Advances in Mathematical
Sciences and Applications
Vol. 29, No. 2 (2020), pp. 403-418

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PARABOLIC QUASI-VARIATIONAL INEQUALITIES (II)

– REMARKS ON CONTINUITY OF SOLUTIONS –

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Abstract. This paper is concerned with a parabolic variational obstacle problem (VI) for a semimonotone operator coupled with a semi-linear heat equation (H). Denoting the solutions of (VI) and (H) by u and θ, respectively, we suppose that the heat source term of (H) depends on u and an interior obstacle of the form

$$|u| \le \gamma(\theta)$$

is imposed. In case the obstacle function $\gamma(\theta)$ is continuous, but degenerate, namely $\gamma(\theta) = 0$ may happen somewhere, the continuity of u in time has not proved yet for a general class of quasi-variational inequalities, such as $\{(VI), (H)\}$. In this paper we shall discuss it in a typical case of $\{(VI), (H)\}$.

Communicated by Editors; Received September 8, 2020.
AMS Subject Classification: 34G25, 35G45, 35K51, 35K57, 35K59.
Key words: subdifferential, obstacle problems, quasi-variational inequality

1. Introduction

In this paper we treat a typical class of quasi-variational systems. Let Ω be a bounded domain in $\mathbf{R}^N,\ 2 \le N < \infty$, with smooth boundary $\Gamma := \partial\Omega$ and put

$$0 < T < \infty, \quad Q := \Omega \times (0,T), \quad \Sigma := \Gamma \times (0,T),$$

$$H := L^2(\Omega),\ V := W_0^{1,p}(\Omega) \text{ with } 2 \le p < \infty,\ V^* : \text{dual space of } V.$$

In this case, we have

$$V \subset H \subset V^* \ \text{ with dense and compact embeddings;}$$

we denote by $\langle \cdot,\cdot \rangle$ the duality between V^* and V, by $(\cdot,\cdot)_H$ the innner product in H and by $|\cdot|_H,\ |\cdot|_V$ and $|\cdot|_{V^*}$ the norms in $H,\ V$ and V^*, respectively.

Our problem is formulated as follows:

Definition 1.1. A pair of functions $\{u,\theta\}$ is called a weak solution of $QVI(\gamma; f, u_0, \theta_0)$ if it fulfills the following statements (a),(b),(c):

(a) (Regularity) $\theta \in W^{1,2}(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))\ (\subset C(\overline{Q}))$ and $u \in C([0,T];H) \cap L^p(0,T;V)$.

(b) (Heat conduction) θ is the solution of

$$\theta_t - \kappa\Delta\theta = h(x,t,u) \ \text{ in } L^2(\Omega),\ \text{a.e. } t \in (0,T), \tag{1.1}$$

$$\frac{\partial\theta}{\partial n} + n_0\theta = 0 \text{ a.e. on } \Sigma, \quad \theta(\cdot,0) = \theta_0 \text{ on } \Omega, \tag{1.2}$$

where $\kappa,\ n_0$ are positive constants, $\theta_t := \frac{\partial\theta}{\partial t}$ and $\frac{\partial\theta}{\partial n}$ denotes the outward normal derivative of θ on Γ.

(c) (Quasi-variational inequality) u is a solution of

$$|u(x,t)| \le \gamma(\theta(x,t)) \text{ a.e. } x \in \Omega,\ \forall t \in [0,T], \quad u(0) = u_0,$$

$$\begin{gathered}\int_s^t \langle \eta', u-\eta\rangle d\tau + \frac{1}{p}\int_s^t\int_\Omega a(x,\tau,u)(|\nabla u|^p - |\nabla\eta|^p)dxd\tau \\ + \frac{1}{2}|u(t)-\eta(t)|_H^2 \le \int_s^t \langle f, u-\eta\rangle d\tau + \frac{1}{2}|u(s)-\eta(s)|_H^2, \\ \forall \eta \in \mathcal{K}_0(\theta),\ \forall s,\ t \in [0,T],\ s \le t,\end{gathered} \tag{1.3}$$

where

$$\mathcal{K}_0(\theta) := \left\{ \eta \ \middle|\ \begin{array}{l} \eta \in L^p(0,T;V) \cap C(\overline{Q}),\ \eta' \in L^{p'}(0,T;V^*), \\ \mathrm{supp}(\eta) \subset Q_0,\ \eta(t) \in K(\theta;t),\ \forall t \in [0,T] \end{array} \right\}.$$

with $Q_0 := \{(x,t) \in Q \mid \gamma(\theta(x,t)) > 0\}$ and

$$K(\theta;t) := \{z \in V \mid |z| \le \gamma(\theta(\cdot,t)) \text{ a.e. on } \Omega\}, \quad \forall t \in [0,T].$$

Our problem is discussed under the following assumptions:

(C1) $a(x,t,u)$ is a function on $\overline{\Omega} \times [0,T] \times \mathbf{R}$ satisfying the Carathéodory condition (i.e. $a(x,t,u)$ is measurable in (x,t) for each u and continuous in u for a.e. (x,t)), and

$$a_0 \le a(x,t,u) \le a_1 \quad \text{for a.e. } (x,t) \in Q, \ \forall u \in \mathbf{R}, \tag{1.4}$$

where a_0 and a_1 are positive constants.

(C2) $\gamma(\theta)$ is a non-negative, continuous and bounded function of $\theta \in \mathbf{R}$; choose a positive constant γ^* so that

$$0 \le \gamma(\theta) \le \gamma^*, \quad \forall \theta \in \mathbf{R}. \tag{1.5}$$

(C3) $h(x,t,u)$ is a Lipschitz continuous function on $\overline{Q} \times \mathbf{R}$ with Lipschitz constant L_h;

$$|h(x,t,u) - h(\bar{x},\bar{t},\bar{u})| \le L_h(|x-\bar{x}| + |t-\bar{t}| + |u-\bar{u}|), \tag{1.6}$$

(C4) $f \in L^{p'}(0,T;V^*)$, $\frac{1}{p} + \frac{1}{p'} = 1$, and the initial data θ_0 and u_0 satisfy that

$$\theta_0 \in H^2(\Omega), \ \frac{\partial \theta_0}{\partial n} + n_0\theta_0 = 0 \text{ a.e. on } \Gamma, \quad u_0 \in H, \ |u_0| \le \gamma(\theta_0) \text{ a.e. on } \Omega.$$

In the non-degenerate case of γ, namely

$$\varepsilon_0 \le \gamma(\theta) \le \gamma^*, \quad \forall \theta \in \mathbf{R},$$

for a positive constant ε_0, the existence of a weak solution $\{u,\theta\}$ in the sense of Definition 1.1 is already known (see [6; Theorem 5.1]). However, in the degenerate case (1.5) of γ, namely the case of $\varepsilon_0 = 0$, the existence of such a solution is still an open question; especially, the continuity $u : [0,T] \to H$ is not known, yet. The main objective of this paper is to construct a weak solution $\{u,\theta\}$ in the degenerate case of γ, too.

Our main result of this paper is:

Theorem 1.1. *Under assumptions* $(C1) \sim (C4)$, *there exists a weak solution* $\{u,\theta\}$ *of* $QVI(\gamma; f, u_0, \theta_0)$ *in the sense of Definition 1.1.*

The proof of Theorem 1.1 will be performed in several steps.

First γ is approximated by non-degenerate functions

$$\gamma_n(r) := \gamma(r) + \frac{1}{n}, \quad \forall r \in \mathbf{R}.$$

Now we recall the general result [6; Theorem 5.1] in the non-degenerate case, for each $n \in \mathbf{N}$ there exists a pair of functions $\{u_n, \theta_n\}$ such that

(a') (Regularity) $\theta_n \in W^{1,2}(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$ $(\subset C(\overline{Q}))$ and
$u_n \in C([0,T];H) \cap L^p(0,T;V)$.

(b') (Heat conduction) θ_n is the solution of

$$\theta_{n,t} - \kappa\Delta\theta_n = h(x,t,u_n) \text{ in } L^2(\Omega), \text{ a.e. } t \in (0,T),$$
$$\frac{\partial \theta_n}{\partial n} + n_0\theta_n = 0 \text{ a.e. on } \Sigma, \quad \theta_n(\cdot,0) = \theta_0 \text{ on } \Omega,$$

(c') (Quasi-variational inequality) u_n is a solution of

$$|u_n(x,t)| \le \gamma_n(\theta_n(x,t)) \text{ a.e. } x \in \Omega, \ \forall t \in [0,T], \quad u_n(0) = u_0, \tag{1.7}$$

$$\begin{aligned} &\int_s^t \langle \eta', u_n - \eta\rangle d\tau + \int_s^t \int_\Omega a(x,\tau,u_n)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - \eta) dx d\tau \\ &\quad + \frac{1}{2}|u_n(t) - \eta(t)|_H^2 \le \int_s^t \langle f, u_n - \eta\rangle d\tau + \frac{1}{2}|u_n(s) - \eta(s)|_H^2, \end{aligned} \tag{1.8}$$
$$\forall \eta \in \mathcal{K}_0^n(\theta_n), \ \forall s, \ t \in [0,T], \ s \le t,$$

where

$$\mathcal{K}_0^n(\theta_n) := \left\{ \eta \ \middle| \ \begin{array}{l} \eta \in L^p(0,T;V), \ \eta' \in L^{p'}(0,T;V^*), \\ \eta(t) \in K^n(\theta_n;t), \ \forall t \in [0,T] \end{array} \right\}. \tag{1.9}$$

with

$$K^n(\theta_n;t) := \{ z \in V \mid |z| \le \gamma_n(\theta_n(\cdot,t)) \text{ a.e. on } \Omega\}, \quad \forall t \in [0,T]. \tag{1.10}$$

Remark 1.1. Since $|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - \eta) \ge \frac{1}{p}(|\nabla u_n|^p - |\nabla \eta|^p)$, (1.8) implies that

$$\begin{aligned} &\int_s^t \langle \eta', u_n - \eta\rangle d\tau + \frac{1}{p}\int_s^t \int_\Omega a(x,\tau,u_n)(|\nabla u_n|^p - |\nabla\eta|^p) dx d\tau \\ &\quad + \frac{1}{2}|u_n(t) - \eta(t)|_H^2 \le \int_s^t \langle f, u_n - \eta\rangle d\tau + \frac{1}{2}|u_n(s) - \eta(s)|_H^2, \end{aligned} \tag{1.11}$$

Let us give some uniform estimates for $\{u_n, \theta_n\}$ which are easily derived from (1.7)∼(1.10). Since $\eta \equiv 0$ is a trivial test function, it follows from (1.8) with (1.4) that

$$\int_{\Omega\times(0,t)} a_0|\nabla u_n|^p dx d\tau + \frac{1}{2}|u_n(t)|_H^2 \le \int_0^t \langle f, u_n\rangle d\tau + \frac{1}{2}|u_0|_H^2, \quad \forall t \in [0,T],$$

which yields a uniform estimate of the form:

$$|u_n|_{L^p(0,T;V)}^p + |u_n|_{C([0,T];H)}^2 \le N_0(1 + |f|_{L^{p'}(0,T;V^*)}^{p'} + |u_0|_H^2), \quad \forall n, \tag{1.12}$$

for some positive constant N_0. Moreover, by a uniform estimate for semilinear heat equations (cf. [2; Appendix]) we have that

$$|\theta_n|_{W^{1,2}(0,T;H^1(\Omega))} + |\theta_n|_{L^\infty(0,T;H^2(\Omega))} \le N_1(1 + |h|_{L^2(0,T;H^1(\Omega))} + |\theta_0|_{H^2(\Omega)}), \tag{1.13}$$

for some positive constants N_1. Here, by (1.6),

$$|h(\cdot,\cdot,u_n(t))|_{H^1(\Omega)} \le L_h'(1 + |u_n(t)|_V), \quad \text{a.e. } t \in (0,T)$$

for a positive constant L'_h. Therefore, taking account of this estimate, we get from (1.12) and (1.13) that

$$|\theta_n|_{W^{1,2}(0,T;H^1(\Omega))}+|\theta_n|_{L^\infty(0,T;H^2(\Omega))}\le N'_1(1+|f|_{L^{p'}(0,T;V^*)}+|u_0|_H+|\theta_0|_{H^2(\Omega)}),\quad \forall n,\quad (1.14)$$

By virtue of (1.12) and (1.14) we may assume by extracting a subsequence from $\{u_n,\theta_n\}$ if necessary, that

$$\theta_n\to\theta \text{ in } C(\overline{Q}),\ \text{ weakly in } W^{1,2}(0,T;H^1(\Omega)) \text{ and weakly}^* \text{ in } L^\infty(0,T;H^2(\Omega))\quad (1.15)$$

and

$$u_n\to u \text{ in weakly in } L^p(0,T;V) \text{ and weakly}^* \text{ in } L^\infty(0,T;H),\quad (1.16)$$

(as $n\to\infty$) for some function $\theta\in W^{1,2}(0,T;H^1(\Omega))\cap L^\infty(0,T;H^2(\Omega))$ and $u\in L^2(0,T;V)$. Furthermore, since $\gamma_n(\theta_n)\to\gamma(\theta)$ in $C(\overline{Q})$, we see that

$$|u(x,t)|\le\gamma(\theta(x,t))\ \text{ a.e. } x\in\Omega,\ \forall t\in[0,T].\quad (1.17)$$

2. The relative compactness of $\{u_n\}$ in $L^p(Q)$

This section is the key step for the proof of Theorem 1.1. Let $\{u_n,\theta_n\}$ be a sequence of solutions to $QVI(\gamma_n;f,u_0,\theta_0):=\{(a'),(b'),(c')\}$ as we recalled in the previous section, and let $\{u,\theta\}$ be the limit of $\{u_n,\theta_n\}$ satisfying (1.12)$\sim$(1.17).

(Step 1) Now, let δ be any positive number and t_0 be any time in $[0,T)$ and fix them for a moment. Also, let $\Omega_\delta(t_0)$ be a subdomain of Ω such that

$$\gamma(\theta(x,t_0))>2\delta,\quad \forall x\in\Omega_\delta(t_0).$$

Then, by the continuity of γ, there is a time interval $[T_1,T'_1]$ such that $0\le T_1\le t_0<T'_1\le T$ such that

$$\gamma(\theta(x,t))>\delta,\quad \forall(x,t)\in\Omega_\delta(t_0)\times[T_1,T'_1].\quad (2.1)$$

Also, given any number $\varepsilon>0$, by (1.15) there is a positive integer n_ε such that

$$|\gamma(\theta)-\gamma_n(\theta_n)|_{C(\overline{Q})}<\varepsilon,\quad \forall n\ge n_\varepsilon,\quad (2.2)$$

so that

$$\gamma_n(\theta_n)\ge\delta-\varepsilon\ \text{ on } Q_\delta(t_0):=\Omega_\delta(t_0)\times(T_1,T'_1),\ \forall n\ge n_\varepsilon.\quad (2.3)$$

We are going to apply the compactness result for variational inequalities mentioned in the appendix to show the relative compactness of $\{u_n\}$ in $L^p(Q_\delta(t_0))$. To do so, put:

$$H_1:=L^2(\Omega_\delta(t_0)),\quad V_1:=W^{1,p}(\Omega_\delta(t_0)),\quad W_1:=W_0^{1,q}(\Omega_\delta(t_0)),\ N+2<q<\infty.$$

We denote by W_1^* the dual space of W_1, and by $\langle\cdot,\cdot\rangle$ the duality between W_1^* and W_1, too. Then, $V_1 \subset H_1 \subset W_1^*$ with dense and compact embeddings, and the restriction of each function u_n onto $Q_\delta(t_0)$ satisfies that

$$|u_n(x,t)| \le \gamma_n(\theta_n(x,t)) \text{ a.e. } x \in \Omega_\delta(t_0),\ \forall t \in [T_1, T_1'], \tag{2.4}$$

Lemma 2.1. *Let $\varepsilon > 0$ be as above and $\varepsilon_1 > 0$ be any small number. If $0 < \varepsilon < \varepsilon_1\delta$, then for any $\eta \in \mathcal{K}_0(\theta)$,*

$$(1-\varepsilon_1)|\eta(x,t)| \le \gamma_n(\theta_n(x,t)), \quad \forall x \in \Omega_\delta(t_0),\ \forall t \in [T_1, T_1']. \tag{2.5}$$

Proof. We see by (2.2) and (2.3) that for all $x \in \Omega_\delta(t_0)$ and all $t \in [T_1, T_1']$

$$\begin{aligned}
(1-\varepsilon_1)|\eta(x,t)| &\le (1-\varepsilon_1)\gamma(\theta(x,t)) \\
&\le (1-\varepsilon_1)(\gamma_n(\theta_n(x,t)) + \varepsilon) \\
&\le \gamma_n(\theta_n(x,t)) - \varepsilon_1\gamma_n(\theta_n(x,t)) + \varepsilon(1-\varepsilon_1) \\
&\le \gamma_n(\theta_n(x,t)) - \varepsilon_1(\delta-\varepsilon) + \varepsilon(1-\varepsilon_1) \\
&= \gamma_n(\theta_n(x,t)) - \varepsilon_1\delta + \varepsilon < \gamma_n(\theta_n(x,t)), \quad \forall n \ge n_\varepsilon,
\end{aligned}$$

if $\varepsilon > 0$ is chosen so as to be $\varepsilon < \varepsilon_1\delta$. Thus (2.5) holds. □

From (2.1) and $W_1 \subset C(\overline{\Omega_\delta(t_0)})$ with compact embedding it follows that for a positive constant $\nu > 0$ small enough

$$\nu B_{W_1}(0) \subset \left\{ z \in C(\overline{\Omega_\delta(t_0)}) \;\middle|\; |z| \le \frac{\delta}{2} \text{ on } \overline{\Omega_\delta(t_0)} \right\},$$

where $B_{W_1}(0)$ is the closed unit ball of W_1 around the origin. Hence, for any ε with $0 < \varepsilon \le \frac{\delta}{2}$ and for all $n \ge n_\varepsilon$, it follows from (2.3) that

$$\nu B_{W_1}(0) \subset K^n(\theta_n; t), \quad \forall t \in [T_1, T_1']; \tag{2.6}$$

here note that $\eta \in W_1$ can be considered as a function in V by extending it by 0 outside of $\Omega_\delta(t_0)$.

Lemma 2.2. *There exists a positive number C^* such that*

$$\int_{T_1}^{T_1'} \langle \eta', u_n\rangle d\tau \le C^*, \quad \forall \eta \in C_0^1(T_1, T_1'; W_1) \text{ with } \eta(t) \in \nu B_{W_1}(0),\ \forall t \in [T_1, T_1']. \tag{2.7}$$

Proof. Since $\{u_n, \theta_n\}$ is a weak solution to the non-degenerate problem $QVI(\gamma_n; f, u_0, \theta_0)$, we see that

$$\begin{aligned}
&\int_{T_1}^{T_1'} \langle \eta', u_n - \eta\rangle d\tau + \frac{1}{p}\int_{T_1}^{T_1'} \int_\Omega a(x,\tau,u_n)(|\nabla u_n|^p - |\nabla \eta|^p) dx d\tau \\
&\quad + \frac{1}{2}|u_n(T_1') - \eta(T_1')|_H^2 \le \int_{T_1}^{T_1'} \langle f, u_n - \eta\rangle d\tau + \frac{1}{2}|u_n(T_1) - \eta(T_1)|_H^2,
\end{aligned} \tag{2.8}$$

$$\forall \eta \in \mathcal{K}_0^n(\theta_n).$$

By virtue of Lemma 2.1 and (2.6), for any $\eta \in C_0^1(T_1, T_1'; W_1)$ with $\eta(t) \in \nu B_{W_1}(0)$ for all $t \in [T_1, T_1']$ we observe that $(1-\varepsilon_1)\eta \in \mathcal{K}_0^n(\theta_n)$ for all small $\varepsilon_1 > 0$. Taking $(1-\varepsilon_1)\eta$ as the test function of (2.8), we get

$$\begin{aligned}\int_{T_1}^{T_1'} (1-\varepsilon_1)\langle \eta', u_n\rangle d\tau \quad &\le \quad \frac{a_1}{p}(1-\varepsilon_1)^p \int_{T_1}^{T_1'} \int_{\Omega_\delta(t_0)} |\nabla \eta|^p dx d\tau \\ &\quad + \int_{T_1}^{T_1'} |f|_{V^*}(|u_n|_V + (1-\varepsilon_1)|\eta|_V) d\tau + \frac{1}{2}|u_n(T_1)|_H^2,\end{aligned}$$

whence for any ε_1 with $0 < \varepsilon_1 \le \frac{1}{2}$

$$\int_{T_1}^{T_1'} \langle \eta', u_n\rangle d\tau \le a_1 \int_{T_1}^{T_1'} |\nabla \eta|^p dx d\tau + \int_{T_1}^{T_1'} |f|_{V^*}(2|u_n|_V + |\eta|_V) d\tau + |u_n(T_1)|_H^2.$$

The right hand side of the above inequality is dominated by a positive constant C^* independent of $\eta \in C_0^1(T_1, T_1'; W_1)$ with $\eta(t) \in \nu B_{W_1}(0)$ for all $t \in [T_1, T_1']$ as well as independent of $|u_n(T_1)|_H$ by estimate (2.4). Thus (2.7) is obtained. $\square$

By Lemma 2.3 together with (1.12) our compactness result (see the appendix) shows that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k}(t) \to u(t) \text{ weakly in } H_1 \text{ as } k \to \infty, \ \forall t \in [T_1, T_1'],$$

and

$$u_{n_k} \to u \text{ in } L^p(Q_\delta(t_0)) \ \text{ as } k \to \infty.$$

(Step 2) We put

$$\Omega_*(t) := \{x \in \overline{\Omega} \mid \gamma(\theta(x,t)) = 0\}, \ \forall t \in [0,T], \quad Q_* := \cup_{t\in[0,T]} \Omega_*(t) \times \{t\}$$

and for each $\delta > 0$

$$\Omega_\delta(t) := \{x \in \Omega \mid \gamma(\theta(x,t)) > 2\delta\}, \ \forall t \in [0,T], \quad \hat{Q}_\delta := \cup_{t\in[0,T]} \Omega_\delta(t) \times \{t\}.$$

With similar notation, given $\delta > 0$, it is easy to check that for $\delta' := \frac{\delta}{2}$ there exists a finite number of cylindrical domains of the form

$$\hat{Q}_i := \Omega_{\delta'}(t_i) \times [T_i, T_i'], \quad 0 \le T_i \le t_i < T_i' \le T, \ i = 1, 2, \cdots, \ell,$$

such that

$$\hat{Q}_\delta \subset \cup_{i=1}^{\ell} \hat{Q}_i.$$

As was proved in the first step, for every $i = 1, 2, \cdots, \ell$, it is possible to extract a subsequence $\{u_{n_k}\}$ from $\{u_n\}$ so that

$$u_{n_k}(t) \to u(t) \text{ weakly in } L^2(\Omega_{\delta'}(t_i)), \ \forall t \in [T_i, T_i'], \ \forall i = 1, 2, \cdots, \ell,$$

$$u_{n_k} \to u \ \text{ in } L^p(\hat{Q}_i), \ \forall i = 1, 2, \cdots, \ell,$$

(as $k \to \infty$) and additionally

$$u_{n_k}(t) \to u(t) \ \text{ in } L^p(\Omega_{\delta'}(t_i)), \ \text{a.e. } t \in (T_i, T_i'), \ \forall i = 1, 2, \cdots, \ell.$$

Therefore we have

$$u_{n_k}(t) \to u(t) \text{ weakly in } L^2(\Omega_\delta(t)), \ \forall t \in [0, T],$$

and

$$u_{n_k} \to u \ \text{ in } L^p(\hat{Q}_\delta), \quad u_{n_k}(t) \to u(t) \ \text{ in } L^p(\Omega_\delta(t)), \ \text{a.e. } t \in (0, T).$$

Since $\gamma(\theta) \le 2\delta$ on $Q - \hat{Q}_\delta$, we see by (2.2) that

$$|u_{n_k} - u| \le 4\delta + \frac{1}{n_k} \ \text{ a.e. on } Q - \hat{Q}_\delta.$$

Hence

$$\int_{Q - \hat{Q}_\delta} |u_{n_k} - u|^p dx d\tau \to 0 \text{ as } k \to \infty \text{ and } \delta \to 0.$$

As a consequence we have:

Lemma 2.4. *There exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that*

$$u_{n_k}(t) \to u(t) \ \textit{weakly in } H, \ \forall t \in [0, T], \tag{2.9}$$

and

$$u_{n_k} \to u \ \textit{ in } L^p(Q), \quad u_{n_k}(t) \to u(t) \ \text{ in } H, \ \textit{a.e. } t \in (0, T). \tag{2.10}$$

3. Verification of (b) and (c) in Definition 1.1

For simplicity we use the following notation:

$$\Phi_s^t(w; v) := \frac{1}{p} \int_s^t a(x, t, w) |\nabla v|^p dx d\tau, \quad \forall w \in L^p(Q), \ \forall v \in L^p(0, T; V),$$

$$\forall s, \ t \in [0, T], \ s \le t.$$

Let $\{u_{n_k}, \theta_{n_k}\}$ be a subsequence of $\{u_n, \theta_n\}$ satisfying (2.9) and (2.10) of Lemma 2.4 together with (1.15) and (1.16). Then, for each n_k we see from (c') and (1.11), it holds that

$$u_{n_k}(t) \in K^{n_k}(\theta_{n_k}; t), \ \forall t \in [0, T], \quad u_{n_k}(0) = u_0,$$

$$\begin{aligned} &\int_s^t \langle \eta', u_{n_k} - \eta \rangle d\tau + \Phi_s^t(u_{n_k}; u_{n_k}) + \frac{1}{2}|u_{n_k}(t) - \eta(t)|_H^2 \\ &\le \int_s^t \langle f, u_{n_k} - \eta \rangle d\tau + \Phi_s^t(u_{n_k}; \eta) + \frac{1}{2}|u_{n_k}(s) - \eta(s)|_H^2, \end{aligned} \tag{3.1}$$

$$\forall \eta \in \mathcal{K}_0^{n_k}(\theta_{n_k}),\ \forall s,\ t \in [0,T],\ s \le t.$$

Now, put $E_0 := \{s \in [0,T] \mid u_{n_k}(s) \to u(s) \text{ in } H \text{ as } k \to \infty\}$. Then, by our construction of $\{u_{n_k}\}$, $0 \in E_0$ because $u_{n_k}(0) = u_0$, and moreover, by Lemma 2.4, $[0,T] - E_0$ is of linear measure zero.

Lemma 3.1. *The pair of function $\{u, \theta\}$ satisfies*

$$\begin{aligned} &\int_s^t \langle \eta', u - \eta \rangle d\tau + \Phi_s^t(u; u) + \frac{1}{2}|u(t) - \eta(t)|_H^2 \\ &\le \int_s^t \langle f, u - \eta \rangle d\tau + \Phi_s^t(u; \eta) + \frac{1}{2}|u(s) - \eta(s)|_H^2, \end{aligned} \tag{3.2}$$

$$\forall \eta \in \mathcal{K}_0(\theta),\ \mathrm{supp}(\eta) \subset Q_0 := \{(x,t) \in Q \mid \gamma(\theta(x,t)) > 0\},\ \forall s \in E_0,\ \forall t \ge s.$$

Proof. Let δ be any positive number and η be any function in $\mathcal{K}_0(\theta)$ such that

$$\mathrm{supp}(\eta) \subset Q_\delta := \{(x,t) \in Q \mid \gamma(\theta(x,t)) > \delta\}.$$

Then, given $\varepsilon > 0$, by (1.15) there is a positive integer k_ε such that

$$|\gamma(\theta_{n_k}) - \gamma(\theta)| < \varepsilon \ \text{ on } Q, \quad \forall k \ge k_\varepsilon,$$

whence

$$\gamma_{n_k}(\theta_{n_k}) \ge \delta - \varepsilon \text{ on } Q_\delta, \quad \forall k \ge k_\varepsilon,$$

Next, let ε_1 be a small positive number. Then,

$$\begin{aligned} (1-\varepsilon_1)|\eta| &\le (1-\varepsilon_1)\gamma(\theta) \le (1-\varepsilon_1)(\gamma_{n_k}(\theta_{n_k}) + \varepsilon) \\ &\le \gamma_{n_k}(\theta_{n_k}) - \varepsilon_1\delta + \varepsilon \\ &\le \gamma_{n_k}(\theta_{n_k}) \ \text{ on } Q_\delta,\ \forall k \ge k_\varepsilon, \end{aligned}$$

as long as $0 < \varepsilon \le \varepsilon_1\delta$. Therefore, taking $(1-\varepsilon_1)\eta$ as a test function in (3.1), we observe that

$$\begin{aligned} &(1-\varepsilon_1)\int_s^t \langle \eta', u_{n_k} - (1-\varepsilon_1)\eta \rangle d\tau + \Phi_s^t(u_{n_k}; u_{n_k}) + \frac{1}{2}|u_{n_k}(t) - (1-\varepsilon_1)\eta(t)|_H^2 \\ &\le \int_s^t \langle f, u_{n_k} - (1-\varepsilon_1)\eta \rangle d\tau + \Phi_s^t(u_{n_k}; (1-\varepsilon)\eta) + \frac{1}{2}|u_{n_k}(s) - (1-\varepsilon_1)\eta(s)|_H^2, \end{aligned} \tag{3.3}$$

$$\forall s,\ t \in [0,T],\ s \le t,\ \forall k \ge k_\varepsilon.$$

Fixing $\delta > 0$, pass to the limit $k \to \infty$ in (3.3) and use Lemma 2.4 to see that

$$\Phi_s^t(u; u) \le \liminf_{k\to\infty} \Phi_s^t(u_{n_k}; u_{n_k}), \quad \Phi_s^t(u; (1-\varepsilon_1)\eta) = \lim_{k\to\infty} \Phi_s^t(u_{n_k}; (1-\varepsilon_1)\eta),$$

$$|u(t) - (1-\varepsilon_1)\eta(t)|_H^2 \le \liminf_{k\to\infty} |u_{n_k}(t) - (1-\varepsilon_1)\eta(t)|_H^2,\ \forall t \in [0,T],$$

and

$$|u(s) - (1-\varepsilon_1)\eta(s)|_H^2 = \lim_{k\to\infty} |u_{n_k}(s) - (1-\varepsilon_1)\eta(s)|_H^2, \ \forall s \in E_0.$$

By virtue of these facts, finally letting $\varepsilon_1 \to 0$ yields (3.2) for all $\eta \in \mathcal{K}_0$ satisfying $\mathrm{supp}(\eta) \subset Q_0$, since $\delta > 0$ is arbitrary. □

Lemma 3.2. *The pair of functions $\{u, \theta\}$ satisfies (1.1) and (1.2).*

Proof. For each k, the function θ_{n_k} satisfies that

$$\frac{\partial \theta_{n_k}}{\partial t} - \kappa \Delta \theta_{n_k} = h(x, t, u_{n_k}) \ \text{ a.e. in } Q,$$

with

$$\frac{\partial \theta_{n_k}}{\partial n} + n_0 \theta_{n_k} = 0 \ \text{ a.e. on } \Sigma, \quad \theta_{n_k}(0) = \theta_0.$$

Since $h(x,t,u_{n_k}) \to h(x,t,u)$ in $L^2(Q)$ by (2.10) of Lemma 2.4 and $\theta_{n_k} \to \theta$ in $C(\overline{Q})$, weakly in $W^{1,2}(0,T;H^1(\Omega))$ and weakly* in $L^\infty(0,T;H^2(\Omega))$ as $k \to \infty$, it follows that θ is a unique solution of (1.1) and (1.2), and $\theta \in W^{1,2}(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$. □

Next, in order to prove the continuity of $u(t)$ in H with respect to time, we make use of the following family of test functions

$$\eta_{s,\varepsilon,\varepsilon_1,\delta}(x,t) \ \text{ with parameters } s \in [0,T], \ \varepsilon > 0, \ \varepsilon_1 > 0, \ \delta > 0,$$

which is defined as follows:

- Given $\varepsilon > 0$, take a positive number t_ε such that

$$|\gamma(\theta(\cdot,t)) - \gamma(\theta(\cdot,s))|_{C(\overline{\Omega})} < \varepsilon, \quad \forall s, \ t \in [0,T] \text{ with } |s-t| \le t_\varepsilon, \tag{3.4}$$

  and let $\sigma_\varepsilon := \sigma_\varepsilon(t)$ be a smooth non-negative function on $\mathbf{R}$ such that

$$\sigma_\varepsilon(0) = 1, \ \sigma_\varepsilon(t) = \sigma_\varepsilon(-t), \ \forall t \ge 0, \ 0 \le \sigma_\varepsilon \le 1 \text{ on } \mathbf{R}, \ \mathrm{supp}(\sigma_\varepsilon) \subset (-t_\varepsilon, t_\varepsilon), \tag{3.5}$$

$$\sigma'_\varepsilon(t) := \frac{d\sigma_\varepsilon(t)}{dt} \le 0, \ \forall t \ge 0; \text{ hence } \sigma'_\varepsilon(0) = 0.$$

- Given $\delta > 0$ and $t_0 \in [0,T]$, denote by $U_\delta(t_0)$ the δ-neighborhood of $\Omega_*(t_0) := \{x \in \overline{\Omega} \mid \gamma(\theta(x,t_0)) = 0\}$. Choose a smooth function $\alpha_\delta := \alpha_\delta(x,t_0)$ so that

$$0 \le \alpha_\delta(\cdot,t_0) \le 1 \text{ on } \Omega, \quad \alpha_\delta(x,t_0) = \begin{cases} 0, & \forall x \in U_\delta(t_0), \\ 1, & \forall x \in \Omega - U_{2\delta}(t_0). \end{cases} \tag{3.6}$$

- Given $\varepsilon > 0$, choose a smooth function $z_\varepsilon(\cdot,s)$ in V for every $s \in [0,T]$ so that

$$|u(s) - z_\varepsilon(s)|_H < \varepsilon, \quad |z_\varepsilon(\cdot,s)| \le \gamma(\theta(\cdot,s)) \text{ on } \Omega, \ |\nabla z_\varepsilon(\cdot,s)| \le C(\varepsilon) \text{ on } \Omega, \tag{3.7}$$

  where $C(\varepsilon)$ is a positive constant depending on ε, but not on s.

We note that for any $t_0 \in [0,T]$ and any $\delta > 0$ there is a positive constant $C_\delta > 0$ such that

$$\gamma(\theta(x,t_0)) \geq C_\delta, \quad \forall x \in \Omega - U_\delta(t_0). \tag{3.8}$$

Hence, by (3.4),

$$\gamma(\theta(x,t)) \geq C_\delta - \varepsilon, \text{ if } |t-t_0| < t_\varepsilon \text{ and } x \in \Omega - U_\delta(t_0). \tag{3.9}$$

Lemma 3.3. *Let $t_0 \in (0,T)$, $\varepsilon_1 \in (0,1)$, σ_ε, α_δ and z_ε be as above. Then,*

$$\eta_{s,\varepsilon,\varepsilon_1,\delta}(x,t) := (1-\varepsilon_1)\sigma_\varepsilon(t-t_0)\alpha_\delta(x,t_0)z_\varepsilon(x,s)$$

belongs to $\mathcal{K}_0(\theta)$, if $\varepsilon_1 \in (0,1)$, $0 < \varepsilon < \varepsilon_1 \cdot \frac{C_\delta}{2}$, $|s-t_0| \leq t_\varepsilon$ and $|t-t_0| \leq t_\varepsilon$.

Proof. If $\varepsilon_1 \in (0,1)$, $0 < \varepsilon < \varepsilon_1 \cdot \frac{C_\delta}{2}$, $x \in \Omega - U_\delta(t_0)$, $|s-t_0| \leq t_\varepsilon$ and $|t-t_0| \leq t_\varepsilon$, then we observe by (3.5)$\sim$(3.9) that

$$\begin{aligned}
|\eta_{s,\varepsilon,\varepsilon_1,\delta}(x,t)| &:= (1-\varepsilon_1)\sigma_\varepsilon(t-t_0)\alpha_\delta(x,t_0)|z_\varepsilon(x,s)| \\
&\leq (1-\varepsilon_1)\sigma_\varepsilon(t-t_0)\alpha_\delta(x,t_0)\gamma(\theta(x,s)) \\
&\leq (1-\varepsilon_1)\sigma_\varepsilon(t-t_0)\alpha_\delta(x,t_0)(\gamma(\theta(x,t))+2\varepsilon) \\
&\leq (1-\varepsilon_1)\sigma_\varepsilon(t-t_0)\alpha_\delta(x,t_0)\gamma(\theta(x,t)) + 2\varepsilon(1-\varepsilon_1)\sigma_\varepsilon(t-t_0)\alpha_\delta(x,t_0) \\
&\leq \gamma(\theta(x,t)) - \sigma_\varepsilon(t-t_0)\alpha_\delta(x,t_0)(\varepsilon_1 \cdot C_\delta - 2\varepsilon) \\
&\leq \gamma(\theta(x,t)).
\end{aligned}$$

Since $\eta_{s,\varepsilon,\varepsilon_1,\delta}(x,t) = 0$ for $x \in U_\delta(t_0)$ or $|t-t_0| \geq t_\varepsilon$, it follows from the above inequalities that $\eta_{s,\varepsilon,\varepsilon_1,\delta} \in \mathcal{K}_0(\theta)$. $\square$

We hereafter denote by $|S|$ the N-dimensional Lebesgue measure of any measurable set S.

Lemma 3.4. *Let $t_0 \in (0,T]$ and $s \in [0,T]$ with $|s-t_0| < t_\varepsilon$. Then,*

$$\begin{aligned}
|u(t_0) - \eta_{s,\varepsilon,\varepsilon_1,\delta}(t_0)|_H \geq |u(t_0)-u(s)|_H - \varepsilon(1+|\Omega|^{\frac{1}{2}}) - \varepsilon_1\gamma^*|\Omega|^{\frac{1}{2}} \\
-\gamma^*|U_{2\delta}(t_0) - \Omega_*(t_0)|^{\frac{1}{2}}
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
|u(s) - \eta_{s,\varepsilon,\varepsilon_1,\delta}(s)|_H \leq \varepsilon(1+|\Omega|^{\frac{1}{2}}) + \gamma^*|U_{2\delta}(t_0) - \Omega_*(t_0)|^{\frac{1}{2}} \\
+(1-\sigma_\varepsilon(s-t_0))\gamma^*|\Omega|^{\frac{1}{2}} + \varepsilon_1\gamma^*|\Omega|^{\frac{1}{2}}.
\end{aligned} \tag{3.11}$$

Proof. We have

$$\begin{aligned}
& |u(t_0) - \eta_{s,\varepsilon,\varepsilon_1,\delta}(t_0)|_H \\
= \;& |u(t_0) - (1-\varepsilon_1)\alpha_\delta(t_0)z_\varepsilon(s)|_H \\
\geq \;& |u(t_0) - z_\varepsilon(s)|_H - |z_\varepsilon(s) - (1-\varepsilon_1)\alpha_\delta(t_0)z_\varepsilon(s)|_H \\
\geq \;& |u(t_0)-u(s)|_H - \varepsilon - |(1-\alpha_\delta(t_0))z_\varepsilon(s)|_H - \varepsilon_1|\alpha_\delta(t_0)z_\varepsilon(s)|_H \\
\geq \;& |u(t_0)-u(s)|_H - \varepsilon - |(1-\alpha_\delta(t_0))\gamma(\theta(s))|_H - \varepsilon_1|\gamma(\theta(s))|_H.
\end{aligned}$$

Since $\varepsilon_1|\gamma(\theta(s))|_H \le \varepsilon_1\gamma^*|\Omega|^{\frac{1}{2}}$ and

$$|(1-\alpha_\delta(t_0))\gamma(\theta(s))|_H \le |(1-\alpha_\delta(t_0))(\gamma(\theta(t_0))+\varepsilon)|_H \le \gamma^*|U_{2\delta}(t_0)-\Omega_*(t_0)|^{\frac{1}{2}} + \varepsilon|\Omega|^{\frac{1}{2}},$$

it follows from the above inequalities that (3.10) is obtained.

Also, we observe that

$$\begin{aligned}
& |u(s)-\eta_{s,\varepsilon,\varepsilon_1,\delta}(s)|_H \\
= \ & |u(s)-(1-\varepsilon_1)\sigma_\varepsilon(s-t_0)\alpha_\delta(t_0)z_\varepsilon(s)|_H \\
\le \ & |u(s)-\sigma_\varepsilon(s-t_0)\alpha_\delta(t_0)z_\varepsilon(s)|_H + \varepsilon_1|\sigma_\varepsilon(s-t_0)\alpha_\delta(t_0)z_\varepsilon(s)|_H \\
\le \ & |u(s)-z_\varepsilon(s)|_H + |(1-\sigma_\varepsilon(s-t_0)\alpha_\delta(t_0))z_\varepsilon(s)|_H + \varepsilon_1|\sigma_\varepsilon(s-t_0)\alpha_\delta(t_0)z_\varepsilon(s)|_H \\
\le \ & |u(s)-z_\varepsilon(s)|_H + |(1-\alpha_\delta(t_0))z_\varepsilon(s)|_H + (1-\sigma_\varepsilon(s-t_0))|\alpha_\delta(t_0)z_\varepsilon(s)|_H \\
& +\varepsilon_1|\sigma_\varepsilon(s-t_0)\alpha_\delta(t_0)z_\varepsilon(s)|_H \\
\le \ & \varepsilon + \gamma^*|U_{2\delta}(t_0)-\Omega_*(t_0)|^{\frac{1}{2}} + \varepsilon|\Omega|^{\frac{1}{2}} + (1-\sigma_\varepsilon(s-t_0))\gamma^*|\Omega|^{\frac{1}{2}} + \varepsilon_1\gamma^*|\Omega|^{\frac{1}{2}}.
\end{aligned}$$

Thus (3.11) is obtained. □

4. Proof of Theorem 1.1

We show first that the variational inequality (3.2) holds for all $s,\ t \in [0,T]$ with $s < t$. To do so, we recall from Lemma 3.1 that for any $t_0 \in (0,T]$ and any $s \in [0,t_0] \cap E_0$

$$\begin{aligned}
& \int_s^{t_0} \langle \eta'_{s,\varepsilon,\varepsilon_1,\delta}, u-\eta_{s,\varepsilon,\varepsilon_1,\delta}\rangle d\tau + \Phi_s^{t_0}(u;u) + \frac{1}{2}|u(t_0)-\eta_{s,\varepsilon,\varepsilon_1,\delta}(t_0)|_H^2 \\
& \le \int_s^{t_0} \langle f, u-\eta_{s,\varepsilon,\varepsilon_1,\delta}\rangle d\tau + \Phi_s^{t_0}(u;\eta_{s,\varepsilon,\varepsilon_1,\delta}) + \frac{1}{2}|u(s)-\eta_{s,\varepsilon,\varepsilon_1,\delta}(s)|_H^2.
\end{aligned} \tag{4.1}$$

We note from (3.5)∼(3.7) that $\eta_{s,\varepsilon,\varepsilon_1,\delta}$ is bounded in $C^1(\overline{Q})$ uniformly in $s \in [0,t_0]$ for fixed $\varepsilon,\ \varepsilon_1$ and δ, so that

$$\lim_{s\uparrow t_0} \int_s^{t_0} \langle \eta'_{s,\varepsilon,\varepsilon_1,\delta}, u-\eta_{s,\varepsilon,\varepsilon_1,\delta}\rangle d\tau = 0, \quad \lim_{s\uparrow t_0} \Phi_s^{t_0}(u;u) = 0,$$

$$\lim_{s\uparrow t_0} \int_s^{t_0} \langle f, u-\eta_{s,\varepsilon,\varepsilon_1,\delta}\rangle d\tau = 0, \quad \lim_{s\uparrow t_0} \Phi_s^{t_0}(u;\eta_{s,\varepsilon,\varepsilon_1,\delta}) = 0.$$

Therefore, letting $s \uparrow t_0,\ s \in E_0$ in (4.1), we see that

$$\limsup_{s\in E_0,\ s\uparrow t_0} |u(t_0)-\eta_{s,\varepsilon,\varepsilon_1,\delta}(t_0)|_H \le \limsup_{s\in E_0,\ s\uparrow t_0} |u(s)-\eta_{s,\varepsilon,\varepsilon_1,\delta}(s)|_H, \tag{4.2}$$

Moreover, using (3.10) and (3.11) of Lemma 3.4, we derive from (4.2) that

$$\limsup_{s\in E_0,\ s\uparrow t_0} |u(t_0)-u(s)|_H \le 2\varepsilon(1+|\Omega|^{\frac{1}{2}}) + 2\varepsilon_1\gamma^*|\Omega|^{\frac{1}{2}} + 2\gamma^*|U_{2\delta}(t_0)-\Omega_*(t_0)|^{\frac{1}{2}}, \tag{4.3}$$

whence

$$\lim_{s\in E_0, s\uparrow t_0} |u(t_0)-u(s)|_H = 0, \tag{4.4}$$

since $\varepsilon,\ \varepsilon_1$ and δ are arbitrary as long as $\varepsilon_1 \in (0,1),\ 0<\varepsilon<\frac{\varepsilon_1 C_\delta}{2}$, and $|U_{2\delta}(t_0)-\Omega_*(t_0)| \to 0$ as $\delta \downarrow 0$.

Since t_0 arbitrary in $(0,T]$, the above observation proves:

Lemma 4.1. *For each $s' \in (0,T]$ there exists a sequence $\{s_n\}$ in E_0 such that*

$$s_n \uparrow s', \quad u(s_n) \to u(s') \ \text{in}\ H\ (\text{as}\ n\to\infty.) \tag{4.5}$$

and the variational inequality (1.3) in Definition 1.1 is fulfilled.

Proof. The convergence (4.5) is immediately seen by taking s' as t_0 in the above observation (4.4), and we have by Lemma 3.1 that

$$\begin{aligned} &\int_{s_n}^t \langle \eta', u-\eta\rangle d\tau + \varPhi_{s_n}^t(u;u) + \frac{1}{2}|u(t)-\eta(t)|_H^2 \\ &\le \int_{s_n}^t \langle f, u-\eta\rangle d\tau + \varPhi_{s_n}^t(u;\eta) + \frac{1}{2}|u(s_n)-\eta(s_n)|_H^2. \end{aligned} \tag{4.6}$$

for all $\eta \in \mathcal{K}_0(\theta)$. Now, passing to the limit in (4.6) as $n\to\infty$ and using (4.5), we see that the variational inequality (1.3) holds for any $s=s',\ t\in(0,T]$ with $s'\le t$; (1.3) for $s'=0$ and $t>0$ automatically holds, since $0\in E_0$. □

Corollary 4.1. *$u(t)$ is continuous in H from the left with respect to time t.*

Proof. In Lemma 4.1 we showed that the variational inequality (1.3) holds for every $s,\ t=t_0\in[0,T],\ s<t_0$. Hence, by repeating the same argument as above to get $\lim_{s\uparrow t_0}|u(t_0)-u(s)|_H=0$ without the restriction $s\in E_0$. Thus u is continuous in H from the the left in time. □

Lemma 4.2. *$u(t)$ is continuous in H from the right with respect to time t.*

Proof. Let t_0 be any time in $[0,T)$. We make use again the test functions $\eta_{t_0,\varepsilon,\varepsilon_1,\delta}(x,t) := (1-\varepsilon_1)\sigma_\varepsilon(t-t_0)\alpha_\delta(x,t_0)z_\varepsilon(x,t_0)$ for our proof. By Lemma 4.1, we have

$$\begin{aligned} &\int_{t_0}^t \langle \eta'_{t_0,\varepsilon,\varepsilon_1,\delta}, u-\eta_{t_0,\varepsilon,\varepsilon_1,\delta}\rangle d\tau + \varPhi_{t_0}^t(u;u) + \frac{1}{2}|u(t)-\eta_{t_0,\varepsilon,\varepsilon_1,\delta}(t)|_H^2 \\ &\le \int_{t_0}^t \langle f, u-\eta_{t_0,\varepsilon,\varepsilon_1,\delta}\rangle d\tau + \varPhi_{t_0}^t(u;\eta_{t_0,\varepsilon,\varepsilon_1,\delta}) + \frac{1}{2}|u(t_0)-\eta_{t_0,\varepsilon,\varepsilon_1,\delta}(t_0)|_H^2. \end{aligned} \tag{4.7}$$

By Lemma 3.4,

$$\begin{aligned} |u(t)-\eta_{t_0,\varepsilon,\varepsilon_1,\delta}(t)|_H \ge |u(t)-u(t_0)|_H - \varepsilon(1+|\Omega|^{\frac{1}{2}}) - \varepsilon_1\sigma_\varepsilon(t-t_0)\gamma^*|\Omega|^{\frac{1}{2}} \\ - \gamma^*|U_{2\delta}(t_0)-\Omega_*(t_0)|^{\frac{1}{2}} \end{aligned}$$

and

$$|u(t_0) - \eta_{t_0,\varepsilon,\varepsilon_1,\delta}(t_0)|_H \le \varepsilon(1 + |\Omega|^{\frac{1}{2}}) + \gamma^*|U_{2\delta}(t_0) - \Omega_*(t_0)|^{\frac{1}{2}} + \varepsilon_1\gamma^*|\Omega|^{\frac{1}{2}}.$$

Just as in the proof of (4.2) and (4.3), by using these inequalities we drive from (4.7) that

$$\limsup_{t\downarrow t_0} |u(t) - u(t_0)|_H \le 2\varepsilon(1 + |\Omega|^{\frac{1}{2}}) + 2\gamma^*|U_{2\delta}(t_0) - \Omega_*(t_0)|^{\frac{1}{2}} + 2\varepsilon_1\gamma^*|\Omega|^{\frac{1}{2}},$$

as long as $0 < \varepsilon < \frac{\varepsilon_1 C_\delta}{2}$ and $0 < \varepsilon_1 < 1$. By the arbitrariness of such ε, ε_1 and δ we conclude that $u(t) \to u(t_0)$ in H as $t \downarrow t_0$. Thus $u(t)$ is continuous in H from the right in time. $\square$

The proof of Theorem 1.1 is now complete. Indeed, by Corollary 4.1 and Lemma 4.2, u is continuous in H on $[0, T]$ and $\{u, \theta\}$ possesses the other regularity properties required in statement (a) by (1.15) and (1.16). The statement (b) was proved in Lemma 3.2, and (c) was proved in Lemma 4.1 together with (1.17).

Remark 4.1. In this paper we proved the continuity of $u(t)$ in $H := L^2(\Omega)$ for a concrete parabolic quasi-variational inequality under obstacle $|u| \le \gamma(\theta)$. It is expected to discuss a similar problem under gradient constraint, $|\nabla u| \le \gamma(\theta)$, in the degenerate case of γ. This is a future question. See [1,9,11,13,14,15] for some related works of obstacle or gradient constraint problems.

Remark 4.2. There has not been a general abstract set-up of parabolic quasi-variational inequalities, taking a class of semimonotone operators with time-derivative operators into account. Especially, it seems quite difficult how to describe "degenerate case of constraints" in the abstract set-up, which is one of challenging problems. See [6,7,8,10,12] for some abstract parabolic quasi-variational formulations.

Appendix

Let H be a general Hilbert space with inner product $(\cdot,\cdot)_H$, V and W be *separable* and reflexive Banach spaces such that V and W are dense subspaces of H with compact embeddings and W is a closed subspace of V with continuous embedding. We denote the dual spaces of V and W by V^* and W^*, respectively, and the duality between V and V^* or W and W^* by $\langle\cdot,\cdot\rangle$. In this case we have

$$V \subset H \subset W^* \ \text{ with dense and compact embeddings.}$$

Here we recall the concept of bounded variation of function $w : [0, T] \to W^*$. The total variation of w, denoted by $\mathrm{Var}_{W^*}(w)$, is defined by

$$\mathrm{Var}_{W^*}(w) := \sup_{\substack{\eta \in C_0^1(0,T;W),\\ |\eta|_{L^\infty(0,T;W)} \le 1}} \int_0^T \langle w, \eta'\rangle_{W^*,W} dt.$$

Our compactness theorem is based on the uniform total variation estimate for all functions in the class $Z(M_0)$ as stated in the follwoing theorem.

Theorem (cf. [9; Theorem 3.1, Lemma 3.2]) *Let M_0 be any positive number and let*

$$Z(M_0) := \{w \mid |w|_{L^p(0,T;V)} \leq M_0,\ |w|_{L^\infty(0,T;H)},\ \mathrm{Var}_{W^*}(w) \leq M_0\},$$

where $2 \leq p < \infty$. Then, given any sequence $\{u_n\}$ in $Z(M_0)$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with a function $u \in L^p(0,T;V) \cap L^\infty(0,T;H)$ such that

(i) u is of bounded variation from $[0,T]$ into W^* such that

$$u_{n_k}(t) \to u(t) \quad \textit{weakly in } W^*,\ \forall t \in [0,T]\ (\textit{as } k \to \infty);$$

hence $u_{n_k}(t) \to u(t)$ weakly in H for every $t \in [0,T]$.

(ii) $\{u_{n_k}\}$ converges (strongly) to u in $L^p(0,T;H)$.

On account of the above theorem, the set $Z(M_0)$ is compact in $L^p(0,T;H)$.

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