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NUMERICAL REGULARITY MAP FOR BLOW-UP SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In the paper, nonlinear ordinary differential equations with blow-up solutions are investigated numerically. The blow-up time is obtained by using our numerical method that consists of the Runge-Kutta method, the bounding transform and the numerical limit. The regularity of the solution in a certain interval is investigated numerically by using the spectral collocation method. From these numerical results we propose a map on the regularity of the solution.

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1 Introduction

Information on the regularity of solutions of differential equations is important in the proof of the existence of solutions. This is because it is related to the setting of functional spaces. From a viewpoint of this situation we have carried out numerical simulations on the regularity of solutions and investigated existence of solutions numerically [4, 9, 10]. In our numerical simulations, we have used the spectral collocation methods, which allow us to investigate the regularity of functions [1, 2]. We have also used the multiple-precision arithmetic to eliminate effects of rounding errors [7, 11].

On the other hand, blow-up problems have been investigated extensively so far. We recall the seminal work of Fujita [5] for the semilinear heat equation. The solution of the blow-up problem becomes unbounded in a finite time such as $\lim_{t \uparrow T} \|u(t)\| = \infty$ with $\|\cdot\|$ being a suitable norm. $T(< \infty)$ is called the blow-up time. From a viewpoint of numerical analysis, numerical estimation of the blow-up time is the most interesting. Explicit numerical methods with the uniform temporal mesh, e.g. the explict Euler scheme, usually fail to compute the blow-up time, if they do not involve special techniques. Therefore, Nakagawa [12] proposed a numerical method where an adaptive temporal mesh control is adopted. He showed that the numerical blow-up time converges to the exact one. Cho [3] proposed a nice scheme to estimate the blow-up time. He used the uniform temporal mesh size and the special quantity. He showed the convergence property as well. We proposed a simple numerical method with the uniform temporal mesh size [13]. Our method consists of the Runge-Kutta method, the bounding transform [8], the numerical limit and the multiple-precision arithmetic. Our method was applied to a nonlinear system of ordinary differential equations, which is related to default risk [13].

In the paper, we consider merging the above two types of research results. The following two sample problems with blow-up solutions [3] are solved numerically.

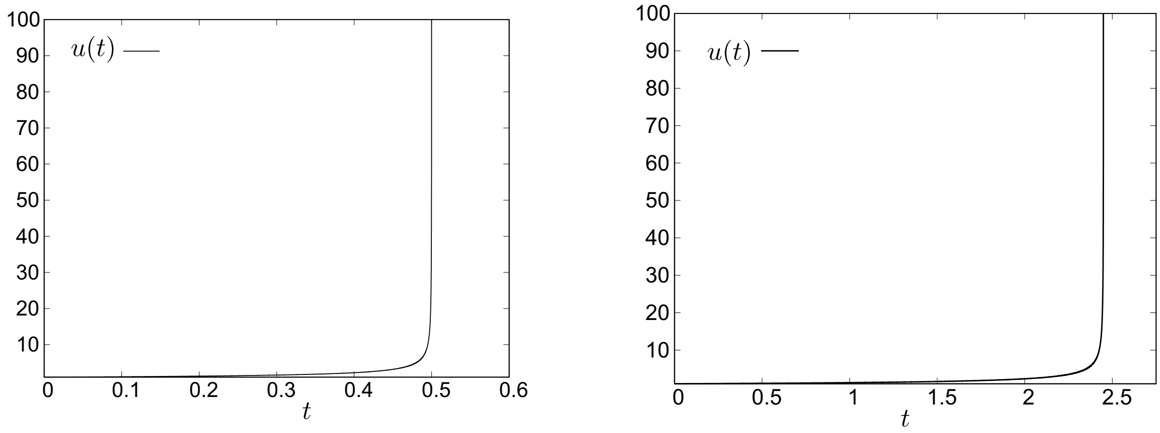
Problem 1.
$$u'(t) = u^3, \quad u(0) = 1. \tag{1}$$

The exact solution is $u(t) = \dfrac{1}{\sqrt{1-2t}}$. The blow-up time T_b is $\dfrac{1}{2}$.

Problem 2.
$$u''(t) = u^2, \quad u(0) = 1, \quad u'(0) = \sqrt{\frac{2}{3}}\,. \tag{2}$$

The exact solution is $u(t) = \left(1 - \dfrac{1}{\sqrt{6}}\,t\right)^{-2}$. The blow-up time T_b is $\sqrt{6} = 2.449489\cdots$.

To these Problems we apply our method [13] to estimate the blow-up time. By using information about the estimated blow-up time, we also investigate the regularity of the solution in a certain interval by using the spectral collocation method. Then, we make a numerical regularity map by combining these data.



(1) Problem 1. (2) Problem 2.

Fig.1. Profiles of exact solutions.

2 Numerical blow-up time

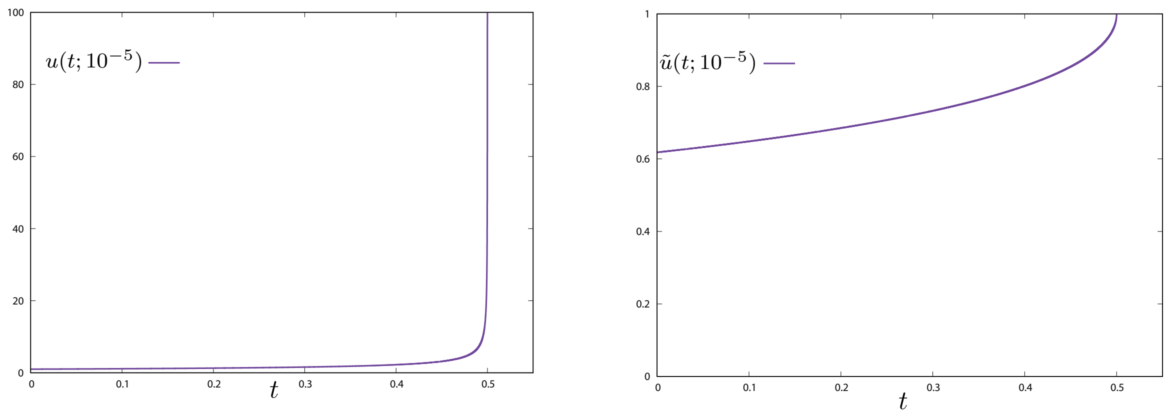
We apply our method [13] to estimate the blow-up time. To remove the difficulty of infinity, we consider the following bounding transform [8] of u into $\tilde{u}$.

$$\tilde{u} = \frac{-1 + \sqrt{1 + 4u^2}}{2u} \quad \left(u = \frac{\tilde{u}}{1 - \tilde{u}^2} \right). \tag{3}$$

The numerical blow-up time is the time when the value of the transformed solution $\tilde{u}$ becomes 1. It is estimated by using the numerical limit, i.e. the extrapolation. Numrical computation is performed in multiple precision (100 digits) by using exflib [6].

2.1 Numerical blow-up time for Problem 1

First, we apply the 4th order Runge-Kutta method. Then, we apply the bounding transform.



(1) Profile of $u(t; \Delta t = 10^{-5})$. (2) Profile of $\tilde{u}(t; \Delta t = 10^{-5})$.

Fig.2. Profiles of u and $\tilde{u}$ of Problem 1 for $\Delta t = 10^{-5}$.

Fig. 2 shows profiles of a numerical solution u of Problem 1 by the Runge-Kutta method with $\Delta t = 10^{-5}$ and its transformed solution $\tilde{u}$.

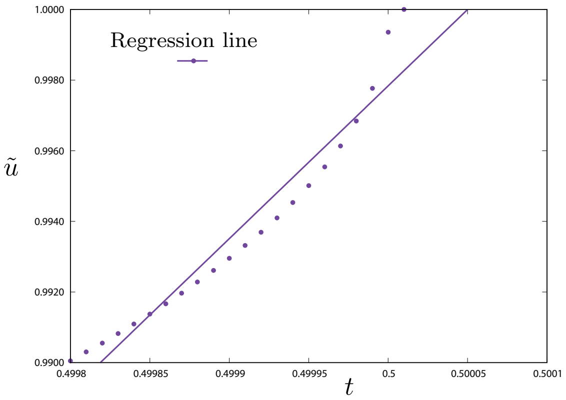
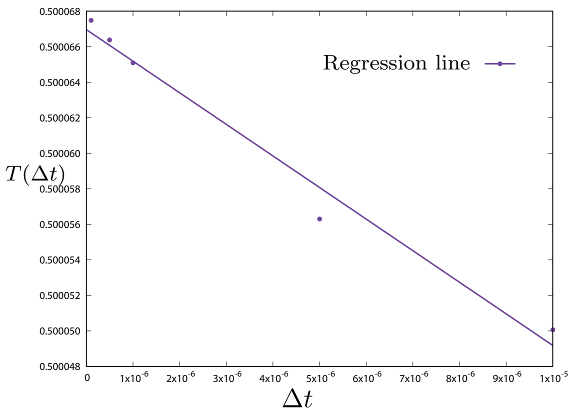
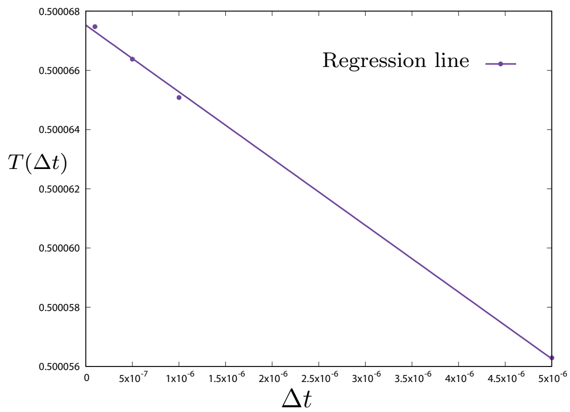


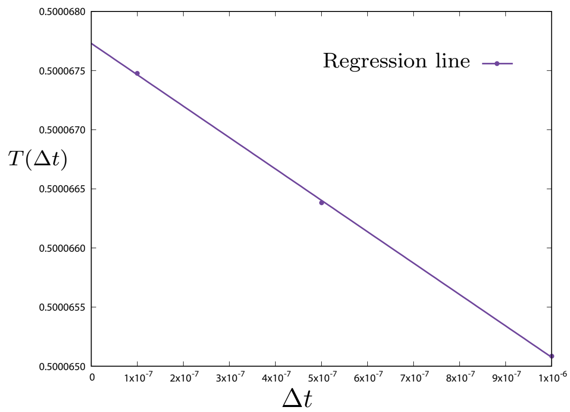
Fig.3. Numerical blow-up time $T_a(\Delta t)$ for $\Delta t = 10^{-5}$ by using data $\tilde{u} \in [0.99,\ 1]$.
($T_a(\Delta t = 10^{-5}) = 0.50005 \cdots$.)



(1) Regression line for 5 data.
($\Delta t = 10^{-5},\ 5 \times 10^{-6},\ 10^{-6},\ 5 \times 10^{-7},\ 10^{-7}$,
$T_a = 0.500066 \cdots$.)



(2) Regression line for 4 data.
($\Delta t = 5 \times 10^{-6},\ 10^{-6},\ 5 \times 10^{-7},\ 10^{-7}$,
$T_a = 0.500067 \cdots$.)

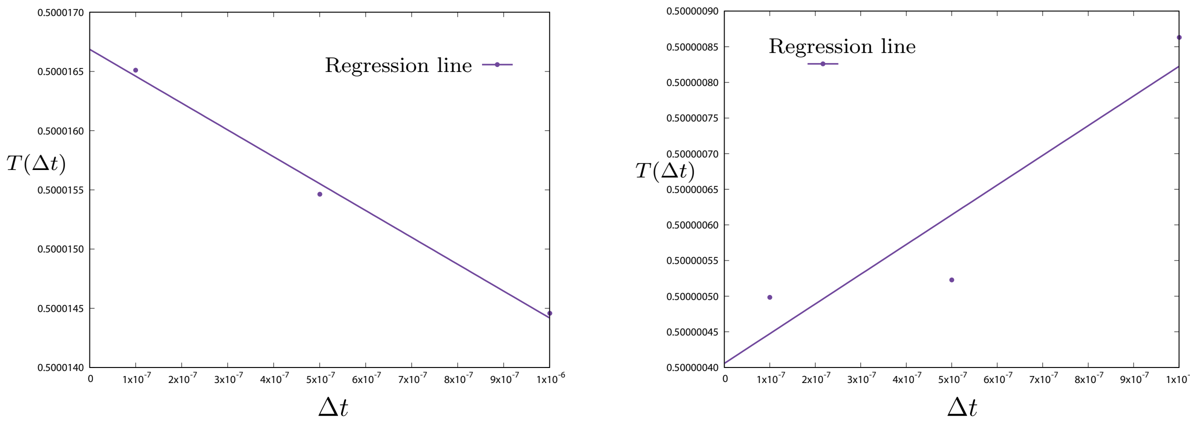


(3) Regression line for 3 data.
($\Delta t = 10^{-6},\ 5 \times 10^{-7},\ 10^{-7}$, $T_a = 0.500067 \cdots$.)

Fig.4. Numerical blow-up time T_a for Problem 1 by using data $\tilde{u} \in [0.99,\ 1]$.

To obtain the numerical blow-up time $T_a(\Delta t)$ for given Δt, data of $\tilde{u}$ near 1 are necessary. Fig. 3 shows how to obtain it. Data of $\tilde{u}(t; \Delta t = 10^{-5}) \in [0.99, 1]$ and the regression line are shown. From Fig. 3, it is found that $T_a(\Delta t = 10^{-5}) = 0.500050 \cdots$. Fig. 4 shows how to obtain the numerical blow-up time T_a. We consider the regression line passing through points $(\Delta t,\ T_a(\Delta t))$. Intersection of the vertical axis and the regression line indicates the numerical blow-up time as $\lim_{\Delta t \to 0} T_a(\Delta t) (\equiv T_a)$.

Fig. 5 shows numerical blow-up times by using data $\tilde{u}$ closer to 1 than that in Figs. 3 and 4.

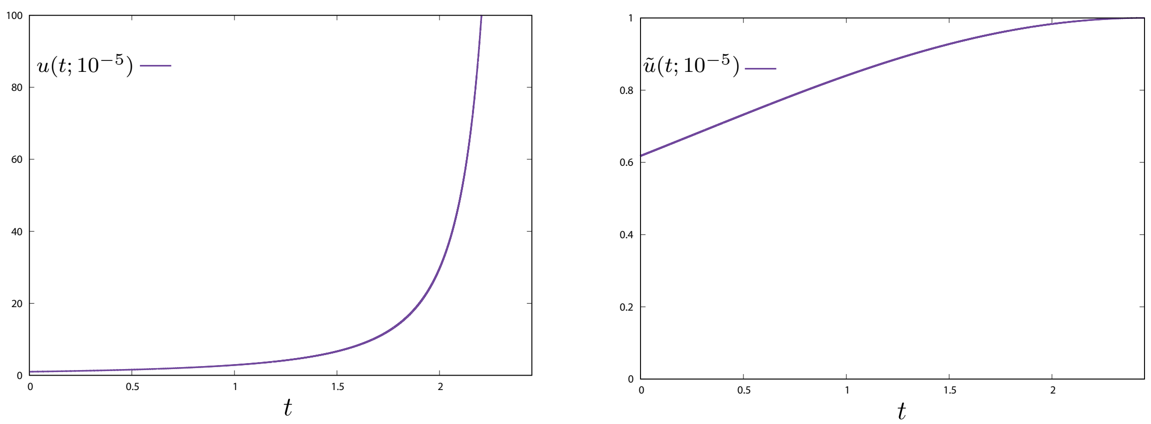


(1) $\tilde{u} \in [0.995,\ 1]$ ($T_a = 0.500016 \cdots$).　　(2) $\tilde{u} \in [0.999,\ 1]$ ($T_a = 0.500000 \cdots$).

Fig.5. Numerical blow-up time T_a for Problem 1 by using data $\tilde{u}$ closer to 1.

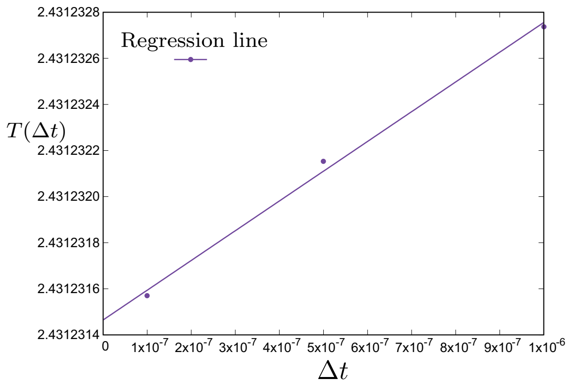
2.2 Numerical blow-up time for Problem 2

We apply the same procedure. Fig. 6 shows profiles of a numerical solution u of Problem 2 by the Runge-Kutta method with $\Delta t = 10^{-5}$ and its transformed function $\tilde{u}$. Fig. 7 shows the numerical blow-up time T_a.

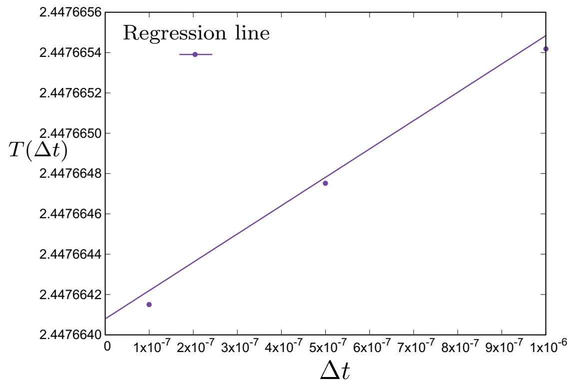


(1) Profile of $u(t; \Delta t = 10^{-5})$.　　(2) Profile of $\tilde{u}(t; \Delta t = 10^{-5})$.

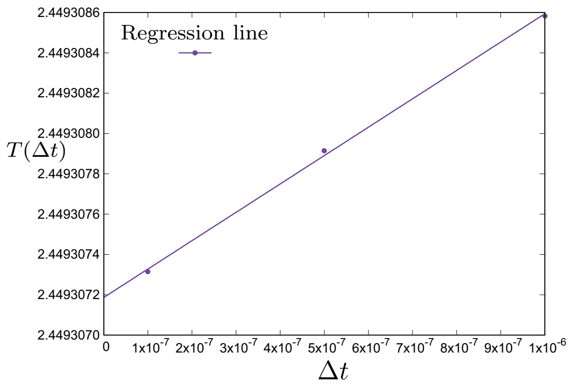
Fig.6. Profiles of u and $\tilde{u}$ of Problem 2 for $\Delta t = 10^{-5}$.



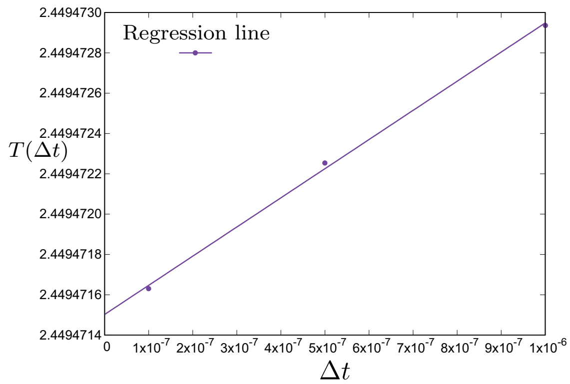
(1) $\tilde{u} \in [0.999,\ 1]$ ($T_a = 2.431231\cdots$).



(2) $\tilde{u} \in [0.99999,\ 1]$ ($T_a = 2.447664\cdots$).



(3) $\tilde{u} \in [0.9999999,\ 1]$ ($T_a = 2.449307\cdots$).



(4) $\tilde{u} \in [0.999999999,\ 1]$ ($T_a = 2.449471\cdots$).

Fig.7. Numerical blow-up time T_a for Problem 2.

3 Numerical regularity map

Numerical results in the previous section suggest the existence of blow-up solutions, and give approximate blow-up times. In this section we investigate the regularity of solutions numerically. To do so, it is convenient to use the spectral method [1, 2]. It is well known that the rate of error convergence is related to the regulairty of the solution. Here, the calculation of the error is performed by using numerical solutions, considering the general situation where the exact solution is unknown.

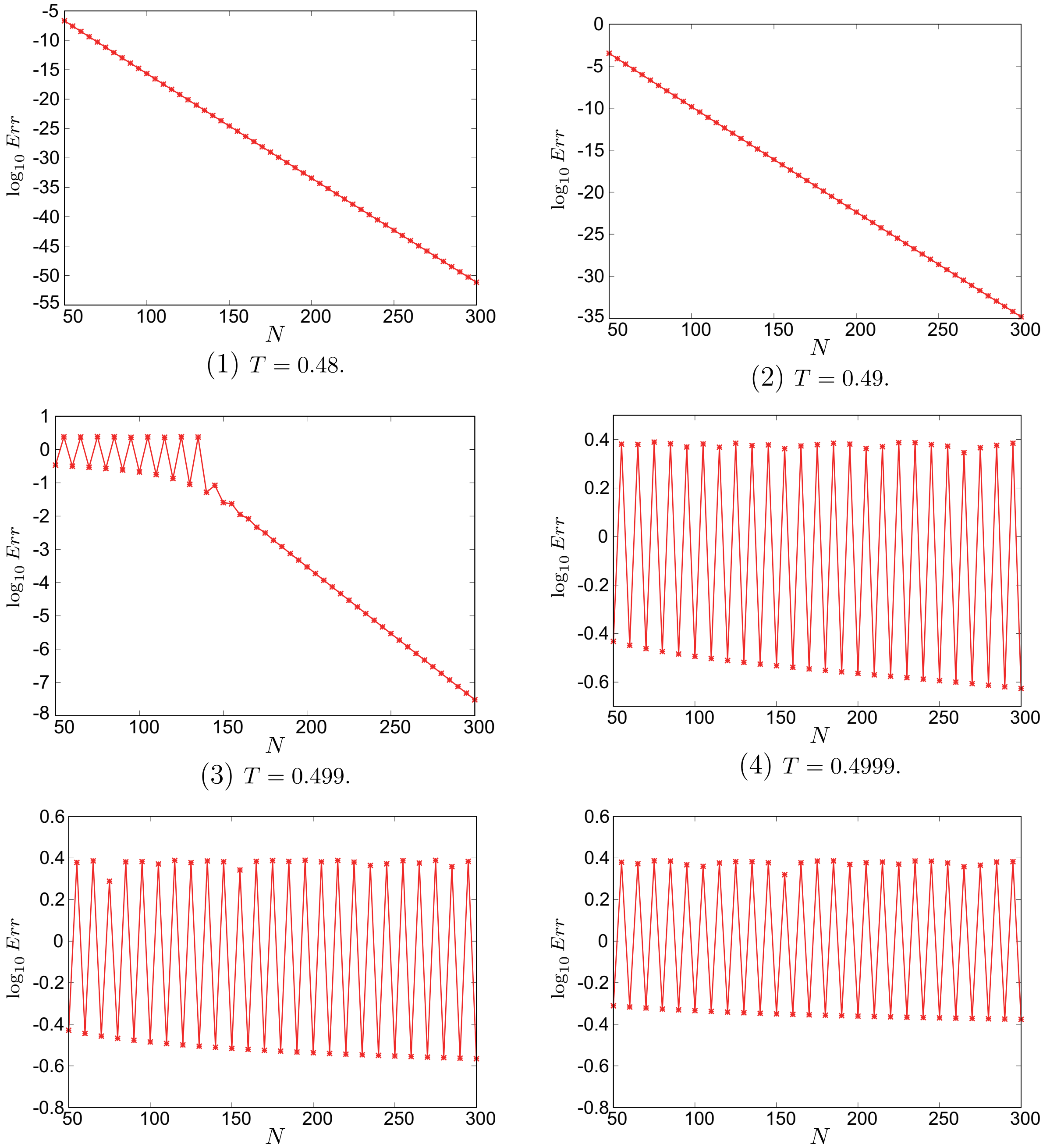
Among the spectral methods, the spectral collocation method is convenient and easily applicable to nonlinear problems. Here, we use the Chebyshev spectral collocation method (C-SCM). Discretized nonlinear equations are solved by using Newton's method. To apply C-SCM, we need to convert the interval $[0, T]$ to the interval $[-1, 1]$. The inequality $T < T_b$ must hold. However, the blow-up time T_b is generally unknown, so this inequality is ignored in our numerical computation. Thus, Problems are transformed according to the variable transformation: $x = 1 - 2\,\dfrac{T-t}{T}$. We also consider the following definition of the error:

$$Err = \max_{0 \le i \le K_{max}} |\bar{u}_N(x_i) - \bar{u}_{N_1}(x_i)|, \quad N_1 = N + 2$$

where $\bar{u}_N(x)$ is a numerical solution of the transformed problem with the approximation order N,

$$x_i = 1 - 2\,\frac{T - t_i}{T}, \quad t_i = i\,h, \quad 0 \le i \le K_{max}, \quad h = \frac{1}{100}, \quad K_{max} = \lfloor \frac{T}{h} \rfloor.$$

Numrical computation is performed in multiple precision (100 digits) by using exflib [6].



(1) $T = 0.48$.

(2) $T = 0.49$.

(3) $T = 0.499$.

(4) $T = 0.4999$.

(5) $T = 0.5$.

(6) $T = 0.51$.

Fig. 8. Error behaviors of Problem 1.

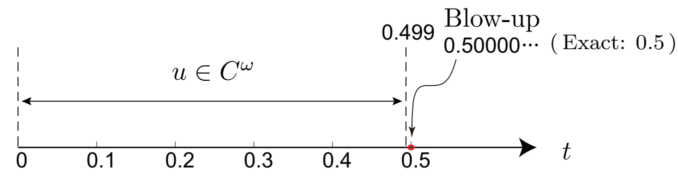
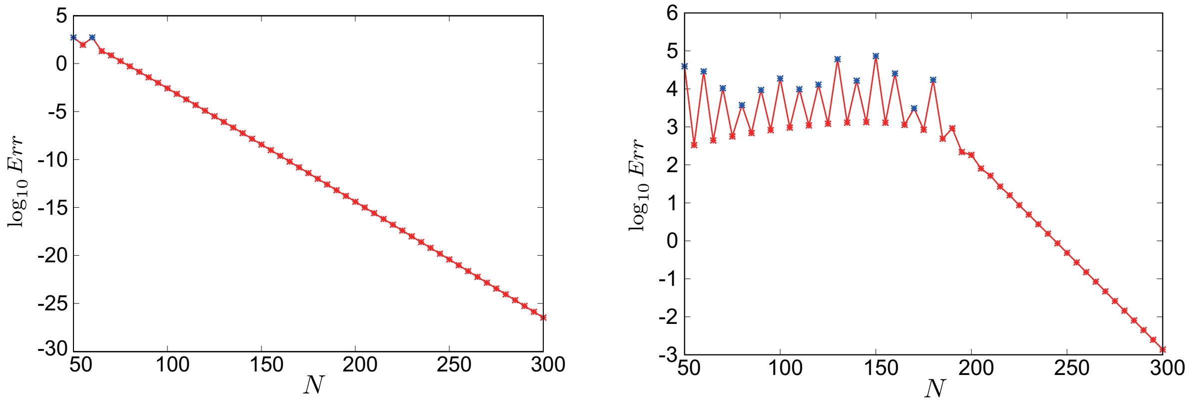
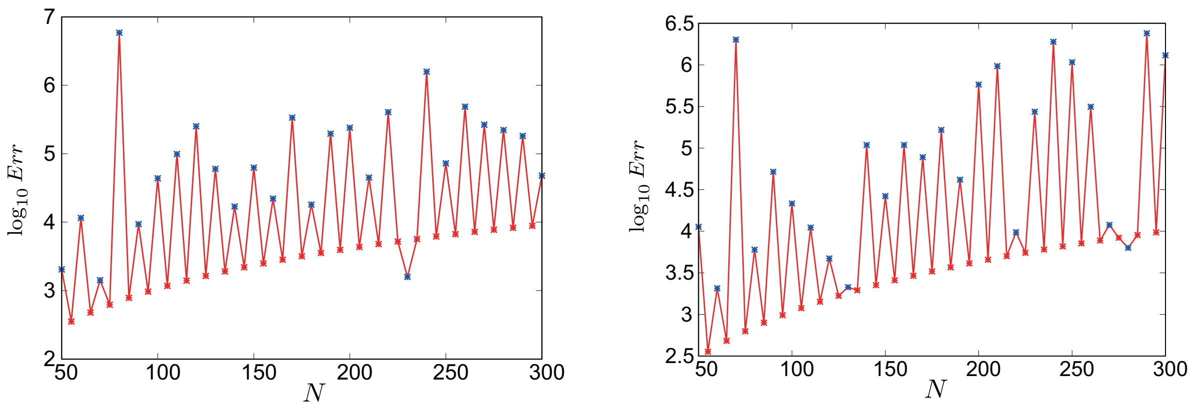


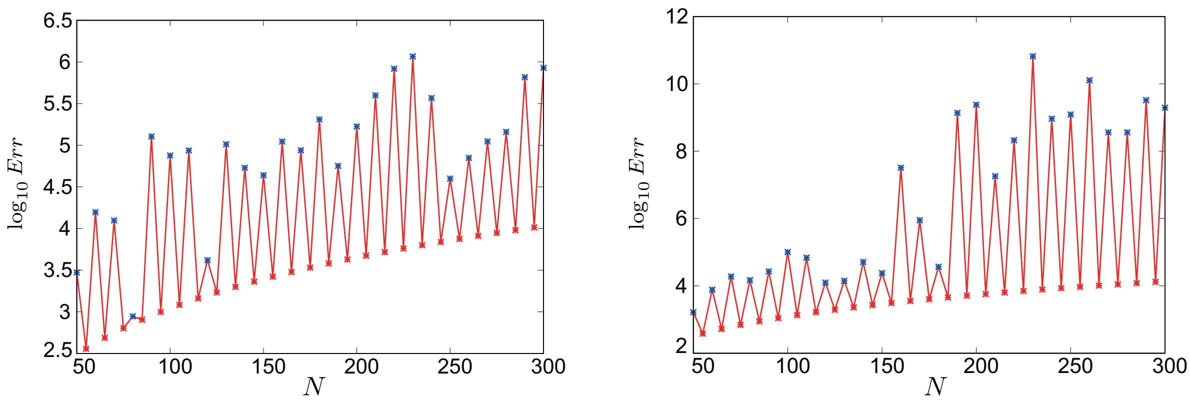
Fig. 9. Numerical regularity map for the solution of Problem 1.



(1) $T = 2.4$. (2) $T = 2.44$.



(3) $T = 2.448$. (4) $T = 2.449$.



(5) $T = 2.45$. (6) $T = 2.46$.

Fig. 10. Error behaviors of Problem 2.

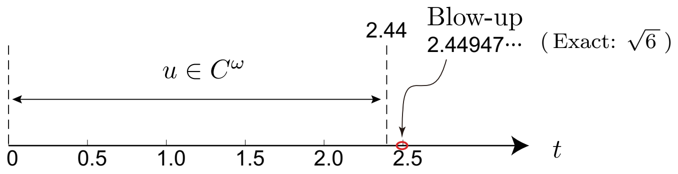


Fig. 11. Numerical regularity map for the solution of Problem 2.

In Newton's method, the maximum number of iterations is set to 10000. Newton's method is determined to have converged when the relative value of the update amount becomes 10^{-95} or less.

First, we investigate the regularity of the solution of Problem 1. The numerical results in the previous section show the existence of the blow-up solution and indicate that the blow-up time is about 0.5. By using this information, we set several T. Fig. 8 shows error behaviors of Problem 1 for these T. Figs. 8(1)∼8(3) show that the error decreases exponentially. This suggests the existence of the analytic solution in the interval [0, 0.499]. Since the spectral method is sensitive to the singularity, errors do not converge when the singularity exists in the interval or outside but near the interval. So, Fig. 8(3) shows that the singlarity exists outside the interval and near $t = 0.499$. Figs. 2∼5 suggest that this singularity comes from the blow-up phenomenon of the solution. On the other hand, Figs. 8(4)∼ 8(6) show that there may be no analytic solution in thsee interval. Thus, we can obtain a numerical regularity map of the solution of Problem 1 as Fig. 9.

We apply the same argument to Problem 2. Fig. 10 shows error behaviors of Problem 2 for several T. The points plotted in blue indicate that the numerical solution when the number of iterations of Newton's method reaches the upper limit is used to calculate the error. Figs. 10(1)∼10(2) show that the error decreases exponentially. This suggests the existence of the analytic solution in the interval [0, 2.44]. Fig. 10(2) shows that the singlarity exists outside the interval and near $t = 2.44$. Figs. 6∼7 suggest that this singularity comes from the blow-up phenomenon of the solution. On the other hand, Figs. 10(3)∼10(6) show that there may be no analytic solution in the interval. Thus, we can obtain a numerical regularity map of the solution of Problem 2 as Fig. 11.

Comparing Fig. 9 and Fig. 11, the gaps of time between the numerical blow-up time and the interval where the analytic solution exists are different. This seems to correspond to the different blow-up rates.

4 Conclusion

In the paper, nonlinear ordinary differential equations with blow-up solutions are investigated numerically. Numerical calculation of the blow-up time is carried out by using our method which is simple and versatile. Our numerical blow-up times are very accurate. Regularity of the solution is also investigated numerically by using the spectral collocation method. We propose a new concept called the regularity map by combining these results. This is one of the visualization of numerical results, and we think that it will bring useful information to theoretical analysis.

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