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NUMERICAL SCHEMES FOR ORDINARY DIFFERENTIAL EQUATIONS DESCRIBING SHRINKING AND STRETCHING MOTION OF ELASTIC MATERIALS

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Abstract. In this paper, we discuss shrinking and stretching motion of elastic materials by a mathematical model given as a system of ordinary differential equations. First, we propose the model in which the stress is given by a function having a singularity. By using the singularity we can obtain a uniform estimate for the strain from below. Due to the estimates, we establish existence and uniqueness of solutions to the model. Moreover, we construct a numerical scheme preserving the energy, and show existence and convergence of numerical solutions.

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1 Introduction

The aim of this article is to propose a mathematical model describing shrinking and stretching motion of elastic materials. This research is strongly motivated by the mathematical analysis for dynamics of shape memory alloy, for instance, Brokate-Sprekels [1].

In this article, we consider dynamics of the one-dimensional elastic material in $\mathbb{R}^2$ and assume that it is given by a polygon having N vertexes for each time t and the natural length of each side is l_{N*}, where $N \in \mathbb{Z}_{>0} := \{n \in \mathbb{Z} | n > 0\}$. Let $X_i = X_i(t) \in \mathbb{R}^2$ be the position of each vertex of the polygon (see Figure 1.1). Hence, the strain ε_i of ith side is given by

$$\varepsilon_i = \frac{l_i - l_{N*}}{l_{N*}}, \quad l_i = |X_{i+1} - X_i| \quad \text{for } i = 1, 2, \cdots, N,$$

where $X_{N+1} = X_1$.

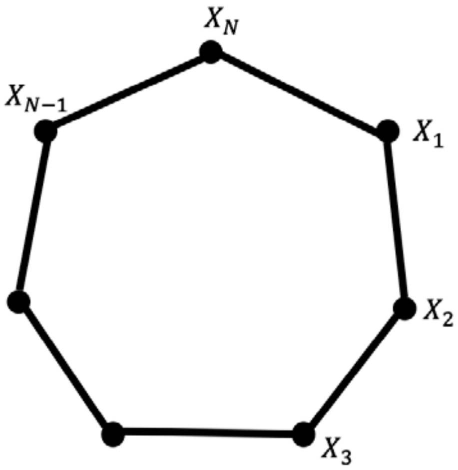

Figure 1.1:

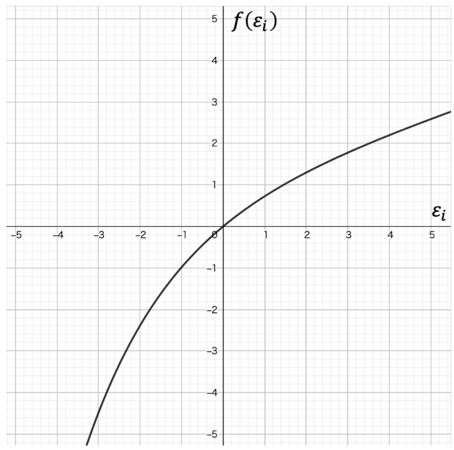

Figure 1.2:

Next, we list our physical assumption as follows

- The mass of each side concentrates at the vertex, namely, the mass of X_i is $\dfrac{M}{N}$, where M is the total mass of the material.
- For each vertex X_i only two tensions from X_{i-1} and X_{i+1} are acted. The tension F_{i+1} from X_{i+1} is represented by

$$F_{i+1} = f(\varepsilon_i)\frac{X_{i+1} - X_i}{l_i}.$$

Here, the stress function $f(\varepsilon_i)$ and $\dfrac{X_{i+1} - X_i}{l_i}$ correspond to the magnitude and the trend of the tension, respectively.

- We assume that

$$f(\varepsilon) = \frac{\kappa}{2}\left(\varepsilon + \frac{1}{2} - \frac{1}{2(\varepsilon+1)^2}\right) \quad \text{for the strain } \varepsilon > -1, \tag{1.1}$$

  where $\kappa > 0$.

One of purposes of this article is to construct a numerical scheme by applying the structure preserving numerical method (see Furihata and Matsuo [2]). By existence of the trend $\dfrac{X_{i+1}-X_i}{l_i}$, we could not show that the scheme has solutions, when the stress function is linear, namely, Hooke's law is adapted. Hence, in order to overcome this difficulty we propose the nonlinear stress function f having a singularity at $\varepsilon = -1$. The singularity of f at $\varepsilon = -1$ means that the magnitude tends to infinity as the strain goes to -1. Moreover, we choose coefficients appearing in (1.1) such that $f'(0) = \kappa$, namely, for small ε the behavior of f is similar to the function defined by Hooke's law.

Under the physical assumption as above, Newton's law implies the following initial value problem P for the system of second order ordinary differential equations

$$\begin{aligned} m\frac{d^2X_i}{dt^2} &= f(\varepsilon_i)\frac{X_{i+1}-X_i}{l_i} - f(\varepsilon_{i-1})\frac{X_i - X_{i-1}}{l_{i-1}} \quad \text{on } [0,T], && (1.2)\\ \frac{dX_i}{dt} &= V_i, && (1.3)\\ X_i(0) &= X_{0i}, \quad \frac{dX_i(0)}{dt} = V_{0i} \quad \text{for } i = 1,2,\cdots,N, && (1.4) \end{aligned}$$

where $X_N = X_0$ and $X_{N+1} = X_1$, X_{0i} is the initial position and V_{0i} is the initial velocity for $i = 1,2,\cdots,N$.

One of purposes of this article is to discuss a numerical scheme to obtain approximate solutions of P such that the following energy $G(X,V)$ is conserved

$$G(X,V) = \sum_{i=1}^{N}\left\{\frac{m}{2}|V_i|^2 + l_{N*}\hat{f}(\varepsilon_i)\right\} \qquad \text{for } (X,V) \in \mathbb{R}^{4N},$$

where $\hat{f}(\varepsilon_i) = \dfrac{\kappa}{4}\left(\varepsilon_i^2 + \varepsilon_i + \dfrac{1}{1+\varepsilon_i}\right)$ for $i = 1,2,\cdots,N$, $X = (X_1, X_2, \cdots, X_N)$ and $V = (V_1, V_2, \cdots, V_N)$. By applying ideas in [2], we get the following numerical scheme (NS) $=$ (NS)$(\Delta t, X^{(n)}, V^{(n)})$, where $\Delta t = \dfrac{T}{K}$ for $K \in \mathbb{Z}_{>0}$. Find $X^{(n+1)} = \left(X_1^{(n+1)}, X_2^{(n+1)}, \cdots, X_N^{(n+1)}\right)$ and $V^{(n+1)} = \left(V_1^{(n+1)}, V_2^{(n+1)}, \cdots, V_N^{(n+1)}\right)$ such that

$$\begin{aligned} &\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t}\\ &= -\frac{\kappa}{4m}\left\{\left\{\varepsilon_{i-1}^{(n+1)} + \varepsilon_{i-1}^{(n)} + 1 - \frac{1}{\left(1+\varepsilon_{i-1}^{(n+1)}\right)\left(1+\varepsilon_{i-1}^{(n)}\right)}\right\}\frac{X_i^{(n+1)} - X_{i-1}^{(n+1)} + X_i^{(n)} - X_{i-1}^{(n)}}{\left|X_i^{(n+1)} - X_{i-1}^{(n+1)}\right| + \left|X_i^{(n)} - X_{i-1}^{(n)}\right|}\right. \end{aligned}$$

$$-\left\{\varepsilon_i^{(n+1)}+\varepsilon_i^{(n)}+1-\frac{1}{\left(1+\varepsilon_i^{(n+1)}\right)\left(1+\varepsilon_i^{(n)}\right)}\right\}\frac{X_{i+1}^{(n+1)}-X_i^{(n+1)}+X_{i+1}^{(n)}-X_i^{(n)}}{\left|X_{i+1}^{(n+1)}-X_i^{(n+1)}\right|+\left|X_{i+1}^{(n)}-X_i^{(n)}\right|}\Bigg\}, \quad (1.5)$$

$$\frac{X_i^{(n+1)}-X_i^{(n)}}{\Delta t}=\frac{V_i^{(n+1)}+V_i^{(n)}}{2} \quad \text{for } n=0,1,\cdots,K-1 \text{ and } i=1,2,\cdots,N, \quad (1.6)$$

where $X^{(n)}, V^{(n)}$ and $\varepsilon^{(n)}$ are given.

In this article, we establish existence and uniqueness of solutions to P by applying Banach's fixed point theorem in Section 3. Also, we discuss existence of solutions to the scheme (NS) in Section 4. Finally, we show convergence of the numerical solution to the solution of P and its rate.

Our system has two advantages. In mathematical analysis for dynamics of elastic materials, it is important to estimate lower bounds for the strain ε, since $\varepsilon=-1$ means that different points overlap. In this paper, by applying the singularity of the stress function, we can get the lower bound of ε (see (3.6) in Lemma 3.2). This is the first advantage of this research. As a next step of this research, we will try to get a similar estimate in the system of partial differential equations. The second one is that we can construct the numerical scheme such that the energy is conserved, even though it is not easy to handle the difference of $\dfrac{X_{i+1}-X_i}{l_i}$. For shape memory alloy which is an elastic material Yoshikawa [3, 4] already applied the structure preserving numerical method and discussed error estimates. Numerical results to P will be shown in our forthcoming paper.

2 Main results

The first mathematical result is concerned with existence and uniqueness of solutions to P.

Theorem 2.1. *Let $X_{0i}\in\mathbb{R}^2$ and $V_{0i}\in\mathbb{R}^2$ for $i=1,2,\cdots,N$. If $X_{0i}\neq X_{0j}$ for $i\neq j$, then P has a unique solution $X\in C^2([0,T])^{4N}$.*

Theorem 2.2 represents existence of a unique solution for the numerical scheme (NS).

Theorem 2.2. *Let $X_{0i}\in\mathbb{R}^2$ and $V_{0i}\in\mathbb{R}^2$ for $i=1,2,\cdots,N$. If $X_{0i}\neq X_{0j}$ for $i\neq j$, then there exists $K_0\in\mathbb{Z}_{>0}$ such that (NS)$\left(\Delta t, X^{(n)}, V^{(n)}\right)$ has a unique solution $\left(X^{(n+1)}, V^{(n+1)}\right)\in\mathbb{R}^4$ for $n=0,1,\cdots,K-1$ and $K\geq K_0$, where $\Delta t=\dfrac{T}{K}$ and $X^{(0)}=(X_{01},X_{02},\cdots,X_{0N})$, $V^{(0)}=(V_{01},V_{02},\cdots,V_{0N})$.*

The third result guarantees convergence of numerical solutions to the solution of P.

Theorem 2.3. *Assume $X_{0i}\in\mathbb{R}^2$, $V_{0i}\in\mathbb{R}^2$ for $i=1,2,\cdots,N$, and $X_{0i}\neq X_{0j}$ for $i\neq j$. Let K_0 be a positive integer defined in Theorem 2.2 and $\left(X_K^{(n+1)}, V_K^{(n+1)}\right)$ be a solution of (NS) $\left(\Delta t, X_K^{(n)}, V_K^{(n)}\right)$ for $n=0,1,\cdots,K-1$ and $K\geq K_0$, where $\Delta t=\dfrac{T}{K}$, $X_K^{(0)}=(X_{01},X_{02},\cdots,X_{0N})$ and $V_K^{(0)}=(V_{01},V_{02},\cdots,V_{0N})$. Moreover, put*

$$X^K(t) \quad = \quad V_K^{(n)}\left(t-\frac{n}{K}\right)+X_K^{(n)} \qquad \textit{for } \frac{n}{K}<t\leq\frac{n+1}{K} \textit{ and } n=0,1,\cdots,K-1,$$

$$X^K(0) = X_0.$$

There exists a positive constant C such that

$$\left|X(t) - X^K(t)\right| \le C\left|\Delta t\right| \quad for\ 0 \le t \le T\ and\ K \ge K_0,$$

where X is a solution of P.

3 Proof of Theorem 2.1

In this section, we prove Theorem 2.1. For its proof we need several steps. From (1.2) and (1.3) we can obtain the following integral equations

$$X_i(t_1) = \int_0^{t_1} V_i(t)dt + X_{0i}, \tag{3.1}$$

$$V_i(t_1) = \frac{1}{m}\int_0^{t_1}\left(f(\varepsilon_i)\frac{X_{i+1}(t) - X_i(t)}{l_i(t)} - f(\varepsilon_{i-1})\frac{X_i(t) - X_{i-1}(t)}{l_{i-1}(t)}\right)dt + V_{0i}, \tag{3.2}$$

for $i = 1, 2, \cdots, N$ and $t_1 \in [0, T]$.

Let $F : C([0,T])^{4N} \to C([0,T])^{4N}$ be the mapping defined by the following

$$F(Z) = (f_1(Z), f_2(Z), \cdots, f_N(Z), g_1(Z), g_2(Z), \cdots, g_N(Z)) \text{ for } Z = (X, V) \in C([0,T])^{4N},$$

where

$$f_i(Z)(t) = \int_0^t V_i(\tau)d\tau + X_{0i}, \tag{3.3}$$

$$g_i(Z)(t) = \frac{1}{m}\left(\int_0^t f(\varepsilon_i)\frac{X_{i+1}(\tau) - X_i(\tau)}{l_i(\tau)} - f(\varepsilon_{i-1})\frac{X_i(\tau) - X_{i-1}(\tau)}{l_{i-1}(\tau)}d\tau\right) + V_{0i}, \tag{3.4}$$

$$X = (X_1, X_2, \cdots, X_N), V = (V_1, V_2, \cdots, V_N).$$

Moreover, we define the energy function G as follows

$$G(X, V) = \sum_{i=1}^{N}\left\{\frac{m}{2}\left|V_i\right|^2 + l_{N*}\hat{f}(\varepsilon_i)\right\} \text{ for } (X, V) \in \mathbb{R}^{4N}, \tag{3.5}$$

where $\hat{f}(\varepsilon_i) = \frac{\kappa}{4}\left(\varepsilon_i^2 + \varepsilon_i + \frac{1}{1+\varepsilon_i}\right), \varepsilon_i = \frac{l_i - l_{N*}}{l_{N*}}, l_i = |X_{i+1} - X_i|$ for $i = 1, 2, \cdots, N$, $X = (X_1, X_2, \cdots, X_N)$ and $V = (V_1, V_2, \cdots, V_N)$.

The first lemma guarantees that the energy is preserved.

Lemma 3.1. *If X is a solution of P, then G is preserved, namely, $\frac{d}{dt}G(X(t), V(t)) = 0$ for $t \in [0, T]$, where $V = \frac{dX}{dt}$ on $[0, T]$.*

Proof. Let $t \in [0,T]$. By multiplying $\dfrac{d}{dt}X_i(t)$ on both sides of (1.2) for $i = 1, 2, \cdots, N$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{m}{2}\sum_{i=1}^{N}\left|\frac{d}{dt}X_i(t)\right|^2\right) \\
&= \sum_{i=1}^{N} f(\varepsilon_i(t))\frac{X_{i+1}(t)-X_i(t)}{l_i(t)}\left(\frac{d}{dt}X_i(t)\right) - \sum_{i=1}^{N} f(\varepsilon_{i-1}(t))\frac{X_i(t)-X_{i-1}(t)}{l_{i-1}(t)}\left(\frac{d}{dt}X_i(t)\right) \\
&= \sum_{i=1}^{N} f(\varepsilon_i(t))\frac{X_{i+1}(t)-X_i(t)}{l_i(t)}\left(\frac{d}{dt}X_i(t)\right) - \sum_{i=1}^{N} f(\varepsilon_i(t))\frac{X_{i+1}(t)-X_i(t)}{l_i(t)}\left(\frac{d}{dt}X_{i+1}(t)\right) \\
&= -\sum_{i=1}^{N}\frac{f(\varepsilon_i(t))}{2l_i(t)}\frac{d}{dt}\left|X_{i+1}(t)-X_i(t)\right|^2 .
\end{aligned}$$

In the calculation above, by using the condition $X_N = X_0$ and $X_{N+1} = X_1$, we obtain the second equation. Moreover, by using $l_i(t) = |X_{i+1}(t) - X_i(t)|$ and $\varepsilon_i(t) = \dfrac{l_i(t) - l_{N*}}{l_{N*}}$ for $i = 1, , 2, \cdots, N$ and $t \in [0,T]$, we have

$$\frac{f(\varepsilon_i(t))}{2l_i(t)}\frac{d}{dt}\left|X_{i+1}(t)-X_i(t)\right|^2 = l_{N*}f(\varepsilon_i)\frac{d}{dt}\varepsilon_i(t) \quad \text{for } i = 1, 2, \cdots, N \text{ and } t \in [0,T].$$

Since $\hat{f}(\varepsilon) = \dfrac{\kappa}{4}\left(\varepsilon^2 + \varepsilon + \dfrac{1}{1+\varepsilon}\right)$ is the primitive of f, we have

$$\frac{d}{dt}\left\{\sum_{i=1}^{N}\left(\frac{m}{2}\left|\frac{dX_i}{dt}(t)\right|^2 + l_{N*}\frac{d}{dt}\hat{f}(\varepsilon_i)\right)\right\} = 0 \quad \text{for } t \in [0,T],$$

and

$$\frac{d}{dt}\left\{\sum_{i=1}^{N}\left(\frac{m}{2}\left|V_i(t)\right|^2 + l_{N*}\frac{d}{dt}\hat{f}(\varepsilon_i)\right)\right\} \;=\; 0 \quad \text{for } t \in [0,T],$$

namely,

$$\frac{d}{dt}G(X,V) \;=\; 0 \quad \text{on } [0,T].$$

Hence, G is preserved. Thus, Lemma 3.1 is proved. □

Next, we give some estimates for X and V obtained from the energy G.

Lemma 3.2. *Let $(X,V) \in \mathbb{R}^{4N}$. If $d_0 = G(X,V) \in \mathbb{R}$, then the following inequalities hold*

$$|X_{i+1} - X_i| \;\geq\; \frac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}}, \tag{3.6}$$

$$|X_{i+1} - X_i| \;\leq\; \frac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right), \tag{3.7}$$

$$|V_i| \;\leq\; \sqrt{\frac{1}{2m}\left(4d_0 + N\kappa l_{N*}\right)} \quad \textit{for } i = 1, 2, \cdots, N, \tag{3.8}$$

where $X = (X_1, X_2, \cdots, X_N)$, $V = (V_1, V_2, \cdots, V_N)$, $l_i = |X_{i+1} - X_i|$, $\varepsilon_i = \dfrac{l_i - l_{N*}}{l_{N*}}$ and $X_{N+1} = X_1$.

Proof. First, let us show (3.6). By using (3.5) and $\varepsilon_i > -1$ for $i = 1, 2, \cdots, N$, we have

$$\begin{aligned} d_0 &= G(X, V) \\ &\geq l_{N*} \sum_{i=1}^{N} \hat{f}\left(\varepsilon_i\right) \\ &\geq \frac{\kappa l_{N*}}{4} \sum_{i=1}^{N} \left(\varepsilon_i + \frac{1}{1+\varepsilon_i}\right) \\ &\geq \frac{\kappa l_{N*}}{4} \left(-N + \frac{l_{N*}}{l_i}\right) \qquad \text{for } i = 1, 2, \cdots, N. \end{aligned}$$

Therefore, we obtain

$$\frac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}} \leq l_i = |X_{i+1} - X_i| \qquad \text{for } i = 1, 2, \cdots, N.$$

It means that (3.6) holds.

Next, we show (3.7). Since $\dfrac{l_i}{l_{N*}} > 0$ for $i = 1, 2, \cdots, N$, we have

$$\begin{aligned} d_0 &= G(X, V) \\ &\geq l_{N*} \sum_{i=1}^{N} \hat{f}\left(\varepsilon_i\right) \\ &\geq \frac{\kappa l_{N*}}{4} \sum_{i=1}^{N} \varepsilon_i \\ &= \frac{\kappa l_{N*}}{4} \left(\sum_{i=1}^{N} \frac{l_i}{l_{N*}} - \sum_{i=1}^{N} 1\right) \\ &\geq \frac{\kappa l_{N*}}{4} \left(\frac{l_i}{l_{N*}} - N\right) \qquad \text{for } i = 1, 2, \cdots, N. \end{aligned}$$

Therefore, we obtain

$$|X_{i+1} - X_i| = l_i \leq \frac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right) \qquad \text{for } i = 1, 2, \cdots, N,$$

so that (3.7) holds.

Finally, we show (3.8). By the similar way to (3.6), we have

$$\begin{aligned} d_0 &= G\left(X, V\right) \\ &\geq \frac{m}{2}\left|V_i\right|^2 + l_{N*} \sum_{i=1}^{N} \hat{f}\left(\varepsilon_i(t)\right) \\ &\geq \frac{m}{2}\left|V_i\right|^2 + \frac{\kappa l_{N*}}{4} \sum_{i=1}^{N} \varepsilon_i(t) \\ &\geq \frac{m}{2}\left|V_i\right|^2 + \frac{\kappa l_{N*}}{4} \sum_{i=1}^{N} (-1) \end{aligned}$$

$$\geq \quad \frac{m}{2}|V_i|^2 - \frac{N\kappa l_{N*}}{4} \qquad \text{for } i = 1, 2, \cdots, N.$$

Therefore, we obtain (3.8) as follows

$$|V_i| \quad \leq \quad \sqrt{\frac{1}{2m}\left(4d_0 + N\kappa l_{N*}\right)} \qquad \text{for } i = 1, 2, \cdots, N.$$

Thus, Lemma 3.2 holds. $\square$

The third lemma is concerned with existence of a set on which F is a contraction mapping.

Lemma 3.3. *For $T_1 > 0$, $\delta > 0, M > 0$ and $M' > 0$ we put*

$$\begin{aligned} W(T_1, \delta, M', M) \quad = \quad & \Big\{(X, V) \in C([0, T_1])^{4N} \mid \delta \leq |X_{i+1}(t) - X_i(t)| \leq M', \\ & |V_{i+1}(t) - V_i(t)| \leq M \textit{ for } i = 1, 2, \cdots, N \textit{ and } t \in [0, T_1]\Big\}. \end{aligned}$$

If $X_0 = (X_{01}, X_{02}, \cdots, X_{0N}) \in \mathbb{R}^{2N}$, $V_0 = (V_{01}, V_{02}, \cdots, V_{0N}) \in \mathbb{R}^{2N}$ and $X_{0i} \neq X_{0j}$ for $i \neq j$, then there exists $T_0 \in (0, T]$ such that $F : W(T_0, \delta, M', M) \to W(T_0, \delta, M', M)$ is a contraction.

Proof. First, we prove that F is a function from $W(T_1, \delta, M', M)$ to $W(T_1, \delta, M', M)$ for some $T_1 \in (0, T]$ and δ, M, $M' > 0$. Let $T_1 \in (0, T]$ and $(X, V) \in W(T_1, \delta, M', M)$. Since $X_{0i} \neq X_{0j}$ for $i \neq j$, it follows that $d_0 = G(X_0, V_0) \in \mathbb{R}$. Clearly, Lemma 3.2 implies

$$\begin{aligned} |f_{i+1}((X, V))(t) - f_i((X, V))(t)| \quad & \geq \quad |X_{0(i+1)} - X_{0i}| - \int_0^t |V_{i+1}(\tau) - V_i(\tau)| d\tau \\ & \geq \quad |X_{0(i+1)} - X_{0i}| - MT_1 \\ & \geq \quad \frac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}} - MT_1 \qquad \text{for } i = 1, 2, \cdots, N \text{ and } t \in [0, T_1]. \end{aligned}$$

Now, we choose $\delta > 0$ such that $\dfrac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}} \geq 2\delta$, we have

$$|f_{i+1}((X, V))(t) - f_i((X, V))(t)| \quad \geq \quad 2\delta - MT_1 \qquad \text{for } i = 1, 2, \cdots, N \text{ and } t \in [0, T_1].$$

And we choose $T_1 > 0$ such that $MT_1 \leq \delta$, we have

$$|f_{i+1}((X, V))(t) - f_i((X, V))(t)| \quad \geq \quad \delta \qquad \text{for } i = 1, 2, \cdots, N \text{ and } t \in [0, T_1].$$

In the same way, by using (3.7), we have

$$\begin{aligned} & |f_{i+1}((X, V))(t) - f_i((X, V))(t)| \\ \leq \quad & \int_0^t |V_{i+1}(\tau) - V_i(\tau)| d\tau + |X_{0(i+1)} - X_{0i}| \\ \leq \quad & MT_1 + |X_{0(i+1)} - X_{0i}| \\ \leq \quad & MT_1 + \frac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right) \qquad \text{for } i = 1, 2, \cdots, N \text{ and } t \in [0, T_1]. \end{aligned}$$

Now, we choose $M' > 0$ such that $\dfrac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right) \leq \dfrac{M'}{2}$, we have

$$|f_{i+1}((X, V))(t) - f_i((X, V))(t)| \quad \leq \quad MT_1 + \frac{M'}{2} \qquad \text{for } i = 1, 2, \cdots, N \text{ and } t \in [0, T_1].$$

And we choose $T_1 > 0$ such that $MT_1 \leq \dfrac{M'}{2}$, we have

$$|f_{i+1}((X,V))(t) - f_i((X,V))(t)| \ \leq \ M' \quad \text{for } i = 1, 2, \cdots, N \text{ and } t \in [0, T_1].$$

Hence, we choose $T_1 > 0$ such that $T_1 = \min\left\{\dfrac{\delta}{M}, \dfrac{M'}{2M}\right\}$, we can obtain

$$\delta \leq |f_{i+1}((X,V))(t) - f_i((X,V))(t)| \leq M' \quad \text{for } i = 1, 2, \cdots, N \text{ and } t \in [0, T_1]. \tag{3.9}$$

Next, by using (3.4) and (3.8), we have

$$\begin{aligned}
&|g_{i+1}((X,V))(t) - g_i((X,V))(t)| \\
&\quad \leq \frac{1}{m}\left|\int_0^t f(\varepsilon_{i+1})\frac{X_{i+2}(\tau) - X_{i+1}(\tau)}{l_{i+1}(\tau)}d\tau\right| + \frac{2}{m}\left|\int_0^t f(\varepsilon_i)\frac{X_{i+1}(\tau) - X_i(\tau)}{l_i(\tau)}d\tau\right| \\
&\qquad + \frac{1}{m}\left|\int_0^t f(\varepsilon_{i-1})\frac{X_i(\tau) - X_{i-1}(\tau)}{l_{i-1}(\tau)}d\tau\right| + |V_{0(i+1)}| + |V_{0i}| \\
&\quad \leq \frac{1}{m}\left|\int_0^t f(\varepsilon_{i+1})\frac{X_{i+2}(\tau) - X_{i+1}(\tau)}{l_{i+1}(\tau)}d\tau\right| + \frac{2}{m}\left|\int_0^t f(\varepsilon_i)\frac{X_{i+1}(\tau) - X_i(\tau)}{l_i(\tau)}d\tau\right| \\
&\qquad + \frac{1}{m}\left|\int_0^t f(\varepsilon_{i-1})\frac{X_i(\tau) - X_{i-1}(\tau)}{l_{i-1}(\tau)}d\tau\right| + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})} \\
&\quad =: A_1 + A_2 + A_3 + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})} \quad \text{for } i = 1, 2, \cdots, N \text{ and } t \in [0, T].
\end{aligned}$$

First, we choose $M > 0$ such that $2\sqrt{\dfrac{1}{2m}(4d_0 + N\kappa l_{N*})} \leq \dfrac{M}{5}$. Uniform estimates for A_i for $i = 1, 2, 3$ can be obtained in the same way, so we would like to show you how to get the uniform estimate for only A_1. By using (1.1) and the definition of the strain, we have

$$\begin{aligned}
|A_1| &= \frac{\kappa}{4m}\left|\int_0^t \left\{2\left(\frac{l_{i+1}(\tau) - l_{N*}}{l_{N*}}\right) + 1 + \left(\frac{l_{N*}}{l_{i+1}(\tau)}\right)^2\right\}\frac{X_{i+2}(\tau) - X_{i+1}(\tau)}{|X_{i+2}(\tau) - X_{i+1}(\tau)|}d\tau\right| \\
&\leq \frac{\kappa}{4m}\int_0^t \left\{2\left(\frac{|l_{i+1}(\tau)| + l_{N*}}{l_{N*}}\right) + 1 + \left|\frac{l_{N*}}{l_{i+1}(\tau)}\right|^2\right\}d\tau.
\end{aligned}$$

Since $(X,V) \in W(T_1, \delta, M', M)$, we obtain

$$|A_1| \leq \frac{\kappa}{4m}\left\{\frac{2}{l_{N*}}\left(M' + l_{N*}\right) + 1 + \frac{l_{N*}^2}{\delta^2}\right\}T_1.$$

Hence, we choose $T_1 > 0$ such that $\dfrac{\kappa}{4m}\left\{\dfrac{2}{l_{N*}}\left(M' + l_{N*}\right) + 1 + \dfrac{l_{N*}^2}{\delta^2}\right\}T_1 \leq \dfrac{M}{5}$, we have

$$|A_1| \leq \frac{M}{5}.$$

In the same way, by choosing small $T_1 > 0$, we can obtain

$$|A_2| \leq \frac{2M}{5}, \quad |A_3| \leq \frac{M}{5}.$$

Hence, we have

$$|g_{i+1}((X,V))(t)-g_i((X,V))(t)| \le \frac{M}{5}+\frac{2M}{5}+\frac{M}{5}+\frac{M}{5}=M \quad \text{for } i=1,2,\cdots,N \text{ and } t\in[0,T].$$

By using the above uniform estimate and (3.9), we can prove $F((X,V)) \in W(T_1,\delta,M',M)$, namely, F is the mapping from $W(T_1,\delta,M',M)$ to $W(T_1,\delta,M',M)$.

Finally, we prove that $F: W(T_0,\delta,M',M) \to W(T_0,\delta,M',M)$ is a contraction mapping for some $T_0 \in (0,T_1]$. Let $(X,V),(X',V') \in W(T,\delta,M',M)$. Since X_i, V_i, $f_i((X,V))$ and $g_i((X,V)) \in \mathbb{R}^2$ for $i=1,2,\cdots,N$, we can put

$$\begin{aligned} X_i &= (X_{i1},X_{i2}), \quad V_i=(V_{i1},V_{i2}), \quad f_i((X,V))=(f_{i1}((X,V)),f_{i2}((X,V))) \\ g_i((X,V)) &= (g_{i1}((X,V)),g_{i2}((X,V))) \qquad\qquad \text{for } i=1,2,\cdots,N. \end{aligned}$$

And we regard the norm of $C([0,T])^{4N}$ as follows

$$\begin{aligned} &\left\|F((X,V))-F((X',V'))\right\|_{C([0,T])^{4N}} \\ &= \Bigg(\sum_{i=1}^N \left\|f_{i1}((X,V))-f_{i1}((X',V'))\right\|^2_{C([0,T])}+\sum_{i=1}^N \left\|f_{i2}((X,V))-f_{i2}((X',V'))\right\|^2_{C([0,T])} \\ &\quad +\sum_{i=1}^N \left\|g_{i1}((X,V))-g_{i1}((X',V'))\right\|^2_{C([0,T])}+\sum_{i=1}^N \left\|g_{i2}((X,V))-g_{i2}((X',V'))\right\|^2_{C([0,T])}\Bigg)^{\frac{1}{2}}. \end{aligned} \tag{3.10}$$

By using (3.3), we have

$$\begin{aligned} &\left|f_{ij}((X,V))(t)-f_{ij}((X',V'))(t)\right| \\ \le\ & \int_0^t \left|V_{ij}(\tau)-V'_{ij}(\tau)\right| d\tau \\ \le\ & T_1 \left\|V_{ij}-V'_{ij}\right\|_{C([0,T_1])} \\ \le\ & T_1 \left\|(X,V)-(X',V')\right\|_{C([0,T_1])^{4N}} \qquad \text{for } i=1,2,\cdots,N, j=1,2 \text{ and } t\in[0,T_1]. \end{aligned}$$

Therefore, we can obtain

$$\left|f_{ij}((X,V))(t)-f_{ij}((X',V'))(t)\right| \le T_1 \left\|(X,V)-(X',V')\right\|_{C([0,T_1])^{4N}}, \tag{3.11}$$

$$\text{for } i=1,2,\cdots,N \text{ and } t_1\in[0,T].$$

Next, by using (3.4), we have

$$\begin{aligned} &\left|g_{ij}((X,V))(t)-g_{ij}((X',V'))(t)\right| \\ \le\ & \frac{1}{m}\int_0^t \left| f(\varepsilon_i)\frac{X_{(i+1)j}(\tau)-X_{ij}(\tau)}{l_i(\tau)} - f(\varepsilon'_i)\frac{X_{(i+1)j}(\tau)-X_{ij}(\tau)}{l_i(\tau)}\right| d\tau \\ & +\frac{1}{m}\int_0^t \left| f(\varepsilon'_i)\frac{X_{(i+1)j}(\tau)-X_{ij}(\tau)}{l_i(\tau)} - f(\varepsilon'_i)\frac{X'_{(i+1)j}(\tau)-X'_{ij}(\tau)}{l_i(\tau)}\right| d\tau \\ & +\frac{1}{m}\int_0^t \left| f(\varepsilon'_i)\frac{X'_{(i+1)j}(\tau)-X'_{ij}(\tau)}{l_i(\tau)} - f(\varepsilon'_i)\frac{X'_{(i+1)j}(\tau)-X'_{ij}(\tau)}{l'_i(\tau)}\right| d\tau \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{m}\int_0^t \left| f(\varepsilon_{i-1})\frac{X_{ij}(\tau)-X_{(i-1)j}(\tau)}{l_{i-1}(\tau)} - f(\varepsilon'_{i-1})\frac{X_{ij}(\tau)-X_{(i-1)j}(\tau)}{l_{i-1}(\tau)}\right| d\tau \\
&+\frac{1}{m}\int_0^t \left| f(\varepsilon'_{i-1})\frac{X_{ij}(\tau)-X_{(i-1)j}(\tau)}{l_{i-1}(\tau)} - f(\varepsilon'_{i-1})\frac{X'_{ij}(\tau)-X'_{(i-1)j}(\tau)}{l_{i-1}(\tau)}\right| d\tau \\
&+\frac{1}{m}\int_0^t \left| f(\varepsilon'_{i-1})\frac{X'_{ij}(\tau)-X'_{(i-1)j}(\tau)}{l_{i-1}(\tau)} - f(\varepsilon'_{i-1})\frac{X'_{ij}(\tau)-X'_{(i-1)j}(\tau)}{l'_{i-1}(\tau)}\right| d\tau \\
=:\ & B_1+B_2+B_3+B_4+B_5+B_6 \qquad \text{for } i=1,2,\cdots,N,\ j=1,2 \text{ and } t\in[0,T_1].
\end{aligned}
$$

First, by using (1.1) and the definition of ε_i and l_i for $i=1,2,\cdots,N$, we have

$$
\begin{aligned}
B_1 &= \int_0^t \left| \{f(\varepsilon_i)-f(\varepsilon'_i)\}\frac{X_{(i+1)j}(\tau)-X_{ij}(\tau)}{l_i(\tau)}\right| d\tau \\
&\le \frac{\kappa}{2m}\int_0^t \left|\varepsilon_i(\tau)-\varepsilon'_i(\tau)\right| \left|1+\frac{2+\varepsilon_i(\tau)+\varepsilon'_i(\tau)}{2(1+\varepsilon_i(\tau))^2(1+\varepsilon'_i(\tau))^2}\right| \frac{\left|X_{(i+1)j}(\tau)-X_{ij}(\tau)\right|}{|l_i(\tau)|}d\tau \\
&\le \frac{\kappa}{2m}\int_0^t \left|\frac{l_i(\tau)-l'_i(\tau)}{l_{N*}}\right| \left|1+\frac{l_{N*}^3(l_i(\tau)+l'_i(\tau))}{2l_i^2(\tau)l_i'^2(\tau)}\right| \frac{\left|X_{(i+1)j}(\tau)-X_{ij}(\tau)\right|}{|l_i(\tau)|}d\tau \\
&\le \frac{\kappa}{2ml_{N*}}\int_0^t \left(\left|X_{(i+1)j}(\tau)-X'_{(i+1)j}(\tau)\right|+\left|X_{ij}(\tau)-X'_{ij}(\tau)\right|\right) \\
&\qquad\qquad \times\left(1+\frac{l_{N*}^3(|l_i(\tau)|+|l'_i(\tau)|)}{2|l_i(\tau)|^2|l'_i(\tau)|^2}\right)\frac{\left|X_{(i+1)j}(\tau)-X_{ij}(\tau)\right|}{|l_i(\tau)|}d\tau.
\end{aligned}
$$

Since $(X,V)\in W(T_1,\delta,M',M)$, we have

$$
\begin{aligned}
B_1 &\le \frac{\kappa}{2ml_{N*}}\left(1+\frac{l_{N*}^3M'}{\delta^4}\right)\frac{M'}{\delta}\int_0^t \left(\left|X_{(i+1)j}(\tau)-X'_{(i+1)j}(\tau)\right|+\left|X_{ij}(\tau)-X'_{ij}(\tau)\right|\right)d\tau \\
&\le \frac{\kappa M'}{m\delta l_{N*}}\left(1+\frac{l_{N*}^3M'}{\delta^4}\right)T_1\left\|(X,V)-(X',V')\right\|_{C([0,T_1])^{4N}}. \qquad (3.12)
\end{aligned}
$$

In the same way, we can obtain

$$
B_4 \le \frac{\kappa M'}{m\delta l_{N*}}\left(1+\frac{l_{N*}^3M'}{\delta^4}\right)T_1\|(X,V)-(X',V')\|_{C([0,T_1])^{4N}}, \qquad (3.13)
$$

Similarly, we see that

$$
\begin{aligned}
B_2 &= \frac{1}{m}\int_0^t \left|\frac{f(\varepsilon'_i)}{l_i(\tau)}\left(X_{(i+1)j}(\tau)-X_{ij}(\tau)-X'_{(i+1)j}(\tau)+X'_{ij}(\tau)\right)\right| d\tau \\
&\le \frac{\kappa}{2m}\int_0^t \left(\frac{|l'_i(\tau)|+l_{N*}}{l_{N*}}+\frac{1}{2}+\frac{l_{N*}^2}{|l'_i(\tau)|^2}\right)\frac{1}{|l_i(\tau)|} \\
&\qquad\qquad \times\left(\left|X_{(i+1)j}(\tau)-X'_{(i+1)j}(\tau)\right|+\left|X_{ij}(\tau)-X'_{ij}(\tau)\right|\right)d\tau \\
&\le \frac{\kappa}{2m\delta}\left(\frac{M'+l_{N*}}{l_{N*}}+\frac{1}{2}+\frac{l_{N*}^2}{2\delta^2}\right)\int_0^t \left(\left|X_{(i+1)j}(\tau)-X'_{(i+1)j}(\tau)\right|+\left|X_{ij}(\tau)-X'_{ij}(\tau)\right|\right)d\tau
\end{aligned}
$$

$$\leq \quad \frac{\kappa}{m\delta}\left(\frac{M'+l_{N*}}{l_{N*}}+\frac{1}{2}+\frac{l_{N*}^2}{2\delta^2}\right)T_1\left\|(X,V)-(X',V')\right\|_{C([0,T_1])^{4N}}. \tag{3.14}$$

Also, we can obtain

$$B_5 \quad \leq \quad \frac{\kappa}{m\delta}\left(\frac{M'+l_{N*}}{l_{N*}}+\frac{1}{2}+\frac{l_{N*}^2}{2\delta^2}\right)T_1\left\|(X,V)-(X',V')\right\|_{C([0,T_1])^{4N}}. \tag{3.15}$$

Similarly to the argument above, we have

$$\begin{aligned}
B_3 \quad &= \quad \frac{1}{m}\int_0^t \left|f(\varepsilon_i')\left(X'_{(i+1)j}(\tau)-X'_{ij}(\tau)\right)\left(\frac{1}{l_i(\tau)}-\frac{1}{l_i'(\tau)}\right)\right|d\tau \\
&\leq \quad \frac{\kappa}{2m}\int_0^t\left(|\varepsilon_i'|+\frac{1}{2}+\frac{1}{2}\left|\frac{1}{(1+\varepsilon_i')}\right|^2\right)\left|X'_{(i+1)j}(\tau)-X'_{ij}(\tau)\right|\frac{|l_i'(\tau)-l_i(\tau)|}{|l_i(\tau)||l_i'(\tau)|}d\tau \\
&\leq \quad \frac{\kappa}{2m}\int_0^t\left(\frac{|l_i'(\tau)|+l_{N*}}{l_{N*}}+\frac{1}{2}+\frac{1}{2}\frac{l_{N*}^2}{|l_i'(\tau)|^2}\right)\left|X'_{(i+1)j}(\tau)-X'_{ij}(\tau)\right|\frac{|l_i'(\tau)-l_i(\tau)|}{|l_i(\tau)||l_i'(\tau)|}d\tau \\
&\leq \quad \frac{\kappa M'}{2m\delta^2}\left(\frac{M'+l_{N*}}{l_{N*}}+\frac{1}{2}+\frac{1}{2}\frac{l_{N*}^2}{\delta^2}\right)\int_0^t\left|l_i'(\tau)-l_i(\tau)\right|d\tau \\
&\leq \quad \frac{\kappa M'}{2m\delta^2}\left(\frac{M'+l_{N*}}{l_{N*}}+\frac{1}{2}+\frac{1}{2}\frac{l_{N*}^2}{\delta^2}\right)\int_0^t\left(\left|X'_{(i+1)j}(\tau)-X_{(i+1)j}(\tau)\right|+\left|X'_{ij}(\tau)-X_{ij}(\tau)\right|\right)d\tau \\
&\leq \quad \frac{\kappa M'}{m\delta^2}\left(\frac{M'+l_{N*}}{l_{N*}}+\frac{1}{2}+\frac{1}{2}\frac{l_{N*}^2}{\delta^2}\right)T_1\left\|(X,V)-(X',V')\right\|_{C([0,T_1])^{4N}}. \qquad (3.16)
\end{aligned}$$

In the same way, we can obtain

$$B_6 \quad \leq \quad \frac{\kappa M'}{m\delta^2}\left(\frac{M'+l_{N*}}{l_{N*}}+\frac{1}{2}+\frac{1}{2}\frac{l_{N*}^2}{\delta^2}\right)T_1\left\|(X,V)-(X',V')\right\|_{C([0,T_1])^{4N}}. \tag{3.17}$$

By using (3.12) - (3.17), we have

$$|g_{ij}((X,V))(t)-g_{ij}((X',V'))(t)| \quad \leq \quad \alpha T_1\left\|(X,V)-(X',V')\right\|_{C([0,T_1])^{4N}}, \tag{3.18}$$

for $i=1,2,\cdots,N,\ j=1,2$ and $t\in[0,T_1]$, where α is the positive constant given by

$$\alpha=\left\{\frac{2\kappa M'}{m\delta l_{N*}}\left(1+\frac{l_{N*}^3M'}{\delta^4}\right)+\frac{2\kappa}{m\delta}\left(\frac{M'+l_{N*}}{l_{N*}}+\frac{1}{2}+\frac{l_{N*}^2}{2\delta^2}\right)+\frac{2\kappa M'}{m\delta^2}\left(\frac{M'+l_{N*}}{l_{N*}}+\frac{1}{2}+\frac{1}{2}\frac{l_{N*}^2}{\delta^2}\right)\right\}.$$

From (3.10), (3.11) and (3.18), it follows that

$$\begin{aligned}
&\left\|F((X,V))-F((X',V'))\right\|_{C([0,T_1])^{4N}} \\
&= \quad \left(\sum_{i=1}^N\left\|f_{i1}((X,V))-f_{i1}((X',V'))\right\|_{C([0,T_1])}^2+\sum_{i=1}^N\left\|f_{i2}((X,V))-f_{i2}((X',V'))\right\|_{C([0,T_1])}^2\right. \\
&\quad + \left.\sum_{i=1}^N\left\|g_{i1}((X,V))-g_{i1}((X',V'))\right\|_{C([0,T_1])}^2+\sum_{i=1}^N\left\|g_{i2}((X,V))-g_{i2}((X',V'))\right\|_{C([0,T_1])}^2\right)^{\frac{1}{2}} \\
&\leq \quad T_1\sqrt{2N\left(1+\alpha^2\right)}\left\|(X,V)-(X',V')\right\|_{C([0,T_1])^{4N}}.
\end{aligned}$$

Therefore, by choosing $T_0 \in (0, T_1]$ such that $T_0 < \dfrac{1}{\sqrt{2N(1+\alpha^2)}}$, we see that $F : W(T_0, \delta, M', M) \to W(T_0, \delta, M', M)$ is a contraction. Hence, Lemma 3.1 is proved.

□

We can easily show the following lemma. So, we omit its proof.

Lemma 3.4. *(1.2), (1.3), (1.4) and $X \in C^2([0,T])^{2N}$ hold if and only if (3.1), (3.2) and $(X, V) \in C^2([0,T])^{4N}$ hold.*

Proof of Theorem 2.1. By Lemma 3.3, there exists $T_1 > 0$ such that $F : W(T_1, \delta, M', M) \to W(T_1, \delta, M', M)$ is a contraction. Banach's fixed point theorem implies that there exists one and only one $Z \in W(T_1, \delta, M', M)$ such that $Z = F(Z)$. Thus, by Lemma 3.4 P has a unique solution on $[0, T_1]$.

Let $t \in [T_1, T_1 + T_2]$ where $T_2 \in (0, T - T_1]$. Let us consider the following initial value problem $\mathrm{P}^{(1)}$

$$
\begin{aligned}
m\frac{d^2 X_i^{(1)}}{dt^2}(t) &= f\left(\varepsilon_i^{(1)}\right)\frac{X_{i+1}^{(1)}(t) - X_i^{(1)}(t)}{l_i^{(1)}(t)} - f\left(\varepsilon_{i-1}^{(1)}\right)\frac{X_i^{(1)}(t) - X_{i-1}^{(1)}(t)}{l_{i-1}^{(1)}(t)} \quad \text{on } [T_1, T_1 + T_2],\\
X_i^{(1)}(T_1) &= X_i^{(0)}(T_1), \quad \frac{dX_i^{(1)}}{dt}(T_1) = \frac{dX_i^{(0)}}{dt}(T_1) \quad \text{for } i = 1, 2, \cdots, N,
\end{aligned}
$$

where $X_i^{(0)}$ is the unique solution of P on $[0, T_1]$. Let $F : C([T_1, T_1+T_2])^{4N} \to C([T_1, T_1+T_2])^{4N}$ be the mapping defined by the following

$$
F\left(Z^{(1)}\right) = \left(f_1\left(Z^{(1)}\right), f_2\left(Z^{(1)}\right), \cdots, f_N\left(Z^{(1)}\right), g_1\left(Z^{(1)}\right), g_2\left(Z^{(1)}\right), \cdots, g_N\left(Z^{(1)}\right)\right),
$$

for $Z^{(1)} = \left(X^{(1)}, V^{(1)}\right) \in C([T_1, T_1 + T_2])^{4N}$, where

$$
\begin{aligned}
f_i\left(Z^{(1)}\right)(t) &= \int_{T_1}^{t} V_i^{(1)}(\tau)d\tau + X_i^{(1)}(T_1),\\
g_i\left(Z^{(1)}\right)(t) &= \frac{1}{m}\left(\int_{T_1}^{t} f\left(\varepsilon_i^{(1)}\right)\frac{X_{i+1}^{(1)}(\tau) - X_i^{(1)}(\tau)}{l_i^{(1)}(\tau)} - f\left(\varepsilon_{i-1}^{(1)}\right)\frac{X_i^{(1)}(\tau) - X_{i-1}^{(1)}(\tau)}{l_{i-1}^{(1)}(\tau)}d\tau\right)\\
&\qquad + V_i^{(1)}(T_1),\\
X^{(1)} &= \left(X_i^{(1)}, X_2^{(1)}, \cdots, X_N^{(1)}\right), \quad V^{(1)} = \left(V_i^{(1)}, V_2^{(1)}, \cdots, V_N^{(1)}\right).
\end{aligned}
$$

For $\delta > 0, M > 0$ and $M' > 0$ we put

$$
\begin{aligned}
W_1(T_2, \delta, M', M) = \Big\{&(X, V) \in C([T_1, T_1 + T_2])^{4N} \mid \delta \le |X_{i+1}(t) - X_i(t)| \le M',\\
&|V_{i+1}(t) - V_i(t)| \le M \text{ for } i = 1, 2, \cdots, N, t \in [T_1, T_1 + T_2]\Big\}.
\end{aligned}
$$

Here, since Lemma 3.1 implies $G(X(T_1), V(T_1)) = d_0$, we can show that $F : W_1(T_1, \delta, M', M) \to W_1(T_1, \delta, M', M)$ is a contraction. Hence, by applying Banach's fixed point theorem, again, we know that P has a unique solution on $[0, 2T_1]$.

By repeating the discussion above, we obtain a solution of P on $[0, T]$. Consequently, we can prove Theorem 2.1. □

4 Proof of Theorem 2.2

In this section, we prove Theorem 2.2. For its proof we need several steps. In order to handle (1.5) and (1.6) easily, for $K \in \mathbb{Z}_{>0}$ and $\Delta t = \dfrac{T}{K}$, we define $\hat{g}_{Ki}\Big(X^{(n+1)}, X^{(n)}, V^{(n)}\Big)$ and $\hat{f}_{Ki}\Big(V^{(n+1)}, V^{(n)}, X^{(n)}\Big)$ as follows

$$
\begin{aligned}
&V_i^{(n+1)} \\
&= \frac{\kappa \Delta t}{4m} \left\{ \left\{ \varepsilon_{i-1}^{(n+1)} + \varepsilon_{i-1}^{(n)} + 1 - \frac{1}{\left(1+\varepsilon_{i-1}^{(n+1)}\right)\left(1+\varepsilon_{i-1}^{(n)}\right)} \right\} \frac{X_i^{(n+1)} - X_{i-1}^{(n+1)} + X_i^{(n)} - X_{i-1}^{(n)}}{\left|X_i^{(n+1)} - X_{i-1}^{(n+1)}\right| + \left|X_i^{(n)} - X_{i-1}^{(n)}\right|} \right. \\
&\qquad - \left\{ \varepsilon_i^{(n+1)} + \varepsilon_i^{(n)} + 1 - \frac{1}{\left(1+\varepsilon_i^{(n+1)}\right)\left(1+\varepsilon_i^{(n)}\right)} \right\} \\
&\qquad\qquad \left. \times \frac{X_{i+1}^{(n+1)} - X_i^{(n+1)} + X_{i+1}^{(n)} - X_i^{(n)}}{\left|X_{i+1}^{(n+1)} - X_i^{(n+1)}\right| + \left|X_{i+1}^{(n)} - X_i^{(n)}\right|} \right\} + V_i^{(n)} \\
&=: \hat{g}_{Ki}\left(X^{(n+1)}, X^{(n)}, V^{(n)}\right), \qquad (4.1) \\
&X_i^{(n+1)} \\
&= \frac{\Delta t}{2}\left(V_i^{(n+1)} + V_i^{(n)}\right) + X_i^{(n)} =: \hat{f}_{Ki}\left(V^{(n+1)}, V^{(n)}, X^{(n)}\right), \qquad (4.2)
\end{aligned}
$$

for $i = 1, 2, \cdots, N, n = 0, 1, \cdots, K$.

$X(0) = X_0, V(0) = V_0$. For given $X^{(n)}$ and $V^{(n)}$, let $\widehat{F}_K : \mathbb{R}^{4N} \to \mathbb{R}^{4N}$ be the following mapping

$$
\begin{aligned}
\widehat{F}_K(Z) = \Big(&\hat{f}_{K_1}\left(V, V^{(n)}, X^{(n)}\right),\ \hat{f}_{K_2}\left(V, V^{(n)}, X^{(n)}\right),\ \cdots,\ \hat{f}_{K_N}\left(V, V^{(n)}, X^{(n)}\right), \\
&\hat{g}_{K_1}\left(X, X^{(n)}, V^{(n)}\right),\ \hat{g}_{K_2}\left(X, X^{(n)}, V^{(n)}\right),\ \cdots,\ \hat{g}_{K_N}\left(X, X^{(n)}, V^{(n)}\right)\Big),
\end{aligned}
$$

for $Z = (X, V) \in \mathbb{R}^{4N}$.

The first lemma guarantees that the energy is preserved by (NS).

Lemma 4.1. *If (NS)$\Big(\Delta t, X^{(n)}, V^{(n)}\Big)$ has a solution $\Big(X^{(n+1)}, V^{(n+1)}\Big)$ for $n = 0, 1, \cdots, K-1$ and $K \in \mathbb{Z}_{>0}$, where $\Delta t = \dfrac{T}{K}$, then $G\Big(X^{(n+1)}, V^{(n+1)}\Big) = G\Big(X^{(n)}, V^{(n)}\Big)$ for $n = 0, 1, \cdots, K-1$.*

Proof. For $n = 0, 1, \cdots, K-1$, easily, we get

$$
\begin{aligned}
&G\left(X^{(n+1)}, V^{(n+1)}\right) - G\left(X^{(n)}, V^{(n)}\right) \\
&= \frac{m}{2}\sum_{i=1}^{N}\left(\left|V_i^{(n+1)}\right|^2 - \left|V_i^{(n)}\right|^2\right) + l_{N*}\sum_{i=1}^{N}\left(\hat{f}\left(\varepsilon_i^{(n+1)}\right) - \hat{f}\left(\varepsilon_i^{(n)}\right)\right) \\
&= \frac{m}{2}\sum_{i=1}^{N}\left(V_i^{(n+1)} - V_i^{(n)}\right)\left(V_i^{(n+1)} + V_i^{(n)}\right) + l_{N*}\sum_{i=1}^{N}\left(\hat{f}\left(\varepsilon_i^{(n+1)}\right) - \hat{f}\left(\varepsilon_i^{(n)}\right)\right).
\end{aligned}
$$

By using (1.6), we have

$$\frac{m}{2}\sum_{i=1}^{N}\left(V_i^{(n+1)}-V_i^{(n)}\right)\left(V_i^{(n+1)}+V_i^{(n)}\right)=m\sum_{i=1}^{N}\left(V_i^{(n+1)}-V_i^{(n)}\right)\frac{X_i^{(n+1)}-X_i^{(n)}}{\Delta t}. \tag{4.3}$$

Here, by using the definition of $\hat{f}$, we have

$$\begin{aligned}
&\hat{f}\left(\varepsilon_i^{(n+1)}\right)-\hat{f}\left(\varepsilon_i^{(n)}\right)\\
&= \frac{\kappa}{4}\left\{\left(\left(\varepsilon_i^{(n+1)}\right)^2-\left(\varepsilon_i^{(n)}\right)^2\right)+\left(\varepsilon_i^{(n+1)}-\varepsilon_i^{(n)}\right)+\left(\frac{1}{1+\varepsilon_i^{(n+1)}}-\frac{1}{1+\varepsilon_i^{(n)}}\right)\right\}\\
&=: \frac{\kappa}{4}\left(C_1+C_2+C_3\right).
\end{aligned}$$

Now, by using the definition of ε_i and l_i for $i=1,2,\cdots,N$, we see that

$$\begin{aligned}
&C_1\\
&=\left(\varepsilon_i^{(n+1)}+\varepsilon_i^{(n)}\right)\left(\varepsilon_i^{(n+1)}-\varepsilon_i^{(n)}\right)\\
&=\frac{1}{l_{N*}}\left(\varepsilon_i^{(n+1)}+\varepsilon_i^{(n)}\right)\left(l_i^{(n+1)}-l_i^{(n)}\right)\\
&=\frac{1}{l_{N*}}\left(\varepsilon_i^{(n+1)}+\varepsilon_i^{(n)}\right)\frac{\left(X_{i+1}^{(n+1)}-X_i^{(n+1)}+X_{i+1}^{(n)}-X_i^{(n)}\right)\left(X_{i+1}^{(n+1)}-X_i^{(n+1)}-X_{i+1}^{(n)}+X_i^{(n)}\right)}{\left|X_{i+1}^{(n+1)}-X_i^{(n+1)}\right|+\left|X_{i+1}^{(n)}-X_i^{(n)}\right|}.
\end{aligned}$$

Similarly, we observe that

$$\begin{aligned}
C_2 &= \frac{1}{l_{N*}}\left(l_i^{(n+1)}-l_i^{(n)}\right)\\
&= \frac{1}{l_{N*}}\frac{\left(X_{i+1}^{(n+1)}-X_i^{(n+1)}+X_{i+1}^{(n)}-X_i^{(n)}\right)\left(X_{i+1}^{(n+1)}-X_i^{(n+1)}-X_{i+1}^{(n)}+X_i^{(n)}\right)}{\left|X_{i+1}^{(n+1)}-X_i^{(n+1)}\right|+\left|X_{i+1}^{(n)}-X_i^{(n)}\right|},\\
C_3 &= -\frac{1}{\left(1+\varepsilon_i^{(n+1)}\right)\left(1+\varepsilon_i^{(n)}\right)}\left(\varepsilon_i^{(n+1)}-\varepsilon_i^{(n)}\right)\\
&= -\frac{1}{l_{N*}}\frac{\left(X_{i+1}^{(n+1)}-X_i^{(n+1)}+X_{i+1}^{(n)}-X_i^{(n)}\right)\left(X_{i+1}^{(n+1)}-X_i^{(n+1)}-X_{i+1}^{(n)}+X_i^{(n)}\right)}{\left(1+\varepsilon_i^{(n+1)}\right)\left(1+\varepsilon_i^{(n)}\right)\left|X_{i+1}^{(n+1)}-X_i^{(n+1)}\right|+\left|X_{i+1}^{(n)}-X_i^{(n)}\right|}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&l_{N*}\sum_{i=1}^{N}\left\{\hat{f}\left(\varepsilon_i^{(n+1)}\right)-\hat{f}\left(\varepsilon_i^{(n)}\right)\right\}\\
&=\frac{\kappa}{4}\sum_{i=1}^{N}\left\{\varepsilon_i^{(n+1)}+\varepsilon_i^{(n)}+1-\frac{1}{\left(1+\varepsilon_i^{(n+1)}\right)\left(1+\varepsilon_i^{(n)}\right)}\right\}\\
&\qquad\qquad\times\frac{X_{i+1}^{(n+1)}-X_i^{(n+1)}+X_{i+1}^{(n)}-X_i^{(n)}}{\left|X_{i+1}^{(n+1)}-X_i^{(n+1)}\right|+\left|X_{i+1}^{(n)}-X_i^{(n)}\right|}\left(X_{i+1}^{(n+1)}-X_{i+1}^{(n)}\right)\\
&\quad-\frac{\kappa}{4}\sum_{i=1}^{N}\left\{\varepsilon_i^{(n+1)}+\varepsilon_i^{(n)}+1-\frac{1}{\left(1+\varepsilon_i^{(n+1)}\right)\left(1+\varepsilon_i^{(n)}\right)}\right\}
\end{aligned}$$

$$\times \frac{X_{i+1}^{(n+1)} - X_i^{(n+1)} + X_{i+1}^{(n)} - X_i^{(n)}}{\left|X_{i+1}^{(n+1)} - X_i^{(n+1)}\right| + \left|X_{i+1}^{(n)} - X_i^{(n)}\right|} \left(X_i^{(n+1)} - X_i^{(n)}\right).$$

Since $X_0 = X_N$ and $X_1 = X_{N+1}$, we have

$$\begin{aligned}
& l_{N*} \sum_{i=1}^{N} \left\{ \hat{f}\left(\varepsilon_i^{(n+1)}\right) - \hat{f}\left(\varepsilon_i^{(n)}\right) \right\} \\
&= \frac{\kappa}{4} \sum_{i=1}^{N} \left\{ \left\{ \varepsilon_{i-1}^{(n+1)} + \varepsilon_{i-1}^{(n)} + 1 - \frac{1}{\left(1+\varepsilon_{i-1}^{(n+1)}\right)\left(1+\varepsilon_{i-1}^{(n)}\right)} \right\} \frac{X_i^{(n+1)} - X_{i-1}^{(n+1)} + X_i^{(n)} - X_{i-1}^{(n)}}{\left|X_i^{(n+1)} - X_{i-1}^{(n+1)}\right| + \left|X_i^{(n)} - X_{i-1}^{(n)}\right|} \right. \\
&\quad \left. - \left\{ \varepsilon_i^{(n+1)} + \varepsilon_i^{(n)} + 1 - \frac{1}{\left(1+\varepsilon_i^{(n+1)}\right)\left(1+\varepsilon_i^{(n)}\right)} \right\} \frac{X_{i+1}^{(n+1)} - X_i^{(n+1)} + X_{i+1}^{(n)} - X_i^{(n)}}{\left|X_{i+1}^{(n+1)} - X_i^{(n+1)}\right| + \left|X_{i+1}^{(n)} - X_i^{(n)}\right|} \right\} \\
&\qquad \times \left(X_i^{(n+1)} - X_i^{(n)}\right).
\end{aligned}$$

By using (1.5), we have

$$l_{N*} \sum_{i=1}^{N} \left\{ \hat{f}\left(\varepsilon_i^{(n+1)}\right) - \hat{f}\left(\varepsilon_i^{(n)}\right) \right\} = -m \sum_{i=1}^{N} \frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t} \left(X_i^{(n+1)} - X_i^{(n)}\right). \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$\begin{aligned}
& G\left(X^{(n+1)}, V^{(n+1)}\right) - G\left(X^{(n)}, V^{(n)}\right) \\
&= m \sum_{i=1}^{N} \left(V_i^{(n+1)} - V_i^{(n)}\right) \frac{X_i^{(n+1)} - X_i^{(n)}}{\Delta t} - m \sum_{i=1}^{N} \frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t} \left(X_i^{(n+1)} - X_i^{(n)}\right) \\
&= 0.
\end{aligned}$$

It means that (3.5) is the preserved energy for scheme (NS). □

The second lemma gives the estimates for $X_i^{(n)}$ and $V_i^{(n)}$ for $n \in \mathbb{Z}_{\geq 0}, i = 1, 2, \cdots, N$.

Lemma 4.2. *Let $K \in \mathbb{Z}_{>0}$ and $\left(X^{(n)}, V^{(n)}\right)$ be a solution of (NS)$\left(\Delta t, X^{(n-1)}, V^{(n-1)}\right)$ for $n = 1, 2, \cdots, K$, where $\Delta t = \dfrac{T}{K}$. If $\left(X^{(0)}, V^{(0)}\right) \in \mathbb{R}^{4N}$ and $X_i^{(0)} \neq X_j^{(0)}$ for $i \neq j$, then the following inequalities hold*

$$\left|X_{i+1}^{(n)} - X_i^{(n)}\right| \geq \frac{\kappa l_{N*}^2}{4d_0 + N\kappa l_{N*}}, \tag{4.5}$$

$$\left|X_{i+1}^{(n)} - X_i^{(n)}\right| \leq \frac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right), \tag{4.6}$$

$$\left|V_i^{(n)}\right| \leq \sqrt{\frac{1}{2m}\left(4d_0 + N\kappa l_{N*}\right)}, \tag{4.7}$$

for $i = 1, 2, \cdots, N, n = 1, 2, \cdots, K$, where $d_0 = G\left(X^{(0)}, V^{(0)}\right)$, $X^{(n)} = \left(X_1^{(n)}, X_2^{(n)}, \cdots, X_N^{(n)}\right)$, $V^{(n)} = \left(V_1^{(n)}, V_2^{(n)}, \cdots, V_N^{(n)}\right)$ for $n = 0, 1, \cdots, K$.

Proof. By Lemma 4.1 we have $G\left(X^{(n)}, V^{(n)}\right) = d_0$ for $n = 0, 1, \cdots, K-1$, and then Lemma 3.2 guarantees (4.5), (4.6) and (4.7). Thus, we have proved this lemma. $\square$

The last lemma in this section is concerned with existence of a set on which $\widehat{F}_K$ is a contraction mapping.

Lemma 4.3. *For $\delta > 0$, $M' > 0$, $M > 0$, put*

$$\widehat{W}(\delta, M', M) = \left\{ z \in \mathbb{R}^{4N} \;\middle|\; \delta \leq |X_{i+1} - X_i| \leq M',\ |V_{i+1} - V_i| \leq M \text{ for } i = 1, 2, \cdots, N \right\}$$

Then there exists $K_0 \in \mathbb{Z}_{>0}$ such that for $K \geq K_0$, $\widehat{F}_K : \widehat{W}(\delta, M', M) \to \widehat{W}(\delta, M', M)$ is a contraction mapping.

Proof. First, we prove that $\widehat{F}_K$ is a function from $\widehat{W}(\delta, M', M)$ to $\widehat{W}(\delta, M', M)$. Let $K \in \mathbb{Z}_{>0}, \Delta t = \dfrac{T}{K}$, $(X, V) \in \widehat{W}(\delta, M', M)$ and put $\widehat{F}_K((X, V)) = (X', V')$. We shall prove $(X', V') \in \widehat{W}(\delta, M', M)$ for some $K \in \mathbb{Z}_{>0}$, $\delta > 0$, $M' > 0$ and $M > 0$. By using (4.2), (4.5) and (4.7), we have

$$\begin{aligned}
\left|X'_{i+1} - X'_i\right| &= \left|\hat{f}_{K(i+1)}\left(V, V^{(n)}, X^{(n)}\right) - \hat{f}_{Ki}\left(V, V^{(n)}, X^{(n)}\right)\right| \\
&\geq \left|X^{(n)}_{i+1} - X^{(n)}_i\right| - \frac{\Delta t}{2}\left(\left|V_{i+1} - V_i\right| + \left|V^{(n)}_{i+1}\right| + \left|V^{(n)}_i\right|\right) \\
&\geq \frac{\kappa {l_{N*}}^2}{4d_0 + N\kappa l_{N*}} - \frac{\Delta t}{2}\left(M + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})}\right) \qquad \text{for } i = 1, 2, \cdots, N.
\end{aligned}$$

In the calculation above, since $(X, V) \in \widehat{W}(\delta, M', M)$, the third inequality holds. Here, we choose $\delta > 0$ such that $2\delta \leq \dfrac{\kappa {l_{N*}}^2}{4d_0 + N\kappa l_{N*}}$, then we have

$$\left|X'_{i+1} - X'_i\right| \geq 2\delta - \frac{\Delta t}{2}\left(M + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})}\right) \qquad \text{for } i = 1, 2, \cdots, N.$$

Moreover, we choose $K \in \mathbb{Z}_{>0}$ such that

$$\frac{\Delta t}{2}\left(M + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})}\right) \leq \delta, \tag{4.8}$$

then we have

$$\left|X'_{i+1} - X'_i\right| \geq \delta \qquad \text{for } i = 1, 2, \cdots, N.$$

In the same way, by using (4.6) and (4.7), we have

$$\begin{aligned}
\left|X'_{i+1} - X'_i\right| &\leq \frac{m\Delta t}{2}\left(\left|V_{i+1} - V_i\right| + \left|V^{(n)}_{i+1}\right| + \left|V^{(n)}_i\right|\right) + \left|X^{(n)}_{i+1} + X^{(n)}_i\right| \\
&\leq \frac{m\Delta t}{2}\left(M + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})}\right) + \frac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right) \qquad \text{for } i = 1, 2, \cdots, N.
\end{aligned}$$

Here, we choose $M' > 0$ such that $\dfrac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right) \leq \dfrac{M'}{2}$. Moreover, we take $\Delta t > 0$ such that

$$\frac{m\Delta t}{2}\left(M + 2\sqrt{\frac{1}{2m}(4d_0 + N\kappa l_{N*})}\right) \leq \frac{M'}{2}, \tag{4.9}$$

we have

$$\left|X'_{i+1} - X'_i\right| \;\leq\; \frac{M'}{2} + \frac{M'}{2} = M' \quad \text{for } i = 1, 2, \cdots, N.$$

Next, we give an estimate for V'_i for $i = 1, 2, \cdots, N$. By using (4.1), we have

$$\begin{aligned}
\left|V'_{i+1} - V'_i\right| &= \left|g_{i+1}\left(X, X^{(n)}, V^{(n)}\right) - g_i\left(X, X^{(n)}, V^{(n)}\right)\right| \\
&\leq \frac{\kappa \Delta t}{4}\left|\frac{\left(\varepsilon_i + \varepsilon_i^{(n)} + 1\right)\left(X_{i+1} - X_i + X_{i+1}^{(n)} - X_i^{(n)}\right)}{\left|X_{i+1} - X_i\right| + \left|X_{i+1}^{(n)} - X_i^{(n)}\right|}\right| \\
&\quad + \frac{\kappa \Delta t}{4}\left|\frac{1}{\left(1 + \varepsilon_i\right)\left(1 + \varepsilon_i^{(n)}\right)} \frac{X_{i+1} - X_i + X_{i+1}^{(n)} - X_i^{(n)}}{\left|X_{i+1} - X_i\right| + \left|X_{i+1}^{(n)} - X_i^{(n)}\right|}\right| \\
&\quad + \frac{\kappa \Delta t}{4}\left|\frac{\left(\varepsilon_{i+1} + \varepsilon_{i+1}^{(n)} + 1\right)\left(X_{i+2} - X_{i+1} + X_{i+2}^{(n)} - X_{i+1}^{(n)}\right)}{\left|X_{i+2} - X_{i+1}\right| + \left|X_{i+2}^{(n)} - X_{i+1}^{(n)}\right|}\right| \\
&\quad + \frac{\kappa \Delta t}{4}\left|\frac{1}{\left(\varepsilon_{i+1} + 1\right)\left(\varepsilon_{i+1}^{(n)} + 1\right)} \frac{X_{i+2} - X_{i+1} + X_{i+2}^{(n)} - X_{i+1}^{(n)}}{\left|X_{i+2} - X_{i+1}\right| + \left|X_{i+2}^{(n)} - X_{i+1}^{(n)}\right|}\right| \\
&\quad + \frac{\kappa \Delta t}{4}\left|\frac{\left(\varepsilon_{i-1} + \varepsilon_{i-1}^{(n)} + 1\right)\left(X_i - X_{i-1} + X_i^{(n)} - X_{i-1}^{(n)}\right)}{\left|X_i - X_{i-1}\right| + \left|X_i^{(n)} - X_{i-1}^{(n)}\right|}\right| \\
&\quad + \frac{\kappa \Delta t}{4}\left|\frac{1}{\left(1 + \varepsilon_{i-1}\right)\left(1 + \varepsilon_{i-1}^{(n)}\right)} \frac{X_i - X_{i-1} + X_i^{(n)} - X_{i-1}^{(n)}}{\left|X_i - X_{i-1}\right| + \left|X_i^{(n)} - X_{i-1}^{(n)}\right|}\right| \\
&\quad + \frac{\kappa \Delta t}{4}\left|\frac{\left(\varepsilon_i + \varepsilon_i^{(n)} + 1\right)\left(X_{i+1} - X_i + X_{i+1}^{(n)} - X_i^{(n)}\right)}{\left|X_{i+1} - X_i\right| + \left|X_{i+1}^{(n)} - X_i^{(n)}\right|}\right| \\
&\quad + \frac{\kappa \Delta t}{4}\left|\frac{1}{\left(\varepsilon_i + 1\right)\left(\varepsilon_i^{(n)} + 1\right)} \frac{X_{i+1} - X_i + X_{i+1}^{(n)} - X_i^{(n)}}{\left|X_{i+1} - X_i\right| + \left|X_{i+1}^{(n)} - X_i^{(n)}\right|}\right| \\
&\quad + \left|V_{i+1}^{(n)} - V_i^{(n)}\right| \\
&=: \sum_{j=1}^{9} D_j \quad \text{for } i = 1, 2, \cdots, N.
\end{aligned}$$

Since the estimates for D_1, D_3, D_5 and D_7 can be obtained in the same way, so we shall estimate D_1, here. By using (4.6) and $(X, V) \in \widehat{W}(\delta, M', M)$, we have

$$\begin{aligned}
|D_1| &= \frac{\kappa \Delta t}{4}\left|\frac{\left(\varepsilon_i + \varepsilon_i^{(n)} + 1\right)\left(X_{i+1} - X_i + X_{i+1}^{(n)} - X_i^{(n)}\right)}{\left|X_{i+1} - X_i\right| + \left|X_{i+1}^{(n)} - X_i^{(n)}\right|}\right| \\
&\leq \frac{\kappa \Delta t}{4 l_{N*}}\left(\left|l_i\right| + \left|l_i^{(n)}\right| + l_{N*}\right) \\
&\leq \frac{\kappa \Delta t}{4 l_{N*}}\left(\left|X_{i+1} - X_i\right| + \left|X_{i+1}^{(n)} - X_i^{(n)}\right| + l_{N*}\right)
\end{aligned}$$

$$\leq \frac{\kappa\Delta t}{4l_{N*}}\left(M' + \frac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right) + l_{N*}\right).$$

Similarly, we have

$$\begin{aligned}
|D_3| &\leq \frac{\kappa\Delta t}{4l_{N*}}\left(M' + \frac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right) + l_{N*}\right),\\
|D_5| &\leq \frac{\kappa\Delta t}{4l_{N*}}\left(M' + \frac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right) + l_{N*}\right),\\
|D_7| &\leq \frac{\kappa\Delta t}{4l_{N*}}\left(M' + \frac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right) + l_{N*}\right).
\end{aligned}$$

Next, since the estimates for D_2, D_4, D_6 and D_8 can be obtained in the same way, so we shall estimate D_2, here. By using (4.5) and $(X, V) \in \widehat{W}(\delta, M', M)$, we have

$$\begin{aligned}
|D_2| &= \frac{\kappa\Delta t}{4}\left|\frac{1}{\left(1+\varepsilon_i\right)\left(1+\varepsilon_i^{(n)}\right)}\frac{X_{i+1} - X_i + X_{i+1}^{(n)} - X_i^{(n)}}{\left|X_{i+1} - X_i\right| + \left|X_{i+1}^{(n)} - X_i^{(n)}\right|}\right|\\
&\leq \frac{\kappa l_{N*}\Delta t}{4}\left(\frac{1}{\left|l_i\right|}\frac{1}{\left|l_i^{(n)}\right|}\right)\\
&= \frac{\kappa l_{N*}^2\Delta t}{4}\left(\frac{1}{\left|X_{i+1} - X_i\right|}\frac{1}{\left|X_{i+1}^{(n)} - X_i^{(n)}\right|}\right)\\
&\leq \Delta t\frac{(4d_0 + N\kappa l_{N*})}{4\delta}.
\end{aligned}$$

Similarly, we have

$$|D_4| \leq \Delta t\frac{(4d_0 + N\kappa l_{N*})}{4\delta}, \quad |D_6| \leq \Delta t\frac{(4d_0 + N\kappa l_{N*})}{4\delta}, \quad |D_8| \leq \Delta t\frac{(4d_0 + N\kappa l_{N*})}{4\delta}.$$

Finally, by using (4.7), we have

$$|D_9| \leq 2\sqrt{\frac{1}{2m}\left(4d_0 + N\kappa l_{N*}\right)}.$$

Therefore, we choose $M > 0$ and $\Delta t > 0$ such that $2\sqrt{\frac{1}{2m}\left(4d_0 + N\kappa l_{N*}\right)} \leq \frac{M}{9}$ and

$$\Delta t = \min\left\{\frac{4l_{N*}M}{9\kappa\left\{M' + \frac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right) + l_{N*}\right\}}, \quad \frac{4\delta M}{9\left(4d_0 + N\kappa l_{N*}\right) + l_{N*}}\right\}, \tag{4.10}$$

then we have

$$\left|V_{i+1}' - V_i'\right| \leq \sum_{j=1}^{9}|D_j| \leq M \quad \text{for } i = 1, 2, \cdots, N.$$

By combining (4.8),(4.9) and (4.10), we take $\Delta t > 0$ as follows, again

$$\Delta t = \min\left\{\frac{2\delta}{m\left(M + 2\sqrt{\frac{1}{m}\left(4d_0 + N\kappa l_{N*}\right)}\right)}, \frac{2M'}{m\left\{M + \sqrt{\frac{1}{2m}\left(4d_0 + N\kappa l_{N*}\right)}\right\} + \frac{1}{\kappa}\left(4d_0 + N\kappa l_{N*}\right)},\right.$$

$$\left.\frac{4l_{N*}M}{9\kappa\left\{M'+\frac{1}{\kappa}\left(4d_0+N\kappa l_{N*}\right)+l_{N*}\right\}},\quad \frac{4\delta M}{9\left(4d_0+N\kappa l_{N*}\right)+l_{N*}}\right\},$$

and then we see that $(X',V')\in\widehat{W}(\delta,M',M)$. Thus, we have proved that $\widehat{F}_K$ is a function from $\widehat{W}(\delta,M',M)$ to $\widehat{W}(\delta,M',M)$.

Finally, we shall show that $\widehat{F}_K$: $\widehat{W}(\delta,M',M)\to\widehat{W}(\delta,M',M)$ is a contraction mapping. Let $Z,Z\in\widehat{W}(\delta,M',M)$ and put $Z=(X,V)$, $Z'=(X',V')$. Easily, we have

$$\begin{aligned}\left\|\widehat{F}_K(Z)-\widehat{F}_K(Z')\right\|^2_{\mathbb{R}^{4N}} &= \sum_{i=1}^{N}\left|\hat{f}_{Ki}\left(V,V^{(n)},X^{(n)}\right)-\hat{f}_{Ki}\left(V',V^{(n)},X^{(n)}\right)\right|^2\\ &\qquad+\sum_{i=1}^{N}\left|\hat{g}_{Ki}\left(X,X^{(n)},V^{(n)}\right)-\hat{g}_{Ki}\left(X',X^{(n)},V^{(n)}\right)\right|^2\\ &\le \sum_{i=1}^{N}\left|\hat{f}_{Ki}\left(V,V^{(n)},X^{(n)}\right)-\hat{f}_{Ki}\left(V',V^{(n)},X^{(n)}\right)\right|\\ &\qquad+\sum_{i=1}^{N}\left|\hat{g}_{Ki}\left(X,X^{(n)},V^{(n)}\right)-\hat{g}_{Ki}\left(X',X^{(n)},V^{(n)}\right)\right|.\end{aligned}$$

Here, by using (4.2), we have

$$\begin{aligned}\left|\hat{f}_{Ki}\left(V,V^{(n)},X^{(n)}\right)-\hat{f}_{Ki}\left(V',V^{(n)},X^{(n)}\right)\right| &= \frac{\Delta t}{2}\left|V_i-V_i'\right|\\ &\le \frac{\Delta t}{2}\left\|Z-Z'\right\|_{\mathbb{R}^{4N}}\ \text{ for } i=1,2,\cdots,N.\end{aligned}\tag{4.11}$$

Next, by using (4.1), we have

$$\begin{aligned}&\left|\hat{g}_{Ki}\left(X,X^{(n)},V^{(n)}\right)-\hat{g}_{Ki}\left(X',X^{(n)},V^{(n)}\right)\right|\\ &\le\frac{\kappa\Delta t}{4m}\left|\frac{\left(\varepsilon_{i-1}+\varepsilon_{i-1}^{(n)}\right)\left(X_i-X_{i-1}+X_i^{(n)}-X_{i-1}^{(n)}\right)}{\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}-\frac{\left(\varepsilon_{i-1}'+\varepsilon_{i-1}^{(n)}\right)\left(X_i'-X_{i-1}'+X_i^{(n)}-X_{i-1}^{(n)}\right)}{\left|X_i'-X_{i-1}'\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}\right|\\ &\quad+\frac{\kappa\Delta t}{4m}\left|\frac{\left(\varepsilon_{i}+\varepsilon_{i}^{(n)}\right)\left(X_{i+1}-X_{i}+X_{i+1}^{(n)}-X_{i}^{(n)}\right)}{\left|X_{i+1}-X_{i}\right|+\left|X_{i+1}^{(n)}-X_{i}^{(n)}\right|}-\frac{\left(\varepsilon_{i}'+\varepsilon_{i}^{(n)}\right)\left(X_{i+1}'-X_{i}'+X_{i+1}^{(n)}-X_{i}^{(n)}\right)}{\left|X_{i+1}'-X_{i}'\right|+\left|X_{i+1}^{(n)}-X_{i}^{(n)}\right|}\right|\\ &\quad+\frac{\kappa\Delta t}{4m}\left|\frac{X_i-X_{i-1}+X_i^{(n)}-X_{i-1}^{(n)}}{\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}-\frac{X_i'-X_{i-1}'+X_i^{(n)}-X_{i-1}^{(n)}}{\left|X_i'-X_{i-1}'\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}\right|\\ &\quad+\frac{\kappa\Delta t}{4m}\left|\frac{X_{i+1}-X_{i}+X_{i+1}^{(n)}-X_{i}^{(n)}}{\left|X_{i+1}-X_{i}\right|+\left|X_{i+1}^{(n)}-X_{i}^{(n)}\right|}-\frac{X_{i+1}'-X_{i}'+X_{i+1}^{(n)}-X_{i}^{(n)}}{\left|X_{i+1}'-X_{i}'\right|+\left|X_{i+1}^{(n)}-X_{i}^{(n)}\right|}\right|\\ &\quad+\frac{\kappa\Delta t}{4m}\left|\frac{1}{\left(1+\varepsilon_{i-1}\right)\left(1+\varepsilon_{i-1}^{(n)}\right)}\frac{X_i-X_{i-1}+X_i^{(n)}-X_{i-1}^{(n)}}{\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}\right.\\ &\qquad\qquad\left.-\frac{1}{\left(1+\varepsilon_{i-1}'\right)\left(1+\varepsilon_{i-1}^{(n)}\right)}\frac{X_i'-X_{i-1}'+X_i^{(n)}-X_{i-1}^{(n)}}{\left|X_i'-X_{i-1}'\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}\right|\end{aligned}$$

$$+\frac{\kappa\Delta t}{4m}\left|\frac{1}{\left(1+\varepsilon_i\right)\left(1+\varepsilon_i^{(n)}\right)}\frac{X_{i+1}-X_i+X_{i+1}^{(n)}-X_i^{(n)}}{\left|X_{i+1}-X_i\right|+\left|X_{i+1}^{(n)}-X_i^{(n)}\right|}\right.$$
$$\left.-\frac{1}{\left(1+\varepsilon_i'\right)\left(1+\varepsilon_i^{(n)}\right)}\frac{X_{i+1}'-X_i'+X_{i+1}^{(n)}-X_i^{(n)}}{\left|X_{i+1}'-X_i'\right|+\left|X_{i+1}^{(n)}-X_i^{(n)}\right|}\right|$$
$$=: E_1+E_2+E_3+E_4+E_5+E_6 \quad \text{for } i=1,2,\cdots,N.$$

For E_1 we have

$$\begin{aligned}
&E_1\\
&\le \frac{\kappa\Delta t}{4m}\frac{1}{H}\left\{\left|\left(\varepsilon_{i-1}-\varepsilon_{i-1}'\right)\left(X_i-X_{i-1}+X_i^{(n)}-X_{i-1}^{(n)}\right)\left(\left|X_i'-X_{i-1}'\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\right)\right|\right.\\
&\qquad +\left|\left(\varepsilon_{i-1}'+\varepsilon_{i-1}^{(n)}\right)\left(X_i-X_{i-1}+X_i^{(n)}-X_{i-1}^{(n)}\right)\left(\left|X_i'-X_{i-1}'\right|-\left|X_i-X_{i-1}\right|\right)\right|\\
&\qquad \left.+\left|\left(\varepsilon_{i-1}'+\varepsilon_{i-1}^{(n)}\right)\left(X_i-X_{i-1}-X_i'+X_{i-1}'\right)\left(\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\right)\right|\right\}\\
&=: \frac{\kappa\Delta t}{4m}\frac{1}{H}\left(E_{1,1}+E_{1,2}+E_{1,3}\right),
\end{aligned}$$

where $H=\dfrac{1}{\left(\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\right)\left(\left|X_i'-X_{i-1}'\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\right)}$. Immediately, we have

$$\begin{aligned}
\frac{E_{1,1}}{H} &= \frac{\left|\left(\varepsilon_{i-1}-\varepsilon_{i-1}'\right)\left(X_i-X_{i-1}+X_i^{(n)}-X_{i-1}^{(n)}\right)\right|}{\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}\\
&\le \frac{\left|\left(\varepsilon_{i-1}-\varepsilon_{i-1}'\right)\right|\left(\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\right)}{\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}\\
&\le \frac{1}{l_{N*}}\left|X_i-X_{i-1}-X_i'+X_{i-1}'\right|\\
&\le \frac{1}{l_{N*}}\left(\left|X_i-X_i'\right|+\left|X_{i-1}-X_{i-1}'\right|\right)\\
&\le \frac{2}{l_{N*}}\left\|Z-Z'\right\|_{\mathbb{R}^{4N}}.
\end{aligned}$$

Since it holds

$$\begin{aligned}
E_{1,2} &= \left|\left(\varepsilon_{i-1}'+\varepsilon_{i-1}^{(n)}\right)\left(X_i-X_{i-1}+X_i^{(n)}-X_{i-1}^{(n)}\right)\left(\left|X_i'-X_{i-1}'\right|-\left|X_i-X_{i-1}\right|\right)\right|\\
&\le \left|\varepsilon_{i-1}'+\varepsilon_{i-1}^{(n)}\right|\left(\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\right)\left(\left|X_i-X_i'\right|+\left|X_{i-1}-X_{i-1}'\right|\right)\\
&\le 2\left|\varepsilon_{i-1}'+\varepsilon_{i-1}^{(n)}\right|\left(\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\right)\left\|Z-Z'\right\|_{\mathbb{R}^{4N}},
\end{aligned}$$

we have

$$\frac{E_{1,2}}{H} \leq \frac{2\left|\varepsilon'_{i-1}+\varepsilon^{(n)}_{i-1}\right|}{\left|X'_i-X'_{i-1}\right|+\left|X^{(n)}_i-X^{(n)}_{i-1}\right|}\left\|Z-Z'\right\|_{\mathbb{R}^{4N}}.$$

Here, it is easy to see that

$$\left|\varepsilon'_{i-1}+\varepsilon^{(n)}_{i-1}\right| \leq \frac{1}{l_{N*}}\left(\left|l'_{i-1}\right|+\left|l^{(n)}_{i-1}\right|+2l_{N*}\right) \qquad \text{for } i=1,2,\cdots,N,$$

$$\frac{\left|l'_{i-1}\right|}{\left|X'_i-X'_{i-1}\right|+\left|X^{(n)}_i-X^{(n)}_{i-1}\right|}<1, \quad \frac{\left|l^{(n)}_{i-1}\right|}{\left|X'_i-X'_{i-1}\right|+\left|X^{(n)}_i-X^{(n)}_{i-1}\right|}<1 \quad \text{for } i=1,2,\cdots,N.$$

Moreover, by using (4.5) in Lemma 4.2, we see that

$$\frac{1}{\left|X^{(n)}_{i+1}-X^{(n)}_i\right|}\leq\frac{4d_0+N\kappa l_{N*}}{\kappa l^2_{N*}} \qquad \text{for } i=1,2,\cdots,N. \tag{4.12}$$

From these inequalities, we have

$$\frac{E_{1,2}}{H} \leq 4\left(\frac{1}{l_{N*}}+\frac{4d_0+N\kappa l_{N*}}{\kappa l^2_{N*}}\right)\left\|Z-Z'\right\|_{\mathbb{R}^{4N}}.$$

Next, we have

$$\begin{aligned}
E_{1,3} &= \left|\left(\varepsilon'_{i-1}+\varepsilon^{(n)}_{i-1}\right)\left(X_i-X_{i-1}-X'_i+X'_{i-1}\right)\left(\left|X_i-X_{i-1}\right|+\left|X^{(n)}_i-X^{(n)}_{i-1}\right|\right)\right| \\
&\leq \left|\varepsilon'_{i-1}+\varepsilon^{(n)}_{i-1}\right|\left(\left|X_i-X'_i\right|+\left|X_{i-1}-X'_{i-1}\right|\right)\left(\left|X_i-X_{i-1}\right|+\left|X^{(n)}_i-X^{(n)}_{i-1}\right|\right) \\
&\leq 2\left|\varepsilon'_{i-1}+\varepsilon^{(n)}_{i-1}\right|\left(\left|X_i-X_{i-1}\right|+\left|X^{(n)}_i-X^{(n)}_{i-1}\right|\right)\left\|Z-Z'\right\|_{\mathbb{R}^{4N}}.
\end{aligned}$$

In the same way for $\dfrac{E_{1,2}}{H}$, we have

$$\frac{E_{1,3}}{H} \leq 4\left(\frac{1}{l_{N*}}+\frac{4d_0+N\kappa l_{N*}}{\kappa l^2_{N*}}\right)\left\|Z-Z'\right\|_{\mathbb{R}^{4N}}.$$

Therefore, we obtain

$$E_1\leq\frac{\kappa\Delta t}{2m}\left\{\frac{5}{l_{N*}}+\frac{4}{\kappa l^2_{N*}}\left(4d_0+N\kappa l_{N*}\right)\right\}\left\|Z-Z'\right\|_{\mathbb{R}^{4N}}.$$

Similarly, we have

$$E_2\leq\frac{\kappa\Delta t}{2m}\left\{\frac{5}{l_{N*}}+\frac{4}{\kappa l^2_{N*}}\left(4d_0+N\kappa l_{N*}\right)\right\}\left\|Z-Z'\right\|_{\mathbb{R}^{4N}}.$$

Next, we have

$$E_3 = \frac{\kappa\Delta t}{4m}\left|\frac{X_i-X_{i-1}+X^{(n)}_i-X^{(n)}_{i-1}}{\left|X_i-X_{i-1}\right|+\left|X^{(n)}_i-X^{(n)}_{i-1}\right|}-\frac{X'_i-X'_{i-1}+X^{(n)}_i-X^{(n)}_{i-1}}{\left|X'_i-X'_{i-1}\right|+\left|X^{(n)}_i-X^{(n)}_{i-1}\right|}\right|$$

$$
\begin{aligned}
&\leq \frac{\kappa \Delta t}{4m} \frac{1}{H} \left\{ \left| \left(X_i - X_{i-1} - X_i' + X_{i-1}' \right) \left(\left| X_i' - X_{i-1}' \right| + \left| X_i^{(n)} - X_{i-1}^{(n)} \right| \right) \right| \right. \\
&\qquad \left. + \left| \left(X_i' - X_{i-1}' + X_i^{(n)} - X_{i-1}^{(n)} \right) \left(\left| X_i' - X_{i-1}' \right| + \left| X_i - X_{i-1} \right| \right) \right| \right\} \\
&=: \frac{\kappa \Delta t}{4m} \frac{1}{H} \left(E_{3,1} + E_{3,2} \right).
\end{aligned}
$$

Since

$$
\begin{aligned}
E_{3,1} &\leq \left(\left| X_i - X_i' \right| + \left| X_{i-1} - X_{i-1}' \right| \right) \left(\left| X_i' - X_{i-1}' \right| + \left| X_i^{(n)} - X_{i-1}^{(n)} \right| \right) \\
&\leq 2 \left\| Z - Z' \right\| \left(\left| X_i' - X_{i-1}' \right| + \left| X_i^{(n)} - X_{i-1}^{(n)} \right| \right),
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{E_{3,1}}{H} &\leq = \frac{2 \left\| Z - Z' \right\|_{\mathbb{R}^{4N}} \left(\left| X_i' - X_{i-1}' \right| + \left| X_i^{(n)} - X_{i-1}^{(n)} \right| \right)}{\left(\left| X_i - X_{i-1} \right| + \left| X_i^{(n)} - X_{i-1}^{(n)} \right| \right) \left(\left| X_i' - X_{i-1}' \right| + \left| X_i^{(n)} - X_{i-1}^{(n)} \right| \right)} \\
&= \frac{2 \left\| Z - Z' \right\|_{\mathbb{R}^{4N}}}{\left| X_i - X_{i-1} \right| + \left| X_i^{(n)} - X_{i-1}^{(n)} \right|} \\
&\leq \frac{2 \left\| Z - Z' \right\|_{\mathbb{R}^{4N}}}{\left| X_i^{(n)} - X_{i-1}^{(n)} \right|}.
\end{aligned}
$$

In the same way for $E_{3,1}$, we have

$$
\frac{E_{3,2}}{H} \leq \frac{2 \left\| Z - Z' \right\|_{\mathbb{R}^{4N}}}{\left| X_i^{(n)} - X_{i-1}^{(n)} \right|}.
$$

Therefore, we have

$$
E_3 \leq \frac{\kappa \Delta t}{m} \frac{\left\| Z - Z' \right\|_{\mathbb{R}^{4N}}}{\left| X_i^{(n)} - X_{i-1}^{(n)} \right|}.
$$

By using (4.12), we have

$$
E_3 \leq \frac{\Delta t}{m l_{N*}^2} \left(4 d_0 + N \kappa l_{N*} \right) \left\| Z - Z' \right\|_{\mathbb{R}^{4N}}.
$$

In the same way for E_3, we have

$$
E_4 \leq \frac{\Delta t}{m l_{N*}^2} \left(4 d_0 + N \kappa l_{N*} \right) \left\| Z - Z' \right\|_{\mathbb{R}^{4N}}.
$$

Next, we have

$$
E_5 \leq \frac{\kappa \Delta t}{4m} \left| \frac{1}{1 + \varepsilon_{i-1}^{(n)}} \left(\frac{1}{1 + \varepsilon_{i-1}} - \frac{1}{1 + \varepsilon_{i-1}'} \right) \left(\frac{X_i - X_{i-1} + X_i^{(n)} - X_{i-1}^{(n)}}{\left| X_i - X_{i-1} \right| + \left| X_i^{(n)} - X_{i-1}^{(n)} \right|} \right) \right|
$$

$$
\begin{aligned}
&+\frac{\kappa\Delta t}{4m}\left|\frac{1}{1+\varepsilon_{i-1}^{(n)}}\frac{1}{1+\varepsilon'_{i-1}}\left(\frac{X_i-X_{i-1}-X'_i+X'_{i-1}}{\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}\right)\right| \\
&+\frac{\kappa\Delta t}{4m}\left|\frac{\left(X'_i-X'_{i-1}+X_i^{(n)}-X_{i-1}^{(n)}\right)}{\left(1+\varepsilon_{i-1}^{(n)}\right)\left(1+\varepsilon'_{i-1}\right)}\right. \\
&\qquad\left.\times\left\{\frac{\left|X'_i-X'_{i-1}\right|-\left|X_i-X_{i-1}\right|}{\left(\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\right)\left(\left|X'_i-X'_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\right)}\right\}\right| \\
=:&\ \frac{\kappa\Delta t}{4m}\left(E_{5,1}+E_{5,2}+E_{5,3}\right).
\end{aligned}
$$

We have

$$
\begin{aligned}
E_{5,1} &= \left|\frac{1}{1+\varepsilon_{i-1}^{(n)}}\frac{\varepsilon'_{i-1}-\varepsilon_{i-1}}{\left(1+\varepsilon_{i-1}\right)\left(1+\varepsilon'_{i-1}\right)}\left(\frac{X_i-X_{i-1}+X_i^{(n)}-X_{i-1}^{(n)}}{\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}\right)\right| \\
&\le \left|\frac{l_{N*}}{l_{i-1}^{(n)}}\left(\frac{l_{N*}}{l_{i-1}}\right)\left(\frac{l_{N*}}{l'_{i-1}}\right)\frac{l'_{i-1}-l_{i-1}}{l_{N*}}\right| \\
&= l_{N*}^2\left|\frac{1}{l_{i-1}^{(n)}}\left(\frac{l'_{i-1}-l_{i-1}}{l_{i-1}l'_{i-1}}\right)\right| \\
&\le \frac{l_{N*}^2\left(\left|X_i-X'_i\right|+\left|X_{i-1}-X'_{i-1}\right|\right)}{\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\left|X_i-X_{i-1}\right|\left|X'_i-X'_{i-1}\right|} \\
&\le \frac{2l_{N*}^2\|Z-Z'\|_{\mathbb{R}^{4N}}}{\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\left|X_i-X_{i-1}\right|\left|X'_i-X'_{i-1}\right|}.
\end{aligned}
$$

By using(4.12), we have

$$
E_{5,1} \le \frac{2\left(4d_0+N\kappa l_{N*}\right)}{\kappa}\frac{\|Z-Z'\|_{\mathbb{R}^{4N}}}{\left|X_i-X_{i-1}\right|\left|X'_i-X'_{i-1}\right|}.
$$

Since $Z, Z' \in \widehat{W}(\delta, M', M)$, we have

$$
E_{5,1} \le \frac{2\left(4d_0+N\kappa l_{N*}\right)}{\kappa\delta^2}\|Z-Z'\|_{\mathbb{R}^{4N}}.
$$

In the same way for $E_{5,1}$, by using (4.12) and $Z, Z' \in \widehat{W}(\delta, M', M)$, we have

$$
\begin{aligned}
E_{5,2} &= \left|\frac{1}{1+\varepsilon_{i-1}^{(n)}}\frac{1}{1+\varepsilon'_{i-1}}\left(\frac{X_i-X_{i-1}-X'_i+X'_{i-1}}{\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}\right)\right| \\
&\le l_{N*}^2\left|\frac{1}{l_{i-1}^{(n)}l'_{i-1}}\right|\left(\frac{\left|X_i-X'_i\right|+\left|X_{i-1}-X'_{i-1}\right|}{\left|X_i-X_{i-1}\right|+\left|X_i^{(n)}-X_{i-1}^{(n)}\right|}\right) \\
&\le \frac{l_{N*}^2}{\left|X_i^{(n)}-X_{i-1}^{(n)}\right|\left|X'_i-X'_{i-1}\right|}\frac{2\|Z-Z'\|_{\mathbb{R}^{4N}}}{\left|X_i-X_{i-1}\right|}
\end{aligned}
$$

$$\leq \quad \frac{2\left(4d_0+N\kappa l_{N*}\right)}{\kappa\delta^2}\left\|Z-Z'\right\|_{\mathbb{R}^{4N}}.$$

Finally, in the same way for $E_{5,1}$ and $E_{5,2}$, by using (4.12) and $Z, Z' \in \widehat{W}(\delta, M', M)$, we have

$$\begin{aligned}
E_{5,3} \quad = \quad & \left| \frac{\left(X'_i - X'_{i-1} + X_i^{(n)} - X_{i-1}^{(n)}\right)}{\left(1+\varepsilon_{i-1}^{(n)}\right)\left(1+\varepsilon'_{i-1}\right)} \right. \\
& \left. \times \left\{ \frac{\left|X'_i - X'_{i-1}\right| - \left|X_i - X_{i-1}\right|}{\left(\left|X_i - X_{i-1}\right| + \left|X_i^{(n)} - X_{i-1}^{(n)}\right|\right)\left(\left|X'_i - X'_{i-1}\right| + \left|X_i^{(n)} - X_{i-1}^{(n)}\right|\right)} \right\} \right| \\
\leq \quad & l_{N*}^2 \left| \frac{1}{l_{i-1}^{(n)} l'_{i-1}} \right| \frac{\left|X_i - X'_i\right| + \left|X_{i-1} - X'_{i-1}\right|}{\left|X_i - X_{i-1}\right| + \left|X_i^{(n)} - X_{i-1}^{(n)}\right|} \\
\leq \quad & \frac{l_{N*}^2}{\left|X_i^{(n)} - X_{i-1}^{(n)}\right| \left|X'_i - X'_{i-1}\right|} \frac{2\|Z-Z'\|_{\mathbb{R}^{4N}}}{\left|X_i - X_{i-1}\right|} \\
\leq \quad & \frac{2\left(4d_0+N\kappa l_{N*}\right)}{\kappa\delta^2}\|Z-Z'\|_{\mathbb{R}^{4N}}.
\end{aligned}$$

Accordingly, we obtain

$$E_5 \quad \leq \quad \frac{3\Delta t}{2m\delta^2}\left(4d_0+N\kappa l_{N*}\right)\|Z-Z'\|_{\mathbb{R}^{4N}}.$$

In the same way for E_5, we obtain

$$E_6 \quad \leq \quad \frac{3\Delta t}{2m\delta^2}\left(4d_0+N\kappa l_{N*}\right)\|Z-Z'\|_{\mathbb{R}^{4N}}.$$

Therefore, we obtain

$$\begin{aligned}
& \left|\hat{g}_{Ki}\left(X, X^{(n)}, V^{(n)}\right) - \hat{g}_{Ki}\left(X', X^{(n)}, V^{(n)}\right)\right| \\
\leq \quad & \frac{\kappa\Delta t}{m}\left\{\frac{5\kappa}{l_{N*}} + \frac{3}{\kappa}\left(\frac{2}{l_{N*}^2}+\frac{1}{\delta^2}\right)\left(4d_0+N\kappa l_{N*}\right)\right\}\|Z-Z'\|_{\mathbb{R}^{4N}} \qquad \text{for } i = 1, 2, \cdots, N.
\end{aligned}$$

By using the above inequality and (4.11), we obtain

$$\begin{aligned}
& \left|\|\widehat{F}_K(Z) - \widehat{F}_K(Z')\right\|_{\mathbb{R}^{4N}} \\
\leq \quad & \sum_{i=1}^{N}\frac{\Delta t}{2}\left\|Z-Z'\right\|_{\mathbb{R}^{4N}} + \sum_{i=1}^{N}\frac{\kappa\Delta t}{m}\left\{\frac{5\kappa}{l_{N*}}+\frac{3}{\kappa}\left(\frac{2}{l_{N*}^2}+\frac{1}{\delta^2}\right)\left(4d_0+N\kappa l_{N*}\right)\right\}\|Z-Z'\|_{\mathbb{R}^{4N}} \\
\leq \quad & \frac{N\Delta t}{2m}\left\{m + 2\kappa\left\{\frac{5\kappa}{l_{N*}}+\frac{3}{\kappa}\left(\frac{2}{l_{N*}^2}+\frac{1}{\delta^2}\right)\left(4d_0+N\kappa l_{N*}\right)\right\}\right\}\|Z-Z'\|_{\mathbb{R}^{4N}} \\
& \hspace{20em} \text{for } Z, Z' \in \widehat{W}(\delta, M', M).
\end{aligned}$$

Hence, by choosing $\Delta t > 0$ such that

$$\frac{N\Delta t}{2m}\left\{m + 2\kappa\left\{\frac{5\kappa}{l_{N*}}+\frac{3}{\kappa}\left(\frac{2}{l_{N*}^2}+\frac{1}{\delta^2}\right)\left(4d_0+N\kappa l_{N*}\right)\right\}\right\} < 1,$$

$\widehat{F}_K : \widehat{W}(\delta, M', M) \to \widehat{W}(\delta, M', M)$ is a contraction. □

Proof of Theorem 2.2. By Lemma 4.3, there exists $K_0 \in \mathbb{Z}_{>0}$ such that for $K \geq K_0$ $\widehat{F}_K$: $\widehat{W}(\delta, M', M) \to \widehat{W}(\delta, M', M)$ is a contraction. From Banach's fixed point theorem, F_K has a unique fixed point $\widehat{Z}$ for $K \geq K_0$. Consequently, Theorem 2.2 has been proved. □

5 Proof of the convergence

In this section, we prove Theorem 2.3. For its proof we need several steps. Let $K \geq K_0$, $\Delta t = \dfrac{T}{K}$ and $\left(X^{(n)}, V^{(n)}\right) \in \mathbb{R}^4$ be a solution of (NS)$\left(\Delta t, X^{(n-1)}, V^{(n-1)}\right)$ for $n = 1, 2, \cdots, K$. Also, we define the approximate solution $X^{(K)}\colon [0,T] \to \mathbb{R}^{2N}$ and the approximation of the derivative $V^{(K)}\colon [0,T] \to \mathbb{R}^{2N}$ of P in the following way

$$X_i^{(K)}(t) = \frac{X_i^{(n+1)} - X_i^{(n)}}{\Delta t}(t - n\Delta t) + X_i^{(n)} \quad \text{for } t \in [n\Delta t, (n+1)\Delta t], n = 0, 1, \cdots, K-1, \tag{5.1}$$

$$V_i^{(K)}(t) = \frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t}(t - n\Delta t) + V_i^{(n)} \quad \text{for } t \in [n\Delta t, (n+1)\Delta t], n = 0, 1, \cdots, K-1, \tag{5.2}$$

and $X^{(K)} = \left(X_1^{(K)}, X_2^{(K)}, \cdots, X_N^{(K)}\right),\ V^{(K)} = \left(V_1^{(K)}, V_2^{(K)}, \cdots, V_N^{(K)}\right)$.

Now, we see that $X_i^{(K)}, V_i^{(K)} \in W^{1,2}(0,T)^2$ for $i = 1, 2, \cdots, N$ and

$$\frac{dX_i^{(K)}}{dt} = \sum_{n=0}^{K-1} \frac{X_i^{(n+1)} - X_i^{(n)}}{\Delta t} \chi_{[n\Delta t, (n+1)\Delta t)}, \tag{5.3}$$

$$\frac{dV_i^{(K)}}{dt} = \sum_{n=0}^{K-1} \frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t} \chi_{[n\Delta t, (n+1)\Delta t)} \quad \text{for } i = 1, 2, \cdots, N, \tag{5.4}$$

where χ_I is the characteristic function for the interval $I \subset [0,T]$.

The first lemma is concerned with the uniform estimates for the derivative of the approximation solution.

Lemma 5.1. *There exists $M_1 > 0$ such that*

$$\left| \frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t} \right| \leq M_1, \quad \left| V_i^{(n)} \right| \leq M_1,$$

for any $K \in \mathbb{Z}_{>0}$ with $K \geq K_0$, $i = 1, 2, \cdots, N$ and $n = 0, 1, \cdots, N-1$.

Proof. Let $K \in \mathbb{Z}_{>0}$ with $K \geq K_0$, $n = 0, 1, \cdots, K-1$ and $i = 1, 2, \cdots, N$. By using (1.5), we have

$$\begin{aligned} \left| \frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t} \right| \leq\ & \frac{\kappa}{4m} \left\{ \left| \varepsilon_{i-1}^{(n+1)} \right| + \left| \varepsilon_{i-1}^{(n)} \right| + 1 + \frac{1}{\left| 1 + \varepsilon_{i-1}^{(n+1)} \right| \left| 1 + \varepsilon_{i-1}^{(n)} \right|} \right\} \\ & + \frac{\kappa}{4m} \left\{ \left| \varepsilon_i^{(n+1)} \right| + \left| \varepsilon_i^{(n)} \right| + 1 + \frac{1}{\left| 1 + \varepsilon_i^{(n+1)} \right| \left| 1 + \varepsilon_i^{(n)} \right|} \right\}. \end{aligned}$$

By using (4.5) and (4.6) in Lemma 4.2, we have

$$\left| \frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t} \right| \leq \frac{\kappa}{2m} \left\{ \frac{2(4d_0 + N\kappa l_{N*})}{\kappa l_{N*}} + 3 + \frac{(4d_0 + N\kappa l_{N*})^2}{\kappa^2 l_{N*}^2} \right\}.$$

We put $M_1 > 0$ by

$$M_1 = \max\left\{\frac{\kappa}{2m}\left\{\frac{2(4d_0+N\kappa l_{N*})}{\kappa l_{N*}} + 3 + \frac{(4d_0+N\kappa l_{N*})^2}{\kappa^2 l_{N*}^2}\right\}, \sqrt{\frac{1}{2m}(4d_0+N\kappa l_{N*})}\right\},$$

and then Lemma 4.2 implies

$$\left|\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t}\right| \leq M_1, \quad \left|V_i^{(n)}\right| \leq M_1 \quad \text{for } K \geq K_0, n = 0, 1, \cdots, K-1 \text{ and } i = 1, 2, \cdots, N.$$

This guarantees that Lemma 5.1 holds. □

Now, we put

$$\xi_i^K(t) \;\; = \;\; X_i^K(t) - X_i^K(0) - \int_0^t V_i(\tau)d\tau, \tag{5.5}$$

for $t \in [n\Delta t, (n+1)\Delta t], n = 0, 1, \cdots, K-1, K \geq K_0$ and $i = 1, 2, \cdots, N$. The second lemma shows the convergence of ξ_i^K as $K \to \infty$ for $i = 1, 2, \cdots, N$.

Lemma 5.2. *The following uniform estimate holds*

$$\left|\xi_i^K(t)\right| \leq 2M_1(\Delta t)^2 \quad \textit{for } K \geq K_0 \textit{ and } i = 1, 2, \cdots, N,$$

where M_1 is the same constant as in Lemma 5.1, *namely, $\xi_i^K \to 0$ in $C([0,T])^2$ as $K \to \infty$ for $i = 1, 2, \cdots, N$.*

Proof. Put $X_i^K = \left(X_{i,1}^K, X_{i,2}^K\right)$, $V_i^K = \left(V_{i,1}^K, V_{i,2}^K\right)$ and $\xi_i^K = \left(\xi_{i,1}^K, \xi_{i,2}^K\right)$ for $i = 1, 2, \cdots, N$. By using (5.1) and (1.6), we have

$$\frac{dX_{i,j}^K}{dt} = \sum_{n=0}^{K-1} \frac{V_{i,j}^{(n+1)} + V_{i,j}^{(n)}}{2} \chi_{[n\Delta t,(n+1)\Delta t)} \quad \text{for } t \in [n\Delta t, (n+1)\Delta t), n = 0, 1, \cdots, K-1.$$

By integrating both sides of this equation, we have

$$\int_0^t \frac{dX_{i,j}^K}{dt}(\tau)d\tau = \sum_{n=0}^{K-1} \int_0^t \frac{V_{i,j}^{(n+1)}(\tau) + V_{i,j}^{(n)}(\tau)}{2} \chi_{[n\Delta t,(n+1)\Delta t)}(\tau)d\tau,$$
$$\text{for } K \geq K_0, i = 1, 2, \cdots, N, j = 1, 2 \text{ and } t \in [0, T].$$

For the left hand side of the equation, we see that

$$\int_0^t \frac{dX_{i,j}^K}{dt}(\tau)d\tau \;\; = \;\; X_{i,j}^K(t) - X_{i,j}^K(0) \quad \text{for } K \geq K_0, i = 1, 2, \cdots, N, j = 1, 2 \text{ and } t \in [0, T],$$

and for the right hand side, we have

$$\begin{aligned}
&\int_0^t \sum_{n=0}^{K-1} \frac{V_{i,j}^{(n+1)} + V_{i,j}^{(n)}}{2} \chi_{[n\Delta t,(n+1)\Delta t)} d\tau \\
&= \int_0^{n\Delta t} \sum_{m=0}^{n-1} \frac{V_{i,j}^{(m+1)} + V_{i,j}^{(m)}}{2} \chi_{[m\Delta t,(m+1)\Delta t)} d\tau + \frac{V_{i,j}^{(n+1)} + V_{i,j}^{(n)}}{2} (t - n\Delta t) \\
&= \sum_{m=0}^{n-1} \frac{V_{i,j}^{(m+1)} + V_{i,j}^{(m)}}{2} \Delta t + \frac{V_{i,j}^{(n+1)} + V_{i,j}^{(n)}}{2} (t - n\Delta t),
\end{aligned}$$

for $K \geq K_0, i = 1, 2, \cdots, N, j = 1, 2, t \in [n\Delta t, (n+1)\Delta t]$ and $n = 0, 1, \cdots, K-1$.

Therefore, we have

$$X_{i,j}^K(t) - X_{i,j}^K(0) = \sum_{m=0}^{n-1} \frac{V_{i,j}^{(m+1)} + V_{i,j}^{(m)}}{2}\Delta t + \frac{V_{i,j}^{(n+1)} + V_{i,j}^{(n)}}{2}(t - n\Delta t), \tag{5.6}$$

for $K \geq K_0, i = 1, 2, \cdots, N, j = 1, 2, t \in [n\Delta t, (n+1)\Delta t]$ and $n = 0, 1, \cdots, K-1$.

By using(5.2), we have

$$V_{i,j}^K = \sum_{n=0}^{K-1} \left\{ \frac{V_{i,j}^{(n+1)} - V_{i,j}^{(n)}}{\Delta t}(t - n\Delta t) + V_{i,j}^{(n)} \right\} \chi_{[n\Delta t,(n+1)\Delta t)},$$

for $K \geq K_0, i = 1, 2, \cdots, N, j = 1, 2$ and $t \in [0, T]$.

By integrating both sides of the equation, we have

$$\begin{aligned}
&\int_0^t V_{i,j}^K(\tau)d\tau \\
&= \int_0^t \sum_{n=0}^{K-1} \left\{ \frac{V_{i,j}^{(n+1)} - V_{i,j}^{(n)}}{\Delta t}(\tau - n\Delta t) + V_{i,j}^{(n)} \right\} \chi_{[n\Delta t,(n+1)\Delta t)} d\tau \\
&= \int_0^{n\Delta t} \sum_{m=0}^{n-1} \left\{ \frac{V_{i,j}^{(m+1)} - V_{i,j}^{(m)}}{\Delta t}(\tau - m\Delta t) + V_{i,j}^{(m)} \right\} \chi_{[m\Delta t,(m+1)\Delta t)} d\tau \\
&\qquad + \int_{n\Delta t}^t \left\{ \frac{V_{i,j}^{(n+1)} - V_{i,j}^{(n)}}{\Delta t}(\tau - n\Delta t) + V_{i,j}^{(n)} \right\} \chi_{[n\Delta t,(n+1)\Delta t)} d\tau \\
&= \sum_{m=0}^{n-1} \left\{ \frac{V_{i,j}^{(m+1)} - V_{i,j}^{(m)}}{\Delta t} \int_{m\Delta t}^{(m+1)\Delta t} (\tau - m\Delta t)d\tau + V_{i,j}^{(m)}\Delta t \right\} \\
&\qquad + \frac{V_{i,j}^{(n+1)} - V_{i,j}^{(n)}}{\Delta t} \int_{n\Delta t}^t (\tau - n\Delta t)d\tau + V_{i,j}^{(n)}\Delta t \\
&= \sum_{m=0}^{n-1} \frac{V_{i,j}^{(m+1)} + V_{i,j}^{(m)}}{2}\Delta t + \left\{ \frac{V_{i,j}^{(n+1)} - V_{i,j}^{(n)}}{\Delta t}\frac{1}{2}(t - n\Delta t) + V_{i,j}^{(n)} \right\}(t - n\Delta t),
\end{aligned} \tag{5.7}$$

for $t \in [n\Delta t, (n+1)\Delta t), n = 0, 1, \cdots, K-1$.

By using (5.5), (5.6) and (5.7), we have

$$\begin{aligned}
\left|\xi_{i,j}^K(t)\right| &= \left| X_{i,j}^K(t) - X_{i,j}^K(0) - \int_0^t V_{i,j}^K d\tau \right| \\
&= (t - n\Delta t)\left| \frac{V_{i,j}^{(n+1)} + V_{i,j}^{(n)}}{2} - \left\{ \left(\frac{V_{i,j}^{(n+1)} - V_i^{(n)}}{\Delta t}\frac{1}{2}(t - n\Delta t) \right) + V_{i,j}^{(n)} \right\} \right| \\
&= (t - n\Delta t)\left| \frac{V_{i,j}^{(n+1)} - V_{i,j}^{(n)}}{2} - \frac{V_{i,j}^{(n+1)} - V_i^{(n)}}{\Delta t}\frac{1}{2}(t - n\Delta t) \right| \\
&\leq (t - n\Delta t)\left(\frac{\Delta t}{2}\left| \frac{V_{i,j}^{(n+1)} - V_{i,j}^{(n)}}{\Delta t} \right| + \frac{1}{2}(t - n\Delta t)\left| \frac{V_{i,j}^{(n+1)} - V_{i,j}^{(n)}}{\Delta t} \right| \right),
\end{aligned}$$

for $K \geq K_0, i = 1, 2, \cdots, N, j = 1, 2, t \in [n\Delta t, (n+1)\Delta t]$ and $n = 0, 1, \cdots, K-1$.

Moreover, since $t \in [n\Delta t, (n+1)\Delta t]$, by Lemma 5.1 we have

$$\left|\xi_{i,j}^K(t)\right| \leq M_1 (\Delta t)^2 = \frac{M_1 T^2}{K^2} \quad \text{for } K \geq K_0, i = 1, 2, \cdots, N, j = 1, 2 \text{ and } t \in [0, T].$$

Therefore, we obtain

$$\xi_{i,j}^K \to 0 \text{ in } C([0,T]) \text{ as } K \to \infty \quad \text{for } i = 1, 2, \cdots, N \text{ and } j = 1, 2.$$

We see that $\xi_i^K \in C([0,T])^2$ for $K \geq K_0, i = 1, 2, \cdots, N$ and $\left|\xi_i^K\right| \leq 2M_1 (\Delta t)^2$ on $[0,T]$ for $K \geq K_0$ and $i = 1, 2, \cdots, N$. This guarantees that Lemma 5.2 holds. $\square$

Proof of Theorem 2.3. From Theorem 2.1, we have the following integral equation

$$X_i(t) = \int_0^t V_i(\tau) d\tau + X_{0i} \quad \text{for } t \in [0,T] \text{ and } i = 1, 2, \cdots, N.$$

By using this equation and (5.5), we have

$$X_i(t) - X_i^K(t) = \int_0^t \left(V_i(\tau) - V_i^K(\tau)\right) d\tau - \xi_i^K(t), \tag{5.8}$$

for $t \in [n\Delta t, (n+1)\Delta t], n = 0, 1, \cdots, K-1, K \geq K_0$ and $i = 1, 2, \cdots, N$.

Now, $V_i(0) = V_i^K(0)$ holds for $K \geq K_0$ and $i = 1, 2, \cdots, N$, so we have

$$\begin{aligned} V_i(t) - V_i^K(t) &= V_i(t) - V_i(0) + V_i^K(0) - V_i^K(t) \\ &= \int_0^{n\Delta t} \left(\frac{dV_i}{d\tau}(\tau) - \frac{dV_i^K}{d\tau}(\tau)\right) d\tau + \int_{n\Delta t}^t \left(\frac{dV_i}{d\tau}(\tau) - \frac{dV_i^K}{d\tau}(\tau)\right) d\tau \\ &= \sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left(\frac{dV_i}{d\tau}(\tau) - \frac{dV_i^K}{d\tau}(\tau)\right) d\tau + \int_{n\Delta t}^t \left(\frac{dV_i}{d\tau}(\tau) - \frac{dV_i^K}{d\tau}(\tau)\right) d\tau, \end{aligned} \tag{5.9}$$

for $i = 1, 2, \cdots, N, K \geq K_0, n = 0, 1, \cdots, K-1$ and $t \in [n\Delta t, (n+1)\Delta t]$.

By using (1.2) and (1.5), then we have

$$\begin{aligned} &\frac{dV_i}{d\tau}(\tau) - \frac{dV_i^K}{d\tau}(\tau) \\ &= \frac{1}{m}\left\{f(\varepsilon_i)\frac{X_{i+1}(\tau) - X_i(\tau)}{l_i(\tau)} - f(\varepsilon_{i-1})\frac{X_i(\tau) - X_{i-1}(\tau)}{l_{i-1}(\tau)}\right\} \\ &\quad -\frac{\kappa}{4m}\left\{\left\{\varepsilon_i^{(p)} + \varepsilon_i^{(p-1)} + 1 - \frac{1}{\left(1+\varepsilon_i^{(p)}\right)\left(1+\varepsilon_i^{(p-1)}\right)}\right\}\right. \\ &\qquad \left.\times \frac{X_{i+1}^{(p)}(\tau) - X_i^{(p)}(\tau) + X_{i+1}^{(p-1)}(\tau) - X_i^{(p-1)}(\tau)}{\left|X_{i+1}^{(p)}(\tau) - X_i^{(p)}(\tau)\right| + \left|X_{i+1}^{(p-1)}(\tau) - X_i^{(p-1)}(\tau)\right|}\right\} \\ &\quad +\frac{\kappa}{4m}\left\{\left\{\varepsilon_{i-1}^{(p)} + \varepsilon_{i-1}^{(p-1)} + 1 - \frac{1}{\left(1+\varepsilon_{i-1}^{(p)}\right)\left(1+\varepsilon_{i-1}^{(p-1)}\right)}\right\}\right. \end{aligned}$$

$$\times \frac{X_i^{(p)}(\tau) - X_{i-1}^{(p)}(\tau) + X_i^{(p-1)}(\tau) - X_{i-1}^{(p-1)}(\tau)}{\left|X_i^{(p)}(\tau) - X_{i-1}^{(p)}(\tau)\right| + \left|X_i^{(p-1)}(\tau) - X_{i-1}^{(p-1)}(\tau)\right|} \Bigg\}$$

for $i = 1, 2, \cdots, N, K \geq K_0, \tau \in [(p-1)\Delta t, p\Delta t]$ and $p = 1, 2, \cdots, n$.

Accordingly, we have

$$\left|\frac{dV_i}{d\tau}(\tau) - \frac{dV_i^K}{d\tau}(\tau)\right| \leq F_1 + F_2,$$

for $i = 1, 2, \cdots, N, K \geq K_0, \tau \in [(p-1)\Delta t, p\Delta t]$ and $p = 1, 2, \cdots, n$, where

$$\begin{aligned} F_1 \;&:=\; \Bigg|\frac{1}{m} f(\varepsilon_i) \frac{X_{i+1}(\tau) - X_i(\tau)}{l_i(\tau)} - \frac{\kappa}{4m}\left\{\varepsilon_i^{(p)} + \varepsilon_i^{(p-1)} + 1 - \frac{1}{\left(1+\varepsilon_i^{(p)}\right)\left(1+\varepsilon_i^{(p-1)}\right)}\right\} \\ &\qquad \times \frac{X_{i+1}^{(p)}(\tau) - X_i^{(p)}(\tau) + X_{i+1}^{(p-1)}(\tau) - X_i^{(p-1)}(\tau)}{\left|X_{i+1}^{(p)}(\tau) - X_i^{(p)}(\tau)\right| + \left|X_{i+1}^{(p-1)}(\tau) - X_i^{(p-1)}(\tau)\right|}\Bigg|, \\ F_2 \;&:=\; \Bigg|\frac{1}{m} f(\varepsilon_{i-1}) \frac{X_i(\tau) - X_{i-1}(\tau)}{l_{i-1}(\tau)} - \frac{\kappa}{4m}\left\{\varepsilon_{i-1}^{(p)} + \varepsilon_{i-1}^{(p-1)} + 1 - \frac{1}{\left(1+\varepsilon_{i-1}^{(p)}\right)\left(1+\varepsilon_{i-1}^{(p-1)}\right)}\right\} \\ &\qquad \times \frac{X_i^{(p)}(\tau) - X_{i-1}^{(p)}(\tau) + X_i^{(p-1)}(\tau) - X_{i-1}^{(p-1)}(\tau)}{\left|X_i^{(p)}(\tau) - X_{i-1}^{(p)}(\tau)\right| + \left|X_i^{(p-1)}(\tau) - X_{i-1}^{(p-1)}(\tau)\right|}\Bigg|. \end{aligned}$$

For F_1 we have

$$\begin{aligned} F_1 \;&\leq\; \frac{\kappa}{4m}\left|\frac{X_{i+1}(\tau) - X_i(\tau)}{l_i(\tau)}\left\{\left(2\varepsilon_i + 1 - \frac{1}{(\varepsilon_i+1)^2}\right)\right.\right. \\ &\qquad \left.\left. - \left(\varepsilon_i^{(p)} + \varepsilon_i^{(p-1)} + 1 - \frac{1}{\left(1+\varepsilon_i^{(p)}\right)\left(1+\varepsilon_i^{(p-1)}\right)}\right)\right\}\right| \\ &\quad + \frac{\kappa}{4m}\left|\left\{\frac{X_{i+1}(\tau) - X_i(\tau)}{l_i(\tau)} - \frac{X_{i+1}^{(p)}(\tau) - X_i^{(p)}(\tau) + X_{i+1}^{(p-1)}(\tau) - X_i^{(p-1)}(\tau)}{\left|X_{i+1}^{(p)}(\tau) - X_i^{(p)}(\tau)\right| + \left|X_{i+1}^{(p-1)}(\tau) - X_i^{(p-1)}(\tau)\right|}\right\}\right. \\ &\qquad \left. \times \left(\varepsilon_i^{(p)} + \varepsilon_i^{(p-1)} + 1 - \frac{1}{\left(1+\varepsilon_i^{(p)}\right)\left(1+\varepsilon_i^{(p-1)}\right)}\right)\right| \\ &=: \; F_{1,1} + F_{1,2}. \end{aligned}$$

For $F_{1,1}$ we have

$$\begin{aligned} &F_{1,1} \\ &\leq \frac{\kappa}{4m}\left\{\left|\varepsilon_i - \varepsilon_i^{(p)}\right| + \left|\varepsilon_i - \varepsilon_i^{(p-1)}\right| + \left|\frac{\left(\varepsilon_i+1\right)^2 - \left(1+\varepsilon_i^{(p)}\right)\left(1+\varepsilon_i^{(p-1)}\right)}{\left(\varepsilon_i+1\right)^2\left(1+\varepsilon_i^{(p)}\right)\left(1+\varepsilon_i^{(p-1)}\right)}\right|\right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\kappa}{4m}\left\{\left|\frac{l_i(\tau)-l_i^{(p)}(\tau)}{l_{N*}}\right| + \left|\frac{l_i(\tau)-l_i^{(p-1)}(\tau)}{l_{N*}}\right|\right.\\
&\qquad\left. + \left|\frac{l_{N*}^2}{l_i^2(\tau)l_i^{(p)}(\tau)l_i^{(p-1)}(\tau)}\left\{l_i^{(p)}(\tau)\Big(l_i(\tau)-l_i^{(p-1)}(\tau)\Big) + l_i(\tau)\Big(l_i(\tau)-l_i^{(p)}(\tau)\Big)\right\}\right|\right\}\\
&\le \frac{\kappa}{4m}\left\{\left(\frac{1}{l_{N*}} + \frac{l_{N*}}{\Big|l_i(\tau)\Big|\Big|l_i^{(p)}(\tau)\Big|\Big|l_i^{(p-1)}(\tau)\Big|}\right)\Big|X_{i+1}(\tau)-X_{i+1}^{(p)}(\tau)\Big|\right.\\
&\qquad\left. + \left(\frac{1}{l_{N*}} + \frac{l_{N*}}{\Big|l_i(\tau)\Big|\Big|l_i^{(p)}(\tau)\Big|\Big|l_i^{(p-1)}(\tau)\Big|}\right)\Big|X_i(\tau)-X_i^{(p)}(\tau)\Big|\right\}\\
&\quad + \frac{\kappa}{4m}\left\{\left(\frac{1}{l_{N*}} + \frac{l_{N*}}{\Big|l_i(\tau)\Big|^2\Big|l_i^{(p-1)}(\tau)\Big|}\right)\Big|X_{i+1}(\tau)-X_{i+1}^{(p)}(\tau)\Big|\right.\\
&\qquad\left. + \left(\frac{1}{l_{N*}} + \frac{l_{N*}}{\Big|l_i(\tau)\Big|^2\Big|l_i^{(p-1)}(\tau)\Big|}\right)\Big|X_i(\tau)-X_i^{(p)}(\tau)\Big|\right\}.
\end{aligned}
$$

By using (4.5) and Lemmas 3.1 and 3.2, we have

$$
\begin{aligned}
F_{1,1} \le \frac{\kappa}{4m}\left\{\frac{1}{l_{N*}} + \frac{(4d_0+N\kappa l_{N*})^3}{\kappa^3 l_{N*}^6}\right\}&\left\{\Big|X_{i+1}(\tau)-X_{i+1}^{(p)}(\tau)\Big| + \Big|X_i(\tau)-X_i^{(p)}(\tau)\Big|\right.\\
&\left. + \Big|X_{i+1}(\tau)-X_{i+1}^{(p-1)}(\tau)\Big| + \Big|X_i(\tau)-X_i^{(p-1)}(\tau)\Big|\right\},
\end{aligned}
$$

where $d_0 = G\left(X_0, V_0\right)$. Next, for $F_{1,2}$ we have

$$
\begin{aligned}
F_{1,2} &= \frac{\kappa}{4m}\left|\left\{\frac{X_{i+1}(\tau)-X_i(\tau)}{l_i(\tau)} - \frac{X_{i+1}^{(p)}(\tau)-X_i^{(p)}(\tau)+X_{i+1}^{(p-1)}(\tau)-X_i^{(p-1)}(\tau)}{\Big|X_{i+1}^{(p)}(\tau)-X_i^{(p)}(\tau)\Big| + \Big|X_{i+1}^{(p-1)}(\tau)-X_i^{(p-1)}(\tau)\Big|}\right\}\right.\\
&\qquad\left.\times\left(\varepsilon_i^{(p)}+\varepsilon_i^{(p-1)}+1-\frac{1}{\Big(1+\varepsilon_i^{(p)}\Big)\Big(1+\varepsilon_i^{(p-1)}\Big)}\right)\right|\\
&\le \frac{\kappa}{4m}\left\{\Big|\varepsilon_i^{(p)}\Big| + \Big|\varepsilon_i^{(p-1)}\Big| + 1 + \left|\frac{1}{\Big(1+\varepsilon_i^{(p)}\Big)\Big(1+\varepsilon_i^{(p-1)}\Big)}\right|\right\} \times \frac{\Big|F_{1,2,1}\Big| + \Big|F_{1,2,2}\Big|}{Q},
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{1}{Q} &= \frac{1}{2\Big|X_{i+1}(\tau)-X_i(\tau)\Big|\left(\Big|X_{i+1}^{(p)}(\tau)-X_i^{(p)}(\tau)\Big| + \Big|X_{i+1}^{(p-1)}(\tau)-X_i^{(p-1)}(\tau)\Big|\right)},\\
F_{1,2,1} &= 2\Big(X_{i+1}(\tau)-X_i(\tau)\Big)\\
&\qquad \times\left(\Big|X_{i+1}^{(p)}(\tau)-X_i^{(p)}(\tau)\Big| + \Big|X_{i+1}^{(p-1)}(\tau)-X_i^{(p-1)}(\tau)\Big| - 2\Big|X_{i+1}(\tau)-X_i(\tau)\Big|\right),\\
F_{1,2,2} &= 2\Big|X_{i+1}(\tau)-X_i(\tau)\Big|
\end{aligned}
$$

$$\times\left(2X_{i+1}(\tau)-2X_i(\tau)-X_{i+1}^{(p)}(\tau)+X_i^{(p)}-X_{i+1}^{(p-1)}(\tau)+X_i^{(p-1)}(\tau)\right).$$

Easily, we get

$$\begin{aligned}\left|F_{1,2,1}\right| \;\le\; & 2\left|X_{i+1}(\tau)-X_i(\tau)\right|\left\{\left|X_{i+1}(\tau)-X_{i+1}^{(p)}(\tau)\right|+\left|X_i(\tau)-X_i^{(p)}(\tau)\right|\right.\\ & \left.+\left|X_{i+1}(\tau)-X_{i+1}^{(p-1)}(\tau)\right|+\left|X_i(\tau)-X_i^{(p-1)}(\tau)\right|\right\},\end{aligned}$$

$$\begin{aligned}\left|F_{1,2,2}\right| \;\le\; & 2\left|X_{i+1}(\tau)-X_i(\tau)\right|\left(\left|X_{i+1}(\tau)-X_{i+1}^{(p)}\right|+\left|X_i(\tau)-X_i^{(p)}(\tau)\right|\right.\\ & \left.+\left|X_{i+1}(\tau)-X_{i+1}^{(p-1)}(\tau)\right|+\left|X_i(\tau)-X_i^{(p-1)}(\tau)\right|\right).\end{aligned}$$

By using (4.12), (4.6) in Lemma 4.2 and Lemmas 3.1 and 3.2, we have

$$\begin{aligned}& F_{1,2}\\ &\le \frac{\kappa}{4m}\left\{\frac{\left|l_i^{(p)}(\tau)\right|+l_{N*}}{l_{N*}}+\frac{\left|l_i^{(p-1)}(\tau)\right|+l_{N*}}{l_{N*}}+1+\frac{l_{N*}^2}{\left|l_i^{(p)}(\tau)\right|\left|l_i^{(p-1)}(\tau)\right|}\right\}\\ &\quad\times\frac{(4d_0+N\kappa l_{N*})^2}{\kappa^2 l_{N*}^4}\left(\left|X_{i+1}(\tau)-X_{i+1}^{(p)}\right|+\left|X_i(\tau)-X_i^{(p)}(\tau)\right|\right.\\ &\qquad\left.+\left|X_{i+1}(\tau)-X_{i+1}^{(p-1)}(\tau)\right|+\left|X_i(\tau)-X_i^{(p-1)}(\tau)\right|\right)\\ &\le \frac{4d_0+N\kappa l_{N*}}{4m\kappa^2 l_{N*}^5}\left\{2\kappa l_{N*}^2\left(4d_0+N\kappa l_{N*}\right)+3\kappa^2 l_{N*}^3+(4d_0+N\kappa l_{N*})^2\right\}\\ &\quad\times\left(\left|X_{i+1}(\tau)-X_{i+1}^{(p)}(\tau)\right|+\left|X_i(\tau)-X_i^{(p)}(\tau)\right|+\left|X_{i+1}(\tau)-X_{i+1}^{(p-1)}(\tau)\right|+\left|X_i(\tau)-X_i^{(p-1)}(\tau)\right|\right).\end{aligned}$$

Therefore, we have

$$\begin{aligned}F_1 \;\le\; & \beta\left(\left|X_{i+1}(\tau)-X_{i+1}^{(p)}(\tau)\right|+\left|X_i(\tau)-X_i^{(p)}(\tau)\right|\right.\\ & \left.+\left|X_{i+1}(\tau)-X_{i+1}^{(p-1)}(\tau)\right|+\left|X_i(\tau)-X_i^{(p-1)}(\tau)\right|\right)\qquad\text{for } i=1,2,\cdots,N.\end{aligned}$$

where

$$\begin{aligned}\beta=\frac{1}{4m\kappa^2 l_{N*}^6}&\left\{\left(\kappa^3 l_{N*}^5+(4d_0+N\kappa l_{N*})^3\right)\right.\\ &\left.+l_{N*}\left(4d_0+N\kappa l_{N*}\right)\left\{2\kappa l_{N*}^2\left(4d_0+N\kappa l_{N*}\right)+3\kappa^2 l_{N*}^3+(4d_0+N\kappa l_{N*})^2\right\}\right\}.\end{aligned}$$

In the same way for F_1, we have

$$\begin{aligned}F_2 \;\le\; & \beta\left(\left|X_i(\tau)-X_i^{(p)}(\tau)\right|+\left|X_{i-1}(\tau)-X_{i-1}^{(p)}(\tau)\right|\right.\\ & \left.+\left|X_i(\tau)-X_i^{(p-1)}(\tau)\right|+\left|X_{i-1}(\tau)-X_{i-1}^{(p-1)}(\tau)\right|\right)\qquad\text{for } i=1,2,\cdots,N.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\left|\frac{dV_i}{d\tau}(\tau)-\frac{dV_i^K}{d\tau}(\tau)\right| \le 2\beta&\left(\left|X_{i+1}(\tau)-X_{i+1}^{(p)}(\tau)\right|+\left|X_{i+1}(\tau)-X_{i+1}^{(p-1)}(\tau)\right|\right.\\ &+\left|X_i(\tau)-X_i^{(p)}(\tau)\right|+\left|X_i(\tau)-X_i^{(p-1)}(\tau)\right|\\ &\left.+\left|X_{i-1}(\tau)-X_{i-1}^{(p)}(\tau)\right|+\left|X_{i-1}(\tau)-X_{i-1}^{(p-1)}(\tau)\right|\right),\end{aligned}\tag{5.10}$$

for $K \geq K_0, i = 1, 2, \cdots, N, \tau \in [(p-1)\Delta t, p\Delta t]$ and $p = 1, 2, \cdots, K$.

By using (5.8), (5.9) and (5.10), we have

$$
\begin{aligned}
&\left|X_i(t) - X_i^K(t)\right| \\
\leq\ & \int_0^t \left(\sum_{p=1}^n \int_{(p-1)}^{p\Delta t} \left| \frac{dV_i}{d\tau}(\tau) - \frac{dV_i^K}{d\tau}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| \frac{dV_i}{d\tau}(\tau) - \frac{dV_i^K}{d\tau}(\tau) \right| d\tau \right) ds + \left|\xi_i^K(t)\right| \\
\leq\ & 2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{i+1}(\tau) - X_{i+1}^{(p)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{i+1}(\tau) - X_{i+1}^{(n+1)}(\tau) \right| d\tau \right) ds \\
& +2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{i+1}(\tau) - X_{i+1}^{(p-1)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{i+1}(\tau) - X_{i+1}^{(n)}(\tau) \right| d\tau \right) ds \\
& +2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{i}(\tau) - X_{i}^{(p)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{i}(\tau) - X_{i}^{(n+1)}(\tau) \right| d\tau \right) ds \\
& +2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{i}(\tau) - X_{i}^{(p-1)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{i}(\tau) - X_{i}^{(n)}(\tau) \right| d\tau \right) ds \\
& +2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{i-1}(\tau) - X_{i-1}^{(p)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{i-1}(\tau) - X_{i-1}^{(n+1)}(\tau) \right| d\tau \right) ds \\
& +2\beta \int_0^t \left(\sum_{p=1}^n \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{i-1}(\tau) - X_{i-1}^{(p-1)}(\tau) \right| d\tau + \int_{n\Delta t}^t \left| X_{i-1}(\tau) - X_{i-1}^{(n)}(\tau) \right| d\tau \right) ds \\
& +\left|\xi_i^K(t)\right| \\
& =: R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + \left|\xi_i^K(t)\right|, \qquad (5.11)
\end{aligned}
$$

for $K \geq K_0, i = 1, 2, \cdots, N, t \in [n\Delta t, (n+1)\Delta t]$ and $n = 0, 1, \cdots, K-1$.

For R_1 we have

$$
\begin{aligned}
\left| X_{i+1}(\tau) - X_{i+1}^{(p)}(\tau) \right| &\leq \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| + \left| X_{i+1}^K(\tau) - X_{i+1}^K(p\Delta t) \right| \\
&\leq \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| + \int_\tau^{p\Delta t} \left| \frac{dX_{i+1}^K}{du}(u) \right| du,
\end{aligned}
$$

for $K \geq K_0, i = 1, 2, \cdots, N, p = 0, 1, \cdots, K-1$ and $\tau \in [(p-1)\Delta t, p\Delta t]$.

By using (5.3), (1.6) and Lemma 5.1, we have

$$
\left| \frac{dX_{i+1}^K}{du}(u) \right| \leq M_1 \quad \text{for } u \in [\tau, p\Delta t], K \geq K_0 \text{ and } i = 1, 2, \cdots, N. \qquad (5.12)
$$

Therefore, we have

$$
\left| X_{i+1}(\tau) - X_{i+1}^{(p)}(\tau) \right| \leq \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| + M_1 \left| p\Delta t - \tau \right|,
$$

for $K \geq K_0, i = 1, 2, \cdots, N, p = 0, 1, \cdots, K-1$ and $\tau \in [(p-1)\Delta t, p\Delta t]$.

By using these inequalities, we have

$$\begin{aligned} R_1 &\leq 2\beta \int_0^t \sum_{p=1}^{n} \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| d\tau ds + 2\beta M_1 \int_0^t \sum_{p=1}^{n} \int_{(p-1)\Delta t}^{p\Delta t} |p\Delta t - \tau| \, d\tau ds \\ &\quad + 2\beta \int_0^t \int_{n\Delta t}^{t} \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| d\tau ds + 2\beta M_1 \int_0^t \int_{n\Delta t}^{t} ((n+1)\Delta t - \tau) \, d\tau ds \\ &\leq 2\beta \int_0^t \int_0^t \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| d\tau ds + (2\beta M_1 T^2)\Delta t \\ &\quad + 2\beta M_1 T \left((n+1)\Delta t(t - n\Delta t) - \frac{1}{2}(t - n\Delta t)(t + n\Delta t) \right) \\ &\leq 2\beta T \int_0^t \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| d\tau + (2\beta M_1 T^2)\Delta t + 2\beta M_1 T (\Delta t)^2. \end{aligned}$$

In the same way for R_1, we have

$$\begin{aligned} R_3 &\leq 2\beta T \int_0^t \left| X_i(\tau) - X_i^K(\tau) \right| d\tau + (2\beta M_1 T^2)\Delta t + 2\beta M_1 T (\Delta t)^2, \\ R_5 &\leq 2\beta T \int_0^t \left| X_{i-1}(\tau) - X_{i-1}^K(\tau) \right| d\tau + (2\beta M_1 T^2)\Delta t + 2\beta M_1 T (\Delta t)^2. \end{aligned}$$

Next, for R_2 by using (5.12), we have

$$\begin{aligned} \left| X_{i+1}(\tau) - X_{i+1}^{(p-1)}(\tau) \right| &\leq \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| + \left| X_{i+1}^K(\tau) - X_{i+1}^K((p-1)\Delta t) \right| \\ &\leq \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| + \int_{(p-1)\Delta t}^{\tau} \left| \frac{dX_{i+1}^K}{du}(u) \right| du \\ &\leq \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| + M_1 \left| \tau - (p-1)\Delta t \right|, \end{aligned}$$

for $K \geq K_0, i = 1, 2, \cdots, N, p = 0, 1, \cdots, K-1$ and $\tau \in [(p-1)\Delta t, p\Delta t]$).

By using these inequalities, we have

$$\begin{aligned} & R_2 \\ & \leq 2\beta \int_0^t \sum_{p=1}^{n} \int_{(p-1)\Delta t}^{p\Delta t} \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| d\tau ds + 2\beta M_1 \int_0^t \sum_{p=1}^{n} \int_{(p-1)\Delta t}^{p\Delta t} |\tau - (p-1)\Delta t| \, d\tau ds \\ &\quad + 2\beta \int_0^t \int_{n\Delta t}^{t} \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| d\tau ds + 2\beta M_1 \int_0^t \int_{n\Delta t}^{t} (\tau - n\Delta t) \, d\tau ds \\ & \leq 2\beta \int_0^t \int_0^t \left| X_{i+1}(\tau) - X_{i+1}^K(\tau) \right| d\tau ds + \left(2\beta M_1 T^2 \right) \Delta t \\ &\quad + 2\beta M_1 T \left(\frac{1}{2} (t + n\Delta t)(t - n\Delta t) - n\Delta t(t - n\Delta t) \right) \end{aligned}$$

$$\leq 2\beta T \int_0^t \left|X_{i+1}(\tau) - X_{i+1}^K(\tau)\right| d\tau + \left(2\beta M_1 T^2\right) \Delta t + \frac{2\beta M_1 T}{2} (\Delta t)^2 .$$

In the same way for R_2, we have

$$\begin{aligned} R_4 &\leq 2\beta T \int_0^t \left|X_i(\tau) - X_i^K(\tau)\right| d\tau + \left(2\beta M_1 T^2\right) \Delta t + \frac{2\beta M_1 T}{2} (\Delta t)^2 , \\ R_6 &\leq 2\beta T \int_0^t \left|X_{i-1}(\tau) - X_{i-1}^K(\tau)\right| d\tau + \left(2\beta M_1 T^2\right) \Delta t + \frac{2\beta M_1 T}{2} (\Delta t)^2 . \end{aligned}$$

Therefore, from the estimates for R_i for $i = 1, 2, 3, 4, 5, 6$, Lemma 5.2 and (5.11) it follows

$$\begin{aligned} &\left|X_i(t) - X_i^K(t)\right| \\ &\leq 4\beta T \left\{ \int_0^t \left|X_{i+1}(\tau) - X_{i+1}^K(\tau)\right| d\tau + \int_0^t \left|X_i(\tau) - X_i^K(\tau)\right| d\tau + \int_0^t \left|X_{i-1}(\tau) - X_{i-1}^K(\tau)\right| d\tau \right\} \\ &\qquad +12\beta M_1 T^2 \Delta t + 9\beta M_1 T (\Delta t)^2 + 2M_1 (\Delta t)^2 , \qquad (5.13) \end{aligned}$$

for $K \geq K_0, i = 1, 2, \cdots, N$ and $t \in [0, T]$.

By taking the sum of both side of (5.13) from $i = 1$ to N, we obtain

$$\begin{aligned} \sum_{i=1}^N \left|X_i(t) - X_i^K(t)\right| &\leq 12\beta T \int_0^t \sum_{i=1}^N \left|X_i(\tau) - X_i^K(\tau)\right| d\tau \\ &\quad +N \left\{ 12\beta M_1 T^2 \Delta t + 9\beta M_1 T (\Delta t)^2 + 2M_1 (\Delta t)^2 \right\} , \end{aligned}$$

for $K \geq K_0$ and $t \in [0, T]$.

Put $E_K(t) = \sum_{i=1}^N \left|X_i(t) - X_i^K(t)\right|$ for $K \geq K_0$ and $t \in [0, T]$. Easily, we get

$$E_K(t) \leq 12\beta T \int_0^t E_K(\tau) d\tau + N \left\{ 12\beta M_1 T^2 \Delta t + 9\beta M_1 T (\Delta t)^2 + 2M_1 (\Delta t)^2 \right\} , \qquad (5.14)$$

for $K \geq K_0$ and $t \in [0, T]$.

By using Gronwall's inequality, we obtain

$$\begin{aligned} E_K(t) &\leq N \left(12\beta T^2 e^{12\beta T} + 1\right) \left\{ 12\beta M_1 T^2 \Delta t + 9\beta M_1 T (\Delta t)^2 + 2M_1 (\Delta t)^2 \right\} \qquad (5.15) \\ &\to 0 \quad \text{as } K \to \infty \quad \text{for } t \in [0, T]. \end{aligned}$$

We see that $E_K \to 0$ in $C([0, T])$ as $K \to \infty$, namely,

$$X_i^K \to X_i \text{ in } C([0, T])^2 \text{ as } K \to \infty \quad \text{for } i = 1, 2, \cdots, N.$$

Moreover, by using (5.15), we see that

$$\left|X_i(t) - X_i^K(t)\right| \leq N \left(12\beta T^2 e^{12\beta T} + 1\right) \left\{ 12\beta M_1 T^2 + 9\beta M_1 T (\Delta t) + 2M_1 (\Delta t) \right\} \Delta t,$$

for $K \geq K_0, i = 1, 2, \cdots, N$ and $t \in [0, T]$.

Thus, Theorem 2.3 has been proved. □

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