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APPROXIMATE METHODS FOR SINGULAR OPTIMAL CONTROL PROBLEMS OF NONLINEAR EVOLUTION INCLUSIONS WITH QUASI-VARIATIONAL STRUCTURE

Dedicated to Professor Mitsuharu Ôtani on the Occasion of his 70th Birthday

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Abstract. In this paper, we consider abstract doubly quasi-variational evolution inclusions governed by time-dependent subdifferentials, and we are interested in a general class of singular optimal control problems that are set up for non-well-posed state systems. Our main objective of this paper is to establish an approximate procedure for such singular optimal control problems by considering parameter-dependent evolution inclusions. Also, we apply our abstract results to quasi-variational inequalities with time-dependent gradient constraints: interior obstacle problems and Navier-Stokes type problem.

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1 Introduction

This paper is concerned with singular optimal control problems for doubly nonlinear quasi-variational evolution inclusions governed by time-dependent subdifferentials.

Throughout this paper, let H be a (real) Hilbert space and V be a uniformly convex Banach space such that V is dense in H and the injection from V into H is compact, supposing also that the dual space V^* of V is uniformly convex; the norms in V and V^* are denoted by $|\cdot|_V$ and $|\cdot|_{V^*}$, respectively, and the duality pairing between V and V^* is denoted by $\langle\cdot,\cdot\rangle$; similarly the norm and the inner product in H are denoted by $|\cdot|_H$ and $(\cdot,\cdot)_H$, respectively. In this case, identifying H with its dual, we have

$$V \hookrightarrow H \hookrightarrow V^* \ \text{ (with dense and compact embeddings);}$$

note that

$$\langle u,v\rangle = (u,v)_H, \ \text{ if } u \in H \text{ and } v \in V.$$

Recently, N. Kenmochi–K. Shirakawa–N. Yamazaki [25] established the abstract solvability theory of the following doubly nonlinear quasi-variational evolution inclusions governed by time-dependent subdifferentials in the Banach space V^*:

$$\begin{cases} \partial_*\psi^t(u;u'(t)) + \partial_*\varphi^t(u;u(t)) + g(u;t,u(t)) \ni f(t) \ \text{ in } V^* \text{ for a.a. } t \in (0,T), \\ u(0) = u_0 \ \text{ in } V, \end{cases} \tag{1.1}$$

where $0 < T < \infty$, $u' = \frac{du}{dt}$ in V, f is a given V^*-valued function, and $u_0 \in V$ is a given initial datum.

For each parameter v and $t \in [0,T]$, $\psi^t(v;z)$ is a proper (i.e., not identically equal to infinity), l.s.c. (i.e., lower semi-continuous), convex function in $z \in V$, $\varphi^t(v;z)$ is a non-negative, continuous convex function in $z \in V$, and $g(v;t,z)$ is a single-valued Lipschitz operator from V into V^* (see Section 2 for their precise definitions). Note that (t,v) is a parameter that determines the convex functions $\psi^t(v;\cdot)$, $\varphi^t(v;\cdot)$, and the perturbation $g(v;t,\cdot)$.

In the present paper, we discuss optimal control problems for (1.1). Note that problem (1.1) has multiple solutions, in general (see in [25, Example 3.1] for instance). Therefore, the optimal control problem associated with the state equation (1.1) is singular, namely, control problems formulated for non-well-posed state system. By a similar approach as in [24], we consider a singular optimal control problem formulated for (1.1). To this end, let $\mathcal{F}_M$ be a control space with constant $M > 0$, defined by:

$$\mathcal{F}_M := \left\{ f \in W^{1,2}(0,T;V^*) \cap L^2(0,T;H) \ ; \ \begin{array}{c} |f|_{W^{1,2}(0,T;V^*)} \le M, \\ |f|_{L^2(0,T;H)} \le M \end{array} \right\}, \tag{1.2}$$

where $|\cdot|_{W^{1,2}(0,T;V^*)}$ (resp. $|\cdot|_{L^2(0,T;H)}$) is the norm of $W^{1,2}(0,T;V^*)$ (resp. $L^2(0,T;H)$). Then, we formulate a singular optimal control problem for (1.1) as follows:

Problem (OP): Find a control $f^* \in \mathcal{F}_M$ such that

$$J(f^*) = \inf_{f \in \mathcal{F}_M} J(f);$$

such a function f^* is called an optimal control. Here, $J(f)$ is a functional defined by

$$J(f) := \inf_{u \in \mathcal{S}(f)} \pi_f(u), \tag{1.3}$$

where $f \in \mathcal{F}_M$ is any control, and $\mathcal{S}(f)$ is the set of all solutions to (1.1) associated with control f. In addition, $\pi_f(u)$ is a functional of $u \in \mathcal{S}(f)$ defined by:

$$\pi_f(u) := \frac{1}{2}\int_0^T |u(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f(t)|_{V^*}^2 dt, \tag{1.4}$$

where $u_{ad} \in L^2(0,T;V)$ is a given target profile.

There is a vast amount of literature on optimal control problems for (parabolic or elliptic) variational inequalities. For instance, see [8, 15, 16, 18, 27, 28, 31, 32, 33, 38]. In particular, P. Neittaanmäki–J. Sprekels–D. Tiba [33, Section 3.1.3.1] discussed singular control problems for linear elliptic equations of second-order with the homogeneous Dirichlet boundary condition.

The theory of nonlinear evolution inclusions is useful in any systematic study of variational inequalities. For instance, many mathematicians have studied nonlinear evolution inclusions of the form:

$$u'(t) + \partial\varphi^t(u(t)) \ni f(t) \text{ in } H \text{ for a.a. } t \in (0,T), \tag{1.5}$$

where $\varphi^t : H \to \mathbb{R} \cup \{\infty\}$ is a time-dependent proper, l.s.c., and convex function for each $t \in [0,T]$. For fundamental results on (1.5), we refer to [16, 19, 20, 34, 37]. In particular, S. Hu–N. S. Papageorgiou [16] treated some optimal control problems for (1.5). Furthermore, the optimal control of parameter-dependent evolution inclusions for (1.5) has previously been considered in several previous works (cf. [16, 35]).

Doubly nonlinear evolution inclusions have been studied, for instance, by N. Kenmochi–I. Pawlow [22], in which nonlinear evolution inclusions of the following type were discussed:

$$\frac{d}{dt}\partial\psi(u(t)) + \partial\varphi^t(u(t)) \ni f(t) \text{ in } H \text{ for a.a. } t \in (0,T), \tag{1.6}$$

where $\psi : H \to \mathbb{R} \cup \{\infty\}$ is a proper, l.s.c., and convex function. The abstract results for (1.6) can be applied to elliptic-parabolic equations. From the viewpoint of (1.6), K.-H. Hoffmann–M. Kubo–N. Yamazaki [15] studied optimal control problems for quasi-linear elliptic-parabolic variational inequalities with time-dependent constraints. Additionally, A. Kadoya–N. Kenmochi [17] touched on the optimal shape design of elliptic-parabolic equations.

G. Akagi [2], T. Arai [3], M. Aso–M. Frémond–N. Kenmochi [4], M. Aso–N. Kenmochi [5], P. Colli [11], P. Colli–A. Visintin [12], O. Grange–F. Mignot [14], and T. Senba [36] investigated the following type of doubly nonlinear evolution inclusions:

$$\partial\psi^t(u'(t)) + \partial\varphi(u(t)) \ni f(t) \text{ in } H \text{ for a.a. } t \in (0,T). \tag{1.7}$$

As mentioned above, one interesting feature is that (1.7) is not generally well-posed; namely, it lacks the uniqueness of solutions. In this respect, M. H. Farshbaf-Shaker–N.

Yamazaki [13] studied the optimal control problem without the uniqueness of solutions in the state system (1.7) with single-valued Lipschitz perturbation g by employing the idea of Y. Murase–A. Kadoya–N. Kenmochi [31].

In the recent works of N. Kenmochi–K. Shirakawa–N. Yamazaki [23, 24], a more general approach was proposed to treat

$$\partial_* \psi^t(u'(t)) + \partial_* \varphi^t(u; u(t)) + g(t, u(t)) \ni f(t) \quad \text{in } V^* \quad \text{for a.a. } t \in (0, T). \tag{1.8}$$

Moreover, singular optimal control problems for (1.8) were discussed, and as applications of abstract results, parabolic quasi-variational inequalities with gradient constraint for the time derivatives were studied (cf. [24, Section 11]).

In this paper, we systematically investigate (OP). The novelties of this work are as follows:

(a) We show the existence of optimal control for (OP);

(b) We propose an approximate procedure to investigate the singular optimal control problem for (1.1) by considering parameter-dependent evolution inclusions.

The remainder of this paper is organized as follows. In Section 2, we recall the abstract result of the existence of solutions to (1.1) for each $f \in L^2(0, T; V^*)$ and $u_0 \in V$. In Section 3, we give abstract results of existence-uniqueness and convergence of solutions to parameter-dependent abstract evolution inclusions. In Section 4, we consider the singular optimal control problem for (1.1), and state the main result (Theorem 4.1), corresponding to item (a). In Section 5, we propose an approximate procedure to construct an optimal control for (OP) and give the main results (Theorems 5.1 and 5.2), corresponding to item (b). In the final Section 6, we apply our general results to some model problems: quasi-variational inequalities with time-dependent gradient constraints (interior obstacle problems and Navier-Stokes type problem).

Notation

We now list some notation and definitions of subdifferentials of convex functions. Let $\phi : V \to \mathbb{R} \cup \{\infty\}$ be a proper, l.s.c., and convex function. Then, the effective domain $D(\phi)$ is defined by

$$D(\phi) := \{z \in V;\ \phi(z) < \infty\}.$$

The subdifferential $\partial_* \phi : V \to V^*$ of ϕ is a possibly multi-valued operator from V into V^*, and is defined by $z^* \in \partial_* \phi(z)$ if and only if

$$z \in D(\phi) \quad \text{and} \quad \langle z^*, y - z \rangle \leq \phi(y) - \phi(z) \quad \text{for all } y \in V.$$

Its graph is the set $G(\partial_* \phi) := \{[z, z^*] \in V \times V^*\ ;\ z^* \in \partial_* \phi(z)\}$, which is often identified with $\partial_* \phi$, namely, $z^* \in \partial_* \phi(z)$ is denoted by $[z, z^*] \in \partial_* \phi$.

In particular, let $\phi : H \to \mathbb{R} \cup \{\infty\}$ be proper, l.s.c., and convex. Then we denote by $\partial \phi$ the subdifferential of ϕ from H into H.

For various properties and related notions of a proper, l.s.c., convex function ϕ and its subdifferential $\partial_* \phi$, we refer to the monographs by V. Barbu [7, 9]. In particular, for those in Hilbert spaces, we refer to the monographs by H. Brézis [10].

We also recall a notion of convergence for convex functions, developed by U. Mosco [29, 30] and H. Attouch [6].

Let ϕ, ϕ_n ($n \in \mathbb{N}$) be proper, l.s.c., and convex functions on V. Then, we say that ϕ_n converges to ϕ on V in the sense of U. Mosco [29] as $n \to \infty$ if the following two conditions are satisfied:

(M1) for any subsequence $\{\phi_{n_k}\}_{k\in\mathbb{N}} \subset \{\phi_n\}_{n\in\mathbb{N}}$, if $z_k \to z$ weakly in V as $k \to \infty$, then

$$\liminf_{k\to\infty} \phi_{n_k}(z_k) \geq \phi(z);$$

(M2) for any $z \in D(\phi)$, there is a sequence $\{z_n\}_{n\in\mathbb{N}}$ in V such that

$$z_n \to z \text{ in } V \text{ as } n \to \infty \quad \text{ and } \quad \lim_{n\to\infty} \phi_n(z_n) = \phi(z).$$

For some important properties of the Mosco convergence of convex functions, we refer to the monographs by H. Attouch [6] and N. Kenmochi [21]. Especially the following ones are often used. Let ϕ, ϕ_n ($n \in \mathbb{N}$) be proper, l.s.c., and convex functions on V. Assume ϕ_n converges to ϕ on V in the sense of Mosco as $n \to \infty$. Then:

(Fact 1) $z_n^* \in \partial_*\phi_n(z_n)$, $z_n \to z$ weakly in V, $z_n^* \to z^*$ weakly in V^*, and $\langle z_n^*, z_n\rangle \to \langle z^*, z\rangle$ (as $n \to \infty$), then $z^* \in \partial_*\phi(z)$.

(Fact 2) **(Graph convergence)** If $z^* \in \partial_*\phi(z)$, then there are sequences $\{z_n\}_{n\in\mathbb{N}} \subset V$ and $\{z_n^*\}_{n\in\mathbb{N}} \subset V^*$ such that

$$z_n^* \in \partial_*\phi_n(z_n),\ z_n \to z \text{ in } V, \text{ and } z_n^* \to z^* \text{ in } V^*.$$

2 Solvability of (1.1)

We begin with the precise formulation of our problem (1.1). For this purpose, we use a time-dependent proper, l.s.c., and convex function $\psi_0^t(\cdot)$ on V such that there are positive constants C_1 and C_2 satisfying

$$\psi_0^t(z) \geq C_1|z|_V^2 - C_2, \quad \forall z \in V,\ \forall t \in [0,T],\ 0 < T < \infty, \tag{2.1}$$

and $t \to \psi_0^t(\cdot)$ is continuous on V in the sense of Mosco.

Our doubly nonlinear evolution inclusions are formulated in terms of two functionals $\psi^t(v; z)$, $\varphi^t(v; z)$, and a mapping $g(v; t, z)$ together with a prescribed initial-value u_0 and a forcing term f, as follows:

$$\begin{cases} \partial_*\psi^t(u; u'(t)) + \partial_*\varphi^t(u; u(t)) + g(u; t, u(t)) \ni f(t) \text{ in } V^*, \text{ a.a. } t \in (0,T), \\ u(0) = u_0 \text{ in } V, \end{cases} \tag{2.2}$$

and their solutions are constructed in the class

$$D_0 := \left\{ v \in W^{1,2}(0,T;V) \;\middle|\; \int_0^T \psi_0^t(v'(\tau))d\tau < \infty,\ v(0) = u_0 \right\}, \tag{2.3}$$

depending on the functional ψ_0^t and the initial-value u_0.

We suppose that $D_0 \neq \emptyset$ and the following assumptions are fulfilled:

(*Assumption* (ψ))

The functional $\psi^t(v;z)$ is defined for each $(t,v,z) \in [0,T] \times D_0 \times V$ so that $\psi^t(v;z)$ is proper, l.s.c., and convex in $z \in V$ for any $t \in [0,T]$ and any $v \in D_0$, and

$$\psi^t(v_1;z) = \psi^t(v_2;z), \ \forall z \in V, \text{ if } v_1 = v_2 \text{ on } [0,t],$$

for $v_i \in D_0, \ i = 1,2$. Furthermore, assume:

(ψ1) If $t_n \in [0,T]$, $v_n \in D_0$, $\sup_{n\in\mathbb{N}} \int_0^T \psi_0^t(v_n'(t))dt < \infty$, $t_n \to t$, and $v_n \to v$ in $C([0,T];H)$ (hence $v \in D_0$ and $v \in W^{1,2}(0,T;V)$ by (2.1)) as $n \to \infty$, then

$$\psi^{t_n}(v_n;\cdot) \to \psi^t(v;\cdot) \text{ on } V \text{ in the sense of Mosco as } n \to \infty.$$

(ψ2) $D(\psi^t(v;\cdot)) \subset D(\psi_0^t)$ for all $v \in D_0$ and $t \in [0,T]$, and

$$\psi^t(v;z) \geq \psi_0^t(z), \quad \forall t \in [0,T], \ \forall v \in D_0, \ \forall z \in D(\psi^t(v;\cdot)).$$

(ψ3) $\partial_*\psi^t(v;0) \ni 0$ for all $t \in [0,T]$ and $v \in D_0$, and there is a non-negative function $c_\psi(\cdot) \in L^1(0,T)$ such that

$$\psi^t(v;0) \leq c_\psi(t), \quad \forall t \in [0,T], \ \forall v \in D_0.$$

(*Assumption* (ϕ))

Let $\varphi^t : [0,T] \times D_0 \times V \to \mathbb{R}$ be a function such that $\varphi^t(v;z)$ is non-negative, finite, continuous, and convex in $z \in V$ for any $t \in [0,T]$ and any $v \in D_0$, and

$$\varphi^t(v_1;z) = \varphi^t(v_2;z), \ \forall z \in V, \text{ if } v_1 = v_2 \text{ on } [0,t],$$

for $v_i \in D_0, \ i = 1,2$. Besides, assume the following:

(ϕ1) The subdifferential $\partial_*\varphi^t(v;z)$ of $\varphi^t(v;z)$ with respect to $z \in V$ is linear and bounded from $D(\partial_*\varphi^t(v;\cdot)) = V$ into V^* for each $t \in [0,T]$ and $v \in D_0$, and there is a positive constant C_3 such that

$$|\partial_*\varphi^t(v;z)|_{V^*} \leq C_3|z|_V, \ \ \forall z \in V, \ \forall t \in [0,T], \ \forall v \in D_0.$$

(ϕ2) If $\{v_n\}_{n\in\mathbb{N}} \subset D_0$, $\sup_{n\in\mathbb{N}} \int_0^T \psi_0^t(v_n'(t))dt < \infty$, $v \in D_0$, and $v_n \to v$ in $C([0,T];H)$ (as $n \to \infty$), then

$$\varphi^t(v_n;\cdot) \to \varphi^t(v;\cdot) \ \text{ on } V \text{ in the sense of Mosco}, \ \forall t \in [0,T].$$

(ϕ3) $\varphi^0(v;0) = 0$ for all $v \in D_0$. Moreover, there is a positive constant C_4 such that

$$\varphi^0(v;z) \geq C_4|z|_V^2, \ \ \forall z \in V, \ \forall v \in D_0.$$

($\phi 4$) There is a function $\alpha \in W^{1,1}(0,T)$ such that

$$|\varphi^t(v;z) - \varphi^s(v;z)| \leq |\alpha(t) - \alpha(s)|\varphi^s(v;z), \quad \forall z \in V,\ \forall v \in D_0,\ \forall s,t \in [0,T].$$

(*Assumption* (g))

Let $g := g(v;t,z)$ be a single-valued operator from $D_0 \times [0,T] \times V$ into V^* such that $g(v;t,z)$ is strongly measurable in $t \in [0,T]$ for each $v \in D_0$ and $z \in V$, and

$$g(v_1;t,z) = g(v_2;t,z), \ \ \forall z \in V, \text{ if } v_1, v_2 \in D_0 \text{ and } v_1 = v_2 \text{ on } [0,t],\ \forall t \in [0,T].$$

Moreover, assume:

($g1$) Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence in D_0 such that $\sup_{n\in\mathbb{N}} \int_0^T \psi_0^t(v_n'(t))dt < \infty$ and $v_n \to v$ in $C([0,T];H)$ (as $n \to \infty$), and $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in V such that $z_n \to z$ weakly in V. Then,

$$g(v_n;t,z_n) \to g(v;t,z) \ \text{ in } V^*,\ \forall t \in [0,T].$$

($g2$) $g(v;\cdot,0) \in L^2(0,T;V^*)$ for any $v \in D_0$, and $g(v;t,\cdot)$ is uniformly Lipschitz from V into V^*, i.e., there is a constant $L_g > 0$ such that

$$|g(v;t,z_1) - g(v;t,z_2)|_{V^*} \leq L_g|z_1 - z_2|_V,\ \forall z_i \in V\ (i=1,2),\ \forall v \in D_0,\ \forall t \in [0,T].$$

($g3$) There is a non-negative function $g_0 \in L^2(0,T)$ such that

$$|g(v;t,0)|_{V^*} \leq g_0(t), \ \text{ a.a. } t \in (0,T),\ \forall v \in D_0.$$

Next, we give the definition of solutions to evolution inclusion (2.2).

Definition 2.1. Given data $f \in L^2(0,T;V^*)$ and $u_0 \in V$, a function $u : [0,T] \to V$ is called a solution to $CP(\psi^t, \varphi^t, g; f, u_0)$ or $CP(f,u_0)$ or simply CP, if and only if the following conditions are fulfilled:

(i) $u \in W^{1,2}(0,T;V)$.

(ii) There exists a function $\xi \in L^2(0,T;V^*)$ such that

$$\xi(t) \in \partial_*\psi^t(u;u'(t)) \ \text{ in } V^*,\ \text{a.a. } t \in (0,T),$$

and

$$\xi(t) + \partial_*\varphi^t(u;u(t)) + g(u;t,u(t)) = f(t) \ \text{ in } V^*,\ \text{a.a. } t \in (0,T).$$

(iii) $u(0) = u_0$ in V.

We here recall the abstract existence result of evolution inclusion $CP(f,u_0)$.

Proposition 2.1 ([25, Theorem 2.1]). *Assume that $u_0 \in V$ and assumptions (ψ), (ϕ), and (g) are fulfilled. Let f be any function in $L^2(0,T;V^*)$. Then, $CP(f,u_0)$ admits at least one solution u. Moreover, there exists a constant $N_1 > 0$, independent of f and u_0, such that*

$$\int_0^T \psi^t(u;u'(t))dt + \sup_{t\in[0,T]} \varphi^t(u;u(t)) \le N_1 \left(|u_0|_V^2 + |f|^2_{L^2(0,T;V^*)} + 1 \right) \tag{2.4}$$

for any solution u to $CP(f,u_0)$.

The solvability for $CP(f,u_0)$ was performed in several steps. The first approach is to consider an auxiliary approximate problem of the form for each fixed $\varepsilon \in (0,1]$ and $v \in D_0$:

$$\varepsilon F_0 u'_{\varepsilon,v}(t) + \partial_*\psi^t(v;u'_{\varepsilon,v}(t)) + \partial_*\varphi^t(v;u_{\varepsilon,v}(t)) + g(v;t,u_{\varepsilon,v}(t)) \ni f(t) \text{ in } V^*, \ t>0, \tag{2.5}$$

subject to initial condition $u_{\varepsilon,v}(0) = u_0$ in V, where we set $F_0 := \partial_*\varphi^0(v_0;\cdot)$ for the fixed function $v_0 \in D_0$ (cf. (5.17)). In the first step we proved the existence of a unique solution $u_{\varepsilon,v}$ to (2.5) for every $\varepsilon \in (0,1]$ and $v \in D_0$ by applying the general theory of ordinary differential equations. In the second step the convergence of approximate solution $u_{\varepsilon,v}$ was discussed. In the third step, for each $\varepsilon \in (0,1]$ and $v \in D_0$, the solution $u_{\varepsilon,v}$ of (2.5) is denoted by $S_\varepsilon(v)$, namely $u_{\varepsilon,v} := S_\varepsilon(v)$, and then, we found a fixed point of S_ε in D_0, $u_{\varepsilon,v} = S_\varepsilon(u_{\varepsilon,v})$, which is denoted simply by u_ε. It is a solution to

$$\varepsilon F_0 u'_\varepsilon(t) + \partial_*\psi^t(u_\varepsilon;u'_\varepsilon(t)) + \partial_*\varphi^t(u_\varepsilon;u_\varepsilon(t)) + g(u_\varepsilon;t,u_\varepsilon(t)) \ni f(t), \quad u_\varepsilon(0) = u_0. \tag{2.6}$$

In the final step we showed that the solution u_ε converges in $C([0,T];V)$ as $\varepsilon \to 0$ and the limit u is a solution of $CP(f,u_0)$. For the detailed proof of Proposition 2.1, we refer to [25, Theorem 2.1].

As is seen from the counterexample in [25, Example 3.1], the uniqueness of solutions to doubly nonlinear evolution inclusions is not expected, in general. However, in a restricted class for ψ^t and φ^t we have the uniqueness of solution.

Proposition 2.2 ([25, Proposition 3.1]). *Suppose that ψ^t and g are independent of $v \in D_0$, namely $\psi^t(v;z) = \psi^t(z)$ and $g(v;t,z) = g(t,z)$ for all $v \in D_0$ and $z \in V$; in this case $\psi_0^t := \psi^t$ on V. In addition, assume that ψ^t is uniformly monotone for any $t \in [0,T]$, namely, there exists a positive constant $C_5 > 0$ such that*

$$\langle \xi_1 - \xi_2, z_1 - z_2 \rangle \ge C_5 |z_1 - z_2|_V^2,$$
$$\forall z_i \in D(\partial_*\psi^t), \ \xi_i \in \partial_*\psi^t(z_i) \ (i=1,2), \ \forall t \in [0,T].$$

Furthermore, suppose that $u_0 \in V$ and assumptions (ψ), (ϕ), and (g) are fulfilled. Also, assume that $\partial_*\varphi^t(v;\cdot)$ is Lipschitz in $v \in D_0$, more precisely, there exists a positive constant $C_6 > 0$ such that

$$|\partial_*\varphi^t(v_1;z) - \partial_*\varphi^t(v_2;z)|_{V^*} \le C_6 |v_1(t) - v_2(t)|_V \left(1 + |z|_V\right),$$
$$\forall v_i \in D_0 \ (i=1,2), \ \forall z \in D_0, \ \forall t \in [0,T].$$

Let f be any function in $L^2(0,T;V^*)$. Then, $CP(f,u_0)$ has at most one solution.

3 Some additional results

In this section, we recall the previous works in [25], and set up some auxiliary results. In particular, the convergence result of solutions to parameter-dependent evolution inclusions is key to considering (OP).

The following lemma is derived directly from assumption (ϕ).

Lemma 3.1 (cf. [25, Lemma 2.1]). *Suppose that assumption (ϕ) is fulfilled. Then, the following inequality holds: for all $t \in [0,T]$, $z \in V$, and $v \in D_0$,*

$$\frac{C_4}{|\alpha'|_{L^1(0,T)}+1}|z|_V^2 \le \varphi^t(v;z) \le \langle \partial_* \varphi^t(v;z), z\rangle \le C_3|z|_V^2. \tag{3.1}$$

For the detailed proof of Lemma 3.1, we refer to [25, Lemma 2.1].

Now we recall the convergence result of solutions to parameter-dependent evolution inclusions established in [25, Section 2.3].

Given $u_0 \in V$, $v \in D_0$, and $f \in L^2(0,T;V^*)$, we denote by $CP_v(\psi^t, \varphi^t, g; f, u_0)$ or $CP_v(f,u_0)$ the problem to find a function $u \in W^{1,2}(0,T;V)$ with $\xi \in L^2(0,T;V^*)$ satisfying that

$$\begin{cases} \xi(t) \in \partial_* \psi^t(v;u'(t)) \text{ in } V^*, \ \text{a.a. } t \in (0,T), \\ \xi(t) + \partial_*\varphi^t(v;u(t)) + g(v;t,u(t)) = f(t) \text{ in } V^*, \ \text{a.a. } t \in (0,T), \\ u(0) = u_0 \text{ in } V. \end{cases}$$

Note that the existence of solutions to $CP_v(f,u_0)$ can be proved by applying the abstract theory established in [23, Theorems 1].

Proposition 3.1 (cf. [23, Theorem 1]). *Suppose that $u_0 \in V$ and assumptions (ψ), (ϕ), and (g) are fulfilled. Then, for each $v \in D_0$ and $f \in L^2(0,T;V^*)$, there exists at least one solution u to $CP_v(f,u_0)$. Moreover, for any fixed $v \in D_0$, any solution u to $CP_v(f,u_0)$ satisfies the bound: there exists a constant $N_1 > 0$, independent of $v \in D_0$ and $f \in L^2(0,T;V^*)$, such that*

$$\int_0^T \psi^t(v;u'(t))dt + \sup_{t\in[0,T]} \varphi^t(v;u(t)) \le N_1(|u_0|_V^2 + |f|^2_{L^2(0,T;V^*)} + 1), \tag{3.2}$$

where N_1 is the same constant as in (2.4) *of Proposition 2.1.*

As is seen from the counterexample in [25, Example 3.1], the uniqueness of solutions to doubly nonlinear evolution inclusions $CP_v(f,u_0)$ is not expected, in general. However, by arguments similar to [23, Theorem 2], we can show the uniqueness of solution to $CP_v(f,u_0)$ in a restricted class for ψ^t and φ^t.

Proposition 3.2 (cf. [23, Theorem 2]). *Suppose that $u_0 \in V$ and assumptions* (ψ), (ϕ), *and* (g) *are fulfilled. In addition, assume that $\psi^t(v;\cdot)$ is uniformly monotone for any $t \in [0,T]$ and $v \in D_0$, namely, there exists a positive constant $C_5 > 0$ such that*

$$\begin{gathered} \langle \xi_1 - \xi_2, z_1 - z_2\rangle \ge C_5|z_1 - z_2|_V^2, \\ \forall z_i \in D(\partial_*\psi^t), \ \xi_i \in \partial_*\psi^t(v;z_i), \ i=1,2, \ \forall v \in D_0, \ \forall t \in [0,T]. \end{gathered} \tag{3.3}$$

Let f be any function in $L^2(0,T;V^*)$. Then, problem $CP_v(f,u_0)$ admits a unique solution u on $[0,T]$.

Remark 3.1 (cf. [25, Remark 3.1])**.** Assume the following conditions:

$$\begin{gathered}\langle \xi_1-\xi_2, z_1-z_2\rangle \ge C_5|z_1-z_2|_H^2,\\ \forall z_i\in D(\partial_*\psi^t),\ \xi_i\in\partial_*\psi^t(v;z_i),\ i=1,2,\ \forall v\in D_0,\ \forall t\in[0,T],\end{gathered} \tag{3.4}$$

and g satisfies that

$$|g(v;t,z_1)-g(v;t,z_2)|_H\le L'_g|z_1-z_2|_H,\ \forall z_i\in V,\ i=1,2,\ \forall v\in D_0,\ \forall t\in[0,T] \tag{3.5}$$

for a positive constant L'_g. Then, Proposition 3.2 guarantees that for any fixed $v\in D_0$, problem $CP_v(f,u_0)$ admits a unique solution u on $[0,T]$.

We now recall the convergence result of solutions to parameter-dependent evolution inclusions $CP_v(f,u_0)$ with respect to the data v and f.

Proposition 3.3 ([25, Proposition 2.2])**.** *Let $u_0\in V$, $\{v_n\}_{n\in\mathbb{N}}$ and $\{f_n\}_{n\in\mathbb{N}}$ be any families in D_0 and $L^2(0,T;V^*)$, respectively, such that*

$$\begin{cases}\displaystyle\sup_{n\in\mathbb{N}}\int_0^T\psi_0^t(v_n'(t))dt<\infty, \quad v_n\to v \ \mathit{in}\ C([0,T];H),\\ f_n\to f\ \mathit{in}\ L^2(0,T;V^*) \quad (\mathit{as}\ n\to\infty),\end{cases} \tag{3.6}$$

and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of solutions to $CP_{v_n}(f_n,u_0)$ on $[0,T]$. Then, there exist a subsequence $\{n_k\}_{k\in\mathbb{N}}\subset\{n\}_{n\in\mathbb{N}}$ and a function $u\in W^{1,2}(0,T;V)$ such that u is a solution to $CP_v(f,u_0)$ on $[0,T]$ and

$$u_{n_k}\to u\ \mathit{in}\ C([0,T];V)\quad \mathit{as}\ k\to\infty.$$

For the detailed proofs of Propositions 3.1–3.3, we refer to [23, Theorems 1 and 2] and [25, Sections 2 and 3].

4 Singular optimal control problem for (QP)

Regarding the state system $CP(f,u_0)$, the existence of a solution was proved in [25, Theorem 2.1], but its uniqueness is not expected. Therefore, in this section, we propose a singular optimal control problem for non-well-posed state systems $CP(f,u_0)$.

Using arguments similar to those in [24, Sections 4 and 8], we study the singular optimal control problem for doubly quasi-variational evolution inclusions $CP(f,u_0)$. Indeed, the following result on the convergence of solutions to $CP(f,u_0)$ is the key to proving the existence of an optimal control for (OP).

Proposition 4.1. *Assume that $u_0 \in V$ and assumptions (ψ), (ϕ), and (g) are fulfilled. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V^*)$ and $f \in L^2(0,T;V^*)$ such that*

$$f_n \to f \ \text{ in } L^2(0,T;V^*) \quad \text{as } n \to \infty, \tag{4.1}$$

and let u_n be a solution to CP(f_n, u_0) on $[0,T]$. Then, there exist a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ and a function $u \in W^{1,2}(0,T;V)$ such that u is a solution to CP(f, u_0) on $[0,T]$ and

$$u_{n_k} \to u \ \text{ in } C([0,T];V) \quad \text{as } k \to \infty. \tag{4.2}$$

Proof. This proposition follows from Proposition 3.3. Indeed, note from (4.1) that $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;V^*)$. Therefore, from the uniform estimate (2.4) with (2.1), (ψ2), Lemma 3.1, and the Ascoli–Arzelà theorem, we can find a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ and a function $u \in W^{1,2}(0,T;V)$ such that

$$\sup_{n\in\mathbb{N}} \int_0^T \psi_0^t(u_{n_k}'(t))dt < \infty,$$
$$u_{n_k} \to u \text{ in } C([0,T];H) \text{ and weakly in } W^{1,2}(0,T;V),$$
$$u_{n_k}(t) \to u(t) \text{ weakly in } V \text{ for all } t \in [0,T]$$

as $k \to \infty$.

Note that $CP(f_{n_k}, u_0)$ can be regarded as the parameter-dependent evolution inclusions $CP_{u_{n_k}}(f_{n_k}, u_0)$. Therefore, by applying Proposition 3.3 that $u_{n_k} \to u$ in $C([0,T];V)$ (as $k \to \infty$) and the limit u is a solution of $CP_u(f, u_0)$, namely u is a solution to $CP(f, u_0)$ on $[0,T]$. Thus, the proof of Proposition 4.1 is complete. □

We now state the main result of this paper, which is directed to the existence of an optimal control for (OP) without the uniqueness of solutions to $CP(f, u_0)$.

Theorem 4.1. *Assume that $u_0 \in V$ and assumptions (ψ), (ϕ), and (g) are fulfilled. Let u_{ad} be a given target function in $L^2(0,T;V)$. Then,* (OP) *has at least one optimal control $f^* \in \mathcal{F}_M$, namely,*

$$J(f^*) = \inf_{f\in\mathcal{F}_M} J(f),$$

where $J(\cdot)$ is the cost functional of (OP)*, which is defined by* (1.3) *and* (1.4)*.*

Taking account of Proposition 4.1, we can prove Theorem 4.1. Indeed, as the proof is quite similar to that of [24, Theorem 4.1] (or Theorem 5.1(I) below), we omit the detailed proof.

5 Approximations for *CP*(f, u_0) and (OP)

In [24, Sections 5 and 9], we established an approximate procedure to investigate the singular optimal control problems for (1.8). In such arguments, we needed the uniqueness of solutions to approximate state systems. However, in doubly quasi-variational evolution inclusions $CP(f, u_0)$, the approximate problem (2.6) is generally a non-well-posed state system (cf. Proposition 2.2 and [25, Example 3.1]). Therefore, we here consider the approximate procedure for $CP(f, u_0)$ and (OP) by parameter-dependent evolution inclusions $CP_v(f, u_0)$.

Now, for each $u_0 \in V$, $w \in D_0$, and $f \in L^2(0,T;V^*)$, the parameter-dependent state problem is of the form:

$$CP_w(f,u_0) \begin{cases} \xi(t) \in \partial_* \psi^t(w; u'(t)) \text{ in } V^*, \text{ a.a. } t \in (0,T), \\ \xi(t) + \partial_* \varphi^t(w; u(t)) + g(w; t, u(t)) = f(t) \text{ in } V^*, \text{ a.a. } t \in (0,T), \\ u(0) = u_0 \text{ in } V. \end{cases}$$

Note from Proposition 3.1 that $CP_w(f,u_0)$ has at least one solution satisfying the uniform estimate (3.2). However, the uniqueness of solutions to $CP_w(f,u_0)$ is not expected, in general (cf. Proposition 3.2 and Remark 3.1). Therefore, we now consider the singular optimal control problem for $CP_w(f,u_0)$. To this end, for each $\delta \in (0,1]$, let E_δ be the cost functional defined by

$$E_\delta(w,f) := \inf_{u \in \mathcal{S}(w,f)} \pi^\delta_{[w,f]}(u), \tag{5.1}$$

where $\mathcal{S}(w,f)$ is the set of all solutions to $CP_w(f,u_0)$ on $[0,T]$, and for any $u \in \mathcal{S}(w,f)$, the value of functional $\pi^\delta_{[w,f]}(u)$ is defined by:

$$\pi^\delta_{[w,f]}(u) := \frac{1}{2}\int_0^T |u(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f(t)|_{V^*}^2 dt + \frac{1}{2\delta}\int_0^T |u(t) - w(t)|_V^2 dt. \tag{5.2}$$

Additionally, for a given positive number M, let $\mathcal{F}_M$ be the control space given by (1.2). Setting $M' := N_1(|u_0|_V^2 + M^2 + 1)$, the set $\mathcal{W}_{M'}$ is defined by

$$\mathcal{W}_{M'} := \left\{ w \in W^{1,2}(0,T;V) \;\middle|\; \int_0^T \psi_0^t(w'(t))dt \le M' \right\},$$

where N_1 is the same constant as in (3.2) (cf. (2.4) of Proposition 2.1).

We now choose the set $\mathcal{W}_{M'}(u_0) \times \mathcal{F}_M$ as the control space, where

$$\mathcal{W}_{M'}(u_0) := \{ w \in \mathcal{W}_{M'} \mid w(0) = u_0 \text{ in } V \}. \tag{5.3}$$

Note that $\mathcal{W}_{M'}(u_0) \subset D_0$. Then, for each $w \in D_0$ and $f \in \mathcal{F}_M$, (3.2) with (ψ2) implies that

$$u \in \mathcal{W}_{M'}(u_0) \text{ for any solution } u \text{ to } CP_w(f,u_0) \text{ on } [0,T]. \tag{5.4}$$

Now, for each $\delta \in (0,1]$, we consider the following singular optimal control problem, denoted by $(\mathrm{OP})_\delta$, with the state problem $CP_w(f,u_0)$ for $w \in D_0$ and $f \in L^2(0,T;V^*)$.

Problem $(\mathbf{OP})_\delta$: Find a control $[w^*_\delta, f^*_\delta] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M$, called an optimal control, such that

$$E_\delta(w^*_\delta, f^*_\delta) = \inf_{[w,f] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M} E_\delta(w,f).$$

Now we mention the following second main result, which is concerned with the existence of an optimal control for $(\mathrm{OP})_\delta$ and the relationship between (OP) and $(\mathrm{OP})_\delta$ with respect to $\delta \in (0,1]$.

Theorem 5.1 (cf. [24, Section 10]). *Suppose that $u_0 \in V$ and assumptions (ψ), (ϕ), and (g) are fulfilled. Let u_{ad} be a given target function in $L^2(0,T;V)$. Then, the following two statements hold.*
(I) *For each $\delta \in (0,1]$, $(\mathrm{OP})_\delta$ has at least one optimal control $[w_\delta^*, f_\delta^*] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M$, namely,*

$$E_\delta(w_\delta^*, f_\delta^*) = \inf_{[w,f]\in\mathcal{W}_{M'}(u_0)\times\mathcal{F}_M} E_\delta(w,f).$$

(II) *Let $[w_\delta^*, f_\delta^*]$ be an optimal control of $(\mathrm{OP})_\delta$ for any $\delta \in (0,1]$, and let $[w^*, f^*]$ be any weak limit of $\{[w_\delta^*, f_\delta^*])\}$ as $\delta \downarrow 0$, namely, there is a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ with $\delta_n \downarrow 0$ (as $n \to \infty$) such that*

$$w_{\delta_n}^* \to w^* \text{ weakly in } L^2(0,T;V), \quad f_{\delta_n}^* \to f^* \text{ weakly in } L^2(0,T;V^*). \tag{5.5}$$

Then, f^* is an optimal control of (OP)*, w^* is a solution of CP(f^*, u_0), and*

$$J(f^*) \ (= \inf_{f\in\mathcal{F}_M} J(f)) = \lim_{\delta\downarrow 0} E_\delta(w_\delta^*, f_\delta^*), \tag{5.6}$$

where J is the functional on $\mathcal{F}_M$ given by (1.3) *and* (1.4).

Proof. Taking account of Proposition 3.3, we can prove (I) of Theorem 5.1. Indeed, note from (5.1) and (5.2) that $E_\delta(w,f) \ge 0$ for all $[w,f] \in \mathcal{W}_{M'}(u_0)\times\mathcal{F}_M$. Let $\{[w_n, f_n]\}_{n\in\mathbb{N}} \subset \mathcal{W}_{M'}(u_0)\times\mathcal{F}_M$ be a minimizing sequence of the functional E_δ on $\mathcal{W}_{M'}(u_0)\times\mathcal{F}_M$, namely,

$$d_\delta^* := \inf_{[w,f]\in\mathcal{W}_{M'}(u_0)\times\mathcal{F}_M} E_\delta(w,f) = \lim_{n\to\infty} E_\delta(w_n, f_n).$$

By the definition in (5.1) of $E_\delta(w_n, f_n)$, for each n, there is a solution $u_n \in \mathcal{S}(w_n, f_n)$ such that

$$\pi^\delta_{[w_n,f_n]}(u_n) < E_\delta(w_n, f_n) + \frac{1}{n}. \tag{5.7}$$

Here, we observe from $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{W}_{M'}(u_0)$ and (5.3) that

$$\sup_{n\in\mathbb{N}} \int_0^T \psi_0^t(w_n'(t))dt \le M', \tag{5.8}$$

hence, by (2.1)

$$\{w_n\}_{n\in\mathbb{N}} \text{ is bounded in } W^{1,2}(0,T;V).$$

Similarly, by $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_M$ and (1.2), we have that

$$\{f_n\}_{n\in\mathbb{N}} \text{ is bounded in } W^{1,2}(0,T;V^*) \cap L^2(0,T;H).$$

Thus, by the Ascoli–Arzelà theorem and the Aubin compactness theorem (cf. [26, Chapter 1, Section 5]), there is a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ and a pair of functions $[w^*, f^*] \in \mathcal{W}_{M'}(u_0)\times\mathcal{F}_M$ such that

$$\left.\begin{array}{ll} w_{n_k} \to w^* & \text{weakly in } W^{1,2}(0,T;V), \\ & \text{in } C([0,T];H) \end{array}\right\} \tag{5.9}$$

$$\left. \begin{array}{ll} f_{n_k} \to f^* & \text{weakly in } W^{1,2}(0,T;V^*), \\ & \text{weakly in } L^2(0,T;H), \\ & \text{in } L^2(0,T;V^*) \end{array} \right\} \tag{5.10}$$

as $k \to \infty$.

Taking a subsequence if necessary, we infer from Proposition 3.3 with (5.8)–(5.10) that there is a solution u^* to $CP_{w^*}(f^*, u_0)$ on $[0,T]$ satisfying

$$u_{n_k} \to u^* \text{ in } C([0,T];V) \quad \text{as } k \to \infty. \tag{5.11}$$

Therefore, it follows from (5.7)–(5.11) and $u^* \in \mathcal{S}(w^*, f^*)$ that

$$\begin{aligned} d^*_\delta &= \inf_{[w,f]\in \mathcal{W}_{M'}(u_0)\times \mathcal{F}_M} E_\delta(w,f) \\ &\le E_\delta(w^*,f^*) = \inf_{u\in \mathcal{S}(w^*,f^*)} \pi^\delta_{[w^*,f^*]}(u) \\ &\le \pi^\delta_{[w^*,f^*]}(u^*) \\ &= \frac{1}{2}\int_0^T |u^*(t)-u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f^*(t)|^2_{V^*} dt \\ &\qquad\qquad\qquad\qquad + \frac{1}{2\delta}\int_0^T |u^*(t)-w^*(t)|^2_V dt \\ &\le \liminf_{k\to\infty} \pi^\delta_{[w_{n_k},f_{n_k}]}(u_{n_k}) \\ &\le \liminf_{k\to\infty} \left\{ E_\delta(w_{n_k},f_{n_k}) + \frac{1}{n_k} \right\} \\ &= \lim_{k\to\infty} E_\delta(w_{n_k},f_{n_k}) = d^*_\delta. \end{aligned}$$

Hence, we have $d^*_\delta = \inf_{[w,f]\in \mathcal{W}_{M'}(u_0)\times \mathcal{F}_M} E_\delta(w,f) = E_\delta(w^*,f^*)$, which implies that $[w^*,f^*] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M$ is an optimal control for $(\mathrm{OP})_\delta$. Thus, the proof of (I) is complete.

Note that assertion (II) provides another method of construction of solutions to the state system $CP(f^*, u_0)$. This is a very important and interesting result. Therefore, we give the detailed proof of (II), although this is a similar manner to the proof of [24, Theorem 10.3].

Let f be any element in $\mathcal{F}_M$, and let $\pi_f(\cdot)$ be the functional defined by (1.4). Additionally, let $\mathcal{S}(f)$ be the set of all solutions to $CP(f,u_0)$ on $[0,T]$. Then, note that $u \in \mathcal{S}(f)$ is also a solution to $CP_u(f,u_0)$ on $[0,T]$. Therefore, since $[w^*_\delta, f^*_\delta] \in \mathcal{W}_{M'}(u_0)\times \mathcal{F}_M$ is an optimal control of $(\mathrm{OP})_\delta$, we observe that

$$E_\delta(w^*_\delta, f^*_\delta) \le \pi^\delta_{[u,f]}(u) = \pi_f(u),\ \forall u \in \mathcal{S}(f),\ \forall f \in \mathcal{F}_M,$$

whence

$$E_\delta(w^*_\delta, f^*_\delta) \le \inf_{f\in\mathcal{F}_M} J(f) =: \tilde{d}^*. \tag{5.12}$$

Now, let u^*_δ be any optimal state corresponding to $E_\delta(w^*_\delta, f^*_\delta)$, namely $u^*_\delta \in \mathcal{S}(w^*_\delta, f^*_\delta)$ and

$$\begin{aligned} E_\delta(w^*_\delta, f^*_\delta) &= \frac{1}{2}\int_0^T |u^*_\delta(t)-u_{ad}(t)|^2_V dt + \frac{1}{2}\int_0^T |f^*_\delta(t)|^2_{V^*} dt + \frac{1}{2\delta}\int_0^T |u^*_\delta(t)-w^*_\delta(t)|^2_V dt \\ &\le \tilde{d}^*. \end{aligned} \tag{5.13}$$

As $w_\delta^* \in \mathcal{W}_{M'}(u_0)$, $u_\delta^* \in \mathcal{W}_{M'}(u_0)$ (cf. (5.4)), and $f_\delta^* \in \mathcal{F}_M$, according to the Ascoli–Arzelà theorem and the Aubin compactness theorem, there exist a subsequence $\{\delta_n\}_{n\in\mathbb{N}} \subset \{\delta\}_{\delta\in(0,1]}$, a pair of functions $[w^*, f^*] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M$, and a function $u^* \in \mathcal{W}_{M'}(u_0)$ such that $\delta_n \to 0$ as $n \to \infty$,

$$w_{\delta_n}^* \to w^* \text{ weakly in } W^{1,2}(0,T;V) \text{ and in } C([0,T];H),$$
$$w_{\delta_n}^*(t) \to w^*(t) \quad \text{weakly in } V, \ \forall t \in [0,T],$$
$$u_{\delta_n}^* \to u^* \text{ weakly in } W^{1,2}(0,T;V) \text{ and in } C([0,T];H),$$
$$u_{\delta_n}^*(t) \to u^*(t) \quad \text{weakly in } V, \ \forall t \in [0,T],$$

and

$$f_{\delta_n}^* \to f^* \quad \text{in } L^2(0,T;V^*) \tag{5.14}$$

as $n \to \infty$.

Therefore, we infer from (5.13) that $u_{\delta_n}^* - w_{\delta_n}^* \to 0$ in $L^2(0,T;V)$ as $n \to \infty$, which implies that

$$u^* = w^* \text{ in } L^2(0,T;V). \tag{5.15}$$

Moreover, by Proposition 3.3, we observe that $u_{\delta_n}^*$ converges in $C([0,T];V)$ to a solution to $CP_{w^*}(f^*, u_0)$ on $[0,T]$, that is equal to u^*, as $n \to \infty$. By (5.15), u^* is a solution to $CP_{u^*}(f^*, u_0)$, hence $CP(f^*, u_0)$, on $[0,T]$.

Now, taking the limit of (5.13) as $\delta := \delta_n \downarrow 0$, we obtain

$$\begin{aligned}
\tilde{d}^* \le J(f^*) &= \inf_{u\in\mathcal{S}(f^*)} \pi_{f^*}(u) \\
&\le \frac{1}{2}\int_0^T |u^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f^*(t)|_{V^*}^2 dt \\
&= \lim_{n\to\infty} \left\{ \frac{1}{2}\int_0^T |u_{\delta_n}^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f_{\delta_n}^*(t)|_{V^*}^2 dt \right\} \\
&\le \liminf_{n\to\infty} \left\{ \frac{1}{2}\int_0^T |u_{\delta_n}^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f_{\delta_n}^*(t)|_{V^*}^2 dt \right. \\
&\qquad\qquad \left. + \frac{1}{2\delta_n}\int_0^T |u_{\delta_n}^*(t) - w_{\delta_n}^*(t)|_V^2 dt \right\} \\
&= \liminf_{n\to\infty} E_{\delta_n}(w_{\delta_n}^*, f_{\delta_n}^*) \le \limsup_{n\to\infty} E_{\delta_n}(w_{\delta_n}^*, f_{\delta_n}^*) \le \tilde{d}^*,
\end{aligned}$$

and hence $\tilde{d}^* = J(f^*)$ with

$$\lim_{n\to\infty} \frac{1}{2\delta_n}\int_0^T |u_{\delta_n}^*(t) - w_{\delta_n}^*(t)|_V^2 dt = 0. \tag{5.16}$$

Hence, we conclude from (5.16) that (5.6) holds. Thus, the proof of (II) is complete. □

Remark 5.1. On account of (5.14)–(5.16), we further observe that the weak limit $[w^*, f^*]$ as in (5.5) is actually the strong limit of the same sequence $\{[w_{\delta_n}^*, f_{\delta_n}^*]\}_{n\in\mathbb{N}}$ in (5.5), in the topology of $L^2(0,T;V) \times L^2(0,T;V^*)$.

Note that Theorem 5.1 can be proved without using the result on (OP) in Section 4 (cf. Theorem 4.1). In addition, we observe that Theorem 5.1 provides the approximate procedure for $CP(f, u_0)$ and (OP) based on the theory of $CP_w(f, u_0)$ and $(\mathrm{OP})_\delta$ via parameter $\delta \in (0, 1]$. Moreover, note that $(\mathrm{OP})_\delta$ is also the singular optimal control problem.

In the rest of this section, we investigate $(\mathrm{OP})_\delta$, hence (OP), from the viewpoint of numerical analysis. To this end, we here use the approximate procedures proposed in [24, Section 5].

Now we choose and fix a function $v_0 \in D_0$, and consider the continuous convex function $\varphi^0(v_0; \cdot)$ on V. For simplicity, we denote by F_0 the subdifferential of $\varphi^0(v_0; \cdot)$:

$$F_0 := \partial_* \varphi^0(v_0; \cdot). \tag{5.17}$$

Then, on account of (3.1) of Lemma 3.1, we observe that $F_0 : D(F_0) = V \to V^*$ satisfies

$$c_0 |z|_V^2 \le \langle F_0 z, z \rangle \le c_0' |z|_V^2, \quad \forall z \in V, \tag{5.18}$$

with $c_0 := \frac{C_4}{|\alpha'|_{L^1(0,T)} + 1}$ and $c_0' = C_3$, so that F_0 is linear, coercive, continuous, and single-valued maximal monotone from V into V^*.

For each $u_0 \in V$, $\varepsilon \in (0, 1]$, $w \in D_0$, $f \in L^2(0, T; V^*)$, and $h \in L^2(0, T; V)$, we now consider the following problem, denoted by $CP_w^\varepsilon(f + \varepsilon F_0 h, u_0)$ (cf. (2.5)):

$$\begin{aligned} \varepsilon F_0 u_\varepsilon'(t) + \xi_\varepsilon(t) + \partial_* \varphi^t(w; u_\varepsilon(t)) + g(w; t, u_\varepsilon(t)) \ni f(t) + \varepsilon F_0 h(t) \ \text{ in } V^*, \\ \text{a.a. } t \in (0, T), \end{aligned} \tag{5.19a}$$

$$\xi_\varepsilon(t) \in \partial_* \psi^t(w; u_\varepsilon'(t)) \text{ in } V^*, \quad \text{a.a. } t \in (0, T), \tag{5.19b}$$

$$u_\varepsilon(0) = u_0 \ \text{ in } V. \tag{5.19c}$$

Proposition 5.1 (cf. [24, Section 5]). *Suppose that $u_0 \in V$ and assumptions (ψ), (ϕ), and (g) are fulfilled. Then, the following two statements hold.*
(I) *Let f be any function in $L^2(0, T; V^*)$, and let h be any function in $L^2(0, T; V)$. Then, for every $\varepsilon \in (0, 1]$ and $w \in D_0$, there is a unique function $u_\varepsilon \in W^{1,2}(0, T; V)$ with $\xi_\varepsilon \in L^2(0, T; V)$ such that* $(5.19) := \{(5.19a), (5.19b), (5.19c)\}$ *holds. Such a function u_ε is called a solution to $CP_w^\varepsilon(f + \varepsilon F_0 h, u_0)$. In addition, the solution u_ε to $CP_w^\varepsilon(f + \varepsilon F_0 h, u_0)$ satisfies the bound: there exists a constant $N_1 > 0$, independent of $\varepsilon > 0$, $w \in D_0$, $f \in L^2(0, T; V^*)$, and $h \in L^2(0, T; V)$ such that*

$$\begin{aligned} & \varepsilon C_4 |u_\varepsilon'|_{L^2(0,T;V)}^2 + \int_0^T \psi^t(w; u_\varepsilon'(t)) dt + \sup_{t \in [0,T]} \varphi^t(w; u_\varepsilon(t)) \\ \le \ & N_1 (|u_0|_V^2 + |f + \varepsilon F_0 h|_{L^2(0,T;V^*)}^2 + 1), \end{aligned} \tag{5.20}$$

where C_4 is the same constant as in ($\phi 3$) *as well as N_1 in* (3.2) *of Proposition 3.1.*
(II) *Let $\{w_n\}_{n \in \mathbb{N}} \subset D_0$ such that*

$$\sup_{n \in \mathbb{N}} \int_0^T \psi_0^t(w_n'(t)) dt < \infty, \quad w_n \to w \ \ \text{in } C([0, T]; H) \ \ (\text{as } n \to \infty).$$

Let $\{f_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V^*)$*,* $f \in L^2(0,T;V^*)$*,* $\{h_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V)$*, and* $h \in L^2(0,T;V)$ *such that*

$$f_n \to f \quad in\ L^2(0,T;V^*) \quad and \quad h_n \to h \quad in\ L^2(0,T;V) \quad as\ n \to \infty. \tag{5.21}$$

For a fixed parameter $\varepsilon \in (0,1]$*, let* u_n *be a unique solution to* $CP^{\varepsilon}_{w_n}(f_n + \varepsilon F_0 h_n, u_0)$ *on* $[0,T]$*. Then, there is a function* $u \in W^{1,2}(0,T;V)$ *such that* u *is a unique solution to* $CP^{\varepsilon}_{w}(f + \varepsilon F_0 h, u_0)$ *on* $[0,T]$ *and*

$$u_n \to u \quad in\ C([0,T];V) \quad as\ n \to \infty.$$

Proof. For each $\varepsilon > 0$ we put

$$\psi^t_{\varepsilon}(w;z) := \varepsilon\varphi^0(v_0;z) + \psi^t(w;z), \quad \forall z \in V. \tag{5.22}$$

As easily checked, the family $\{\psi^t_{\varepsilon}\}$ satisfies assumption (ψ) with ψ^t replaced by ψ^t_{ε}. In addition, from (5.22) and the general theory of maximal monotone operators of the subdifferential type, we observe that $CP^{\varepsilon}_{w}(f + \varepsilon F_0 h, u_0)$ can be regarded as $CP_w(\psi^t_{\varepsilon}, \varphi^t, g; f + \varepsilon F_0 h, u_0)$. Note that $f + \varepsilon F_0 h \in L^2(0,T;V^*)$. Therefore, applying Proposition 3.1 and Proposition 3.2, we observe that there is a unique solution $u_{\varepsilon} \in W^{1,2}(0,T;V)$ to $CP_w(\psi^t_{\varepsilon}, \varphi^t, g; f + \varepsilon F_0 h, u_0)$, i.e., $CP^{\varepsilon}_{w}(f + \varepsilon F_0 h, u_0)$, satisfying

$$\int_0^T \psi^t_{\varepsilon}(w; u'_{\varepsilon}(t))dt + \sup_{t\in[0,T]} \varphi^t(w; u_{\varepsilon}(t)) \le N_1(|u_0|^2_V + |f + \varepsilon F_0 h|^2_{L^2(0,T;V^*)} + 1).$$

Since $\varphi^0(v_0;z) \ge C_4|z|^2_V$ for all $z \in V$ by $(\phi 3)$, estimate (5.20) is immediately obtained from the above inequality. Thus, the proof of (I) is complete.

We show (II). To this end, note from the continuity of F_0 with (5.21) (cf. (5.18)) that

$$f_n + \varepsilon F_0 h_n \to f + \varepsilon F_0 h \ \text{ in } L^2(0,T;V^*) \text{ as } n \to \infty.$$

Since $CP^{\varepsilon}_{w_n}(f_n + \varepsilon F_0 h_n, u_0)$ can be regarded as $CP_{w_n}(\psi^t_{\varepsilon}, \varphi^t, g; f_n + \varepsilon F_0 h_n, u_0)$, by applying Proposition 3.3 and the uniqueness of solution to $CP^{\varepsilon}_{w}(f + \varepsilon F_0 h, u_0)$ (cf. Proposition 3.2), the assertion (II) holds. □

By similar arguments as in [24, Theorem 5.1], we can show the following result on the relationship between $CP_w(f, u_0)$ and $CP^{\varepsilon}_{w}(f + \varepsilon F_0 h, u_0)$ with respect to $w \in D_0$ and $\varepsilon \downarrow 0$.

Proposition 5.2 (cf. [24, Section 5])**.** *Suppose that* $u_0 \in V$ *and assumptions* (ψ)*,* (ϕ)*, and (g) are fulfilled. Then, the following two statements hold.*
(I) *Let* $\varepsilon \in (0,1]$*,* $\{w_{\varepsilon}\}_{\varepsilon\in(0,1]}$ *and* $\{f_{\varepsilon}\}_{\varepsilon\in(0,1]}$ *be any families in* D_0 *and* $L^2(0,T;V^*)$*, respectively, such that*

$$\begin{cases} \sup\limits_{\varepsilon\in(0,1]} \displaystyle\int_0^T \psi^t_0(w'_{\varepsilon}(t))dt < \infty, \quad w_{\varepsilon} \to w \ \ in\ C([0,T];H), \\ f_{\varepsilon} \to f \quad in\ L^2(0,T;V^*) \quad as\ \varepsilon \downarrow 0. \end{cases}$$

Let $\{h_\varepsilon\}_{\varepsilon\in(0,1]}$ be a bounded set in $L^2(0,T;V)$. Additionally, let u_ε be a unique solution to $CP^\varepsilon_{w_\varepsilon}(f_\varepsilon+\varepsilon F_0 h_\varepsilon, u_0)$ on $[0,T]$. Then, there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}\subset\{\varepsilon\}_{\varepsilon\in(0,1]}$ with $\varepsilon_n\to 0$ (as $n\to\infty$) and a function $u\in W^{1,2}(0,T;V)$ such that u is a solution to $CP_w(f,u_0)$ on $[0,T]$ and

$$u_{\varepsilon_n}\to u \quad \text{in } C([0,T];V) \quad \text{as } n\to\infty.$$

(II) *Let u be any solution to $CP_w(f,u_0)$ on $[0,T]$. Then, there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}\subset(0,1]$ with $\varepsilon_n\to 0$ (as $n\to\infty$), $\{w_n\}_{n\in\mathbb{N}}\subset D_0$, $\{f_n\}_{n\in\mathbb{N}}\subset L^2(0,T;V^*)$, $\{h_n\}_{n\in\mathbb{N}}\subset L^2(0,T;V)$, and $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}}\subset W^{1,2}(0,T;V)$ such that u_{ε_n} is a unique solution to $CP^{\varepsilon_n}_{w_n}(f_n+\varepsilon_n F_0 h_n, u_0)$ on $[0,T]$,*

$$\sup_{n\in\mathbb{N}}|h_n|_{L^2(0,T;V)}\le |u'|_{L^2(0,T;V)},$$

$$\sup_{n\in\mathbb{N}}\int_0^T\psi_0^t(w_n'(t))dt<\infty, \quad w_n\to w \quad \text{in } L^2(0,T;V),$$

$$u_{\varepsilon_n}\to u \quad \text{in } C([0,T];V), \quad f_n\to f \quad \text{in } L^2(0,T;V^*) \quad \text{as } n\to\infty.$$

Proof. Now, we show (I). Note from Proposition 5.1 that the uniform estimate (5.20) holds and u_ε satisfies that

$$\varepsilon F_0 u_\varepsilon'(t)+\xi_\varepsilon(t)+\partial_*\varphi^t(w_\varepsilon;u_\varepsilon(t))+g(w_\varepsilon;t,u_\varepsilon(t))=f_\varepsilon(t)+\varepsilon F_0 h_\varepsilon \ \text{ in } V^*, \ \text{a.a. } t\in(0,T),$$

namely

$$\begin{aligned}\xi_\varepsilon(t)+\partial_*\varphi^t(w_\varepsilon;u_\varepsilon(t))+g(w_\varepsilon;t,u_\varepsilon(t))=f_\varepsilon(t)+\varepsilon F_0 h_\varepsilon-\varepsilon F_0 u_\varepsilon'(t) \ \text{ in } V^*,\\ \text{a.a. } t\in(0,T),\end{aligned} \tag{5.23}$$

with

$$\xi_\varepsilon\in L^2(0,T;V^*), \ \xi_\varepsilon(t)\in\partial_*\psi^t(w_\varepsilon;u_\varepsilon'(t)) \text{ in } V^*, \ \text{a.a. } t\in(0,T), \quad u_\varepsilon(0)=u_0 \text{ in } V,$$

namely, u_ε can be regarded as a solution to $CP_{w_\varepsilon}(f_\varepsilon+\varepsilon F_0 h_\varepsilon-\varepsilon F_0 u_\varepsilon', u_0)$.

From (5.18) and the uniform estimate (5.20), we observe that

$$\begin{aligned}&\varepsilon C_4|u_\varepsilon'|^2_{L^2(0,T;V)}+\int_0^T\psi^t(w_\varepsilon;u_\varepsilon'(t))dt+\sup_{t\in[0,T]}\varphi^t(w_\varepsilon;u_\varepsilon(t))\\ \le\ &\tilde N_1(|u_0|_V^2+|f_\varepsilon|^2_{L^2(0,T:V^*)}+\varepsilon^2|h_\varepsilon|^2_{L^2(0,T;V)}+1),\end{aligned} \tag{5.24}$$

where $\tilde N_1$ is a positive constant dependent on N_1 and $c_0'=C_3$. Therefore, it follows from (ψ2) with (2.1), (3.1), (5.24), and the Ascoli–Arzelà theorem, we can find a sequence $\{\varepsilon_n\}_{\varepsilon\in\mathbb{N}}$ with $\varepsilon_n\downarrow 0$ (as $n\to\infty$) and a function $u\in W^{1,2}(0,T;V)$ such that

$$u_{\varepsilon_n}\to u \ \text{ in } C([0,T];H) \text{ and weakly in } W^{1,2}(0,T;V),$$

$$u_{\varepsilon_n}(t)\to u(t) \text{ weakly in } V, \quad \forall t\in[0,T],$$

as $n\to\infty$. Since $f_{\varepsilon_n}+\varepsilon_n F_0 h_{\varepsilon_n}-\varepsilon_n F_0 u'_{\varepsilon_n}\to f$ in $L^2(0,T;V^*)$ (as $n\to\infty$) (cf. (5.18)), it follows from Proposition 3.3 that the limit u is a solution of $CP_w(f,u_0)$. Thus, the proof of (I) is complete.

Next, we show (II). To this end, let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1]$ be a sequence with $\varepsilon_n \to 0$ as $n \to \infty$. Additionally, let u be any solution to $CP_w(f,u_0)$ on $[0,T]$. Note that $u \in W^{1,2}(0,T;V)$ and the following evolution inclusion holds:

$$\partial_*\psi^t(w;u'(t)) + \partial_*\varphi^t(w;u(t)) + g(w;t,u(t)) \ni f(t) \text{ in } V^* \text{ for a.a. } t \in (0,T). \tag{5.25}$$

Adding $\varepsilon_n F_0 u'(t)$ to both sides in (5.25), we observe that the function u is also a solution to $CP_w^{\varepsilon_n}(f+\varepsilon_n F_0 u', u_0)$ on $[0,T]$. Hence, we conclude that assertion (II) holds for $u_{\varepsilon_n} := u$, $w_n := w$, $f_n := f$, and $h_n := u'$.

Thus, Proposition 5.2 has been proved. □

Next, we consider an approximate problem for $(\mathrm{OP})_\delta$, fixing a parameter $\delta \in (0,1]$. Note from $(\psi 2)$, (2.1), and (3.2) that any solution u to $CP_w(f,u_0)$ on $[0,T]$ satisfies the following estimate:

$$\int_0^T |u'(t)|_V^2 dt \le \frac{N_1\left(|u_0|_V^2 + |f|^2_{L^2(0,T;V^*)} + 1\right) + C_2 T}{C_1}. \tag{5.26}$$

Therefore, we take and fix a positive number $N > 0$ so that

$$N^2 \ge \frac{N_1\left(|u_0|_V^2 + M^2 + 1\right) + C_2 T}{C_1}, \tag{5.27}$$

where $M > 0$ is the same positive constant as in the control space $\mathcal{F}_M$ (cf. (1.2)).

For each $\varepsilon \in (0,1]$, we consider a perturbation of the control space $\mathcal{H}_N^\varepsilon$ defined by

$$\mathcal{H}_N^\varepsilon := \left\{ h \in W^{1,2}(0,T;V) \cap L^2(0,T;X) \;\middle|\; \begin{array}{c} |h|_{L^2(0,T;V)} \le N, \\ |h'|_{L^2(0,T;V)} \le \varepsilon^{-1} N, \\ |h|_{L^2(0,T;X)} \le \varepsilon^{-1} N \end{array} \right\}, \tag{5.28}$$

where X is a reflexive Banach space such that X is densely and compactly embedded into V.

Now, for each $\delta \in (0,1]$ and $\varepsilon \in (0,1]$, we study the following control problem for the state system $CP_w^\varepsilon(f+\varepsilon F_0 h, u_0)$, denoted by $(\mathrm{OP})_\delta^\varepsilon$:

Problem $(\mathrm{OP})_\delta^\varepsilon$: Find a triplet of control functions $[w^*_{\delta,\varepsilon}, f^*_{\delta,\varepsilon}, h^*_{\delta,\varepsilon}] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$, called an optimal control, such that

$$E_{\delta,\varepsilon}(w^*_{\delta,\varepsilon}, f^*_{\delta,\varepsilon}, h^*_{\delta,\varepsilon}) = \inf_{[w,f,h]\in\mathcal{W}_{M'}(u_0)\times\mathcal{F}_M\times\mathcal{H}_N^\varepsilon} E_{\delta,\varepsilon}(w,f,h).$$

Here, $E_{\delta,\varepsilon}(w,f,h)$ is the cost functional defined by

$$\begin{aligned} E_{\delta,\varepsilon}(w,f,h) := \; & \frac{1}{2}\int_0^T |u_\varepsilon(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f(t)|_{V^*}^2 dt \\ & + \frac{1}{2\delta}\int_0^T |u_\varepsilon(t) - w(t)|_V^2 dt + \frac{\varepsilon}{2}\int_0^T |h(t)|_V^2 dt, \end{aligned} \tag{5.29}$$

where $[w, f, h]$ is any control in $\mathcal{W}_{M'}(u_0) \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$, u_ε is a unique solution to the state system $CP_w^\varepsilon(f + \varepsilon F_0 h, u_0)$, and $u_{ad} \in L^2(0,T;V)$ is the given target profile.

Note that $(\mathrm{OP})_\delta^\varepsilon$ is the standard optimal control problem, because the state system $CP_w^\varepsilon(f + \varepsilon F_0 h, u_0)$ has a unique solution on $[0,T]$ (cf. Proposition 3.2). Using similar idea as in [24, Theorem 5.2], we show the following result on the relationship between $(\mathrm{OP})_\delta$ and $(\mathrm{OP})_\delta^\varepsilon$ with respect to $\varepsilon \in (0,1]$.

Theorem 5.2 (cf. [24, Theorem 5.2]). *Suppose that $u_0 \in V$ and assumptions (ψ), (ϕ), and (g) are fulfilled. Let $u_{ad} \in L^2(0,T;V)$ and $\delta \in (0,1]$. Then, we have:*
(I) *For each $\varepsilon \in (0,1]$, $(\mathrm{OP})_\delta^\varepsilon$ has at least one optimal control $[w_{\delta,\varepsilon}^*, f_{\delta,\varepsilon}^*, h_{\delta,\varepsilon}^*] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$, namely,*

$$E_{\delta,\varepsilon}(w_{\delta,\varepsilon}^*, f_{\delta,\varepsilon}^*, h_{\delta,\varepsilon}^*) = \inf_{[w,f,h] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon} E_{\delta,\varepsilon}(w, f, h).$$

(II) *Let $\varepsilon \in (0,1]$, and let $[w_{\delta,\varepsilon}^*, f_{\delta,\varepsilon}^*, h_{\delta,\varepsilon}^*] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ be an optimal control of the approximate problem $(\mathrm{OP})_\delta^\varepsilon$. Additionally, assume that*

- (H) *For any function $h \in L^2(0,T;V)$ with $|h|_{L^2(0,T;V)} \le N$, there exists a sequence $\{h_\varepsilon\}_{\varepsilon \in (0,1]}$ of functions $h_\varepsilon \in \mathcal{H}_N^\varepsilon$ such that*

$$h_\varepsilon \to h \quad \text{in } L^2(0,T;V) \text{ as } \varepsilon \to 0.$$

Then, there exists a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset \{\varepsilon\}_{\varepsilon \in (0,1]}$ with $\varepsilon_n \to 0$ $(n \to \infty)$ such that any weak limit function $[w_\delta^*, f_\delta^*]$ of $\{[w_{\delta,\varepsilon_n}^*, f_{\delta,\varepsilon_n}^*]\}_{n \in \mathbb{N}}$ in $L^2(0,T;V) \times L^2(0,T;V^*)$ is an optimal control for $(\mathrm{OP})_\delta$.

Remark 5.2 (cf. [24, Remark 5.1]). The main point of assumption (H) is to guarantee the compactness of $\mathcal{H}_N^\varepsilon$ in $L^2(0,T;V)$. In any application treated in Section 6, assumption (H) is automatically checked by the usual smoothness argument (e.g., the regularization method using the mollifier and the convolution [1, Sections 2.28 and 3.16]). For instance, in the case of $V = W^{1,p}(\Omega)$, $2 \le p < \infty$, assumption (H) is easily verified by choosing $W^{2,p}(\Omega)$ as the space X.

The following convergence result for solutions is a key component in the proof of Theorem 5.2(II).

Proposition 5.3. *Suppose that $u_0 \in V$ and assumptions (ψ), (ϕ), (g), and (H) are fulfilled. Let $w \in \mathcal{W}_{M'}(u_0)$, $f \in \mathcal{F}_M$, and let u be any solution to $CP_w(f, u_0)$ on $[0,T]$. Then, there are sequences $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset (0,1]$ with $\varepsilon_n \to 0$ (as $n \to \infty$), $\{w_n\}_{n \in \mathbb{N}} \subset \mathcal{W}_{M'}(u_0)$, $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_M$, $\{h_n\}_{n \in \mathbb{N}} \subset L^2(0,T;V)$ with $h_n \in \mathcal{H}_N^{\varepsilon_n}$, and $\{u_{\varepsilon_n}\}_{n \in \mathbb{N}} \subset W^{1,2}(0,T;V)$ such that u_{ε_n} is a unique solution to $CP_{w_n}^{\varepsilon_n}(f_n + \varepsilon_n F_0 h_n, u_0)$ on $[0,T]$, and*

$$\sup_{n \in \mathbb{N}} \int_0^T \psi_0^t(w_n'(t))dt < \infty, \quad w_n \to w \quad \text{in } L^2(0,T;V),$$

$$u_{\varepsilon_n} \to u \quad \text{in } C([0,T];V), \quad f_n \to f \quad \text{in } L^2(0,T;V^*) \quad \text{as } n \to \infty.$$

Proof. Let u be any solution to $CP_w(f, u_0)$ on $[0, T]$. Then, note from (5.26) and (5.27) that the solution u satisfies the following:

$$|u'|_{L^2(0,T;V)} \leq N, \tag{5.30}$$

$$\partial_* \psi^t(w; u'(t)) + \partial_* \varphi^t(w; u(t)) + g(w; t, u(t)) \ni f(t) \text{ in } V^* \text{ for a.a. } t \in (0, T). \tag{5.31}$$

Let $\delta \in (0, 1]$ be any constant. Adding $\delta F_0 u'(t)$ to both sides of (5.31), we observe that the function u is also a solution to $CP_w^\delta(f + \delta F_0 u', u_0)$ on $[0, T]$ (cf. Proposition 5.2(II)).

By (5.30) and assumption (H), there exist a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, 1]$ with $\varepsilon_k \to 0$ and a sequence $\{h_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of functions $h_{\varepsilon_k} \in \mathcal{H}_N^{\varepsilon_k}$ such that

$$h_{\varepsilon_k} \to u' \text{ in } L^2(0, T; V) \text{ as } k \to \infty. \tag{5.32}$$

Let $\{\delta_\ell\}_{\ell\in\mathbb{N}}$ be a sequence in $(0, 1]$ so that $\delta_\ell \to 0$ as $\ell \to \infty$. Now, for a fixed number δ_ℓ, we consider the approximate system $CP_w^{\delta_\ell}(f + \delta_\ell F_0 h_{\varepsilon_k}, u_0)$ on $[0, T]$. Then, taking a subsequence if necessary (still denoted by $\{\varepsilon_k\}_{k\in\mathbb{N}}$), we observe from Proposition 5.1(II) with (5.32) that a unique solution $u_{\varepsilon_k}^\ell \in W^{1,2}(0, T; V)$ to $CP_w^{\delta_\ell}(f + \delta_\ell F_0 h_{\varepsilon_k}, u_0)$ on $[0, T]$ converges to the one $\tilde{u}^\ell$ to $CP_w^{\delta_\ell}(f + \delta_\ell F_0 u', u_0)$ on $[0, T]$ in the following sense:

$$u_{\varepsilon_k}^\ell \to \tilde{u}^\ell \text{ in } C([0, T]; V) \text{ as } k \to \infty.$$

As u is also a solution to $CP_w^{\delta_\ell}(f + \delta_\ell F_0 u', u_0)$ on $[0, T]$, we infer from the uniqueness of solutions to $CP_w^{\delta_\ell}(f + \delta_\ell F_0 u', u_0)$ that $u = \tilde{u}^\ell$, and hence

$$u_{\varepsilon_k}^\ell \to u \text{ in } C([0, T]; V) \text{ as } k \to \infty.$$

Note from $h_{\varepsilon_k} \in \mathcal{H}_N^{\varepsilon_k}$ that $\{h_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is bounded in $L^2(0, T; V)$; more precisely,

$$|h_{\varepsilon_k}|_{L^2(0,T;V)} \leq N \text{ for all } k \geq 1.$$

Therefore, from the diagonal argument with respect to the parameters k and ℓ, we verify the validity of Proposition 5.3. Indeed, taking $\delta_n := \varepsilon_n$, we derive the convergence by setting $u_{\varepsilon_n} := u_{\varepsilon_n}^n$, $w_n := w$, $f_n := f$, and $h_n := h_{\varepsilon_n}$. Thus, the proof of Proposition 5.3 is complete. □

Now, let us prove Theorem 5.2, which is concerned with the relationship between $(\mathrm{OP})_\delta$ and $(\mathrm{OP})_\delta^\varepsilon$ with respect to $\varepsilon \in (0, 1]$.

Proof of Theorem 5.2. We first prove (I). Using the standard argument with Proposition 5.1(II), we can show (I) concerning the existence of an optimal control for $(\mathrm{OP})_\delta^\varepsilon$. Indeed, let $\delta \in (0, 1]$ and $\varepsilon \in (0, 1]$ be fixed. Then, we observe from (5.29) that $E_{\delta,\varepsilon}(w, f, h) \geq 0$ for all $[w, f, h] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$. Let $\{[w_n, f_n, h_n]\}_{n\in\mathbb{N}} \subset \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ be a minimizing sequence such that

$$d_{\delta,\varepsilon}^* := \inf_{[w,f,h]\in\mathcal{W}_{M'}(u_0)\times\mathcal{F}_M\times\mathcal{H}_N^\varepsilon} E_{\delta,\varepsilon}(w, f, h) = \lim_{n\to\infty} E_{\delta,\varepsilon}(w_n, f_n, h_n).$$

Here, we observe from $\{[w_n, f_n, h_n]\}_{n\in\mathbb{N}} \subset \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$, (5.3), (1.2), and (5.28) that

$$\sup_{n\in\mathbb{N}} \int_0^T \psi_0^t(w_n'(t))dt \le M', \tag{5.33}$$

$$\{w_n\}_{n\in\mathbb{N}} \text{ is bounded in } W^{1,2}(0,T;V),$$

$$\{f_n\}_{n\in\mathbb{N}} \text{ is bounded in } W^{1,2}(0,T;V^*) \cap L^2(0,T;H),$$

$$\{h_n\}_{n\in\mathbb{N}} \text{ is bounded in } W^{1,2}(0,T;V) \cap L^2(0,T;X).$$

Thus, with the help of the Ascoli–Arzelà theorem and the Aubin compactness theorem (cf. [26, Chapter 1, Section 5]), there exist a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ and a triplet of functions $[w^*, f^*, h^*] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ such that

$$\left. \begin{array}{ll} w_{n_k} \to w^* & \text{weakly in } W^{1,2}(0,T;V), \\ & \text{in } C([0,T];H), \end{array} \right\}, \tag{5.34}$$

$$\left. \begin{array}{ll} f_{n_k} \to f^* & \text{weakly in } W^{1,2}(0,T;V^*), \\ & \text{weakly in } L^2(0,T;H), \\ & \text{in } L^2(0,T;V^*), \end{array} \right\}, \tag{5.35}$$

and

$$\left. \begin{array}{ll} h_{n_k} \to h^* & \text{weakly in } W^{1,2}(0,T;V), \\ & \text{weakly in } L^2(0,T;X), \\ & \text{in } L^2(0,T;V), \end{array} \right\}, \tag{5.36}$$

hence, by the continuity of F_0 (cf. (5.18)),

$$F_0 h_{n_k} \to F_0 h^* \ \text{ in } L^2(0,T;V^*), \tag{5.37}$$

as $k \to \infty$.

Let u_{n_k} be a unique solution to $CP_{w_{n_k}}^\varepsilon(f_{n_k} + \varepsilon F_0 h_{n_k}, u_0)$ on $[0,T]$. Then, we infer from Proposition 5.1(II) with (5.33)–(5.37) that there is a unique solution u^* to $CP_{w^*}^\varepsilon(f^* + \varepsilon F_0 h^*, u_0)$, on $[0,T]$ satisfying

$$u_{n_k} \to u^* \ \text{ in } C([0,T];V) \ \text{ as } k \to \infty. \tag{5.38}$$

Therefore, it follows from (5.34)–(5.38) and the weak lower semicontinuity of $L^2(0,T;V)$-norm, we observe that

$$\begin{aligned} E_{\delta,\varepsilon}(w^*, f^*, h^*) &\le \liminf_{k\to\infty} E_{\delta,\varepsilon}(w_{n_k}, f_{n_k}, h_{n_k}) = \lim_{k\to\infty} E_{\delta,\varepsilon}(w_{n_k}, f_{n_k}, h_{n_k}) \\ &= \inf_{[w,f,h]\in\mathcal{W}_{M'}(u_0)\times\mathcal{F}_M\times\mathcal{H}_N^\varepsilon} E_{\delta,\varepsilon}(w,f,h) \\ &= d_{\delta,\varepsilon}^*, \end{aligned}$$

which implies that $[w^*, f^*, h^*] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ is an optimal control for $(\mathrm{OP})_\delta^\varepsilon$. Thus, the proof of (I) is complete.

Next, we prove (II) by approximating the admissible optimal pair for $(\mathrm{OP})_\delta$.

Define $d_\delta^* := \inf_{[w,f]\in \mathcal{W}_{M'}(u_0)\times \mathcal{F}_M} E_\delta(w,f)$ and let $[\tilde{w}^*, \tilde{f}^*]$ be any optimal control for $(\mathrm{OP})_\delta$ with its optimal state $\tilde{u}^*$, namely, $\tilde{u}^* \in S(\tilde{w}^*, \tilde{f}^*)$ and

$$\begin{aligned} d_\delta^* &= E_\delta(\tilde{w}^*, \tilde{f}^*) \\ &= \frac{1}{2}\int_0^T |\tilde{u}^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |\tilde{f}^*(t)|_{V^*}^2 dt + \frac{1}{2\delta}\int_0^T |\tilde{u}^*(t) - \tilde{w}^*(t)|_V^2 dt. \end{aligned}$$

Now, we approximate the admissible optimal triplet $[\tilde{u}^*, \tilde{w}^*, \tilde{f}^*]$ by applying Proposition 5.3. Indeed, we observe from Proposition 5.3 that there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1]$ with $\varepsilon_n \to 0$ (as $n \to \infty$), $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{W}_{M'}(u_0)$, $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_M$, $\{h_n\}_{n\in\mathbb{N}} \subset L^2(0,T;V)$ with $h_n \in \mathcal{H}_N^{\varepsilon_n}$, and $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}} \subset W^{1,2}(0,T;V)$ such that u_{ε_n} is a unique solution to $CP_{w_n}^{\varepsilon_n}(f_n + \varepsilon_n F_0 h_n, u_0)$ on $[0,T]$,

$$\sup_{n\in\mathbb{N}} \int_0^T \psi_0^t(w_n'(t))dt < \infty, \quad w_n \to \tilde{w}^* \text{ in } L^2(0,T;V), \tag{5.39}$$

$$u_{\varepsilon_n} \to \tilde{u}^* \text{ in } C([0,T];V), \tag{5.40}$$

and

$$f_n \to \tilde{f}^* \text{ in } L^2(0,T;V^*), \tag{5.41}$$

as $n \to \infty$.

Note from $h_n \in \mathcal{H}_N^{\varepsilon_n}$ that $\{h_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;V)$; more precisely,

$$|h_n|_{L^2(0,T;V)} \le N \text{ for all } n \ge 1.$$

Therefore, from (5.1), (5.2), (5.29), and (5.39)–(5.41), it follows that

$$\begin{aligned} d_\delta^* &= E_\delta(\tilde{w}^*, \tilde{f}^*) \\ &= \frac{1}{2}\int_0^T |\tilde{u}^*(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |\tilde{f}^*(t)|_{V^*}^2 dt \\ &\qquad + \frac{1}{2\delta}\int_0^T |\tilde{u}^*(t) - \tilde{w}^*(t)|_V^2 dt \\ &= \lim_{n\to\infty} \left\{ \frac{1}{2}\int_0^T |u_{\varepsilon_n}(t) - u_{ad}(t)|_V^2 dt + \frac{1}{2}\int_0^T |f_n(t)|_{V^*}^2 dt \right. \\ &\qquad \left. + \frac{1}{2\delta}\int_0^T |u_{\varepsilon_n}(t) - w_n(t)|_V^2 dt \right\} \\ &= \lim_{n\to\infty} E_{\delta,\varepsilon_n}(w_n, f_n, h_n) \\ &\ge \limsup_{n\to\infty} d_{\delta,\varepsilon_n}^*, \end{aligned} \tag{5.42}$$

where $d_{\delta,\varepsilon_n}^* := \inf_{[w,f,h]\in \mathcal{W}_{M'}(u_0)\times\mathcal{F}_M\times\mathcal{H}_N^{\varepsilon_n}} E_{\delta,\varepsilon_n}(w,f,h)$.

Now, let $\{[w_{\varepsilon_n}^*, f_{\varepsilon_n}^*, h_{\varepsilon_n}^*]\}_{n\in\mathbb{N}}$ be any sequence of optimal controls $[w_{\varepsilon_n}^*, f_{\varepsilon_n}^*, h_{\varepsilon_n}^*]$ for $(\mathrm{OP})_\delta^{\varepsilon_n}$. In addition, let $u_{\varepsilon_n}^*$ be a unique solution to $CP_{w_{\varepsilon_n}^*}^{\varepsilon_n}(f_{\varepsilon_n}^* + \varepsilon_n F_0 h_{\varepsilon_n}^*, u_0)$ on $[0,T]$. Then, it follows from $[w_{\varepsilon_n}^*, f_{\varepsilon_n}^*, h_{\varepsilon_n}^*] \in \mathcal{W}_{M'}(u_0)\times\mathcal{F}_M\times\mathcal{H}_N^{\varepsilon_n}$, (5.3), (1.2), and (5.28) that

$$\sup_{n\in\mathbb{N}} \int_0^T \psi_0^t((w_{\varepsilon_n}^*)'(t))dt \le M', \tag{5.43}$$

$$\begin{aligned}\{w^*_{\varepsilon_n}\}_{n\in\mathbb{N}} &\text{ is bounded in } W^{1,2}(0,T;V),\\ \{f^*_{\varepsilon_n}\}_{n\in\mathbb{N}} &\text{ is bounded in } W^{1,2}(0,T;V^*)\cap L^2(0,T;H),\\ \{h^*_{\varepsilon_n}\}_{n\in\mathbb{N}} &\text{ is bounded in } L^2(0,T;V).\end{aligned}\tag{5.44}$$

Therefore, by the Ascoli–Arzelà theorem and the Aubin compactness theorem (cf. [26, Chapter 1, Section 5]), there exist a subsequence $\{n_k\}_{k\in\mathbb{N}}\subset\{n\}_{n\in\mathbb{N}}$ and a pair of functions $[w^*,f^*]\in\mathcal{W}_{M'}(u_0)\times\mathcal{F}_M$ such that $\varepsilon_{n_k}\downarrow 0$ and

$$w^*_{\varepsilon_{n_k}}\to w^*\quad \left.\begin{array}{l}\text{weakly in } W^{1,2}(0,T;V),\\ \text{in } C([0,T];H),\end{array}\right\}\tag{5.45}$$

$$f^*_{\varepsilon_{n_k}}\to f^*\quad \left.\begin{array}{l}\text{weakly in } W^{1,2}(0,T;V^*),\\ \text{weakly in } L^2(0,T;H),\\ \text{in } L^2(0,T;V^*),\end{array}\right\}\tag{5.46}$$

as $k\to\infty$. Then, taking a subsequence if necessary, we infer from Proposition 5.2(I) with (5.43)–(5.46) that there is a solution u^* to $CP_{w^*}(f^*,u_0)$ on $[0,T]$ satisfying

$$u^*_{\varepsilon_{n_k}}\to u^*\ \text{ in } C([0,T];V)\ \text{ as } k\to\infty.\tag{5.47}$$

Next, taking a subsequence if necessary, we choose a subsequence of $\{n_k\}_{k\in\mathbb{N}}$ (still denoted by $\{n_k\}_{k\in\mathbb{N}}$) so that

$$\liminf_{n\to\infty} d^*_{\delta,\varepsilon_n}=\lim_{k\to\infty} d^*_{\delta,\varepsilon_{n_k}}.$$

Therefore, it follows from (5.1), (5.2), (5.45)–(5.47), and $u^*\in\mathcal{S}(w^*,f^*)$ that

$$\begin{aligned}&\liminf_{n\to\infty} d^*_{\delta,\varepsilon_n}\\ =&\lim_{k\to\infty} d^*_{\delta,\varepsilon_{n_k}}=\lim_{k\to\infty} E_{\delta,\varepsilon_{n_k}}(w^*_{\varepsilon_{n_k}},f^*_{\varepsilon_{n_k}},h^*_{\varepsilon_{n_k}})\\ =&\liminf_{k\to\infty}\left\{\frac12\int_0^T|u^*_{\varepsilon_{n_k}}(t)-u_{ad}(t)|_V^2dt+\frac12\int_0^T|f^*_{\varepsilon_{n_k}}(t)|_{V^*}^2dt\right.\\ &\qquad\left.+\frac{1}{2\delta}\int_0^T|u^*_{\varepsilon_{n_k}}(t)-w^*_{\varepsilon_{n_k}}(t)|_V^2dt+\frac{\varepsilon_{n_k}}{2}\int_0^T|h^*_{\varepsilon_{n_k}}(t)|_V^2dt\right\}\\ \ge&\frac12\int_0^T|u^*(t)-u_{ad}(t)|_V^2dt+\frac12\int_0^T|f^*(t)|_{V^*}^2dt+\frac{1}{2\delta}\int_0^T|u^*(t)-w^*(t)|_V^2dt\\ =&\pi^\delta_{[w^*,f^*]}(u^*)\\ \ge&E_\delta(w^*,f^*)\\ \ge&d^*_\delta.\end{aligned}\tag{5.48}$$

On account of (5.48) and inequality (5.42), we conclude that

$$d^*_\delta=\lim_{n\to\infty} d^*_{\delta,\varepsilon_n}=E_\delta(w^*,f^*).$$

Hence, $[w^*,f^*]\in\mathcal{W}_{M'}(u_0)\times\mathcal{F}_M$ is an optimal control for $(\mathrm{OP})_\delta$ and u^* is its optimal state. Thus, the proof of Theorem 5.2 is complete. □

Note that Theorem 5.1 is the result of relationship between (OP) and $(\mathrm{OP})_\delta$ with respect to $\delta \in (0,1]$. In addition, Theorem 5.2 is the result of relationship between $(\mathrm{OP})_\delta$ and $(\mathrm{OP})_\delta^\varepsilon$ with respect to $\varepsilon \in (0,1]$. Therefore, it is worth studying $(\mathrm{OP})_\delta^\varepsilon$ for sufficient small $\delta, \varepsilon \in (0,1]$, instead of considering (OP), from a numerical point of view.

Remark 5.3. In this Section 5, without the assumption (3.3) in Proposition 3.2 (or (3.4)–(3.5) in Remark 3.1), we consider the singular optimal control problem for $CP_w(f, u_0)$. If we suppose (3.3) (or (3.4)–(3.5)), then $CP_w(f, u_0)$ has a unique solution, therefore, the corresponding optimal control problem $(\mathrm{OP})_\delta$ is of the standard type for all $\delta \in (0,1]$. Hence, problems $CP(f, u_0)$ and (OP) can be approximated by the well-posed state system $CP_w(f, u_0)$ and the standard optimal control problem $(\mathrm{OP})_\delta$, respectively, namely, we don't have to consider approximate problems $CP_w^\varepsilon(f + \varepsilon F_0 h, u_0)$ and $(\mathrm{OP})_\delta^\varepsilon$ ($\varepsilon \in (0,1]$) under (3.3) (or (3.4)–(3.5)), for instance.

6 Applications

In this section we give applications of general results (Theorems 4.1, 5.1, and 5.2). Indeed, we consider doubly nonlinear quasi-variational inequalities with time-dependent constraints, in which the quasi-variational structure appears in our class of doubly nonlinear evolution inclusions, more concretely, in the v-dependence of $\psi^t(v;\cdot)$, $\varphi^t(v;\cdot)$, and $g(v;\cdot,\cdot)$.

6.1 Doubly nonlinear quasi-variational evolution inclusions

Let Ω be a bounded domain in $\mathbb{R}^N$ ($1 \le N < \infty$) with a smooth boundary $\Gamma := \partial\Omega$, and let $Q := \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$ for $0 < T < \infty$. Put

$$V := H_0^1(\Omega), \quad H := L^2(\Omega), \quad V^* := H^{-1}(\Omega), \quad X := H^2(\Omega);$$

we employ $|z|_V := |\nabla z|_H$ as the norm of V, and $\langle \cdot, \cdot \rangle$ stands for the duality pairing between V^* and V. In this subsection, we treat a quasi-variational inequality with gradient constraint for time derivatives (cf. [25, Section 4]).

Let ρ be a prescribed obstacle function such that

$$\begin{gathered} \rho := \rho(x,t,r) \in C(\overline{Q} \times \mathbb{R}), \\ 0 < \rho_* \le \rho(x,t,r) \le \rho^*, \quad \forall (x,t,r) \in \overline{Q} \times \mathbb{R}, \\ |\rho(x,t_1,r_1) - \rho(x,t_2,r_2)| \le L_\rho(|t_1 - t_2| + |r_1 - r_2|), \\ \forall t_i \in [0,T],\ \forall r_i \in \mathbb{R},\ i = 1,2,\ \forall x \in \overline{\Omega}, \end{gathered} \tag{6.1}$$

where ρ_*, ρ^*, and L_ρ are positive constants.

Now we consider some applications of Theorems 4.1, 5.1, and 5.2.

(Application 1)

For each $t \in [0,T]$ and $v \in C([0,T];H)$, we define a convex constraint set $K(v;t)$ in V by

$$K(v;t) := \{z \in V \ ; \ |\nabla z(x)| \le \rho(x,t,v(x,t)), \ \text{a.a. } x \in \Omega\}, \quad \forall t \in [0,T], \tag{6.2}$$

and a convex subset K_0 of V by

$$K_0 := \{z \in V \ ; \ |\nabla z(x)| \le \rho^*, \ \text{a.a. } x \in \Omega\}. \tag{6.3}$$

Now, consider the following quasi-variational inequality with time-dependent gradient constraint:

$$u'(t) := \frac{\partial u(x,t)}{\partial t} \in K(u;t), \ \text{a.a. } t \in (0,T), \quad u(\cdot,0) = u_0 \text{ in } V, \tag{6.4a}$$

$$\begin{aligned}
&\tau_0 \int_\Omega u'(x,t)(u'(x,t) - z(x))dx \\
&\qquad + \int_\Omega a(x,t,u(x,t))\nabla u(x,t) \cdot \nabla(u'(x,t) - z(x))dx \\
&\qquad + \int_\Omega g(x,t,u(x,t))(u'(x,t) - z(x))dx \\
&\le \int_\Omega f(x,t)(u'(x,t) - z(x))dx, \quad \forall z \in K(u;t), \ \text{a.a. } t \in (0,T),
\end{aligned} \tag{6.4b}$$

where $\tau_0 \ge 0$ is a constant, $g(\cdot,\cdot,\cdot)$ is a Lipschitz continuous function on $\overline{Q} \times \mathbb{R}$, i.e.,

$$\begin{gathered}
|g(x_1,t_1,r_1) - g(x_2,t_2,r_2)| \le L_g(|x_1 - x_2| + |t_1 - t_2| + |r_1 - r_2|), \\
\forall (x_i,t_i) \in \overline{Q}, \ \forall r_i \in \mathbb{R}, \ i = 1,2,
\end{gathered} \tag{6.5}$$

with a positive constant L_g, f is a function given in $L^2(0,T;H)$, u_0 is an initial datum given in V, and $a(\cdot,\cdot,\cdot)$ is a prescribed function in $C(\overline{Q} \times \mathbb{R})$ such that

$$\begin{gathered}
a_* \le a(x,t,r) \le a^*, \quad \forall (x,t) \in \overline{Q}, \ \forall r \in \mathbb{R}, \\
|a(x,t_1,r_1) - a(x,t_2,r_2)| \le L_a(|t_1 - t_2| + |r_1 - r_2|), \\
\forall t_i \in [0,T], \ \forall r_i \in \mathbb{R}, \ i = 1,2, \ \forall x \in \overline{\Omega},
\end{gathered} \tag{6.6}$$

where a_*, a^*, and L_a are positive constants.

A function $u : [0,T] \to V$ is called a solution to (6.4):= $\{(6.4a), (6.4b)\}$, if $u \in W^{1,2}(0,T;V)$ and all of the properties required in (6.4) are fulfilled. In order to reformulate problem (6.4) as the form $CP(\psi^t, \varphi^t, g; f, u_0)$, the functionals $\psi_0^t(\cdot)$, $\psi^t(v;\cdot)$, $\varphi^t(v;\cdot)$ are set up as follows:

$$\psi_0^t(z) := I_{K_0}(z), \quad \forall z \in V, \ \forall t \in [0,T], \tag{6.7}$$

and we define

$$D_0 = \{v \in W^{1,2}(0,T;V) \mid v'(t) \in K_0, \ \text{a.a. } t \in (0,T), \ v(0) = u_0 \text{ in } V\}, \tag{6.8}$$

$$\psi^t(v;z) := \frac{\tau_0}{2} \int_\Omega |z(x)|^2 dx + I_{K(v;t)}(z), \quad \forall z \in V, \ \forall t \in [0,T], \ \forall v \in D_0, \tag{6.9}$$

$$\varphi^t(v;z) := \frac{1}{2}\int_\Omega a(x,t,v(x,t))|\nabla z(x)|^2 dx, \quad \forall z \in V,\ \forall t \in [0,T],\ \forall v \in D_0, \quad (6.10)$$

where $I_{K(v;t)}(\cdot)$ is the indicator function of $K(v;t)$, namely,

$$I_{K(v;t)}(z) := \begin{cases} 0, & \text{if } z \in K(v;t), \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easy to see from the definition of subdifferential that

(i) Let $v \in D_0$ and $t \in [0,T]$. Then $z^* \in \partial_* \psi^t(v;z)$ if and only if $z^* \in V^*$, $z \in K(v;t)$, and

$$\tau_0 \int_\Omega z(x)(z(x) - w(x))dx + \langle -z^*, z - w\rangle \le 0, \quad \forall w \in K(v;t).$$

(ii) Let $v \in D_0$ and $t \in [0,T]$. Then $\partial_* \varphi^t(v;\cdot)$ is singlevalued, linear, and bounded from V into V^* and

$$\langle \partial_* \varphi^t(v;z), w\rangle = \int_\Omega a(x,t,v(x,t))\nabla z(x) \cdot \nabla w(x) dx, \quad \forall w \in V.$$

In addition, for each $v \in D_0$ and $t \in [0,T]$, we define $g(v;t,\cdot) : V \to V^*$ by

$$\langle g(v;t,z), w\rangle := \int_\Omega g(x,t,z(x))w(x)dx, \quad \forall z, w \in V. \quad (6.11)$$

Then, from the above characterizations (i) and (ii) of subdifferentials $\psi^t(v;\cdot)$ and $\varphi^t(v;\cdot)$, it follows that problem (6.4) is reformulated as $CP(\psi^t, \varphi^t, g; f, u_0)$.

Note that the next lemma was shown in [25], therefore, we omit its detailed proof.

Lemma 6.1 (cf. [25, Lemma 4.2]). *Let ψ^t and φ^t be the functionals defined by* (6.9) *and* (6.10)*, respectively. Then, assumptions* (ψ) *and* (ϕ) *are fulfilled.*

The following solvability result for (6.4) was proved in [25, Proposition 4.1].

Proposition 6.1 (cf. [25, Proposition 4.1]). *Assume that* (6.1)*,* (6.5)*, and* (6.6) *are fulfilled by functions $\rho(x,t,r)$, $g(x,t,r)$, and $a(x,t,r)$. Let $\tau_0 \ge 0$, $f \in L^2(0,T;H)$, and $u_0 \in V$. Then, problem* (6.4) *admits at least one solution u in $W^{1,2}(0,T;V)$.*

Proof. On account of Lemma 6.1, $\{\psi^t(v;\cdot)\}$ and $\{\varphi^t(v;\cdot)\}$ given by (6.2), (6.9), and (6.10) fulfill assumptions (ψ) and (ϕ), and (g) is trivially verified. Hence, all the assumptions of Proposition 2.1 are fulfilled. Therefore,

$$\partial_* \psi^t(u;u'(t)) + \partial_* \varphi^t(u;u(t)) + g(u;t,u(t)) \ni f(t) \ \text{ in } V^*, \quad u(0) = u_0 \ \text{ in } V,$$

admits a solution u in $W^{1,2}(0,T;V)$. By the characterization (i) and (ii) of $\partial_* \psi^t(u;\cdot)$ and $\partial_* \varphi^t(u;\cdot)$, and the definition of $g(u;t,\cdot)$, u is a solution to (6.4) on $[0,T]$. □

Note that $\partial_*\psi^t(v;\cdot)$ does not satisfy the assumption in Proposition 2.2, since $\partial_*\psi^t(v;\cdot)$ is dependent on $v \in D_0$. Therefore, problem (6.4) generally has multiple solutions, and the corresponding optimal control problem is of the singular type. We now discuss the singular optimal control problem for (6.4). On account of Proposition 6.1, problem (6.4) can be reformulated as the form $CP(\psi^t, \varphi^t, g; f, u_0)$. Therefore, by applying Theorem 4.1 with $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, and $V^* = H^{-1}(\Omega)$, we have the following result concerning the existence of an optimal control of (OP) for (6.4).

Proposition 6.2. *Let u_{ad} be a given target function in $L^2(0,T;V)$, $u_0 \in V$, and let $M > 0$ and $\tau_0 \geq 0$ be given constants. Then, problem* (OP)*, corresponding to the state system* (6.4)*, has at least one optimal control $f^* \in \mathcal{F}_M$, namely,*

$$J(f^*) = \inf_{f \in \mathcal{F}_M} J(f),$$

where $\mathcal{F}_M$ is the control space defined by (1.2) *with $V^* = H^{-1}(\Omega)$ and $H = L^2(\Omega)$, and $J(\cdot)$ is the cost functional for* (OP)*, which is defined by* (1.3) *and* (1.4).

Next, we discuss the approximate system for (6.4). Let D_0 be the set defined by (6.8). Then, for each $w \in D_0$, we consider the following parameter-dependent variational inequality with time-dependent gradient constraint $K(w;t)$:

$$u'(t) \in K(w;t), \text{ a.a. } t \in (0,T), \quad u(\cdot,0) = u_0 \text{ in } V, \tag{6.12a}$$

$$\begin{aligned} \tau_0 \int_\Omega & u'(x,t)(u'(x,t) - z(x))dx \\ &+ \int_\Omega a(x,t,w(x,t))\nabla u(x,t) \cdot \nabla(u'(x,t) - z(x))dx \\ &+ \int_\Omega g(x,t,u(x,t))(u'(x,t) - z(x))dx \\ \leq \int_\Omega & f(x,t)(u'(x,t) - z(x))dx, \quad \forall z \in K(w;t), \text{ a.a. } t \in (0,T), \end{aligned} \tag{6.12b}$$

where $\tau_0 \geq 0$, $K(\cdot;\cdot)$, $a(\cdot,\cdot,\cdot)$, $g(\cdot,\cdot,\cdot)$, f, and u_0 are the same as in (6.4).

On account of Proposition 3.1 and Remark 3.1, we have the following existence–uniqueness result for problem (6.12):={(6.12a), (6.12b)}.

Proposition 6.3. *Let $\tau_0 \geq 0$ be a constant. Then, for each $u_0 \in V$, $w \in D_0$, and $f \in L^2(0,T;H)$, problem* (6.12) *admits at least one solution u in $W^{1,2}(0,T;V)$. In addition, if $\tau_0 > 0$, then the solution to* (6.12) *is unique.*

Proof. For each $t \in [0,T]$ and $w \in D_0$, the (t,w)-dependent functionals $\psi^t(w;z)$ and $\varphi^t(w;z)$ are defined by (6.9) and (6.10) with $v = w$, respectively. In addition, for each $t \in [0,T]$ and $w \in D_0$, the (t,w)-dependent operator $g(w;t,z)$ is defined by (6.11) with $v = w$. Then, it follows from Lemma 6.1 that assumptions (ψ), (ϕ), and (g) are verified. Therefore, by arguments similar to Proposition 6.1, we easily observe that (6.12) can be reformulated in the abstract form $CP_w(f, u_0)$. Furthermore, if $\tau_0 > 0$, we observe from the characterization (i) of $\partial_*\psi^t(v;\cdot)$ that (3.4) holds. Therefore, by Proposition 3.1 and Remark 3.1, we have shown that problem (6.12) has a solution on $[0,T]$, and if $\tau_0 > 0$, then it is unique. $\square$

In the case $\tau_0 = 0$, problem (6.12) generally has multiple solutions, therefore, the corresponding optimal control problem is of the singular type. Hence, by applying Theorem 5.1(I) with $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$, we can show the following existence result of optimal controls of $(\mathrm{OP})_\delta$ $(\delta \in (0,1])$ for (6.12) and the control space $\mathcal{W}_{M'}(u_0) \times \mathcal{F}_M$ (see Section 5 for the detailed formulation).

Proposition 6.4. *Let u_{ad} be a given target function in $L^2(0,T;V)$, $u_0 \in V$, and let $M > 0$, $M' > 0$ be given constants. Assume $\tau_0 = 0$. Then, for each $\delta \in (0,1]$, problem $(\mathrm{OP})_\delta$, corresponding to the state system* (6.12)*, has at least one optimal control $[w_\delta^*, f_\delta^*] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M$, namely,*

$$E_\delta(w_\delta^*, f_\delta^*) = \inf_{[w,f]\in \mathcal{W}_{M'}(u_0)\times\mathcal{F}_M} E_\delta(w,f),$$

where $\mathcal{W}_{M'}(u_0)$ is the control space defined by (5.3) *with $V = H_0^1(\Omega)$, and $E_\delta(\cdot,\cdot)$ is the cost functional defined by* (5.1) *and* (5.2).

Hence, by applying Theorem 5.1(II), namely, by considering (6.12) and the corresponding approximate optimal control problem $(\mathrm{OP})_\delta$ $(\delta \in (0,1])$, we can approximate (6.4) and its singular optimal control problem (OP) in the case when $\tau_0 = 0$.

We now employ the approximate system to (6.12), as proposed in Section 5, in the case when $\tau_0 = 0$. Indeed, for each $\varepsilon \in (0,1]$ and the fixed function $v_0 \in D_0$, we consider the following parameter-dependent variational inequality with time-dependent gradient constraint $K(w;t)$:

$$u'(t) \in K(w;t), \text{ a.a. } t \in (0,T), \quad u(\cdot,0) = u_0 \text{ in } V, \tag{6.13a}$$

$$\begin{aligned}
&\varepsilon \int_\Omega a(x,0,v_0(x,t))\nabla u'(x,t)(u'(x,t) - z(x))dx \\
&\qquad + \int_\Omega a(x,t,w(x,t))\nabla u(x,t)\cdot\nabla(u'(x,t) - z(x))dx \\
&\qquad + \int_\Omega g(x,t,u(x,t))(u'(x,t) - z(x))dx \\
&\le \int_\Omega f(x,t)(u'(x,t) - z(x))dx \\
&\qquad + \varepsilon \int_\Omega a(x,0,v_0(x,t))\nabla h(x,t)\cdot\nabla(u'(x,t) - z(x))dx \\
&\qquad\qquad \forall z \in K(w;t), \text{ a.a. } t \in (0,T),
\end{aligned} \tag{6.13b}$$

where $K(\cdot;\cdot)$, $a(\cdot,\cdot,\cdot)$, $g(\cdot,\cdot,\cdot)$, f, and u_0 are the same as in (6.4), and h is a given function in $L^2(0,T;V)$.

By the characterization (ii) of subdifferential of $\varphi^t(v;\cdot)$ defined by (6.10), we observe that the subdifferential $F_0 := \partial_*\varphi^0(v_0;\cdot)$ is given by

$$\langle F_0 z, \varpi\rangle = \int_\Omega a(x,0,v_0(x,t))\nabla z(x)\cdot\nabla\varpi(x)dx, \quad \forall \varpi \in V. \tag{6.14}$$

Thus, by (6.6) we easily check that F_0 is linear, coercive, continuous, and single-valued maximal monotone from V into V^* satisfying (5.18) with $V = H^1_0(\Omega)$ and $V^* = H^{-1}(\Omega)$.

From (6.14) and the facts identified in the proof of Proposition 6.3, problem (6.13) can be reformulated in the abstract form $CP^\varepsilon_w(f + \varepsilon F_0 h, u_0)$. Therefore, Proposition 5.1(I) implies the existence–uniqueness of solutions to (6.13) on $[0, T]$. In addition, by applying Proposition 5.2, we get the result on the relationship between (6.12) and (6.13), more precisely, we observe that (6.13) is an approximate problem for (6.12).

Moreover, using the standard smoothness arguments (e.g., regularization by the mollifier and the convolution [1, Sections 2.28 and 3.16]), we can check assumption (H) with $X = H^2(\Omega)$. Therefore, for each $\varepsilon \in (0, 1]$, Theorem 5.2(I) implies that the approximate optimal control problem $(\mathrm{OP})^\varepsilon_\delta$, corresponding to (6.13), has at least one optimal control $[w^*_{\delta,\varepsilon}, f^*_{\delta,\varepsilon}, h^*_{\delta,\varepsilon}] \in \mathcal{W}_{M'}(u_0) \times \mathcal{F}_M \times \mathcal{H}^\varepsilon_N$ (cf. (5.3), (1.2), and (5.28)). Additionally, by Theorem 5.2(II) on the relationship between $(\mathrm{OP})_\delta$ and $(\mathrm{OP})^\varepsilon_\delta$, we see that $(\mathrm{OP})_\delta$ is approximated by $(\mathrm{OP})^\varepsilon_\delta$ as $\varepsilon \downarrow 0$; more precisely, there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$ with $\varepsilon_n \to 0$ $(n \to \infty)$ such that any weak limit function $[w^*_\delta, f^*_\delta]$ of $\{[w^*_{\delta,\varepsilon_n}, f^*_{\delta,\varepsilon_n}]\}_{n\in\mathbb{N}}$ in $L^2(0, T; V) \times L^2(0, T; V^*)$ is an optimal control for $(\mathrm{OP})_\delta$.

Thus, it is worth studying (6.13) and its control problem $(\mathrm{OP})^\varepsilon_\delta$ for sufficient small $\delta, \varepsilon \in (0, 1]$, instead of considering (6.4) with $\tau_0 = 0$ and its singular control problem (OP), from a numerical point of view.

Remark 6.1. In the case $\tau_0 > 0$, note from Proposition 6.3 that the solution to (6.12) is unique, hence, the corresponding optimal control problem $(\mathrm{OP})_\delta$ $(\delta \in (0, 1])$ is of the standard type. Thus, as mentioned in Remark 5.3, the non-well-posed state system (6.4) and its singular optimal control problem (OP) are approximated by the well-posed state system (6.12) and its standard optimal control problem $(\mathrm{OP})_\delta$, respectively, in the case when $\tau_0 > 0$.

(Application 2)

Next, we are going to consider a doubly nonlinear quasi-variational inequality with non-local obstacle function.

We here define an operator $\mathcal{L} : L^2(0, T; H) \to L^2(0, T; H)$ by

$$\mathcal{L}(v; x, t) := \int_0^t \int_\Omega \ell(x, t, y, \tau, v(y, \tau))dyd\tau + \delta_0, \quad \forall (x, t) \in Q, \ \forall v \in L^2(0, T; H),$$

with a given positive constant δ_0 and a given function $\ell \in C(\overline{Q} \times \overline{Q} \times \mathbb{R})$ satisfying the following conditions for positive constants ℓ^* and L_ℓ:

$$\begin{cases} 0 \le \ell(x, t, y, \tau, r) \le \ell^*, \quad \forall (x, t, y, \tau, r) \in \overline{Q} \times \overline{Q} \times \mathbb{R}, \\ |\ell(x, t_1, y, \tau, r_1) - \ell(x, t_2, y, \tau, r_2)| \le L_\ell \left(|t_1 - t_2| + |r_1 - r_2|\right), \\ \qquad\qquad \forall t_i \in [0, T], \ \forall r_i \in \mathbb{R}, \ i = 1, 2, \ \forall (x, y, \tau) \in \overline{\Omega} \times \overline{Q}, \end{cases}$$

With the functional $\mathcal{L}(v; x, t)$, we define a constraint set $K(v; t)$ by

$$K(v; t) := \{z \in V \mid |\nabla z(x)| \le \mathcal{L}(v; x, t), \ \text{a.a. } x \in \Omega\}, \quad \forall t \in [0, T]. \tag{6.15}$$

Now, we consider the following quasi-variational inequality with time-dependent gradient non-local constraint:

$$u'(t) \in K(u;t), \text{ a.a. } t \in (0,T), \quad u(\cdot,0) = u_0 \text{ in } V, \tag{6.16a}$$

$$\begin{aligned} &\tau_0 \int_\Omega u'(x,t)(u'(x,t)-z(x))dx \\ &\quad + \int_\Omega a(x,t,u(x,t))\nabla u(x,t)\cdot\nabla(u'(x,t)-z(x))dx \\ &\quad + \int_\Omega g(x,t,u(x,t))(u'(x,t)-z(x))dx \\ \le &\int_\Omega f(x,t)(u'(x,t)-z(x))dx, \quad \forall z \in K(u;t), \text{ a.a. } t \in (0,T), \end{aligned} \tag{6.16b}$$

where $\tau_0 \ge 0$, $a(\cdot,\cdot,\cdot)$, $g(\cdot,\cdot,\cdot)$, f, and u_0 are the same as in problem (6.4). Similarly define functionals $\psi^t(v;\cdot)$ and $\varphi^t(v;\cdot)$ by (6.9) and (6.10) with (6.8). In addition, define the (t,v)-dependent operator $g(v;t,z)$ by (6.11). Then, we observe that all the assumptions of Proposition 2.1 are fulfilled, and hence, problem (6.16):=$\{$(6.16a), (6.16b)$\}$ has at least one solution u in $W^{1,2}(0,T;V)$. Furthermore, by applying Theorem 4.1 with $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, and $V^* = H^{-1}(\Omega)$, we can prove the existence of an optimal control of (OP), corresponding to (6.16).

Using an approach similar to that employed in Application 1, the state problem (6.16) and its optimal control problem can be approximated by a parameter-dependent variational inequalities similar to (6.12) and (6.13), and their corresponding approximate optimal control problems, respectively.

6.2 Doubly nonlinear evolution inclusions of Navier-Stokes type

In this subsection, we investigate doubly nonlinear evolution inclusions of Navier-Stokes type, which was treated in [25, Section 4.2].

Let Ω be a smooth bounded domain in $\mathbb{R}^3$, with usual notation $Q := \Omega \times (0,T)$, $0 < T < \infty$, $\Gamma := \partial\Omega$, and $\Sigma := \Gamma \times (0,T)$;

$$V := H_0^1(\Omega), \quad H := L^2(\Omega), \quad [V]^3 := V \times V \times V, \quad [H]^3 := H \times H \times H.$$

Now, we define the solenoidal function space as follows:

$$\boldsymbol{\mathcal{D}}_\sigma := \{\boldsymbol{z} = (z^{(1)}, z^{(2)}, z^{(3)}) \in [C_0^\infty(\Omega)]^3 \mid \operatorname{div}\boldsymbol{z} = 0 \text{ in } \Omega\},$$

where $[C_0^\infty(\Omega)]^3 := C_0^\infty(\Omega) \times C_0^\infty(\Omega) \times C_0^\infty(\Omega)$, and put

$$\boldsymbol{H}_\sigma := \text{the closure of } \boldsymbol{\mathcal{D}}_\sigma \text{ in } [H]^3, \quad \boldsymbol{V}_\sigma := \text{the closure of } \boldsymbol{\mathcal{D}}_\sigma \text{ in } [V]^3,$$

with usual norms

$$|\boldsymbol{z}|_{\boldsymbol{H}_\sigma} := \left\{ \sum_{k=1}^3 |z^{(k)}|_H^2 \right\}^{\frac{1}{2}}, \quad |\boldsymbol{z}|_{\boldsymbol{V}_\sigma} := \left\{ \sum_{k=1}^3 |\nabla z^{(k)}|_H^2 \right\}^{\frac{1}{2}}.$$

In addition, we denote the dual space of $\boldsymbol{V}_\sigma$ by $\boldsymbol{V}_\sigma^*$. For simplicity we denote the inner product in $\boldsymbol{H}_\sigma$ by $(\cdot,\cdot)_\sigma$ and the duality pairing between $\boldsymbol{V}_\sigma^*$ and $\boldsymbol{V}_\sigma$ by $\langle\cdot,\cdot\rangle_\sigma$.

Let ρ be a prescribed obstacle function in $C(\overline{Q}\times\mathbb{R}^3)$ such that

$$0<\rho_*\le\rho(x,t,\boldsymbol{r})\le\rho^*,\quad \forall(x,t,\boldsymbol{r})\in\overline{Q}\times\mathbb{R}^3. \tag{6.17}$$

Here ρ_* and ρ^* are positive constants, and $\boldsymbol{K}_0$ is a closed convex subset of $\boldsymbol{V}_\sigma$ given by

$$\boldsymbol{K}_0:=\{\boldsymbol{z}\in\boldsymbol{V}_\sigma \mid |\nabla\boldsymbol{z}(x)|\le\rho^*,\ \text{a.e. on }\Omega\},$$

and put

$$\boldsymbol{D}_0:=\{\boldsymbol{v}\in W^{1,2}(0,T;\boldsymbol{V}_\sigma)\mid \boldsymbol{v}'(t)\in\boldsymbol{K}_0,\ \text{a.a. } t\in(0,T),\ \boldsymbol{v}(0)=\boldsymbol{u}_0 \text{ in } \boldsymbol{V}_\sigma\}, \tag{6.18}$$

where $\boldsymbol{u}_0$ is a given initial datum in $\boldsymbol{V}_\sigma$ (cf. (6.23a)).

Also, for each $\boldsymbol{v}\in\boldsymbol{D}_0$ and $t\in[0,T]$, define

$$\boldsymbol{K}(\boldsymbol{v};t):=\{\boldsymbol{z}\in\boldsymbol{V}_\sigma\mid|\nabla\boldsymbol{z}(x)|\le\rho(x,t,\boldsymbol{v}(x,t)),\ \text{a.e. on }\Omega\},\ \forall t\in[0,T]. \tag{6.19}$$

Now, for each $\boldsymbol{v}\in\boldsymbol{D}_0$ and $t\in[0,T]$, we define functionals $\psi_0^t(\cdot)$, $\psi^t(\boldsymbol{v};\cdot)$, and $\varphi^t(\boldsymbol{v};\cdot)$ on $\boldsymbol{V}_\sigma$ by

$$\psi_0^t(\boldsymbol{z}):=I_{\boldsymbol{K_0}}(\boldsymbol{z}),\quad\forall\boldsymbol{z}\in\boldsymbol{V}_\sigma,\ \forall t\in[0,T],$$

$$\psi^t(\boldsymbol{v};\boldsymbol{z}):=\frac{\tau_0}{2}\int_\Omega|\boldsymbol{z}(x)|^2dx+I_{\boldsymbol{K(v;t)}}(\boldsymbol{z}),\quad\forall\boldsymbol{z}\in\boldsymbol{V}_\sigma,\ \forall t\in[0,T], \tag{6.20}$$

$$\varphi^t(\boldsymbol{v};\boldsymbol{z}):=\frac{1}{2}\int_\Omega a(x,t,\boldsymbol{v}(x,t))|\nabla\boldsymbol{z}(x)|^2dx,\quad\forall\boldsymbol{z}\in\boldsymbol{V}_\sigma,\ \forall t\in[0,T], \tag{6.21}$$

and $\boldsymbol{g}(\boldsymbol{v};t,\cdot):\boldsymbol{V}_\sigma\to\boldsymbol{V}_\sigma^*$ by

$$\begin{aligned}\langle\boldsymbol{g}(\boldsymbol{v};t,\boldsymbol{w}),\boldsymbol{z}\rangle_\sigma&:=\int_\Omega(\boldsymbol{v}(x,t)\cdot\nabla)\boldsymbol{w}(x)\cdot\boldsymbol{z}(x)dx\\&=\sum_{i,j=1}^3\int_\Omega v^{(i)}(x,t)\frac{\partial w^{(j)}(x)}{\partial x_i}z^{(j)}(x)dx,\\&\qquad\forall\boldsymbol{w}=(w^{(1)},w^{(2)},w^{(3)}),\ \boldsymbol{z}=(z^{(1)},z^{(2)},z^{(3)})\in\boldsymbol{V}_\sigma.\end{aligned} \tag{6.22}$$

Now we propose the following quasi-variational inequality of Navier–Stokes type with gradient constraint for time-derivative:

$$\boldsymbol{u}'(t):=\frac{\partial\boldsymbol{u}(x,t)}{\partial t}\in\boldsymbol{K}(\boldsymbol{u};t),\quad\text{a.a. } t\in(0,T),\quad\boldsymbol{u}(\cdot,0)=\boldsymbol{u}_0\text{ in }\boldsymbol{V}_\sigma, \tag{6.23a}$$

$$\begin{aligned}&\tau_0\int_\Omega\boldsymbol{u}'(x,t)\cdot(\boldsymbol{u}'(x,t)-\boldsymbol{z}(x))dx\\&\qquad+\int_\Omega a(x,t,\boldsymbol{u}(x,t))\nabla\boldsymbol{u}(x,t)\cdot\nabla(\boldsymbol{u}'(x,t)-\boldsymbol{z}(x))dx\\&\qquad+\langle\boldsymbol{g}(\boldsymbol{u};t,\boldsymbol{u}(t)),\boldsymbol{u}'(t)-\boldsymbol{z}\rangle_\sigma\\&\le\int_\Omega\boldsymbol{f}(x,t)\cdot(\boldsymbol{u}'(x,t)-\boldsymbol{z}(x))dx,\quad\forall\boldsymbol{z}\in\boldsymbol{K}(\boldsymbol{u};t),\quad\text{a.a. } t\in(0,T),\end{aligned} \tag{6.23b}$$

where $\tau_0 \geq 0$ is a constant, $\boldsymbol{f}$ is a function given in $L^2(0,T;\boldsymbol{H}_\sigma)$, $\boldsymbol{u}_0$ is an initial datum given in $\boldsymbol{V}_\sigma$, and $a(\cdot,\cdot,\cdot)$ is a function in $C(\overline{Q}\times\mathbb{R}^3)$ such that

$$\begin{aligned} &a_* \leq a(x,t,\boldsymbol{r}) \leq a^*, \quad \forall (x,t)\in\overline{Q},\ \forall \boldsymbol{r}\in\mathbb{R}^3,\\ &|a(x,t_1,\boldsymbol{r}_1)-a(x,t_2,\boldsymbol{r}_2)| \leq L_a(|t_1-t_2|+|\boldsymbol{r}_1-\boldsymbol{r}_2|),\\ &\forall t_i\in[0,T],\ \forall \boldsymbol{r}_i\in\mathbb{R}^3,\ i=1,2,\ \forall x\in\overline{\Omega}, \end{aligned} \tag{6.24}$$

where $a_*,\ a^*$, and L_a are positive constants.

In [25, Proposition 4.2] the following result on the solvability of (6.23) := {(6.23a), (6.23b)} was shown.

Proposition 6.5 (cf. [25, Proposition 4.2]). *Let $\rho := \rho(x,t,\boldsymbol{r})$ and $a := a(x,t,\boldsymbol{r})$ be functions for which* (6.17) *and* (6.24) *hold, and let $\boldsymbol{f}\in L^2(0,T;\boldsymbol{H}_\sigma)$ and $\boldsymbol{u}_0\in\boldsymbol{V}_\sigma$. Then, problem* (6.23) := {(6.23a), (6.23b)} *admits at least one solution $\boldsymbol{u}$ in $W^{1,2}(0,T;\boldsymbol{V}_\sigma)$.*

Proof. From (6.20) and (6.21) of the definitions of ψ^t and φ^t we see the following characterizations of their subdifferentials:

(i) Let $\boldsymbol{v}\in\boldsymbol{D}_0$ and $t\in[0,T]$. Then $\boldsymbol{z}^*\in\partial_*\psi^t(\boldsymbol{v};\boldsymbol{z})$ if and only if $\boldsymbol{z}^*\in\boldsymbol{V}_\sigma^*$, $\boldsymbol{z}\in\boldsymbol{K}(\boldsymbol{v};t)$, and

$$\tau_0\int_\Omega \boldsymbol{z}(x)\cdot(\boldsymbol{z}(x)-\boldsymbol{w}(x))dx + \langle -\boldsymbol{z}^*,\boldsymbol{z}-\boldsymbol{w}\rangle_\sigma \leq 0, \quad \forall \boldsymbol{w}\in\boldsymbol{K}(\boldsymbol{v};t).$$

(ii) Let $\boldsymbol{v}\in\boldsymbol{D}_0$ and $t\in[0,T]$. Then $\partial_*\varphi^t(\boldsymbol{v};\cdot)$ is singlevalued, linear, and bounded from $\boldsymbol{V}_\sigma$ into $\boldsymbol{V}_\sigma^*$ and

$$\langle\partial_*\varphi^t(\boldsymbol{v};\boldsymbol{z}),\boldsymbol{w}\rangle_\sigma = \int_\Omega a(x,t,\boldsymbol{v}(x,t))\nabla\boldsymbol{z}(x)\cdot\nabla\boldsymbol{w}(x)dx, \quad \forall \boldsymbol{z},\boldsymbol{w}\in\boldsymbol{V}_\sigma.$$

These characterizations show that the quasi-variational inequality (6.23) is described as

$$\begin{cases} \partial_*\psi^t(\boldsymbol{u};\boldsymbol{u}'(t)) + \partial_*\varphi^t(\boldsymbol{u};\boldsymbol{u}(t)) + \boldsymbol{g}(\boldsymbol{u};t,\boldsymbol{u}(t)) \ni \boldsymbol{f}(t) \ \text{ in } \boldsymbol{V}_\sigma^*, \ \text{a.a. } t\in(0,T),\\ \boldsymbol{u}(0)=\boldsymbol{u}_0 \ \text{ in } \boldsymbol{V}_\sigma. \end{cases} \tag{6.25}$$

In [25, Lemma 4.3] we checked that assumptions (ψ), (ϕ), and (g) are fulfilled by functions ψ^t, φ^t, and $\boldsymbol{g}$ given by (6.20)∼(6.22). Therefore, on account of Proposition 2.1, we obtain Proposition 6.5. □

Note that $\partial_*\psi^t(\boldsymbol{v};\cdot)$ and $\boldsymbol{g}$ do not satisfy the assumptions in Proposition 2.2. Therefore, problem (6.23) generally has multiple solutions, and the corresponding optimal control problem is of the singular type. On account of Proposition 6.5, problem (6.23) can be reformulated as the form (6.25), i.e., $CP(\psi^t,\varphi^t,\boldsymbol{g};\boldsymbol{f},\boldsymbol{u}_0)$. Therefore, by applying Theorem 4.1 with $V=\boldsymbol{V}_\sigma$, $H=\boldsymbol{H}_\sigma$, and $V^* := \boldsymbol{V}_\sigma^*$, we have the following result concerning the existence of an optimal control of (OP) for (6.23).

Proposition 6.6. *Let $\boldsymbol{u}_{ad}$ be a given target function in $L^2(0,T;\boldsymbol{V}_\sigma)$, $\boldsymbol{u}_0 \in \boldsymbol{V}_\sigma$, and let $M > 0$ and $\tau_0 \geq 0$ be given constants. Then, problem* (OP), *corresponding to the state system* (6.23), *has at least one optimal control $\boldsymbol{f}^* \in \mathcal{F}_M$, namely,*

$$J(\boldsymbol{f}^*) = \inf_{\boldsymbol{f}\in\mathcal{F}_M} J(\boldsymbol{f}),$$

where $\mathcal{F}_M$ is the control space defined by (1.2) *with $H = \boldsymbol{H}_\sigma$ and $V^* := \boldsymbol{V}_\sigma^*$, and $J(\cdot)$ is the cost functional for* (OP), *which is similarly defined by* (1.3) *and* (1.4).

Now, using an approach similar to that employed in Subsection 6.1, we approximate (6.23) and its singular optimal control problem (OP). Indeed, let $\boldsymbol{D}_0$ be the set defined by (6.18). Then, for each $\boldsymbol{w} \in \boldsymbol{D}_0$, we consider the following parameter-dependent variational inequality with gradient constraint:

$$\boldsymbol{u}'(t) \in \boldsymbol{K}(\boldsymbol{w};t), \text{ a.a. } t \in (0,T), \quad \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 \text{ in } \boldsymbol{V}_\sigma, \tag{6.26a}$$

$$\begin{aligned}\tau_0 \int_\Omega & \boldsymbol{u}'(x,t)\cdot(\boldsymbol{u}'(x,t)-\boldsymbol{z}(x))dx \\ &+ \int_\Omega a(x,t,\boldsymbol{w}(x,t))\nabla\boldsymbol{u}(x,t)\cdot\nabla(\boldsymbol{u}'(x,t)-\boldsymbol{z}(x))dx \\ &+\langle \boldsymbol{g}(\boldsymbol{w};t,\boldsymbol{u}(t)),\boldsymbol{u}'(t)-\boldsymbol{z}\rangle_\sigma \\ \leq \int_\Omega & \boldsymbol{f}(x,t)\cdot(\boldsymbol{u}'(x,t)-\boldsymbol{z}(x))dx, \quad \forall \boldsymbol{z}\in\boldsymbol{K}(\boldsymbol{w};t), \text{ a.a. } t\in(0,T),\end{aligned} \tag{6.26b}$$

where $\tau_0 \geq 0$, $\boldsymbol{K}(\cdot;\cdot)$, $a(\cdot,\cdot,\cdot)$, $\boldsymbol{g}(\cdot;\cdot,\cdot)$, $\boldsymbol{f}$, and $\boldsymbol{u}_0$ are the same as in (6.23).

On account of Proposition 3.1, we have the following existence result for problem (6.26):=$\{$(6.26a), (6.26b)$\}$.

Proposition 6.7. *Let $\tau_0 \geq 0$ be a constant. Then, for each $\boldsymbol{u}_0 \in \boldsymbol{V}_\sigma$, $\boldsymbol{w} \in \boldsymbol{D}_0$, and $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}_\sigma)$, problem* (6.26) *admits at least one solution $\boldsymbol{u}$ in $W^{1,2}(0,T;\boldsymbol{V}_\sigma)$.*

Proof. From arguments similar to those in Proposition 6.5, we observe that problem (6.26) is described as

$$\begin{cases} \partial_*\psi^t(\boldsymbol{w};\boldsymbol{u}'(t)) + \partial_*\varphi^t(\boldsymbol{w};\boldsymbol{u}(t)) + \boldsymbol{g}(\boldsymbol{w};t,\boldsymbol{u}(t)) \ni \boldsymbol{f}(t) \text{ in } \boldsymbol{V}_\sigma^*, \quad \text{a.a. } t\in(0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \text{ in } \boldsymbol{V}_\sigma. \end{cases}$$

In [25, Lemma 4.3] we checked that assumptions (ψ), (ϕ), and (g) are fulfilled for ψ^t, φ^t, and $\boldsymbol{g}$ given by (6.20)$\sim$(6.22). Therefore, by Proposition 3.1, we have shown that problem (6.26) admits at least one solution $\boldsymbol{u}$ in $W^{1,2}(0,T;\boldsymbol{V}_\sigma)$. □

Note that $\partial_*\psi^t(\boldsymbol{w};\cdot)$ does not satisfy (3.3). In addition, $\boldsymbol{g}$ does not satisfy (3.5). Indeed, we observe from [25, Lemma 4.3] that

$$|\boldsymbol{g}(\boldsymbol{w};t,\boldsymbol{z}_1) - \boldsymbol{g}(\boldsymbol{w};t,\boldsymbol{z}_2)|_{\boldsymbol{V}_\sigma^*} \leq k_4|\boldsymbol{z}_1-\boldsymbol{z}_2|_{\boldsymbol{V}_\sigma}, \quad \forall \boldsymbol{w}\in\boldsymbol{D}_0, \quad z_i\in\boldsymbol{V}_\sigma, (i=1,2),$$

where k_4 is a positive constant, dependent on the constant of the embedding $V \hookrightarrow L^4(\Omega)$. Therefore, problem (6.26) generally has multiple solutions (cf. Proposition 3.2 and Remark 3.1), and the corresponding optimal control problem is of the singular type. By applying Theorem 5.1(I) with $V = \boldsymbol{V}_\sigma$, $H = \boldsymbol{H}_\sigma$, and $V^* := \boldsymbol{V}_\sigma^*$, we can show the following existence result of optimal controls of $(\mathrm{OP})_\delta$ ($\delta \in (0,1]$) for (6.26).

Proposition 6.8. *Let $\boldsymbol{u}_{ad}$ be a given target function in $L^2(0,T;\boldsymbol{V}_\sigma)$, $\boldsymbol{u}_0 \in \boldsymbol{V}_\sigma$, and let $M > 0$, $M' > 0$, and $\tau_0 \geq 0$ be given constants. Then, for each $\delta \in (0,1]$, problem $(\mathrm{OP})_\delta$, corresponding to the state system (6.26), has at least one optimal control $[\boldsymbol{w}_\delta^*, \boldsymbol{f}_\delta^*] \in \mathcal{W}_{M'}(\boldsymbol{u}_0) \times \mathcal{F}_M$, namely,*

$$E_\delta(\boldsymbol{w}_\delta^*, \boldsymbol{f}_\delta^*) = \inf_{[\boldsymbol{w},\boldsymbol{f}] \in \mathcal{W}_{M'}(\boldsymbol{u}_0) \times \mathcal{F}_M} E_\delta(\boldsymbol{w}, \boldsymbol{f}),$$

where $\mathcal{W}_{M'}(\boldsymbol{u}_0)$ is the control space defined by (5.3) with $V = \boldsymbol{V}_\sigma$, and $E_\delta(\cdot,\cdot)$ is the cost functional for $(\mathrm{OP})_\delta$, which is similarly defined by (5.1) and (5.2).

Taking an approach similar to that employed in Section 5, namely, by applying Theorem 5.1(II) with $V = \boldsymbol{V}_\sigma$, $H = \boldsymbol{H}_\sigma$, and $V^* := \boldsymbol{V}_\sigma^*$, the singular optimal control problem (OP) for (6.23) can be approximated by $(\mathrm{OP})_\delta$ ($\delta \in (0,1]$).

Finally, we approximate (6.26) by the well-posed state system. To this end, let $\boldsymbol{v}_0 \in \boldsymbol{D}_0$ be a fixed element. Then, for each $\boldsymbol{w} \in \boldsymbol{D}_0$ and $\varepsilon \in (0,1]$, we consider the following parameter-dependent variational inequality with gradient constraint:

$$\boldsymbol{u}'(t) \in \boldsymbol{K}(\boldsymbol{w};t), \text{ a.a. } t \in (0,T), \quad \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 \text{ in } \boldsymbol{V}_\sigma, \tag{6.27a}$$

$$\begin{aligned}
\varepsilon \int_\Omega & a(x,0,\boldsymbol{v}_0(x,t))\nabla \boldsymbol{u}'(x,t) \cdot \nabla(\boldsymbol{u}'(x,t) - \boldsymbol{z}(x))dx \\
& + \tau_0 \int_\Omega \boldsymbol{u}'(x,t) \cdot (\boldsymbol{u}'(x,t) - \boldsymbol{z}(x))dx \\
& + \int_\Omega a(x,t,\boldsymbol{w}(x,t))\nabla \boldsymbol{u}(x,t) \cdot \nabla(\boldsymbol{u}'(x,t) - \boldsymbol{z}(x))dx \\
& + \langle \boldsymbol{g}(\boldsymbol{w};t,\boldsymbol{u}(t)), \boldsymbol{u}'(t) - \boldsymbol{z}\rangle_\sigma \\
\leq \int_\Omega & \boldsymbol{f}(x,t) \cdot (\boldsymbol{u}'(x,t) - \boldsymbol{z}(x))dx \\
& + \varepsilon \int_\Omega a(x,0,\boldsymbol{v}_0(x,t))\nabla \boldsymbol{h}(x,t) \cdot \nabla(\boldsymbol{u}'(x,t) - \boldsymbol{z}(x))dx \\
& \qquad \forall \boldsymbol{z} \in \boldsymbol{K}(\boldsymbol{w};t), \text{ a.a. } t \in (0,T),
\end{aligned} \tag{6.27b}$$

where $\boldsymbol{h}$ is a given function in $L^2(0,T;\boldsymbol{V}_\sigma)$.

By the characterization (ii) of subdifferential of $\varphi^t(\boldsymbol{v};\cdot)$ defined by (6.21) (cf. Proposition 6.5), we observe that the subdifferential $F_0 := \partial_*\varphi^0(\boldsymbol{v}_0;\cdot)$ is given by

$$\langle F_0 \boldsymbol{z}, \boldsymbol{\varpi}\rangle = \int_\Omega a(x,0,\boldsymbol{v}_0(x,t))\nabla \boldsymbol{z}(x) \cdot \nabla \boldsymbol{\varpi}(x)dx, \quad \forall \boldsymbol{\varpi} \in \boldsymbol{V}_\sigma. \tag{6.28}$$

Thus, by (6.24) we easily check that F_0 is linear, coercive, continuous, and single-valued maximal monotone from $\boldsymbol{V}_\sigma$ into $\boldsymbol{V}_\sigma^*$ satisfying (5.18) with $V = \boldsymbol{V}_\sigma$ and $V^* := \boldsymbol{V}_\sigma^*$.

From (6.28) and the facts identified in the proof of Proposition 6.7, problem (6.27) can be reformulated in the abstract form $CP_{\boldsymbol{w}}^{\varepsilon}(\boldsymbol{f} + \varepsilon F_0 \boldsymbol{h}, \boldsymbol{u}_0)$:

$$\begin{cases} \varepsilon F_0 \boldsymbol{u}'(t) + \partial_* \psi^t(\boldsymbol{w}; \boldsymbol{u}'(t)) + \partial_* \varphi^t(\boldsymbol{w}; \boldsymbol{u}(t)) + \boldsymbol{g}(\boldsymbol{w}; t, \boldsymbol{u}(t)) \ni \boldsymbol{f}(t) + \varepsilon F_0 \boldsymbol{h}(t) \ \text{ in } \boldsymbol{V}_\sigma^*, \\ \hfill \text{a.a. } t \in (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \ \text{ in } \boldsymbol{V}_\sigma. \end{cases}$$

Therefore, Proposition 5.1(I) implies the existence–uniqueness of solutions to (6.27) on $[0,T]$. In addition, by Proposition 5.2, we get the result on the relationship between (6.26) and (6.27). Namely, we observe that (6.26) is approximated by (6.27) as $\varepsilon \downarrow 0$.

Moreover, using the standard smoothness arguments (e.g., regularization by the mollifier and the convolution [1, Sections 2.28 and 3.16]), we can check assumption (H) with $V := \boldsymbol{V}_\sigma$ and $X := [H^2(\Omega)]^3 \cap \boldsymbol{V}_\sigma$, where $[H^2(\Omega)]^3 := H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$. Therefore, Theorem 5.2(I) implies that the approximate optimal control problem $(\mathrm{OP})_\delta^\varepsilon$, corresponding to (6.27), has at least one optimal control $[\boldsymbol{w}_{\delta,\varepsilon}^*, \boldsymbol{f}_{\delta,\varepsilon}^*, \boldsymbol{h}_{\delta,\varepsilon}^*] \in \mathcal{W}_{M'}(\boldsymbol{u}_0) \times \mathcal{F}_M \times \mathcal{H}_N^\varepsilon$ (cf. (5.3), (1.2), and (5.28)). Additionally, by Theorem 5.2(II) on the relationship between $(\mathrm{OP})_\delta$ and $(\mathrm{OP})_\delta^\varepsilon$, we see that $(\mathrm{OP})_\delta$ is approximated by $(\mathrm{OP})_\delta^\varepsilon$ as $\varepsilon \downarrow 0$; more precisely, there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$ with $\varepsilon_n \to 0$ $(n \to \infty)$ such that any weak limit function $[\boldsymbol{w}_\delta^*, \boldsymbol{f}_\delta^*]$ of $\{[\boldsymbol{w}_{\delta,\varepsilon_n}^*, \boldsymbol{f}_{\delta,\varepsilon_n}^*]\}_{n\in\mathbb{N}}$ in $L^2(0,T;\boldsymbol{V}_\sigma) \times L^2(0,T;\boldsymbol{V}_\sigma^*)$ is an optimal control for $(\mathrm{OP})_\delta$.

Consequently, it is worth studying (6.27) and its control problem $(\mathrm{OP})_\delta^\varepsilon$ for sufficient small $\delta, \varepsilon \in (0,1]$, instead of considering quasi-variational inequality (6.23) of Navier-Stokes type and its singular control problem (OP), from a numerical point of view.

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