition of the function γ, we observe from (8.79) and $\theta \in (0,1)$ that

$$
\begin{aligned}
\gamma\left((1-\mu)\big|[d_{0n}, d_{1n}, d_{2n}]\big|_{\mathcal{U}}\right) \ &\leq\ \vartheta\frac{\beta^{\varsigma_n}}{\beta}\left|\left[m_0 a_0^2 d_{0n}, m_1 a_1^2 d_{1n}, m_2 a_2^2 d_{2n}\right]\right|_{\mathcal{U}}, \\
&\leq\ \frac{\beta^{\varsigma_n}}{\beta} M_{max} |[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}},
\end{aligned}
$$

which implies that the inequality (8.76) holds. □

Now, we show our main Theorem 8.1 in this paper, which is concerned with the convergence for numerical algorithm (NA).

Proof of Theorem 8.1. We show (I). By (Step 5) in the numerical algorithm (NA) (cf. (8.71) or (8.82) below), we easily observe that $J^\varepsilon(f_n, h_n, \ell_n)$ is the non-increasing sequence with respect to n. Thus, from the non-negativity of $J^\varepsilon(\cdot,\cdot,\cdot)$ (cf. (1.16)), we infer that $\lim_{n\to\infty} J^\varepsilon(f_n, h_n, \ell_n)$ exists.

We next show (II). Indeed, we first prove (8.2) by contradiction. To this end, we assume that (8.2) does not hold. Then, there exist a constant $\delta > 0$ and a sequence $\{k\}_{k\in\mathbb{N}}$ such that

$$|a_0(p_k + m_0 a_0 f_k)|_{L^2(0,T;H)} \geq \delta \quad \text{for any } k. \tag{8.80}$$

Since $\gamma(\cdot)$ is the increasing function (cf. Definition 8.2), it follows from (8.74) and (8.80) that

$$\gamma\left((1-\mu)\delta\right) \leq \gamma\left((1-\mu)\big|[d_{0k}, d_{1k}, d_{2k}]\big|_{\mathcal{U}}\right) \quad \text{for any } k. \tag{8.81}$$

Then, we observe from (8.71), (8.76), (8.80), and (Step 5) that

$$\begin{aligned}
& J^\varepsilon(f_{k+1}, h_{k+1}, \ell_{k+1}) - J^\varepsilon(f_k, h_k, \ell_k) \\
= \;& J^\varepsilon\left(f_k - \beta^{\varsigma_k} d_{0k}, h_k - \beta^{\varsigma_k} d_{1k}, \ell_k - \beta^{\varsigma_k} d_{2k}\right) - J^\varepsilon(f_k, h_k, \ell_k) \\
\leq \;& -\mu\beta^{\varsigma_k}\big|[d_{0k}, d_{1k}, d_{2k}]\big|_{\mathcal{U}}^2 \\
\leq \;& -\frac{\mu\beta}{M_{max}}\gamma\left((1-\mu)\big|[d_{0k}, d_{1k}, d_{2k}]\big|_{\mathcal{U}}\right)\big|[d_{0k}, d_{1k}, d_{2k}]\big|_{\mathcal{U}} \\
\leq \;& -\frac{\mu\beta}{M_{max}}\gamma\left((1-\mu)\delta\right)\delta \\
< \;& 0 \qquad\qquad\qquad \text{for any } k \in \mathbb{N}.
\end{aligned} \tag{8.82}$$

By repeating this procedure, we observe from (8.82) that

$$\begin{aligned}
J^\varepsilon(f_{k+1}, h_{k+1}, \ell_{k+1}) \;&\leq\; J^\varepsilon(f_k, h_k, \ell_k) - \frac{\mu\beta}{M_{max}}\gamma\left((1-\mu)\delta\right)\delta \\
&\leq\; J^\varepsilon(f_{k-1}, h_{k-1}, \ell_{k-1}) - \frac{2\mu\beta}{M_{max}}\gamma\left((1-\mu)\delta\right)\delta \\
&\leq\; \cdots \\
&\leq\; J^\varepsilon(f_1, h_1, \ell_1) - \frac{k\mu\beta}{M_{max}}\gamma\left((1-\mu)\delta\right)\delta \quad \text{for any } k \in \mathbb{N}.
\end{aligned}$$

Therefore, the above inequality implies that

$$J^\varepsilon(f_{k+1}, h_{k+1}, \ell_{k+1}) \longrightarrow -\infty \quad \text{as} \;\; k \to \infty. \tag{8.83}$$

This contradicts the non-negativity of $J^\varepsilon(\cdot,\cdot,\cdot)$ (cf. (1.16)). Hence, (8.2) holds.

Similarly, we can show (8.3) and (8.4), thus, (II) holds.

Now, we show (III). By Theorem 8.1(I) and the definition of $J^\varepsilon(\cdot,\cdot,\cdot)$ (cf. (1.16)), we observe that $\{[f_n, h_n, \ell_n]\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{U}$. Therefore, there exist a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ and a triplet of functions $[f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**}] \in \mathcal{U}$ such that $n_k \to \infty$ and

$$\left.\begin{aligned}
& f_{n_k} \longrightarrow f^\varepsilon_{**} \;\; \text{weakly in } L^2(0,T;H), \\
& h_{n_k} \longrightarrow h^\varepsilon_{**} \;\; \text{weakly in } L^2(0,T), \\
& \ell_{n_k} \longrightarrow \ell^\varepsilon_{**} \;\; \text{weakly in } L^2(0,T)
\end{aligned}\right\} \tag{8.84}$$

as $k \to \infty$. Then, from Corollary 4.2 concerning the convergence result of solutions to (P;$u_0^\varepsilon, w_0^\varepsilon, f_{n_k}, h_{n_k}, \ell_{n_k})^\varepsilon$, we observe that

$$\begin{aligned} [u_{n_k}, w_{n_k}] = \Lambda^\varepsilon(f_{n_k}, h_{n_k}, \ell_{n_k}) \longrightarrow [u^\varepsilon_{**}, w^\varepsilon_{**}] = \Lambda^\varepsilon(f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**}) \\ \text{in } L^2(0,T;H) \times C([0,T];H) \ \text{ as } k \to \infty. \end{aligned} \tag{8.85}$$

In addition, by (8.85) and the slight modification of the proof of Lemma 8.2, there are functions $p^\varepsilon_{**} \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$, $q^\varepsilon_{**} \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \cap L^\infty(0,T;H)$, and a subsequence of $\{n_k\}_{k\in\mathbb{N}}$ (which we also denote $\{n_k\}_{k\in\mathbb{N}}$ for simplicity) such that $[p_{n_k}, q_{n_k}] = \Lambda^\varepsilon_{ad}(f_{n_k}, h_{n_k}, \ell_{n_k})$ converges to $[p^\varepsilon_{**}, q^\varepsilon_{**}] = \Lambda^\varepsilon_{ad}(f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**})$ in the following sense:

$$p_{n_k} \to p^\varepsilon_{**} \quad \left.\begin{aligned} &\text{in } C([0,T];H), \\ &\text{weakly in } W^{1,2}(0,T;H), \\ &\text{weakly-}* \text{ in } L^\infty(0,T;X), \end{aligned}\right\} \tag{8.86}$$

$$p_{n_k}(\cdot,0) \to p^\varepsilon_{**}(\cdot,0) \ \text{ weakly in } L^2(0,T), \tag{8.87}$$

$$p_{n_k}(\cdot,L) \to p^\varepsilon_{**}(\cdot,L) \ \text{ weakly in } L^2(0,T), \tag{8.88}$$

and

$$q_{n_k} \to q^\varepsilon_{**} \quad \left.\begin{aligned} &\text{in } C([0,T];X'), \\ &\text{in } L^2(0,T;H), \\ &\text{weakly in } W^{1,2}(0,T;X'), \\ &\text{weakly in } L^2(0,T;X), \\ &\text{weakly-}* \text{ in } L^\infty(0,T;H), \end{aligned}\right\} \tag{8.89}$$

as $k \to \infty$. In addition, from arguments similar to (8.51), we observe that

$$p_{n_k} \to p^\varepsilon_{**} \ \text{ in } L^2(0,T;X) \ \text{ as } k \to \infty, \tag{8.90}$$

hence, in particular,

$$p_{n_k}(\cdot,0) \to p^\varepsilon_{**}(\cdot,0) \ \text{ in } L^2(0,T), \quad p_{n_k}(\cdot,L) \to p^\varepsilon_{**}(\cdot,L) \ \text{ in } L^2(0,T) \ \text{ as } k \to \infty. \tag{8.91}$$

Therefore, we infer from (8.2), (8.3), (8.4), (8.84), (8.86), (8.90), and (8.91) that the assertions (8.5)–(8.12) hold. In addition, we conclude from Theorem 6.1 and (8.10)–(8.12) (cf. (8.73)) that (8.13) holds, hence, $[f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**}] \in \mathcal{U}$ is the stationary point of the cost functional J^ε with $\varepsilon > 0$. Thus, the proof of Theorem 8.1 has been completed. □

9 Numerical experiments

In this section, by similar approach as in [35, 37, 38] we perform the simple numerical experiments to (OP)$^\varepsilon$ with some small $\varepsilon > 0$.

9.1 State system and its optimal control problem

For the stability of numerics and the propagation speed of the interfaces, we now rescale (t, x) by the small parameter $\sigma > 0$. Indeed, we change the pair of variables (t, x) into $(s, y) := (\sigma t, \sigma x)$. Then, from the formal calculations we observe that (1.1) and (1.2) are reformulated as follows, respectively:

$$[u + w]_s - \sigma u_{yy} = \frac{a_0 \widetilde{f}(s, y)}{\sigma} \quad \text{in } (s, y) \in Q_\sigma := (0, \sigma T) \times (0, \sigma L)\,, \tag{9.1}$$

$$w_s - \kappa \left(\frac{w_y}{|w_y|} \right)_y + \frac{\partial I_{[-1,1]}(w)}{\sigma} + \frac{g(w)}{\sigma} \ni \frac{u}{\sigma} \quad \text{in } Q_\sigma, \tag{9.2}$$

where we put $\widetilde{f}(s, y) := f(s/\sigma, y/\sigma)$ for $(s, y) \in Q_\sigma$, for simplicity.

Ohtsuka [35] and Ohtsuka–Shirakawa–Yamazaki [37, 38] gave numerical experiments of optimal control problem for the approximate Allen–Cahn type equation associated with total variation energy, in which the singular diffusion term $\left(\frac{w_y}{|w_y|} \right)_y$ was approximated by $\left(\frac{w^\varepsilon_y}{\sqrt{|w^\varepsilon_y|^2 + \varepsilon^2}} \right)_y$ for $\varepsilon > 0$.

In this section, by similar approach as in [35, 37, 38] we perform the simple numerical experiments to $(\mathrm{OP})^\varepsilon$ with some small $\varepsilon > 0$. Indeed, we consider a distributed control problem with the heat source as control, more precisely, $(\mathrm{OP})^\varepsilon$ in the case when $a^\varepsilon(r) = \frac{r}{\sqrt{|r|^2 + \varepsilon^2}}$, $g(r) = r^3 - r$, and $a_1 = a_2 = b_1 = b_2 = c_0 = 0$.

Now, for the fixed rescale parameter $\sigma \in (0, 1]$, we take $T = \widetilde{T}/\sigma$ and $L = \widetilde{L}/\sigma$ for some positive constants $\widetilde{T}$ and $\widetilde{L}$. Then, we give numerical experiments of the optimal control problem for the following state system that is the approximate problem of (9.1) and (9.2):

Problem (P;$\widetilde{f}, 0, 0)^\varepsilon$**.**

$$[u^\varepsilon + w^\varepsilon]_s - \sigma u^\varepsilon_{yy} = \frac{a_0 \widetilde{f}(s, y)}{\sigma} \quad \text{in } (s, y) \in \widetilde{Q}_\sigma := (0, \widetilde{T}) \times (0, \widetilde{L}), \tag{9.3}$$

$$w^\varepsilon_s - \kappa \left(\frac{w^\varepsilon_y}{\sqrt{|w^\varepsilon_y|^2 + \varepsilon^2}} + \varepsilon w^\varepsilon_y \right)_y + \frac{K^\varepsilon(w^\varepsilon)}{\sigma} + \frac{(w^\varepsilon)^3 - w^\varepsilon}{\sigma} = \frac{u^\varepsilon}{\sigma} \quad \text{in } \widetilde{Q}_\sigma, \tag{9.4}$$

$$- u^\varepsilon_y(s, 0) + u^\varepsilon(s, 0) = u^\varepsilon_y(s, \widetilde{L}) + u^\varepsilon(s, \widetilde{L}) = 0, \quad s \in (0, \widetilde{T}), \tag{9.5}$$

$$w^\varepsilon_y(s, 0) = w^\varepsilon_y(s, \widetilde{L}) = 0, \quad s \in (0, \widetilde{T}), \tag{9.6}$$

$$u^\varepsilon(0, y) = u^\varepsilon_0(y), \quad w^\varepsilon(0, y) = w^\varepsilon_0(y), \quad y \in (0, \widetilde{L}). \tag{9.7}$$

In addition, for simplicity, we consider the following distributed control problem with the heat source as control:

Problem (OP)$^\varepsilon$: Find a control function $\widetilde{f}^\varepsilon_* \in L^2(0,\widetilde{T};H)$, call *optimal control*, such that

$$J^\varepsilon(\widetilde{f}^\varepsilon_*) = \inf_{\widetilde{f} \in L^2(0,\widetilde{T};H)} J^\varepsilon(\widetilde{f}). \tag{9.8}$$

Here, $J^\varepsilon(\widetilde{f})$ is the cost functional defined by

$$J^\varepsilon(\widetilde{f}) := \frac{c_1}{2}\int_0^{\widetilde{T}} |(w^\varepsilon - w_d)(s)|_H^2 ds + \frac{m_0}{2}\int_0^{\widetilde{T}} |\widetilde{f}(s)|_H^2 ds, \tag{9.9}$$

where c_1, m_0 are nonnegative constants, w_d is the given desired target profile in $L^2(0,\widetilde{T};H)$, and a couple of functions $[u^\varepsilon, w^\varepsilon]$ is a unique solution to the initial–boundary value state problem (P;$\widetilde{f},0,0)^\varepsilon$ for the control parameter $\widetilde{f} \in L^2(0,\widetilde{T};H)$.

Note that the rescale parameter σ appears in (9.3) and (9.4). However, σ is a fixed positive constant. Hence, by the slight modification of the proof of Theorems 3.1, 4.1, 5.1, 6.1, 7.1, and 8.1, we can prove the solvability of state systems $\{(9.3)\text{–}(9.7)\}$ for any $\varepsilon \in (0,1]$, the existence of optimal controls to (9.8), and so on.

9.2 Discretization

We perform the numerical experiments of (P;$\widetilde{f},0,0)^\varepsilon$ and (OP)$^\varepsilon$ via the standard explicit finite difference scheme. Indeed, let Δt and Δh be the mesh size of time and space, respectively, and set $w_{n,j} := w^\varepsilon(n\Delta t, j\Delta h)$ and $D_y^\pm w_{n,j} := \pm(w_{n,j\pm 1} - w_{n,j})/\Delta h$. Then, the diffusion term in (9.4) is discretized by the following:

$$\mathcal{D}w_{n,j} := \frac{1}{\Delta h}\left[\kappa\left(\frac{D_y^+ w_{n,j}}{\sqrt{|D_y^+ w_{n,j}|^2 + \varepsilon^2}} - \frac{D_y^- w_{n,j}}{\sqrt{|D_y^- w_{n,j}|^2 + \varepsilon^2}} + \varepsilon(D_y^+ w_{n,j} - D_y^- w_{n,j})\right)\right].$$

Other terms are discretized by the standard forms. For instance, we refer to the explicit finite difference scheme used in [35].

9.3 Numerical experiments

In this subsection, we give three numerical experiments of (OP)$^\varepsilon$ with sufficient small parameter ε under the following numerical data:

<u>**Numerical data**</u>

- $\sigma = 0.001$, the domain $\widetilde{Q}_\sigma = (0,\widetilde{T}) \times (0,\widetilde{L})$ with $\widetilde{T} = 0.0025$ and $\widetilde{L} = 1.0$, the space mesh size $\Delta h = 0.005$, the time mesh size $\Delta t = 0.1 \times \Delta h^2 = 0.0000025$, $\kappa = 0.001$, $c_1 = 10.0$, $m_0 = 1.0$, $\varepsilon = 0.001$, the stop parameter $\mu = 0.0001$ for (NA), and the given initial data $[u_0^\varepsilon, w_0^\varepsilon] \equiv [0.0, 0.0]$. In addition, we take $f_0 \equiv 0.0$ as the initial control function for (NA).

(Numerical experiment 1)

In the first experiment, we consider a simple target desired profile w_d such that

$$w_d(s,y) := \begin{cases} 1, & \text{if } y \in [0.30, 0.70], \\ -1, & \text{otherwise}, \end{cases} \qquad \forall s \in [0, \widetilde{T}], \tag{9.10}$$

whose graph is the dotted line in Figure 1.

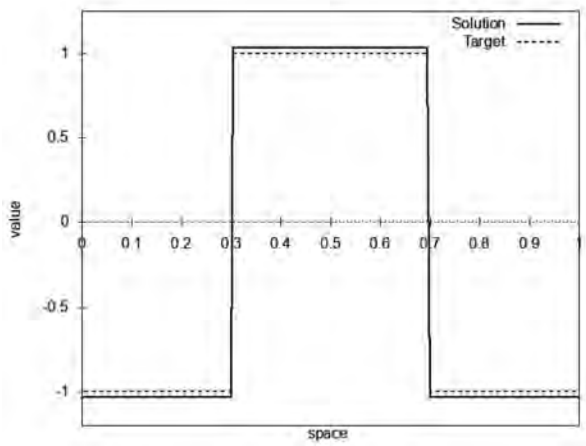

Figure 1: Target profile $w_d(\widetilde{T}, y)$ and solution $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 7$.

We perform a numerical experiment of $(\mathrm{OP})^\varepsilon$ by using the numerical algorithm (NA) proposed in Section 8. Then, (NA) is finished when the iteration number is $n = 7$ as in Figure 2.

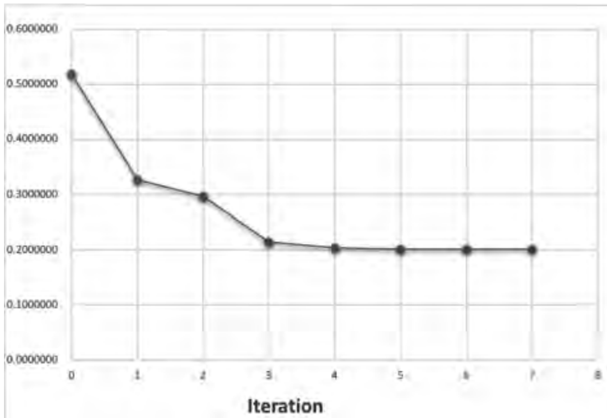

Figure 2: The value of the cost functional J^ε for $(\mathrm{OP})^\varepsilon$.

Figure 3 is the graph of the control function $\widetilde{f}$ found by (NA) in the case of the iteration number $n = 7$.

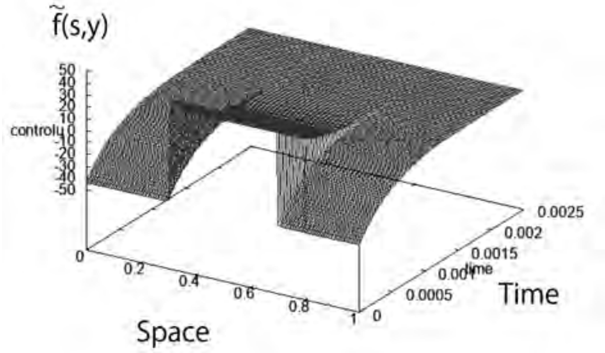

Figure 3: The graph of the control function $\widetilde{f}$ found by (NA) at the iteration number $n = 7$.

Figure 4 is the picture of the solutions u^ε and w^ε for $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ with initial data $[u_0^\varepsilon, w_0^\varepsilon] \equiv [0.0, 0.0]$ in the case of the iteration number $n = 7$.

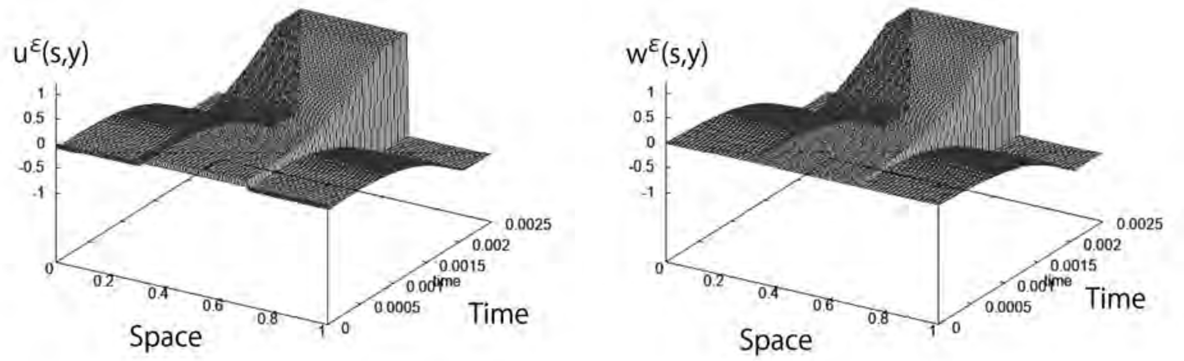

Figure 4: The graph of solution $[u^\varepsilon, w^\varepsilon]$ to $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ at the iteration number $n = 7$: (left) $u^\varepsilon(s,y)$; (right) $w^\varepsilon(s,y)$.

In addition, the real line in Figure 1 means the graph of $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 7$. We observe from Figures 1–4 that the solution $w^\varepsilon(\widetilde{T}, y)$ to $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ has the similar profile to the desired one $w_d(\widetilde{T}, y)$ and the data sequence of cost functional J^ε almost reaches a stationary point.

(Numerical experiment 2)

In the second experiment, we consider a target desired profile w_d such that

$$w_d(s,y) := \begin{cases} 1, & \text{if } y \in [0.00, 0.35], \\ 0, & \text{if } y \in (0.35, 0.70], \\ -1, & \text{if } y \in (0.70, 1.00], \end{cases} \qquad \forall s \in [0, \widetilde{T}], \tag{9.11}$$

whose graph is the dotted line in Figure 5.

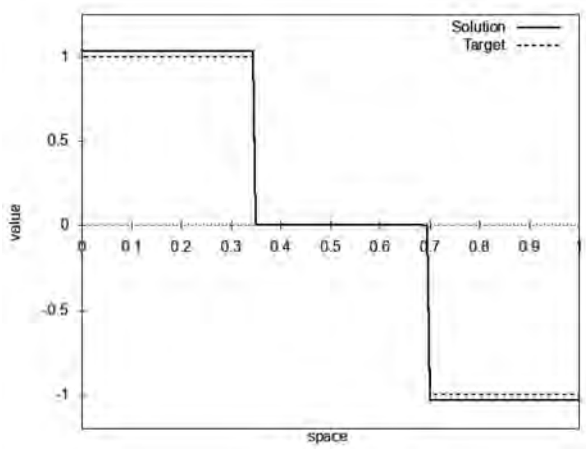

Figure 5: Target profile $w_d(\widetilde{T}, y)$ and solution $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 15$.

We perform a numerical experiment of $(\mathrm{OP})^\varepsilon$ by using the numerical algorithm (NA) proposed in Section 8. Then, (NA) is finished when the iteration number is $n = 15$ as in Figure 6.

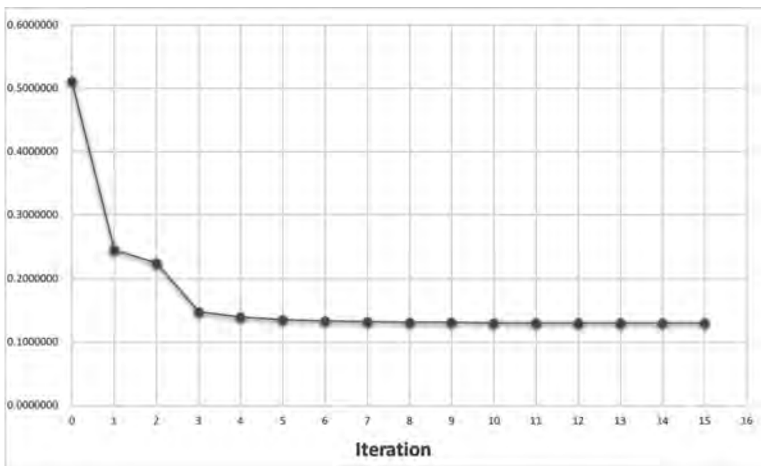

Figure 6: The value of the cost functional J^ε for $(\mathrm{OP})^\varepsilon$.

Figure 7 is the graph of the control function $\widetilde{f}$ found by (NA) in the case of the iteration number $n = 15$.

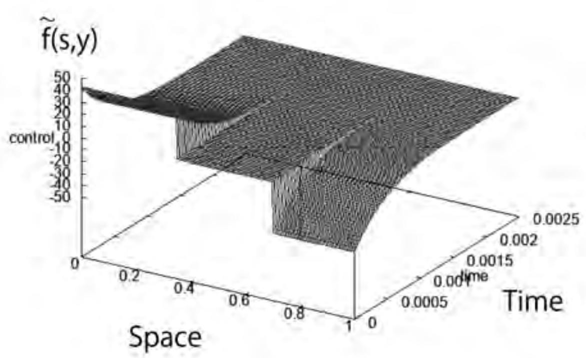

Figure 7: The graph of the control function $\widetilde{f}$ found by (NA) at the iteration number $n = 15$.

Figure 8 is the picture of the solutions u^ε and w^ε for $(\mathrm{P};\widetilde{f}, 0, 0)^\varepsilon$ with initial data $[u_0^\varepsilon, w_0^\varepsilon] \equiv [0.0, 0.0]$ in the case of the iteration number $n = 15$.

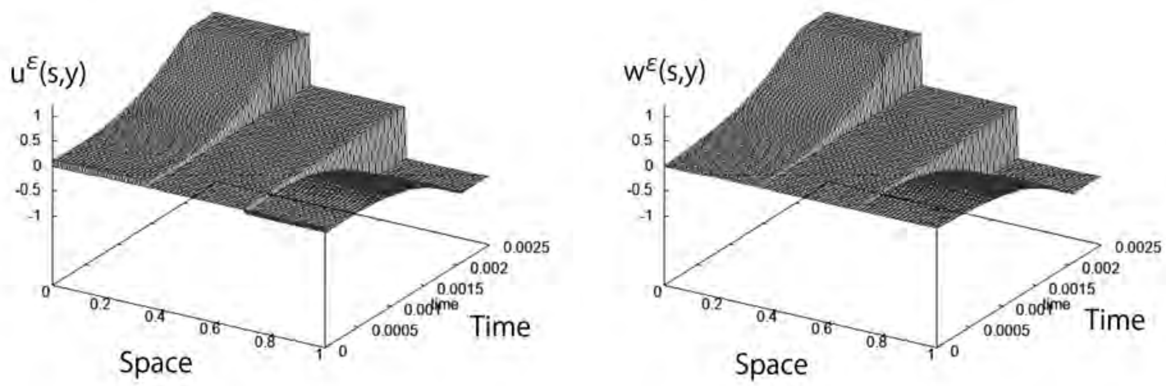

Figure 8: The graph of solution $[u^\varepsilon, w^\varepsilon]$ to $(\mathrm{P};\widetilde{f}, 0, 0)^\varepsilon$ at the iteration number $n = 15$: (left) $u^\varepsilon(s, y)$; (right) $w^\varepsilon(s, y)$.

In addition, the real line in Figure 5 means the graph of $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 15$. We observe from Figures 5–8 that the solution $w^\varepsilon(\widetilde{T}, y)$ to $(\mathrm{P};\widetilde{f}, 0, 0)^\varepsilon$ has the similar profile to the desired one $w_d(\widetilde{T}, y)$ and the data sequence of cost functional J^ε almost reaches a stationary point.

(Numerical experiment 3)

In the final experiment, we consider a target desired profile w_d such that

$$w_d(s,y) := \cos(2\pi y), \quad y \in \overline{\Omega} = [0.0, 1.0], \ \forall s \in [0, \widetilde{T}], \tag{9.12}$$

whose graph is the dotted line in Figure 9.

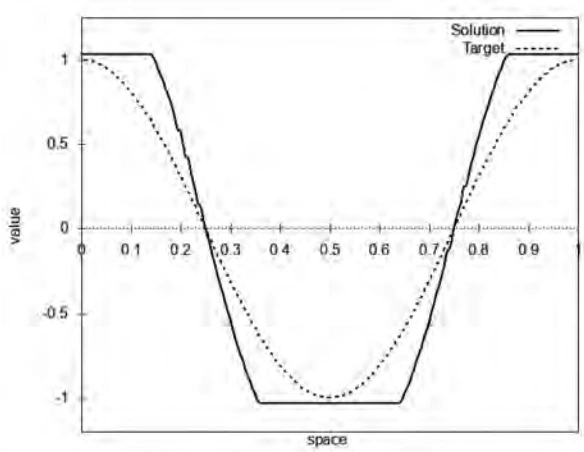

Figure 9: Target profile $w_d(\widetilde{T}, y)$ and solution $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 17$.

Here we take the stop parameter $\mu = 0.00025$ for (NA). Then, we perform a numerical experiment of $(\mathrm{OP})^\varepsilon$ by using the numerical algorithm (NA) proposed in Section 8. Then, (NA) is finished when the iteration number is $n = 17$ as in Figure 10.

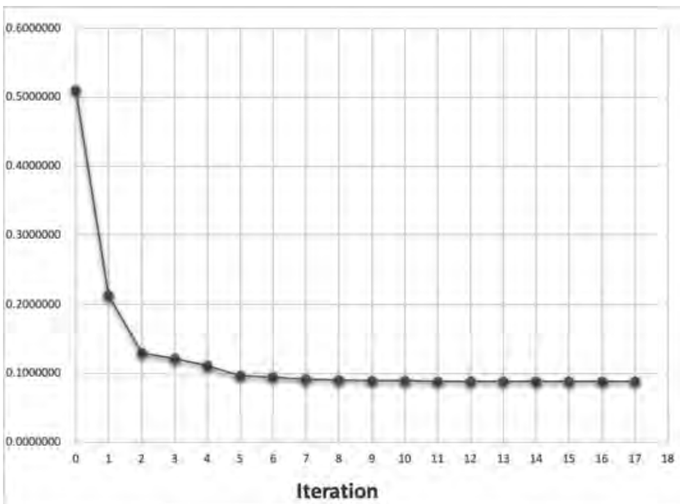

Figure 10: The value of the cost functional J^ε for $(\mathrm{OP})^\varepsilon$.

Figure 11 is the graph of the control function $\widetilde{f}$ found by (NA) in the case of the iteration number $n = 17$.

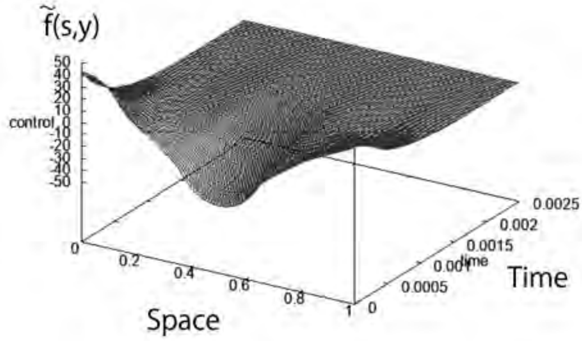

Figure 11: The graph of the control function $\widetilde{f}$ found by (NA) at the iteration number $n = 17$.

Figure 12 is the picture of the solutions u^ε and w^ε for $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ with initial data $[u_0^\varepsilon, w_0^\varepsilon] \equiv [0.0, 0.0]$ in the case of the iteration number $n = 17$.

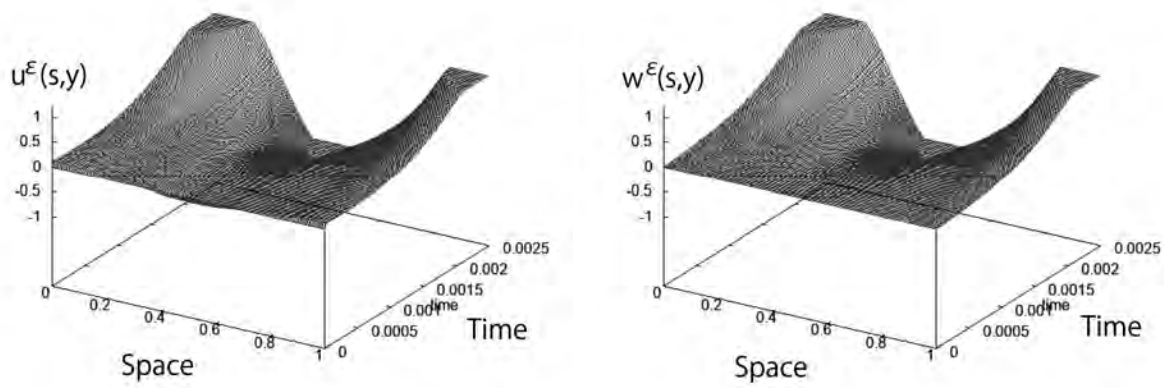

Figure 12: The graph of solution $[u^\varepsilon, w^\varepsilon]$ to $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ at the iteration number $n = 17$: (left) $u^\varepsilon(s,y)$; (right) $w^\varepsilon(s,y)$.

In addition, the real line in Figure 9 means the graph of $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 17$. We observe from Figures 9–12 that the data sequence of cost functional J^ε almost reaches a stationary point, however, there is the slight gap between the solution $w^\varepsilon(\widetilde{T}, y)$ to $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ and the desired profile $w_d(\widetilde{T}, y)$ (see Figure 9). We guess the reason is that the target profile $w_d(\widetilde{T}, y)$ defined by (9.12) is not the stable equilibria for $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$, and there is no desired profile u_d of the temperature in $(\mathrm{OP})^\varepsilon$ (cf. (9.9)).

In the forthcoming paper we will perform numerical experiments for (P;$f, h, \ell)^\varepsilon$ and (OP)$^\varepsilon$ under various situations (cf. Remark 3.1).

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