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ON A PERTURBED FAST DIFFUSION EQUATION WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. This paper discusses finite time extinction for a perturbed fast diffusion equation with dynamic boundary conditions. The fast diffusion equation has the characteristic property of decay, such as the solution decays to zero in a finite amount of time depending upon the initial data. In the target problem, some p-th or q-th order perturbation term may work to blow up within this period. The problem arises from the conflict between the diffusion and the blow up, in the bulk and on the boundary. Firstly, the local existence and uniqueness of the solution are obtained. Finally, a result of finite time extinction for some small initial data is presented.

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1 Introduction

In general, when discussing the well-posedness for some parabolic partial differential equations in a smooth bounded domain, the initial and boundary values are taken as auxiliary conditions. Settings with Dirichlet, Neumann, or Robin boundary conditions are common. Recently, dynamic boundary conditions have also been treated in several studies. Here, the boundary condition includes the time derivative. Moreover, the dynamic boundary condition with surface diffusion, which is the generalized Wentzell (Ventcel') boundary condition [52], is of great interest. The presence of a dynamic boundary condition in evolution problems creates a transmission problem between the dynamics in the bulk and on the boundary.

In this paper, we consider the fast diffusion equation with a dynamic boundary condition of the following form:

$$\partial_t u - \Delta u^m + a u^m = \lambda u^p \quad \text{in } \Omega, \quad t > 0, \tag{1.1}$$

$$\partial_t u + \partial_{\boldsymbol{\nu}} u^m - \Delta_\Gamma u^m + b u^m = \mu u^q \quad \text{on } \Gamma, \quad t > 0, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\Gamma := \partial\Omega$. We set up parameters $0 < m \le 1$, $p, q > 1$, and $(a, b), (\lambda, \mu) \in \{(1, 0), (0, 1)\}$. The symbols ∂_t, Δ, $\partial_{\boldsymbol{\nu}}$, and Δ_Γ denote the time derivative, the Laplacian, the normal derivative with respect to the outward unit normal vector $\boldsymbol{\nu}$ on Γ, and the Laplace–Beltrami operator (see, e.g., [33]), respectively. It is worth noting that Δ_Γ plays an important role in this paper. The second equation (1.2) is called the dynamic boundary condition. It describes the dynamics on the boundary through the appearance of the time derivative.

In general, if $a = \lambda = 0$, then we can categorize the nonlinear parabolic equation (1.1) as a *fast diffusion equation*, as compared with a *heat equation* ($m = 1$) and a *porous medium equation* ($m > 1$) (see, e.g., [51]). The fast diffusion equation has the characteristic property of decay [45]. More precisely, the solution decays to zero in a finite time that depends upon the initial data. This is called *finite time extinction*. In this paper, we consider the finite time extinction for the perturbed fast diffusion equation (1.1)–(1.2) with some initial condition. The p-th or q-th order perturbation term may work to the blow up with in this time. Several studies have been conducted on the perturbed fast diffusion equation with the homogeneous Dirichlet boundary condition,

$$\begin{aligned} \partial_t u - \Delta u^m &= u^p \quad \text{in } \Omega, \quad t > 0, \\ u &= 0 \quad \text{on } \Gamma, \quad t > 0, \\ u(0) &= u_0 \quad \text{in } \Omega, \end{aligned}$$

for example, [5, 27] for $m = 1$. For $0 < m < 1$, we refer to [2, 13, 14, 15, 22, 23, 39, 53] and references therein. The behaviour of the solution differs from the case of $\Omega = \mathbb{R}^3$ (see [37, 41], for example). On the other hand, some studies related to (1.1) considered the nonlinear boundary condition [10, 12, 20, 24, 38, 46, 55], and the dynamic boundary condition [16, 17, 18, 19, 21, 31, 32, 54]. The case of $m > 1$ is also interesting (see [25] for example, that considers equations similar to (1.1)–(1.2)), and has been the subject of several studies. Many of which are related to the pioneering blow up results of [26, 35].

To clarify the setting of our problem, we present the corresponding problems with parameter settings as follows:

$$(a,b,\lambda,\mu)=(1,0,1,0),\qquad \begin{cases}\partial_t u-\Delta u^m+u^m=u^p & \text{in } \Omega,\\ \partial_t u+\partial_{\boldsymbol{\nu}} u^m-\Delta_\Gamma u^m=0 & \text{on } \Gamma;\end{cases} \tag{1.3}$$

$$(a,b,\lambda,\mu)=(0,1,1,0),\qquad \begin{cases}\partial_t u-\Delta u^m=u^p & \text{in } \Omega,\\ \partial_t u+\partial_{\boldsymbol{\nu}} u^m-\Delta_\Gamma u^m+u^m=0 & \text{on } \Gamma;\end{cases} \tag{1.4}$$

$$(a,b,\lambda,\mu)=(1,0,0,1),\qquad \begin{cases}\partial_t u-\Delta u^m+u^m=0 & \text{in } \Omega,\\ \partial_t u+\partial_{\boldsymbol{\nu}} u^m-\Delta_\Gamma u^m=u^q & \text{on } \Gamma;\end{cases} \tag{1.5}$$

$$(a,b,\lambda,\mu)=(0,1,0,1),\qquad \begin{cases}\partial_t u-\Delta u^m=0 & \text{in } \Omega,\\ \partial_t u+\partial_{\boldsymbol{\nu}} u^m-\Delta_\Gamma u^m+u^m=u^q & \text{on } \Gamma,\end{cases} \tag{1.6}$$

where the initial condition is omitted for simplicity. As a remark, we could also mention the case $(a,b)=(1,1)$. However, we do not consider it in the present paper since it is trivial.

In this paper, applying the method of Filo [22], we discuss the local existence and uniqueness of the non-negative solution of (1.1)–(1.2) for some suitable non-negative initial data. Following results by Fila and Filo [14, 15], we obtain a result of finite time extinction for some small initial data.

We present a brief outline of the paper along with a short description of the various items.

In Section 2, we state the main theorems, which are related to the finite time extinction after establishing our notation. Let $(\lambda,\mu)=(0,1)$; for some small initial data, the unique solution of (1.5) or (1.6) decays to zero in a finite time when $1/5<m<1$. This means that we can take any $q>1$. On the other hand, if $(\lambda,\mu)=(1,0)$, we can obtain the same result to (1.3) or (1.4) under the additional assumption $1<p<5m$.

In Section 3, we consider an auxiliary problem. We discuss the well-posedness of some globally Lipschitz perturbation based on Filo [22]. Firstly, we set up a time discretization scheme. Thus, we obtain a solution for an elliptic problem applying the maximal monotone theory. Secondly, correcting the suitable uniform estimates, we prove that a pair of piecewise linear functions converges to a candidate solution to the auxiliary problem. Using fundamental inequalities, we also obtain estimates for time derivatives since the initial data belongs to $H^1\cap L^\infty$. This is a point of emphasis, because the suitable regularity of the time derivative is a special property for the fast diffusion equation (see [23, Theorem 2] and Remark 3.1). Moreover, to obtain a regular solution, we use the bootstrap argument for the dynamic boundary condition. Thanks to surface diffusion, this argument works well. This is another point of emphasis because the equation is treated as a weak or very weak formulation of the porous media equation in general. The solution satisfies the equation in almost everywhere sense. It is a benefit of surface diffusion.

In Section 4, we prove the main theorems step by step. Firstly, we obtain the local existence of the solution to the original problem under a general setting, that is, $0<m\le 1$, $p,q>1$. We use a standard method of the cut off function. The solution also satisfies an energy inequality and equality of conservation with respect to the $L^{(1+m)/m}$-norm. The proof of the main theorems is based on the effective use of this inequality and equality. Next, under the assumption $1/5 < m < 1$, we prove the property of the finite time extinction for $(\lambda, \mu) = (0, 1)$, namely (1.5) or (1.6). FIGURE 1 shows the strategy of the proof of the theorem. To complete the proof of finite time extinction, we need to discuss the invariance of some stable set. The essence of the proof is based upon Fila and Filo [14]. If $(\lambda, \mu) = (1, 0)$, we need additional assumption $1 < p < 5m$.

Here, let us present a detailed index of sections and subsections.

2 Main theorems

In this section, we present the main theorems. We first set up our problem in mathematical fundamental settings.

2.1 Notation

Let $T > 0$ be the finite time and $Q := (0, T) \times \Omega$, $\Sigma := (0, T) \times \Gamma$. We use the following notations: $H := L^2(\Omega)$, $V := H^1(\Omega)$, and $W := H^2(\Omega)$, which are Hilbert spaces with standard norms $|\cdot|_X$ and inner products $(\cdot, \cdot)_X$, where X is the corresponding space. Analogously, $H_\Gamma := L^2(\Gamma)$, $V_\Gamma := H^1(\Gamma)$, and $W_\Gamma := H^2(\Gamma)$. For the pair of functions z on Ω and z_Γ on Γ, we use the bold character $\boldsymbol{z} := (z, z_\Gamma)$. Also, we have the following definitions.

$$\begin{aligned}
&\boldsymbol{H} := H \times H_\Gamma, \\
&\boldsymbol{V} := \bigl\{ \boldsymbol{z} \in V \times V_\Gamma \ : \ z_\Gamma = z_{|_\Gamma} \quad \text{a.e. on } \Gamma \bigr\}, \\
&\boldsymbol{W} := (W \times W_\Gamma) \cap \boldsymbol{V}, \\
&\boldsymbol{L}^\infty := L^\infty(\Omega) \times L^\infty(\Gamma).
\end{aligned}$$

The symbol $z_{|\Gamma}$ denotes the trace of z to the boundary Γ. We remark that for the function $\boldsymbol{z} = (z, z_\Gamma) \in \boldsymbol{H}$, the first component z and the second component z_Γ are completely independent because of the lack of regularity.

Subsequently, we set $\alpha := 1/m$ for simplicity. We define functions $\mathrm{sgn}, \gamma, g, g_\Gamma : \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{sgn} r := \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -1 & \text{if } r < 0, \end{cases} \quad \gamma(r) := |r|^\alpha \mathrm{sgn} r = \begin{cases} r^\alpha & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -(-r)^\alpha & \text{if } r < 0, \end{cases}$$
$$g(r) := |r|^{p-1} r = |r|^p \mathrm{sgn} r, \quad g_\Gamma(r) := |r|^{q-1} r = |r|^q \mathrm{sgn} r.$$

Moreover, we put $\beta := \gamma^{-1}$. Then, γ, β, g, and g_Γ are monotone functions. Now, we can set up the problem of perturbed fast diffusion equation with a dynamic boundary condition as follows: Find $v : Q \to [0, \infty)$, $v_\Gamma : \Sigma \to [0, \infty)$ satisfying the following system

$$\partial_t \gamma(v) - \Delta v + av = \lambda g\bigl(\gamma(v)\bigr) \quad \text{a.e. in } Q, \tag{2.1}$$
$$v_{|\Gamma} = v_\Gamma \quad \text{a.e. on } \Sigma, \tag{2.2}$$
$$\partial_t \gamma(v_\Gamma) + \partial_{\boldsymbol{\nu}} v - \Delta_\Gamma v_\Gamma + b v_\Gamma = \mu g_\Gamma\bigl(\gamma(v_\Gamma)\bigr) \quad \text{a.e. on } \Sigma, \tag{2.3}$$
$$v(0) = v_0 \quad \text{a.e. in } \Omega, \tag{2.4}$$
$$v_\Gamma(0) = v_{\Gamma,0} \quad \text{a.e. on } \Gamma. \tag{2.5}$$

The third equation (2.3) is called the dynamic boundary condition since it includes the time derivative. Therefore, we need two initial data for v and v_Γ, or more specifically, conditions (2.4) and (2.5) with given data $v_0 : \Omega \to [0, \infty)$ and $v_{\Gamma,0} : \Gamma \to [0, \infty)$, respectively. Compared with the previous result of Filo [22], the function $\boldsymbol{v} := (v, v_\Gamma)$ will satisfy the equations in almost everywhere sense in (2.1) and (2.3), respectively, thanks to the presence of surface diffusion. In other words, we can obtain sufficient regularity.

To discuss finite time extinction based on the previous results (see, e.g., [14, 15, 34, 40, 42, 44, 47, 50]), we introduce the stable set $\mathcal{W}$ with corresponding energy J as follows: put

$$p_* := \lambda p + \mu q,$$

namely $p_* = p$ if $(\lambda, \mu) = (1, 0)$ and $p_* = q$ if $(\lambda, \mu) = (0, 1)$

$$\mathcal{W} := \bigl\{ \boldsymbol{z} \in \boldsymbol{V} \setminus \{\boldsymbol{0}\} \ : \ z \geq 0, z_\Gamma \geq 0, J(\boldsymbol{z}) < d, 2\varphi_1(\boldsymbol{z}) > (\alpha p_* + 1)\varphi_2(\boldsymbol{z}) \bigr\} \cup \{\boldsymbol{0}\},$$
$$J(\boldsymbol{z}) := \varphi_1(\boldsymbol{z}) - \varphi_2(\boldsymbol{z}),$$
$$\varphi_1(\boldsymbol{z}) := \frac{1}{2} \int_\Omega |\nabla z|^2 dx + \frac{a}{2} \int_\Omega |z|^2 dx + \frac{1}{2} \int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma + \frac{b}{2} \int_\Gamma |z_\Gamma|^2 d\Gamma,$$
$$\varphi_2(\boldsymbol{z}) := \frac{1}{\alpha p_* + 1} \left(\lambda \int_\Omega |z|^{\alpha p + 1} dx + \mu \int_\Gamma |z_\Gamma|^{\alpha q + 1} d\Gamma \right).$$

In the definition of $\mathcal{W}$, the constant d is called the *depth of the potential well*, and is now defined by

$$d := \inf \bigl\{ J(\boldsymbol{z}) \ : \ \boldsymbol{z} \in \boldsymbol{V} \setminus \{\boldsymbol{0}\}, 2\varphi_1(\boldsymbol{z}) = (\alpha p_* + 1)\varphi_2(\boldsymbol{z}) \bigr\}. \tag{2.6}$$

This constant is characterized by the optimal constant of some estimate between $\varphi_1(\boldsymbol{z})$ and $\varphi_2(\boldsymbol{z})$, which will be discussed in Remark 4.2.

2.2 Main theorems

The main theorems are related to the finite time extinction for the solution $\boldsymbol{v} = (v, v_\Gamma)$ of (2.1)–(2.5). Let $(a,b) \in \{(1,0),(0,1)\}$, $(\lambda,\mu) = (0,1)$. For the cases of (1.5) or (1.6), there are no restrictions for $q > 1$.

Theorem 2.1. *Assume that $1/5 < m < 1$, $q > 1$, and $\boldsymbol{v}_0 := (v_0, v_{\Gamma,0}) \in \mathcal{W} \cap \boldsymbol{L}^\infty$. Then, there exists $T_{\mathrm{ext}} \in (0,\infty)$ depending on $|v_0|_{L^{(1+m)/m}(\Omega)}$ and $|v_{\Gamma,0}|_{L^{(1+m)/m}(\Gamma)}$ such that $v(t) = 0$ a.e. in Ω, $v_\Gamma(t) = 0$ a.e. on Γ for all $t \ge T_{\mathrm{ext}}$. Moreover, there exists a positive constant $C(m) > 0$ depending on m such that*

$$\bigl|v(t)\bigr|_{L^{(1+m)/m}(\Omega)} + \bigl|v_\Gamma(t)\bigr|_{L^{(1+m)/m}(\Gamma)} \le C(m)(T_{\mathrm{ext}} - t)^{m/(1-m)} \tag{2.7}$$

for all $t \in [0, T_{\mathrm{ext}}]$.

Corollary 2.1. *Assume that $1/5 < m < 1$, $q > 1$, $v_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $v_0 \ge 0$, and $0 < |v_0|_V^2 < 2d$, namely $\boldsymbol{v}_0 := (v_0, 0)$. Then, there exists $T_{\mathrm{ext}} \in (0,\infty)$ depending on $|v_0|_{L^{(1+m)/m}(\Omega)}$ such that $v(t) = 0$ a.e. in Ω, $v_\Gamma(t) = 0$ a.e. on Γ for all $t \ge T_{\mathrm{ext}}$. Moreover, the same kind of estimate from above* (2.7) *holds.*

Let $(a,b) \in \{(1,0),(0,1)\}$, $(\lambda,\mu) = (1,0)$. For (1.3) and (1.4) under the restriction of $p > 1$, we can discuss the finite time extinction.

Theorem 2.2. *Assume that $1/5 < m < 1$, $1 < p < 5m$, and $\boldsymbol{v}_0 \in \mathcal{W} \cap \boldsymbol{L}^\infty$. Then, there exists $T_{\mathrm{ext}} \in (0,\infty)$ depending on $|v_0|_{L^{(1+m)/m}(\Omega)}$ and $|v_{\Gamma,0}|_{L^{(1+m)/m}(\Gamma)}$ such that $v(t) = 0$ a.e. in Ω, $v_\Gamma(t) = 0$ a.e. on Γ for all $t \ge T_{\mathrm{ext}}$. Moreover, the same kind of estimate from above* (2.7) *holds.*

As a remark, the well-posedness of the problem is discussed in Proposition 4.1.

3 Global existence for globally Lipschitz perturbations

In this section, we discuss the auxiliary problem for (2.1)–(2.5). Let us replace the perturbations g, g_Γ by globally Lipschitz continuous monotone functions f, f_Γ with Lipschitz constants $L_f, L_{f_\Gamma} > 0$. Furthermore, f and f_Γ satisfy $f(0) = f_\Gamma(0) = 0$: Find $v : Q \to [0,\infty)$, $v_\Gamma : \Sigma \to [0,\infty)$ satisfying the system

$$\partial_t \gamma(v) - \Delta v + av = f\bigl(\gamma(v)\bigr) \quad \text{a.e. in } Q, \tag{3.1}$$

$$v_{|_\Gamma} = v_\Gamma \quad \text{a.e. on } \Sigma, \tag{3.2}$$

$$\partial_t \gamma(v_\Gamma) + \partial_{\boldsymbol{\nu}} v - \Delta_\Gamma v_\Gamma + b v_\Gamma = f_\Gamma\bigl(\gamma(v_\Gamma)\bigr) \quad \text{a.e. on } \Sigma, \tag{3.3}$$

$$v(0) = v_0 \quad \text{a.e. in } \Omega, \tag{3.4}$$

$$v_\Gamma(0) = v_{\Gamma,0} \quad \text{a.e. on } \Gamma. \tag{3.5}$$

In this section, we set $(a,b) \in \{(1,0),(0,1)\}$, and the function γ is the same as it was in the previous section.

We obtain the global existence and uniqueness result for globally Lipschitz perturbations as follows.

Proposition 3.1. *Let $0 < m \le 1$, $0 < T < \infty$. Let us assume that $\boldsymbol{v}_0 := (v_0, v_{\Gamma,0}) \in \boldsymbol{V} \cap \boldsymbol{L}^\infty$ with $v_0 \ge 0$ and $v_{\Gamma,0} \ge 0$. Then, there exists a unique pair of non-negative functions $\boldsymbol{v} := (v, v_\Gamma)$ such that*

$$
\begin{aligned}
& v \in C\big([0,T];H\big) \cap L^\infty\big(0,T;V \cap L^\infty(\Omega)\big) \cap L^2(0,T;W), \\
& v^{(\alpha+1)/2} = v^{(1+m)/2m} \in H^1(0,T;H), \\
& \gamma(v) \in H^1(0,T;H) \cap L^\infty(0,T;V), \\
& v_\Gamma \in C\big([0,T];H_\Gamma\big) \cap L^\infty\big(0,T;V_\Gamma \cap L^\infty(\Gamma)\big) \cap L^2(0,T;W_\Gamma), \\
& v_\Gamma^{(\alpha+1)/2} = v_\Gamma^{(1+m)/2m} \in H^1(0,T;H_\Gamma), \\
& \gamma(v_\Gamma) \in H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma)
\end{aligned}
$$

and (3.1)–(3.5) *hold. Moreover, they satisfy the energy inequality*

$$
\begin{aligned}
& \frac{4\alpha}{(\alpha+1)^2} \int_0^t \left(\int_\Omega \big|\partial_t\big(v^{(\alpha+1)/2}(s)\big)\big|^2 dx + \int_\Gamma \big|\partial_t\big(v_\Gamma^{(\alpha+1)/2}(s)\big)\big|^2 d\Gamma \right) ds \\
& \quad + \varphi_1\big(\boldsymbol{v}(t)\big) - \int_\Omega \widehat{f}_\gamma\big(v(t)\big) dx - \int_\Gamma \widehat{f}_{\Gamma,\gamma}\big(v_\Gamma(t)\big) d\Gamma \\
& \le \varphi_1(\boldsymbol{v}_0) - \int_\Omega \widehat{f}_\gamma(v_0) dx - \int_\Gamma \widehat{f}_{\Gamma,\gamma}(v_{\Gamma,0}) d\Gamma
\end{aligned}
\tag{3.6}
$$

and the L^∞-boundednesses

$$
\big|v(t)\big|_{L^\infty(\Omega)} \le e^{Lt/\alpha}\big(|v_0|_{L^\infty(\Omega)} + |v_{\Gamma,0}|_{L^\infty(\Gamma)}\big), \tag{3.7}
$$

$$
\big|v_\Gamma(t)\big|_{L^\infty(\Gamma)} \le e^{Lt/\alpha}\big(|v_0|_{L^\infty(\Omega)} + |v_{\Gamma,0}|_{L^\infty(\Gamma)}\big) \tag{3.8}
$$

for all $t \in [0,T]$, where $L := \max\{L_f, L_{f_\Gamma}\}$. Furthermore, there exists a positive constant M_0 such that

$$
\big|v(t) - v(s)\big|_H + \big|v_\Gamma(t) - v_\Gamma(s)\big|_H \le M_0 |t-s|^{1/(\alpha+1)} \tag{3.9}
$$

for all $s, t \in [0,T]$.

We present the proof of the proposition in Subsection 3.3. In estimate (3.6), the function $\widehat{f}_\gamma$ is defined as the primitive of $f \circ \gamma$, namely

$$
\widehat{f}_\gamma(r) := \int_0^r (f \circ \gamma)(s) ds = \int_0^r f\big(\gamma(s)\big) ds \quad \text{for all } r \in \mathbb{R}.
$$

The primitive $\widehat{f}_{\Gamma,\gamma}$ of $f_\Gamma \circ \gamma$ is also defined analogously.

To discuss the existence of solutions to the above problem, we employ the argument of Filo [22]. Indeed, the well-posedness for the nonlinear diffusion equation with the dynamic boundary condition without the perturbation, can be solved using idea from [28, 29, 30, 48]. To this problem, see also the abstract approach of evolution equations governed by the difference between two subdifferentials [1, 3, 34, 36, 43]. Also we consider [31] and that author's series of papers on various problems with dynamic boundary conditions.

3.1 Time discretization

The essential idea of Filo [22] was to apply time discretization and suitable fundamental inequalities. Let $n \in \mathbb{N}$, and set $h := T/n$: for each $i = 1, 2, \dots, n$, find v_i and $v_{\Gamma,i}$ satisfying

$$\frac{\gamma(v_i) - \gamma(v_{i-1})}{h} - \Delta v_i + a v_i = f_{i-1} \quad \text{a.e. in } \Omega, \tag{3.10}$$

$$(v_i)_{|\Gamma} = v_{\Gamma,i} \quad \text{a.e. on } \Gamma, \tag{3.11}$$

$$\frac{\gamma(v_{\Gamma,i}) - \gamma(v_{\Gamma,i-1})}{h} + \partial_{\boldsymbol{\nu}} v_i - \Delta_\Gamma v_{\Gamma,i} + b v_{\Gamma,i} = f_{\Gamma,i-1} \quad \text{a.e. on } \Gamma, \tag{3.12}$$

where $f_{i-1} := f(\gamma(v_{i-1}))$ and $f_{\Gamma,i-1} := f_\Gamma(\gamma(v_{\Gamma,i-1}))$. Then, we see that there exists a unique pair $(v_i, v_{\Gamma,i}) \in \boldsymbol{W}$ of non-negative functions such that (3.10)–(3.12) holds for all $i = 1, 2, \dots, n$. Indeed, we define an operator $\boldsymbol{A} : D(\boldsymbol{A}) \to \boldsymbol{H}$ by $\boldsymbol{Az} := (\gamma(z), \gamma(z_\Gamma))$ with $D(\boldsymbol{A}) = L^{2\alpha}(\Omega) \times L^{2\alpha}(\Gamma)$. Then, $\boldsymbol{A}^{-1}$ is monotone and hemi-continuous. Therefore, $\boldsymbol{A}$ is maximal monotone same as $\boldsymbol{A}^{-1}$ (see, e.g., [6, p.36, Theorem 2.4, p.29, Proposition 2.1]). Moreover, we define a proper, lower semi-continuous, and convex functional $\varphi_{\boldsymbol{H}} : \boldsymbol{H} \to [0, \infty]$ by

$$\varphi_{\boldsymbol{H}}(\boldsymbol{z}) := \begin{cases} \varphi_1(\boldsymbol{z}) & \text{if } \boldsymbol{z} \in \boldsymbol{V}, \\ \infty & \text{if } \boldsymbol{z} \in \boldsymbol{H} \setminus \boldsymbol{V}. \end{cases}$$

Then, we see that the subdifferential $\partial\varphi_{\boldsymbol{H}}$ is a maximal monotone operator, characterized by $\partial\varphi_{\boldsymbol{H}}(\boldsymbol{z}) = (-\Delta z + az, \partial_{\boldsymbol{\nu}} z - \Delta_\Gamma z_\Gamma + b z_\Gamma)$ with domain $D(\partial\varphi_{\boldsymbol{H}}) = \boldsymbol{W}$ (see, e.g., [6, 7, 11]). Thanks to the standard maximal monotone theory (see, e.g., [6, p.44, Theorem 2.7]), $A + \partial\varphi_{\boldsymbol{H}}$ is also maximal monotone. Moreover, there exists a positive constant $C_\mathrm{C} > 0$ such that

$$\begin{aligned} \big(\boldsymbol{Az} + \partial\varphi_{\boldsymbol{H}}(\boldsymbol{z}), \boldsymbol{z}\big)_{\boldsymbol{H}} &\geq \int_\Omega |\nabla z|^2 dx + a \int_\Omega |z|^2 dx + \int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma + b \int_\Gamma |z_\Gamma|^2 d\Gamma \\ &\quad \Big(= 2\varphi_1(\boldsymbol{z}) \Big) \\ &\geq C_\mathrm{C} |\boldsymbol{z}|_{\boldsymbol{V}}^2 \end{aligned} \tag{3.13}$$

for all $\boldsymbol{z} \in \boldsymbol{W}$. Indeed, if $(a, b) = (0, 1)$, the Poincaré inequality ensures that there exists a positive constant $C_\mathrm{P} > 0$ such that

$$|z|_V^2 \leq C_\mathrm{P} \left(\int_\Omega |\nabla z|^2 dx + b \int_\Gamma |z_{|\Gamma}|^2 d\Gamma \right) \quad \text{for all } z \in V.$$

If $(a, b) = (1, 0)$, the trace theory between V and H_Γ ensures that there exists a positive constant $C_\mathrm{T} > 0$ such that

$$|z_{|\Gamma}|_{H_\Gamma}^2 \leq C_\mathrm{T} \left(\int_\Omega |\nabla z|^2 dx + a \int_\Omega |z|^2 dx \right) \quad \text{for all } z \in V.$$

Therefore, $\boldsymbol{A} + \partial\varphi_{\boldsymbol{H}}$ is coercive. Thus, the range $R(\boldsymbol{A} + \partial\varphi_{\boldsymbol{H}})$ of $\boldsymbol{A} + \partial\varphi_{\boldsymbol{H}}$ is the whole space $\boldsymbol{H}$ (see, e.g., [6, p.36, Corollary 2.2]). Next, multiplying (3.10) by $\min\{0, v_i\} \in V$ and (3.12) by $\min\{0, v_{\Gamma,i}\} \in V_\Gamma$, respectively, we obtain the non-negativity of the functions v_i and $v_{\Gamma,i}$.

3.2 Uniform estimates

According to [22, Lemma 1.14], we obtain the L^∞-boundedness as follows:

Lemma 3.1. *The functions v_i and $v_{\Gamma,i}$ satisfy*

$$|v_i|_{L^\infty(\Omega)} \le (1+Lh)^{i/\alpha}\big(|v_0|_{L^\infty(\Omega)} + |v_{\Gamma,0}|_{L^\infty(\Gamma)}\big),$$
$$|v_{\Gamma,i}|_{L^\infty(\Gamma)} \le (1+Lh)^{i/\alpha}\big(|v_0|_{L^\infty(\Omega)} + |v_{\Gamma,0}|_{L^\infty(\Gamma)}\big).$$

for all $i = 1, 2, \ldots, n$.

Proof. Let $\kappa > 1$. Multiplying (3.10) by v_i^κ and (3.12) by $v_{\Gamma,i}^\kappa$, using (3.11), and summing the results, we deduce that

$$\begin{aligned}
&\int_\Omega v_i^{\alpha+\kappa} dx + \int_\Gamma v_{\Gamma,i}^{\alpha+\kappa} d\Gamma + \left(h\kappa \int_\Omega v_i^{\kappa-1} |\nabla v_i|^2 dx + h\kappa \int_\Gamma v_{\Gamma,i}^{\kappa-1} |\nabla_\Gamma v_{\Gamma,i}|^2 d\Gamma \right. \\
&\quad \left. + ha \int_\Omega v_i^{\kappa+1} dx + hb \int_\Gamma v_{\Gamma,i}^{\kappa+1} d\Gamma \right) \\
&\le \int_\Omega v_{i-1}^\alpha v_i^\kappa dx + \int_\Gamma v_{\Gamma,i-1}^\alpha v_{\Gamma,i}^\kappa d\Gamma + h \int_\Omega f_{i-1} v_i^\kappa dx + h \int_\Gamma f_{\Gamma,i-1} v_{\Gamma,i}^\kappa d\Gamma \\
&\le (1 + L_f h) \int_\Omega v_{i-1}^\alpha v_i^\kappa dx + (1 + L_{f_\Gamma} h) \int_\Gamma v_{\Gamma,i-1}^\alpha v_{\Gamma,i}^\kappa d\Gamma \\
&\le \frac{\kappa}{\alpha+\kappa} \int_\Omega v_i^{\kappa \cdot \frac{\alpha+\kappa}{\kappa}} dx + \frac{\alpha}{\alpha+\kappa} (1+Lh)^{(\alpha+\kappa)/\alpha} \int_\Omega v_{i-1}^{\alpha \cdot \frac{\alpha+\kappa}{\alpha}} dx \\
&\quad + \frac{\kappa}{\alpha+\kappa} \int_\Gamma v_{\Gamma,i}^{\kappa \cdot \frac{\alpha+\kappa}{\kappa}} d\Gamma + \frac{\alpha}{\alpha+\kappa} (1+Lh)^{(\alpha+\kappa)/\alpha} \int_\Gamma v_{\Gamma,i-1}^{\alpha \cdot \frac{\alpha+\kappa}{\alpha}} d\Gamma
\end{aligned}$$

for all $i = 1, 2, \ldots, n$, where we used the Young inequality. Now, the second terms of the left hand side are non-negative. Therefore, we use the above estimate recurrently:

$$\int_\Omega v_i^{\alpha+\kappa} dx + \int_\Gamma v_{\Gamma,i}^{\alpha+\kappa} d\Gamma \le (1+Lh)^{i(\alpha+\kappa)/\alpha} \left(\int_\Omega v_0^{\alpha+\kappa} dx + \int_\Gamma v_{\Gamma,0}^{\alpha+\kappa} d\Gamma \right).$$

This implies that

$$|v_i|_{L^{\alpha+\kappa}(\Omega)} \le (1+Lh)^{i/\alpha} \left(|v_0|_{L^{\alpha+\kappa}(\Omega)} + |v_{\Gamma,0}|_{L^{\alpha+\kappa}(\Gamma)} \right),$$
$$|v_{\Gamma,i}|_{L^{\alpha+\kappa}(\Gamma)} \le (1+Lh)^{i/\alpha} \left(|v_0|_{L^{\alpha+\kappa}(\Omega)} + |v_{\Gamma,0}|_{L^{\alpha+\kappa}(\Gamma)} \right)$$

for all $i = 1, 2, \ldots, n$. Thus, letting $\kappa \to \infty$ we find the conclusion. □

Using Lemma 3.1, we obtain the following estimates:

Lemma 3.2. *There exist positive constants M_1, M_2, and $M_3 > 0$ independent of $n \in \mathbb{N}$ such that*

$$\sum_{i=1}^n \left| \frac{v_i^{(\alpha+1)/2} - v_{i-1}^{(\alpha+1)/2}}{h} \right|_H^2 h + \sum_{i=1}^n \left| \frac{v_{\Gamma,i}^{(\alpha+1)/2} - v_{\Gamma,i-1}^{(\alpha+1)/2}}{h} \right|_{H_\Gamma}^2 h \le M_1, \tag{3.14}$$

$$|v_i|_V + |v_{\Gamma,i}|_{V_\Gamma} \le M_2 \quad \text{for all } i = 1, 2, \ldots, n, \tag{3.15}$$

$$\sum_{i=1}^n \left| \frac{\gamma(v_i) - \gamma(v_{i-1})}{h} \right|_H^2 h + \sum_{i=}^n \left| \frac{\gamma(v_{\Gamma,i}) - \gamma(v_{\Gamma,i-1})}{h} \right|_{H_\Gamma}^2 h \le M_3 \tag{3.16}$$

for all $n \in \mathbb{N}$.

Proof. Multiplying (3.10) by $v_i - v_{i-1}$ and (3.12) by $v_{\Gamma,i} - v_{\Gamma,i-1}$, using (3.11), and summing these results, we deduce that

$$\begin{aligned}&\int_\Omega \frac{\gamma(v_i) - \gamma(v_{i-1})}{h}(v_i - v_{i-1})dx + \int_\Gamma \frac{\gamma(v_{\Gamma,i}) - \gamma(v_{\Gamma,i-1})}{h}(v_{\Gamma,i} - v_{\Gamma,i-1})d\Gamma \\ &\quad + \varphi_1(\boldsymbol{v}_i) - \varphi_1(\boldsymbol{v}_{i-1}) \\ &\le \int_\Omega \widehat{f}_\gamma(v_i)dx - \int_\Omega \widehat{f}_\gamma(v_{i-1})dx + \int_\Gamma \widehat{f}_{\Gamma,\gamma}(v_{\Gamma,i})d\Gamma - \int_\Gamma \widehat{f}_{\Gamma,\gamma}(v_{\Gamma,i-1})d\Gamma\end{aligned}$$

for all $i = 1, 2, \dots, n$. Now, recall the fundamental inequality

$$\frac{4\alpha}{(\alpha+1)^2}\bigl(r^{(\alpha+1)/2} - s^{(\alpha+1)/2}\bigr)^2 \le (r^\alpha - s^\alpha)(r - s)$$

for all $r, s \ge 0$ (see e.g., [23, Proposition 2] and Appendix). We obtain

$$\begin{aligned}&\frac{4\alpha}{(\alpha+1)^2}\left(\left|\frac{v_i^{(\alpha+1)/2} - v_{i-1}^{(\alpha+1)/2}}{h}\right|_H^2 h + \left|\frac{v_{\Gamma,i}^{(\alpha+1)/2} - v_{\Gamma,i-1}^{(\alpha+1)/2}}{h}\right|_{H_\Gamma}^2 h\right) \\ &\quad + \varphi_1(\boldsymbol{v}_i) - \int_\Omega \widehat{f}_\gamma(v_i)dx - \int_\Gamma \widehat{f}_{\Gamma,\gamma}(v_{\Gamma,i})d\Gamma \\ &\le \varphi_1(\boldsymbol{v}_{i-1}) - \int_\Omega \widehat{f}_\gamma(v_{i-1})dx - \int_\Gamma \widehat{f}_{\Gamma,\gamma}(v_{\Gamma,i-1})d\Gamma \end{aligned} \tag{3.17}$$

for all $i = 1, 2, \dots, n$. Summing (3.17) from $i = 1$ to $i = j \le n$, we obtain

$$\begin{aligned}&\frac{4\alpha}{(\alpha+1)^2}\left(\sum_{i=1}^j\left|\frac{v_i^{(\alpha+1)/2} - v_{i-1}^{(\alpha+1)/2}}{h}\right|_H^2 h + \sum_{i=1}^j\left|\frac{v_{\Gamma,i}^{(\alpha+1)/2} - v_{\Gamma,i-1}^{(\alpha+1)/2}}{h}\right|_{H_\Gamma}^2 h\right) \\ &\quad + \varphi_1(\boldsymbol{v}_j) - \int_\Omega \widehat{f}_\gamma(v_j)dx - \int_\Gamma \widehat{f}_{\Gamma,\gamma}(v_{\Gamma,j})d\Gamma \\ &\le \varphi_1(\boldsymbol{v}_0) - \int_\Omega \widehat{f}_\gamma(v_0)dx - \int_\Gamma \widehat{f}_{\Gamma,\gamma}(v_{\Gamma,0})d\Gamma. \end{aligned} \tag{3.18}$$

Here, using the Lipschitz continuities of f and f_Γ, we have

$$\bigl|\widehat{f}_\gamma(r)\bigr| \le \int_0^r \bigl|f\bigl(\gamma(s)\bigr)\bigr|ds \le L_f \int_0^r s^\alpha\, ds = \frac{L}{\alpha+1}r^{\alpha+1}, \quad \bigl|\widehat{f}_\Gamma(r)\bigr| \le \frac{L}{\alpha+1}r^{\alpha+1}$$

for all $r \ge 0$. Therefore, applying Lemma 3.1, we obtain that there exists a positive constant $\tilde{M}_1 > 0$ depending on $\alpha, T, |v_0|_{L^\infty(\Omega)}, |v_{\Gamma,0}|_{L^\infty(\Gamma)}$, and L, independent of n such that

$$\begin{aligned}&\int_\Omega \widehat{f}_\gamma(v_j)dx + \int_\Gamma \widehat{f}_{\Gamma,\gamma}(v_{\Gamma,j})d\Gamma \\ &\le \frac{L}{\alpha+1}(1 + Lh)^{(\alpha+1)j/\alpha}\bigl(|v_0|_{L^\infty(\Omega)} + |v_{\Gamma,0}|_{L^\infty(\Gamma)}\bigr)^{\alpha+1}\bigl(|\Omega| + |\Gamma|\bigr) \\ &\le \frac{L}{\alpha+1}e^{LT(\alpha+1)/\alpha}\bigl(|v_0|_{L^\infty(\Omega)} + |v_{\Gamma,0}|_{L^\infty(\Gamma)}\bigr)^{\alpha+1}\bigl(|\Omega| + |\Gamma|\bigr) =: \tilde{M}_1. \end{aligned} \tag{3.19}$$

Thus, we deduce that

$$\begin{aligned}
&\sum_{i=1}^{n} \left| \frac{v_i^{(\alpha+1)/2} - v_{i-1}^{(\alpha+1)/2}}{h} \right|_H^2 h + \sum_{i=1}^{n} \left| \frac{v_{\Gamma,i}^{(\alpha+1)/2} - v_{\Gamma,i-1}^{(\alpha+1)/2}}{h} \right|_{H_\Gamma}^2 h \\
&\le \frac{(\alpha+1)^2}{4\alpha} \left(\varphi_1(\boldsymbol{v}_0) + \tilde{M}_1 \right) =: M_1
\end{aligned}$$

for all $n \in \mathbb{N}$; that is, we obtain equation (3.14). Next, using equations (3.13), (3.18), and (3.19), we obtain (3.15). To obtain (3.16), we apply the fundamental inequality

$$|r^\alpha - s^\alpha| \le \frac{2\alpha}{\alpha+1} \max\{r,s\}^{(\alpha-1)/2} \left| r^{(\alpha+1)/2} - s^{(\alpha+1)/2} \right| \tag{3.20}$$

for all $r, s \ge 0$ since $\alpha \ge 1$ (see, for example, [22, Lemma 1.20], a similar method in [40, p.477], and the Appendix). From (3.20) and Lemma 3.1 we have

$$\begin{aligned}
&\int_\Omega \left| \frac{\gamma(v_i) - \gamma(v_{i-1})}{h} \right|^2 dx \\
&\le \max\bigl\{ |v_i|_{L^\infty(\Omega)}, |v_{i-1}|_{L^\infty(\Omega)} \bigr\}^{\alpha-1} \left(\frac{2\alpha}{\alpha+1} \right)^2 \int_\Omega \left| \frac{v_i^{(\alpha+1)/2} - v_{i-1}^{(\alpha+1)/2}}{h} \right|^2 dx \\
&\le e^{(\alpha-1)LT/\alpha} \left(|v_0|_{L^\infty(\Omega)} + |v_{\Gamma,0}|_{L^\infty(\Gamma)} \right)^{\alpha-1} \left(\frac{2\alpha}{\alpha+1} \right)^2 \left| \frac{v_i^{(\alpha+1)/2} - v_{i-1}^{(\alpha+1)/2}}{h} \right|_H^2
\end{aligned}$$

for all $i = 1, 2, \ldots, n$. Thus, we obtain the same estimate for $\gamma(v_{\Gamma,i})$, multiplying h, and summing up them. Moreover, summing these results from $i = 1$ to n, then we apply (3.14) to deduce (3.16). □

Remark 3.1. *The last estimate* (3.16) *can be obtained only in the cases of the fast diffusion equation for* $0 < m < 1$ *or the heat equation for* $m = 1$*. Indeed, the inequality* (3.20) *is true for* $\alpha \ge 1$*. However, it is the advantage from the* L^∞*-bounded initial data. For this reason, the time derivative is treated in the dual space of* V *or* V_Γ *in the case of the porous media equation for* $m > 1$*. In the case of the fast diffusion equation with Dirichlet boundary condition and* $H_0^1 \cap L^{\alpha+1}$*-bounded initial data, Akagi–Kajikiya obtained crucial results in [2].*

3.3 Proof of Proposition 3.1

In the standard manner, we define the following piecewise linear functions and step functions:

$$\begin{aligned}
\hat{v}_h(t) &:= v_{i-1} + \frac{v_i - v_{i-1}}{h}(t - ih) && \text{for } t \in \bigl[(i-1)h, ih\bigr], \\
\bar{v}_h(t) &:= v_i && \text{for } t \in \bigl((i-1)h, ih\bigr], \\
\hat{v}_h^*(t) &:= \gamma(v_{i-1}) + \frac{\gamma(v_i) - \gamma(v_{i-1})}{h}(t - ih) && \text{for } t \in \bigl[(i-1)h, ih\bigr],
\end{aligned}$$

$$
\begin{aligned}
&\bar{v}_h^*(t) := \gamma(v_i) && \text{for } t \in \big((i-1)h, ih\big],\\
&\underline{v}_h^*(t) := \gamma(v_{i-1}) && \text{for } t \in \big[(i-1)h, ih\big),\\
&\hat{v}_h^{**}(t) := v_{i-1}^{(\alpha+1)/2} + \frac{v_i^{(\alpha+1)/2} - v_{i-1}^{(\alpha+1)/2}}{h}(t-ih) && \text{for } t \in \big[(i-1)h, ih\big],\\
&\bar{v}_h^{**}(t) := v_i^{(\alpha+1)/2} && \text{for } t \in \big((i-1)h, ih\big],
\end{aligned}
$$

for $i = 1, 2, \dots, n$, and analogously for $\hat{v}_{\Gamma,h}, \bar{v}_{\Gamma,h}, \hat{v}_{\Gamma,h}^*, \bar{v}_{\Gamma,h}^*, \underline{v}_{\Gamma,h}^*, \hat{v}_{\Gamma,h}^{**}, \bar{v}_{\Gamma,h}^{**}$. According (3.10)–(3.12), these functions satisfy the following equations:

$$
\begin{aligned}
\partial_t \hat{v}_h^* - \Delta \bar{v}_h + a\bar{v}_h &= f(\underline{v}_h^*) \quad \text{a.e. in } \Omega, && (3.21)\\
(v_h)|_\Gamma &= v_{\Gamma,h} \quad \text{a.e. on } \Gamma, && (3.22)\\
\partial_t \hat{v}_{\Gamma,h}^* + \partial_{\boldsymbol{\nu}} \bar{v}_h - \Delta_\Gamma \bar{v}_{\Gamma,h} + b\bar{v}_{\Gamma,h} &= f_\Gamma(\underline{v}_{\Gamma,h}^*) \quad \text{a.e. on } \Gamma, && (3.23)\\
\hat{v}_h(0) &= v_0 \quad \text{a.e. in } \Omega, && (3.24)\\
\hat{v}_{\Gamma,h}(0) &= v_{\Gamma,0} \quad \text{a.e. on } \Gamma. && (3.25)
\end{aligned}
$$

Here, we have the following useful properties:

$$
|\hat{v}_h^*|_{L^2(0,T;X)}^2 \le \frac{h}{2}\big|\gamma(v_0)\big|_X^2 + |\bar{v}_h^*|_{L^2(0,T;X)}^2, \tag{3.26}
$$

$$
|\hat{v}_h^*|_{L^\infty(0,T;X)} = \max\big\{\big|\gamma(v_0)\big|_X, |\bar{v}_h^*|_{L^\infty(0,T;X)}\big\}, \tag{3.27}
$$

$$
|\hat{v}_h^* - \bar{v}_h^*|_{L^2(0,T;X)}^2 = |\hat{v}_h^* - \underline{v}_h^*|_{L^2(0,T;X)}^2 = \frac{h^2}{3}|\partial_t \hat{v}_h^*|_{L^2(0,T;X)}^2, \tag{3.28}
$$

for some suitable function space X.

Now, we prove Proposition 3.1.

Proof. Thanks to (3.15), (3.16), (3.26), and (3.27), we obtain uniform estimates for $\bar{\boldsymbol{v}}_h := (\bar{v}_h, \bar{v}_{\Gamma,h})$, $\hat{\boldsymbol{v}}_h^* := (\hat{v}_h^*, \hat{v}_{\Gamma,h}^*)$, and $\bar{\boldsymbol{v}}_h^* := (\bar{v}_h^*, \bar{v}_{\Gamma,h}^*)$ for example,

$$
\begin{gathered}
|\bar{v}_h|_{L^\infty(0,T;V)} \le M_2,\\
|\partial_t \hat{v}_h^*|_{L^2(0,T;H)}^2 \le M_3,\\
\big|\hat{v}_h(t)\big|_{L^\infty(\Omega)} \le e^{LT/\alpha}\left(|v_0|_{L^\infty(\Omega)} + |v_{\Gamma,0}|_{L^\infty(\Gamma)}\right) \quad \text{for all } t \in [0,T],\\
\big|\hat{v}_h^*(t)\big|_{L^\infty(\Omega)} \le e^{LT}\left(|v_0|_{L^\infty(\Omega)} + |v_{\Gamma,0}|_{L^\infty(\Gamma)}\right)^\alpha \quad \text{for all } t \in [0,T],\\
|\bar{v}_h^*|_{L^\infty(0,T;V)} = \max_{i=1,2,\dots,n}\big|\gamma(v_i)\big|_V \le M_4,\\
|\hat{v}_h^*|_{L^\infty(0,T;V)} = \max_{i=0,1,\dots,n}\big|\gamma(v_i)\big|_V \le M_4,\\
|\hat{v}_h^*|_{L^2(0,T;H)}^2 \le \frac{h}{2}\big|\gamma(v_0)\big|_H^2 + |\bar{v}_h^*|_{L^2(0,T;H)}^2 \le M_5,
\end{gathered}
$$

where M_4 and M_5 are positive constants independent of $n \in \mathbb{N}$. Here, we used Lemmas 3.1 and 3.2, as well as $\nabla\gamma(v_i) = \alpha v_i^{\alpha-1}\nabla v_i$, Then, we see that there exist a subsequence $\{h_k\}_{k\in\mathbb{N}}$ and functions $\boldsymbol{v} = (v, v_\Gamma) \in L^\infty(0,T; \boldsymbol{V} \cap \boldsymbol{L}^\infty)$, $\boldsymbol{v}^* = (v^*, v_\Gamma^*) \in H^1(0,T; \boldsymbol{H}) \cap$

$L^\infty(0,T;\boldsymbol{V}\cap\boldsymbol{L}^\infty)$ such that

$$
\begin{aligned}
\bar{\boldsymbol{v}}_{h_k} &\to \boldsymbol{v} && \text{weakly star in } L^\infty(0,T;\boldsymbol{V}),\\
\hat{v}_{h_k} &\to v && \text{a.e. in } Q,\\
\hat{v}_{\Gamma,h_k} &\to v_\Gamma && \text{a.e. on } \Sigma,\\
\hat{\boldsymbol{v}}^*_{h_k} &\to \boldsymbol{v}^* && \text{weakly in } H^1(0,T;\boldsymbol{H}),\\
& && \text{weakly star in } L^\infty(0,T;\boldsymbol{V}),\\
\bar{\boldsymbol{v}}^*_{h_k} &\to \boldsymbol{v}^* && \text{weakly star in } L^\infty(0,T;\boldsymbol{V}),\\
\hat{v}^*_{h_k} &\to v^* && \text{a.e. in } Q,\\
\hat{v}^*_{\Gamma,h_k} &\to v^*_\Gamma && \text{a.e. on } \Sigma
\end{aligned}
$$

as $k\to\infty$. Moreover, we apply the Aubin–Lions compactness theorem [49, Section 8, Corollary 4] and (3.28) to obtain the strong convergences (not re-labelled):

$$
\begin{aligned}
\hat{v}^*_{h_k} &\to v^* && \text{strongly in } C\big([0,T];L^r(\Omega)\big),\\
\hat{v}^*_{\Gamma,h_k} &\to v^*_\Gamma && \text{strongly in } C\big([0,T];L^r(\Gamma)\big) \quad \text{for } r\in[2,\infty),\\
\bar{\boldsymbol{v}}^*_{h_k}, \underline{\boldsymbol{v}}^*_{h_k} &\to \boldsymbol{v}^* && \text{strongly in } L^2(0,T;\boldsymbol{H})
\end{aligned}
$$

as $k\to\infty$. According Lemma 3.1, we use $V\cap L^\infty(\Omega)\hookrightarrow\hookrightarrow L^r(\Omega)\subset H$ and $V_\Gamma\hookrightarrow\hookrightarrow L^r(\Gamma)\subset H_\Gamma$ for all $r\in[2,\infty)$, where "$\hookrightarrow\hookrightarrow$" stands for the compact imbedding. From the demi-closedness of the maximal monotone operator $\boldsymbol{A}$, we obtain

$$
v^*=\gamma(v) \quad \text{a.e. in } Q, \quad v^*_\Gamma=\gamma(v_\Gamma) \quad \text{a.e. on } \Sigma.
$$

Thus, from (3.21)–(3.25), letting $k\to\infty$ in their weak formulation, we see that $\boldsymbol{v}=(v,v_\Gamma)$ satisfy

$$
\begin{aligned}
&\int_\Omega \partial_t\gamma\big(v(t)\big)zdx+\int_\Gamma \partial_t\gamma\big(v_\Gamma(t)\big)z_\Gamma d\Gamma+\int_\Omega \nabla v(t)\cdot\nabla z dx+a\int_\Omega v(t)zdx\\
&+\int_\Gamma \nabla_\Gamma v_\Gamma(t)\cdot\nabla_\Gamma z_\Gamma d\Gamma+b\int_\Gamma v_\Gamma(t)z_\Gamma d\Gamma=\int_\Omega f\big(\gamma\big(v(t)\big)\big)zdx+\int_\Gamma f_\Gamma\big(\gamma\big(v_\Gamma(t)\big)\big)z_\Gamma d\Gamma
\end{aligned}
$$

for all $\boldsymbol{z}=(z,z_\Gamma)\in\boldsymbol{V}$, for a.a. $t\in(0,T)$, initial conditions $v(0)=v_0$ in H, and $v_\Gamma(0)=v_{\Gamma,0}$ in H_Γ. Let $z\in\mathcal{D}(\Omega)$, then $z_\Gamma=0$ and

$$
-\Delta v(t)=f\big(\gamma\big(v(t)\big)\big)-av(t)-\partial_t\gamma\big(v(t)\big) \quad \text{in } \mathcal{D}'(\Omega).
$$

On the other hand, $f(\gamma(v))-\partial_t\gamma(v)-av\in L^2(0,T;H)$; therefore, we obtain $-\Delta v\in L^2(0,T;H)$ and

$$
\partial_t\gamma\big(v(t)\big)-\Delta v(t)+av(t)=f\big(\gamma\big(v(t)\big)\big) \quad \text{in } H, \tag{3.29}
$$

for a.a. $t\in(0,T)$. Next, for any $\boldsymbol{z}\in\boldsymbol{V}$, we see from (3.29) that

$$
\begin{aligned}
\int_\Gamma \partial_t\gamma\big(v_\Gamma(t)\big)z_\Gamma d\Gamma+\big\langle \partial_{\boldsymbol{\nu}}v(t),z_\Gamma\big\rangle+\int_\Gamma \nabla_\Gamma v_\Gamma(t)\cdot\nabla_\Gamma z_\Gamma d\Gamma+b\int_\Gamma v_\Gamma(t)z_\Gamma d\Gamma&\\
=\int_\Gamma f_\Gamma\big(\gamma\big(v_\Gamma(t)\big)\big)z_\Gamma d\Gamma,&
\end{aligned}
\tag{3.30}
$$

for a.a. $t \in (0,T)$. Here, we apply the bootstrap argument for the dynamic boundary condition with surface diffusion to gain higher regularity (see, e.g., [9, 11, 28, 29]). We have already obtained $-\Delta v \in L^2(0,T;H)$ and $v_\Gamma \in L^2(0,T;V_\Gamma)$. Therefore, from the elliptic regularity theorem (see, e.g., [8, Theorem 3.2, p. 1.79]), we infer

$$v \in L^2\bigl(0,T;H^{3/2}(\Omega)\bigr)$$

and consequently, from the trace theory with elliptic operator type [8, Theorem 2.27, p. 1.64], we obtain $\partial_{\boldsymbol{\nu}} v \in L^2(0,T;H_\Gamma)$. Therefore, from (3.30), we also obtain $\Delta_\Gamma v_\Gamma \in L^2(0,T;H_\Gamma)$ such that

$$\partial_t \gamma\bigl(v_\Gamma(t)\bigr) + \partial_{\boldsymbol{\nu}} v(t) - \Delta_\Gamma v_\Gamma(t) + b v_\Gamma(t) = f_\Gamma\bigl(\gamma\bigl(v_\Gamma(t)\bigr)\bigr) \quad \text{in } H_\Gamma,$$

for a.a. $t \in (0,T)$. Moreover, the information $-\Delta_\Gamma v_\Gamma \in L^2(0,T;H_\Gamma)$ implies $v_\Gamma \in L^2(0,T;W_\Gamma)$ (see, e.g., [33, p. 104]). Finally, this yields $v_\Gamma \in L^2(0,T;H^{3/2}(\Gamma))$. Using the elliptic regularity theorem again, we see that $v \in L^2(0,T;W)$ and $\boldsymbol{v} = (v, v_\Gamma)$ satisfies (3.1)–(3.5).

Next, we obtain the estimates (3.6)–(3.9). Firstly, the estimates (3.7)–(3.8) is the direct consequence of Lemma 3.1. Secondly, we obtain the uniform estimates for $\hat{\boldsymbol{v}}_h^{**} := (\hat{v}_h^{**}, \hat{v}_{\Gamma,h}^{**})$ using(3.14) (3.26), for example

$$\begin{aligned} |\partial_t \hat{v}_h^{**}|_{L^2(0,T;H)}^2 &\le M_1, \\ \bigl|\hat{v}_h^{**}(t)\bigr|_{L^\infty(\Omega)} &\le e^{(\alpha+1)LT/(2\alpha)} \bigl(|v_0|_{L^\infty(\Omega)} + |v_{\Gamma,0}|_{L^\infty(\Gamma)}\bigr)^{(\alpha+1)/2} \quad \text{for all } t \in [0,T], \\ |\hat{v}_h^{**}|_{L^2(0,T;H)}^2 &\le \frac{h}{2}\bigl|v_0^{(\alpha+1)/2}\bigr|_H^2 + |\bar{v}_h^{**}|_{L^2(0,T;H)}^2 \le M_6, \end{aligned}$$

where M_6 is a positive constant independent of $n \in \mathbb{N}$. Then, there exists a subsequence (not re-labelled) and a function $\boldsymbol{v}^{**} = (v^{**}, v_\Gamma^{**}) \in H^1(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V} \cap \boldsymbol{L}^\infty)$ such that

$$\begin{aligned} \hat{\boldsymbol{v}}_{h_k}^{**} \to \boldsymbol{v}^{**} &\quad \text{weakly in } H^1(0,T;\boldsymbol{H}), \\ &\quad \text{weakly star in } L^\infty(0,T;\boldsymbol{V}), \\ \hat{\boldsymbol{v}}_{h_k}^{**} \to \boldsymbol{v}^{**} &\quad \text{strongly in } C\bigl([0,T];\boldsymbol{H}\bigr), \\ \hat{v}_{h_k}^{**} \to v^{**} &\quad \text{a.e. in } Q, \\ \hat{v}_{\Gamma,h_k}^{**} \to v_\Gamma^{**} &\quad \text{a.e. on } \Sigma \end{aligned}$$

as $k \to \infty$. Moreover,

$$\boldsymbol{v}^{**} = \bigl(v^{(\alpha+1)/2}, v_\Gamma^{(\alpha+1)/2}\bigr).$$

Now, for all $t \in [0,T]$ and all $k \in \mathbb{N}$ with $h_k = T/n_k$, there exists $i_k \in \{1,2,\dots,n_k\}$ such that $t \in [(i_k-1)h_k, i_k h_k)$, if $t = T$ then put $i_k = n_k$. Moreover, $(i_k-1)h_k \to t$,

$i_k h_k \to t$ as $k \to \infty$. Therefore, we rewrite (3.18) into the following form

$$\begin{aligned}
&\frac{4\alpha}{(\alpha+1)^2}\int_0^t \left(\int_\Omega \bigl|\partial_t \hat{v}_{h_k}^{**}(s)\bigr|^2 dx + \int_\Gamma \bigl|\partial_t \hat{v}_{\Gamma,h_k}^{**}(s)\bigr|^2 d\Gamma\right) ds \\
&\quad + \varphi_1\bigl(\bar{\boldsymbol{v}}_{h_k}(t)\bigr) - \int_\Omega \widehat{f}\bigl(\bar{v}_{h_k}^{*}(t)\bigr)dx - \int_\Gamma \widehat{f}_\Gamma\bigl(\bar{v}_{\Gamma,h_k}^{*}(t)\bigr)d\Gamma \\
&\le \frac{4\alpha}{(\alpha+1)^2}\int_{(i_k-1)h_k}^t \left(\int_\Omega \bigl|\partial_t \hat{v}_{h_k}^{**}(s)\bigr|^2 dx + \int_\Gamma \bigl|\partial_t \hat{v}_{\Gamma,h_k}^{**}(s)\bigr|^2 d\Gamma\right) ds \\
&\quad + \varphi_1(\boldsymbol{v}_0) - \int_\Omega \widehat{f}_\gamma(v_0)dx - \int_\Gamma \widehat{f}_{\Gamma,\gamma}(v_{\Gamma,0})d\Gamma,
\end{aligned}$$

and take the $\liminf_{k\to\infty}$ of both side, then, applying the lower semi-continuity and the Lebesgue dominated convergence theorem, we obtain the energy estimate (3.6). The L^∞-boundedness (3.7)–(3.8) is a direct consequence of Lemma 3.1. Finally, using the fundamental inequality

$$|r-s| \le \bigl|r^{(\alpha+1)/2} - s^{(\alpha+1)/2}\bigr|^{2/(\alpha+1)}$$

for all $r, s \ge 0$ (see Appendix), we obtain that

$$\begin{aligned}
&\bigl|v(t)-v(s)\bigr|_H^2 + \bigl|v_\Gamma(t)-v_\Gamma(s)\bigr|_H^2 \\
&\le \int_\Omega \bigl|v^{(\alpha+1)/2}(t) - v^{(\alpha+1)/2}(s)\bigr|^{4/(\alpha+1)}dx + \int_\Gamma \bigl|v_\Gamma^{(\alpha+1)/2}(t) - v_\Gamma^{(\alpha+1)/2}(s)\bigr|^{4/(\alpha+1)}d\Gamma \\
&\le \left(|\Omega|^{(\alpha-1)/(\alpha+1)}\bigl|\partial_t v^{(\alpha+1)/2}\bigr|_{L^2(0,T;H)}^{4/(\alpha+1)} + |\Gamma|^{(\alpha-1)/(\alpha+1)}\bigl|\partial_t v_\Gamma^{(\alpha+1)/2}\bigr|_{L^2(0,T;H_\Gamma)}^{4/(\alpha+1)}\right)|t-s|^{2/(\alpha+1)}
\end{aligned}$$

for all $s, t \in [0,T]$. Thus, we obtain the Hölder continuity (3.9).

The proof of uniqueness is quite standard. Let $\boldsymbol{w} := (w, w_\Gamma)$ be the solution starting from the initial data $\boldsymbol{w}_0 := (w_0, w_{\Gamma,0})$ and compare it with $\boldsymbol{v}$. Define some approximation $\sigma_k \in C^1(\mathbb{R})$ of the signum function sgn satisfying $\sigma_k(0) = 0$, $-1 \le \sigma_k(r) \le 1$, $\sigma_k'(r) \ge 0$, and $\sigma_k(r) \to \mathrm{sgn} r$ as $k \to \infty$, for all $r \in \mathbb{R}$. Taking the difference of the equations for $\boldsymbol{v} = (v, v_\Gamma)$ and $\boldsymbol{w} = (w, w_\Gamma)$ we obtain

$$\begin{aligned}
&\int_\Omega \bigl(\partial_t\gamma(v) - \partial_t\gamma(w)\bigr)z dx + \int_\Gamma \bigl(\partial_t\gamma(v_\Gamma) - \partial_t\gamma(w_\Gamma)\bigr)z_\Gamma d\Gamma + \int_\Omega \nabla(v-w)\cdot\nabla z dx \\
&\qquad + a\int_\Omega (v-w)z dx + \int_\Gamma \nabla_\Gamma(v_\Gamma - w_\Gamma)\cdot\nabla_\Gamma z_\Gamma d\Gamma + b\int_\Gamma (v_\Gamma - w_\Gamma)z_\Gamma d\Gamma \\
&\qquad = \int_\Omega \Bigl(f\bigl(\gamma(v)\bigr) - f\bigl(\gamma(w)\bigr)\Bigr)z dx + \int_\Gamma \Bigl(f_\Gamma\bigl(\gamma(v_\Gamma)\bigr) - f_\Gamma\bigl(\gamma(w_\Gamma)\bigr)\Bigr)z_\Gamma d\Gamma
\end{aligned}$$

for all $\boldsymbol{z} = (z, z_\Gamma) \in \boldsymbol{V}$, for a.a. $t \in (0,T)$. Here, we omit the time variables $v = v(t)$ and $w = w(t)$. We take $z := \sigma_k([v-w]^+)$ and $z_\Gamma := \sigma_k([v_\Gamma - w_\Gamma]^+)$, where $[r]^+ := \max\{0, r\}$ for all $r \in \mathbb{R}$. Then, we have

$$\int_\Omega \nabla(v-w)\cdot\nabla\sigma_k\bigl([v-w]^+\bigr)dx = \int_\Omega \sigma_k'\bigl([v-w]^+\bigr)\bigl|\nabla[v-w]^+\bigr|^2 dx \ge 0$$

and the same kind of positivity for the term of $a(v-w)$, $\nabla_\Gamma(v_\Gamma - w_\Gamma)$, and $b(v_\Gamma - w_\Gamma)$, respectively. On the other hand, considering $\mathrm{sgn}([v-w]^+) = \mathrm{sgn}([\gamma(v)-\gamma(w)]^+)$ and letting $k \to \infty$, we obtain

$$
\begin{aligned}
&\int_\Omega \bigl(\partial_t\gamma(v) - \partial_t\gamma(w)\bigr)\sigma_k\bigl([v-w]^+\bigr)dx \\
&\to \int_\Omega \partial_t\bigl(\gamma(v)-\gamma(w)\bigr)\mathrm{sgn}\bigl([v-w]^+\bigr)dx \\
&= \int_\Omega \partial_t\bigl(\gamma(v)-\gamma(w)\bigr)\mathrm{sgn}\bigl(\bigl[\gamma(v)-\gamma(w)\bigr]^+\bigr)dx \\
&= \frac{d}{dt}\int_\Omega \bigl|\bigl[\gamma(v)-\gamma(w)\bigr]^+\bigr|dx
\end{aligned}
$$

and same as $(\partial_t\gamma(v_\Gamma) - \partial_t\gamma(w_\Gamma))\sigma_k([v_\Gamma - w_\Gamma]^+)$. Moreover,

$$
\begin{aligned}
&\int_\Omega \bigl(f\bigl(\gamma(v)\bigr) - f\bigl(\gamma(w)\bigr)\bigr)\sigma_k\bigl([v-w]^+\bigr)dx \\
&\to \int_\Omega \bigl(f\bigl(\gamma(v)\bigr) - f\bigl(\gamma(w)\bigr)\bigr)\mathrm{sgn}\bigl([v-w]^+\bigr)dx \\
&= \int_\Omega \bigl(f\bigl(\gamma(v)\bigr) - f\bigl(\gamma(w)\bigr)\bigr)\mathrm{sgn}\bigl(\bigl[\gamma(v)-\gamma(w)\bigr]^+\bigr)dx \\
&\le L\int_\Omega \bigl|\bigl[\gamma(v)-\gamma(w)\bigr]^+\bigr|dx,
\end{aligned}
$$

same as $(f_\Gamma(\gamma(v_\Gamma)) - f_\Gamma(\gamma(w_\Gamma)))\sigma_k([v_\Gamma - w_\Gamma]^+)$. By applying the Gronwall inequality

$$
\begin{aligned}
&\bigl|\bigl[\gamma\bigl(v(t)\bigr) - \gamma\bigl(w(t)\bigr)\bigr]^+\bigr|_{L^1(\Omega)} + \bigl|\bigl[\gamma\bigl(v_\Gamma(t)\bigr) - \gamma\bigl(w_\Gamma(t)\bigr)\bigr]^+\bigr|_{L^1(\Gamma)} \\
&\le \Bigl(\bigl|\bigl[\gamma(v_0) - \gamma(w_0)\bigr]^+\bigr|_{L^1(\Omega)} + \bigl|\bigl[\gamma(v_{\Gamma,0}) - \gamma(w_{\Gamma,0})\bigr]^+\bigr|_{L^1(\Gamma)}\Bigr) e^{Lt}
\end{aligned}
$$

for all $t \in [0,T]$, this comparison estimate gives us the uniqueness of the solution. □

4 Proof of main theorems

For a convenience, we define

$$
\begin{aligned}
Y(\boldsymbol{z}) &:= \frac{1}{1+m}\int_\Omega z^{(1+m)/m}dx + \frac{1}{1+m}\int_\Gamma z_\Gamma^{(1+m)/m}d\Gamma \\
&= \frac{\alpha}{\alpha+1}\int_\Omega z^{\alpha+1}dx + \frac{\alpha}{\alpha+1}\int_\Gamma z_\Gamma^{\alpha+1}d\Gamma.
\end{aligned}
$$

In the case of locally Lipschitz continuous perturbations, we apply the cut off method to prove local existence.

Proposition 4.1. *Let $0 < m \le 1$. Let us assume that $\boldsymbol{v}_0 := (v_0, v_{\Gamma,0}) \in \boldsymbol{V} \cap \boldsymbol{L}^\infty$ with $v_0 \ge 0$ and $v_{\Gamma,0} \ge 0$. Then there exist $T_{\max} > 0$ depending on the initial data, as well as* *a unique pair of non-negative functions* $\boldsymbol{v} := (v, v_\Gamma)$ *such that they solve* (2.1)–(2.5) *on* $[0, T_{\max})$*. Moreover, the functions* v *and* v_Γ *belong to*

$$
\begin{aligned}
& v \in C\big([0, T_{\max}); L^{\alpha+1}(\Omega)\big) \cap L^\infty\big(0, T; V \cap L^\infty(\Omega)\big) \cap L^2(0, T; W), \\
& v^{(\alpha+1)/2} = v^{(1+m)/2m} \in H^1(0, T; H), \\
& \gamma(v) \in H^1(0, T; H) \cap L^\infty(0, T; V), \\
& v_\Gamma \in C\big([0, T_{\max}); L^{\alpha+1}(\Gamma)\big) \cap L^\infty\big(0, T; V_\Gamma \cap L^\infty(\Gamma)\big) \cap L^2(0, T; W_\Gamma), \\
& v_\Gamma^{(\alpha+1)/2} = v_\Gamma^{(1+m)/2m} \in H^1(0, T; H_\Gamma), \\
& \gamma(v_\Gamma) \in H^1(0, T; H_\Gamma) \cap L^\infty(0, T; V_\Gamma)
\end{aligned}
$$

and satisfy the equality

$$
\frac{d}{dt} Y\big(\boldsymbol{v}(t)\big) + 2\varphi_1\big(\boldsymbol{v}(t)\big) = \lambda \int_\Omega v^{\alpha p+1}(t) dx + \mu \int_\Gamma v_\Gamma^{\alpha q+1}(t) d\Gamma \tag{4.1}
$$

for a.a. $t \in (0, T_{\max})$*. Furthermore, they satisfy the energy inequality*

$$
\frac{4\alpha}{(\alpha+1)^2} \int_s^t \left(\int_\Omega \big|\partial_t\big(v^{(\alpha+1)/2}(\tau)\big)\big|^2 dx + \int_\Gamma \big|\partial_t\big(v_\Gamma^{(\alpha+1)/2}(\tau)\big)\big|^2 d\Gamma \right) d\tau + J\big(\boldsymbol{v}(t)\big) \le J\big(\boldsymbol{v}(s)\big) \tag{4.2}
$$

for all $s, t \in [0, T_{\max})$ *satisfying* $s \le t$.

Proof. Let $M := 2(|v_0|_{L^\infty(\Omega)} + |v_{\Gamma,0}|_{L^\infty(\Gamma)})$. Moreover, define $g_M, g_{\Gamma,M} : \mathbb{R} \to \mathbb{R}$ by

$$
g_M(r) := \begin{cases} -(M+1)^{\alpha p} & \text{if } r < -(M+1)^\alpha, \\ |r|^{p-1} r & \text{if } |r| \le (M+1)^\alpha, \\ (M+1)^{\alpha p} & \text{if } r > (M+1)^\alpha, \end{cases}
$$

$$
g_{\Gamma,M}(r) := \begin{cases} -(M+1)^{\alpha q} & \text{if } r < -(M+1)^\alpha, \\ |r|^{q-1} r & \text{if } |r| \le (M+1)^\alpha, \\ (M+1)^{\alpha q} & \text{if } r > (M+1)^\alpha, \end{cases}
$$

where $p, q > 1$. Then, applying Proposition 3.1, for each $T > 0$ there exists a unique $\boldsymbol{v}_M := (v_M, v_{\Gamma,M})$ such that solves (3.1)–(3.5) with $f := \lambda g_M$, $f_\Gamma := \mu g_{M,\Gamma}$. Moreover, from the L^∞-boundedness (3.7)–(3.8), we have

$$
\big|v_M(t)\big|_{L^\infty(\Omega)} \le M e^{p_*(M+1)^{\alpha(p_*-1)} t/\alpha}, \quad \big|v_{\Gamma,M}(t)\big|_{L^\infty(\Gamma)} \le M e^{p_*(M+1)^{\alpha(p_*-1)} t/\alpha}
$$

for all $t \in [0, T]$, where we recall that $p_* = \lambda p + \mu q$. Now, taking $\delta > 0$ satisfying

$$
M e^{p_*(M+1)^{\alpha(p_*-1)} \delta/\alpha} \le M + 1, \tag{4.3}
$$

$\boldsymbol{v}_M$ solves the original problem (2.1)–(2.5) on $[0, \delta]$. Moreover, we define

$$
T_{\max} := \sup\big\{\delta > 0 \ : \ \text{the problem (2.1)–(2.5) has the unique solution on } [0, \delta]\big\}.
$$

Thus, we have proved the local existence of (2.1)–(2.5) on $[0,T]$ for all $T \in (0, T_{\max})$. Additionally, from equations (2.1)–(2.5), we obtain that $\boldsymbol{v}$ satisfies (4.2). Next, from the characterization of the Sobolev functions and the chain rule, we obtain that

$$\partial_t \gamma(v) = \partial_t v^\alpha = \partial_t \big(v^{(\alpha+1)/2}\big)^{2\alpha/(\alpha+1)} = \frac{2\alpha}{\alpha+1} v^{(\alpha-1)/2} \partial_t v^{(\alpha+1)/2}$$

and also, for v_Γ^α (see, [53, Lemma 2.3]). Therefore, from the equations (2.1)–(2.3), we obtain that the following equality holds:

$$\int_s^t \left(\int_\Omega \partial_t \gamma\big(v(\tau)\big) v(\tau) dx + \int_\Gamma \partial_t \gamma\big(v_\Gamma(\tau)\big) v_\Gamma(\tau) d\Gamma \right) d\tau = Y\big(\boldsymbol{v}(t)\big) - Y\big(\boldsymbol{v}(s)\big) \tag{4.4}$$

for all $s, t \in [0, T_{\max})$, that is, $Y(\boldsymbol{v})$ is absolutely continuous on $[0, T_{\max})$. This implies (4.1) and additional continuities on $[0,T]$ in $L^{\alpha+1}(\Omega)$ and $L^{\alpha+1}(\Gamma)$. □

Also, we obtain that $J(\boldsymbol{v}(t)) \le J(\boldsymbol{v}_0)$ for all $t \in [0, T_{\max})$, and $J(\boldsymbol{v}(t))$ is monotone decreasing as $t \to T_{\max}$. Moreover, we obtain the invariance of $\mathcal{W}$ as follows.

Lemma 4.1. *The set $\mathcal{W} \cap \boldsymbol{L}^\infty$ is invariant; that is, if $\boldsymbol{v}_0 \in \mathcal{W} \cap \boldsymbol{L}^\infty$, then $\boldsymbol{v}(t) \in \mathcal{W} \cap \boldsymbol{L}^\infty$ for all $t \in [0, T_{\max})$.*

The proof of this lemma is given in Subsection 4.2.

4.1 Finite time extinction

Let $(a,b) \in \{(1,0),(0,1)\}$, $(\lambda, \mu) = (0,1)$. The strategy for proving Theorems 2.1 and 2.2 is the same as that in [15, Theorem 2.1] and [44, Proposition 5].

Proof. Assume that $\boldsymbol{v}_0 \in \mathcal{W} \cap \boldsymbol{L}^\infty$. Then, applying [15, Lemma 2.5] we can prove $T_{\max} = \infty$ (see, Remark 4.1). Recalling (4.1), we have the following equality:

$$\begin{aligned} \frac{d}{dt} Y\big(\boldsymbol{v}(t)\big) &= -2\varphi_1\big(\boldsymbol{v}(t)\big) + (\alpha q + 1)\varphi_2\big(\boldsymbol{v}(t)\big) \\ &= -2\varphi_1\big(\boldsymbol{v}(t)\big) + \int_\Gamma v_\Gamma^{\alpha q+1}(t) d\Gamma. \end{aligned} \tag{4.5}$$

for a.a. $t \in (0, \infty)$. From (3.13) and the Sobolev imbedding in 2-dimensions, there exists a positive constant $C_{\mathrm{S}} > 0$ such that

$$\begin{aligned} \left(\int_\Gamma z_\Gamma^{\alpha q+1} d\Gamma \right)^{1/(\alpha q+1)} &= |z_\Gamma|_{L^{\alpha q+1}(\Gamma)} \\ &\le C_{\mathrm{S}} |z_\Gamma|_{V_\Gamma} \\ &\le 2^{1/2} C_{\mathrm{S}} C_{\mathrm{C}}^{-1/2} \varphi_1(\boldsymbol{z})^{1/2}; \end{aligned}$$

that is, there exists a positive constant $C > 0$ such that

$$\varphi_2(\boldsymbol{z}) \le C \varphi_1(\boldsymbol{z})^{(\alpha q+1)/2}. \tag{4.6}$$

for all $\boldsymbol{z} \in \boldsymbol{V}$. From the assumption, we have $J(\boldsymbol{v}(t)) \leq J(\boldsymbol{v}_0) := d_0 < d$ for all $t \in [0, \infty)$ and $2\varphi_1(\boldsymbol{v}_0) > (\alpha q + 1)\varphi_2(\boldsymbol{v}_0)$. Moreover, $J(\boldsymbol{v}(t))$ is monotone decreasing. Hence, there exists $\varepsilon_1 \in (0, 1)$ such that

$$(1 - \varepsilon_1) 2\varphi_1\bigl(\boldsymbol{v}(t)\bigr) \geq (\alpha q + 1)\varphi_2\bigl(\boldsymbol{v}(t)\bigr) \tag{4.7}$$

for all $t \in [0, \infty)$ (see, Remark 4.2 and FIGURE 1). Now, from equations (3.13), (4.5), and (4.7)

$$\begin{aligned} 0 &= \frac{d}{dt} Y\bigl(\boldsymbol{v}(t)\bigr) + 2\varphi_1\bigl(\boldsymbol{v}(t)\bigr) - (\alpha q + 1)\varphi_2\bigl(\boldsymbol{v}(t)\bigr) \\ &\geq \frac{d}{dt} Y\bigl(\boldsymbol{v}(t)\bigr) + 2\varepsilon_1 \varphi_1\bigl(\boldsymbol{v}(t)\bigr) \\ &\geq \frac{d}{dt} Y\bigl(\boldsymbol{v}(t)\bigr) + \varepsilon_1 C_{\mathrm{C}} \bigl|\boldsymbol{v}(t)\bigr|_{\boldsymbol{V}}^2 \end{aligned}$$

that is, under the assumption $1/5 < m$, from the Sobolev imbedding in 3-dimension, there exists a positive constant $C(\alpha)$ depending upon ε_1, C_{C}, and α, such that

$$\frac{d}{dt} Y\bigl(\boldsymbol{v}(t)\bigr) \leq -C(\alpha) Y\bigl(\boldsymbol{v}(t)\bigr)^{2/(\alpha+1)}$$

for a.a. $t \in (0, \infty)$. Recalling the fact that $0 < 2/(\alpha + 1) < 1$, we deduce

$$Y\bigl(\boldsymbol{v}(t)\bigr) \leq \left(\left[Y(\boldsymbol{v}_0)^{(\alpha-1)/(\alpha+1)} - \frac{\alpha - 1}{\alpha + 1} C(\alpha) t \right]^+ \right)^{(\alpha+1)/(\alpha-1)},$$

that is, there exists $T_{\mathrm{ext}} > 0$ depending upon $Y(\boldsymbol{v}_0)$ such that $\boldsymbol{v}(t) \equiv \boldsymbol{0}$ for all $t \geq T_{\mathrm{ext}}$. Moreover, the estimate from above (2.7) holds. □

Remark 4.1. *We know that $\boldsymbol{v}(t)$ is the function in $\boldsymbol{L}^\infty$ for all $t \in [0, T_{\max})$ from the construction of the solution, see Proposition 4.1. However, to obtain the time global estimate with respect to L^∞-norm, we do not apply Lemma 3.1 directly to $\boldsymbol{v}(t)$ any more since g and g_Γ are not global Lipschitz functions. To obtain it, we need the assumptions $1 < p < 5m$ or $q > 1$. Indeed, we can apply the useful proposition with related to the Moser technique. Originally it was obtained by Alikakos [4, Lemma 3.2], extended by Nakao [42, Lemma 3.1] for $m > 1$, and Fila–Filo [15, Proposition 2.6] for $0 < m < 1$. Under the assumption $1 < p < 5m$ or $q > 1$, we can obtain the L^∞-estimate independent of $t \in [0, T_{\max})$ [15, Lemma 2.5] and then we obtain that $T_{\max} = \infty$. Thus, we see that Theorem 2.2 is true also in the critical case $p = 5m$ if we additionally assume that $T_{\max} = \infty$.*

Remark 4.2. *The depth of the potential well d of (2.6) is characterized by*

$$d = \frac{\alpha p_* - 1}{2} \left(\frac{2}{\alpha p_* + 1} \right)^{(\alpha p_* + 1)/(\alpha p_* - 1)} C^{-2/(\alpha p_* - 1)}, \tag{4.8}$$

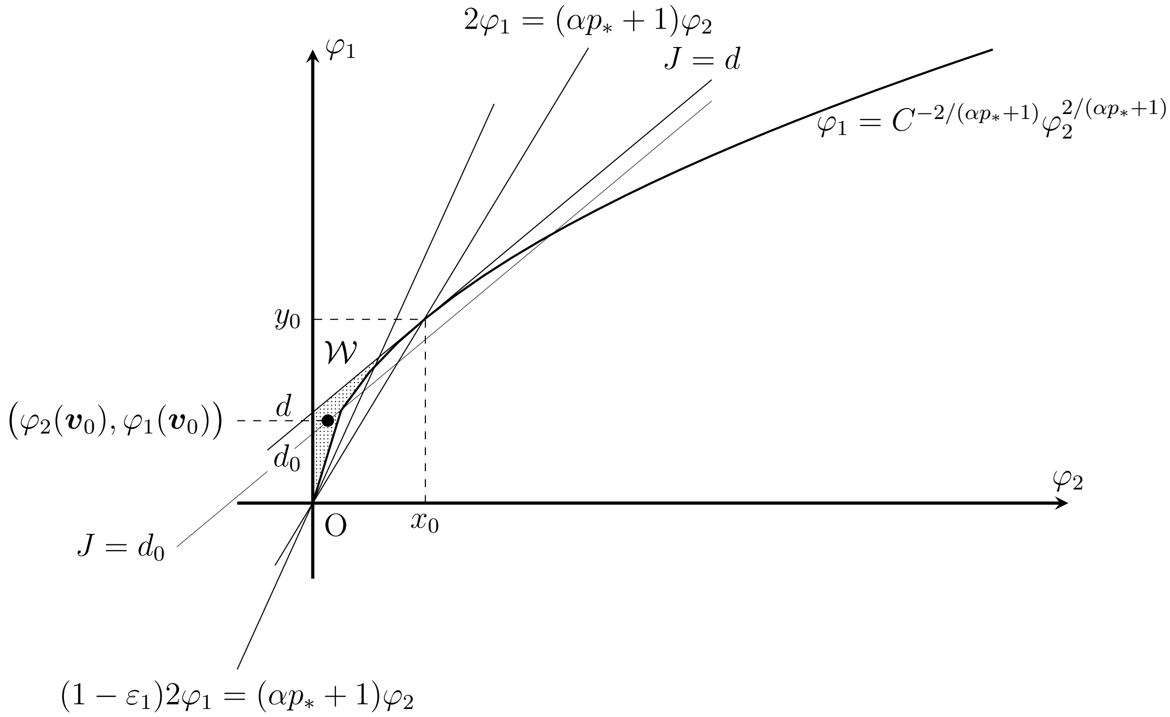

Figure 1: Region of $\mathcal{W}$.

where C is the best constant satisfying (4.6). *Let* $(\varphi_2, \varphi_1) = (x_0, y_0)$ *be the cross point between*

$$\begin{cases} \varphi_1 = C^{-2/(\alpha p_*+1)}\varphi_2^{2/(\alpha p_*+1)}, \\ 2\varphi_1 = (\alpha p_* + 1)\varphi_2. \end{cases}$$

Then, $J = d$ *is the tangential line to the function* $\varphi_1 = C^{-2/(\alpha p_*+1)}\varphi_2^{2/(\alpha p_*+1)}$ *at*

$$(x_0, y_0)$$
$$= \left(\left(\frac{2}{\alpha p_* + 1} \right)^{(\alpha p_*+1)/(\alpha p_*-1)} C^{-2/(\alpha p_*-1)}, \frac{\alpha p_* + 1}{2} \left(\frac{2}{\alpha p_* + 1} \right)^{(\alpha p_*+1)/(\alpha p_*-1)} C^{-2/(\alpha p_*-1)} \right)$$

Thus, we can obtain (4.8) *from* $d = y_0 - x_0$.

4.2 Proof of invariance

It remains to prove Lemma 4.1. The proof of invariance is not difficult if we have $\boldsymbol{v} \in C([0, T_{\max}); \boldsymbol{V})$ (see [14, Chapter 4], for example). However, in general, we do not have such regularity from the nonlinearity of γ. Therefore, we consider an approximation and obtain some information concerning the invariance:

Proof. From the definition of $T_{\max}$, for each $\tau > 0$ there exists $T_\tau \in (T_{\max} - \tau, T_{\max})$ such that the problem (2.1)–(2.5) has a unique solution on $[0, T_\tau]$. Moreover, recalling (4.3) we put $M_\tau > 0$ satisfying

$$M e^{p_*(M+1)^{\alpha(p_*-1)}T_\tau/\alpha} \leq M + M_\tau.$$

For each $\varepsilon > 0$ and $0 < T < \infty$, let us consider the following approximate problem

$$\partial_t \gamma(v_\varepsilon) + \varepsilon \partial_t v_\varepsilon - \Delta v_\varepsilon + a v_\varepsilon = f\big(\gamma(v_\varepsilon)\big) \quad \text{a.e. in } Q, \tag{4.9}$$

$$(v_\varepsilon)_{|\Gamma} = v_{\Gamma,\varepsilon} \quad \text{a.e. on } \Sigma, \tag{4.10}$$

$$\partial_t \gamma(v_{\Gamma,\varepsilon}) + \varepsilon \partial_t v_{\Gamma,\varepsilon} + \partial_{\boldsymbol{\nu}} v_\varepsilon - \Delta_\Gamma v_{\Gamma,\varepsilon} + b v_{\Gamma,\varepsilon} = f_\Gamma\big(\gamma(v_{\Gamma,\varepsilon})\big) \quad \text{a.e. on } \Sigma, \tag{4.11}$$

$$v_\varepsilon(0) = v_0 \quad \text{a.e. in } \Omega, \tag{4.12}$$

$$v_{\Gamma,\varepsilon}(0) = v_{\Gamma,0} \quad \text{a.e. on } \Gamma, \tag{4.13}$$

where $f := \lambda g_{M+M_\tau}$ and $f := \mu g_{\Gamma,M+M_\tau}$ are the same as in the proof of Proposition 4.1. Following Propositions 3.1 and 4.1 with the cut off technique, we prove that there exists a unique pair $\boldsymbol{v}_\varepsilon = (v_\varepsilon, v_{\Gamma,\varepsilon}) \in C([0,T]; \boldsymbol{V})$ of functions that satisfies (4.9)–(4.13) and

$$\begin{aligned}
&\frac{4\alpha}{(\alpha+1)^2} \int_s^t \left(\left| \partial_t \big(\boldsymbol{v}_\varepsilon^{(\alpha+1)/2}(\tau) \big) \right|_{\boldsymbol{H}}^2 + \varepsilon \left| \partial_t \boldsymbol{v}_\varepsilon(\tau) \right|_{\boldsymbol{H}}^2 \right) d\tau \\
&\quad + \varphi_1\big(\boldsymbol{v}_\varepsilon(t)\big) - \int_\Omega \widehat{f}_\gamma\big(v_\varepsilon(t)\big) dx - \int_\Gamma \widehat{f}_{\Gamma,\gamma}\big(v_{\Gamma,\varepsilon}(t)\big) d\Gamma \\
&= \varphi_1\big(\boldsymbol{v}_\varepsilon(s)\big) - \int_\Omega \widehat{f}_\gamma\big(v_\varepsilon(s)\big) dx - \int_\Gamma \widehat{f}_{\Gamma,\gamma}\big(v_{\Gamma,\varepsilon}(s)\big) d\Gamma
\end{aligned} \tag{4.14}$$

for all $s, t \in [0,T]$ with $s \le t$. Moreover

$$\begin{aligned}
\left| v_\varepsilon(t) \right|_{L^\infty(\Omega)} &\le M e^{p_*(M+1)^{\alpha(p_*-1)} t/\alpha} \\
&\le M + M_\tau, \\
\left| v_{\Gamma,\varepsilon}(t) \right|_{L^\infty(\Gamma)} &\le M e^{p_*(M+1)^{\alpha(p_*-1)} t/\alpha} \\
&\le M + M_\tau
\end{aligned}$$

for all $t \in [0, T_\tau]$. Therefore, we can replace g_{M+M_τ} by g and $g_{\Gamma,M+M_\tau}$ by g_Γ on the time interval $[0, T_\tau]$ in equations (4.9) and (4.11). Furthermore, from (4.14) it can be shown that $J(\boldsymbol{v}_\varepsilon(t)) \le J(\boldsymbol{v}_0)$ and $\boldsymbol{v}_\varepsilon(t) \in \mathcal{W} \cap \boldsymbol{L}^\infty$ for all $t \in [0, T_\tau]$, since $\boldsymbol{v}_0 \in \mathcal{W} \cap \boldsymbol{L}^\infty$. Moreover, we obtain the same kind of uniform estimates used in proving Propositions 3.1 and 4.1, such as there exists a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ satisfying $\varepsilon_k \to 0$ as $k \to \infty$ such that

$$\begin{aligned}
\boldsymbol{v}_{\varepsilon_k} \to \boldsymbol{v} \quad & \text{weakly star in } L^\infty(0, T_\tau; \boldsymbol{V}), \\
\gamma(\boldsymbol{v}_{\varepsilon_k}) \to \gamma(\boldsymbol{v}) \quad & \text{weakly in } H^1(0, T_\tau; \boldsymbol{H}), \\
& \text{weakly star in } L^\infty(0, T_\tau; \boldsymbol{V}), \\
\gamma(v_{\varepsilon_k}) \to \gamma(v) \quad & \text{strongly in } C\big([0, T_\tau]; L^r(\Omega)\big), \\
\gamma(v_{\Gamma,\varepsilon_k}) \to \gamma(v_\Gamma) \quad & \text{strongly in } C\big([0, T_\tau]; L^r(\Gamma)\big) \quad \text{for } r \in [2, \infty), \\
\varepsilon_k \partial_t \boldsymbol{v}_{\varepsilon_k} \to \boldsymbol{0} \quad & \text{strongly in } L^2(0, T_\tau; \boldsymbol{H})
\end{aligned}$$

as $k \to \infty$, where $\boldsymbol{v} = (v, v_\Gamma)$ is the unique solution obtained in Proposition 4.1. Thus, we applied the compactness results [49, Section 8, Corollary 4] again to obtain the strong convergences since, the compact imbeddings $V \cap L^\infty(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ and $V_\Gamma \hookrightarrow\hookrightarrow L^r(\Gamma)$ for $r \in [2, \infty)$ hold. Consequently, we deduce

$$\varphi_2\big(\boldsymbol{v}_{\varepsilon_k}(t)\big) \to \varphi_2\big(\boldsymbol{v}(t)\big) \tag{4.15}$$

for all $t \in [0, T_{\max})$ as $k \to \infty$. Recall the definition of $\mathcal{W}$. We already know that $J(\boldsymbol{v}(t)) \le J(\boldsymbol{v}_0) < d$. Therefore, it is sufficient to prove that

$$2\varphi_1\big(\boldsymbol{v}(t)\big) > (\alpha p_* + 1)\varphi_2\big(\boldsymbol{v}(t)\big)$$

for all $t \in [0, T_{\max})$. Let $t \in [0, T_{\max})$. This is clear if $\varphi_2(\boldsymbol{v}(t)) = 0$, therefore we assume $\varphi_2(\boldsymbol{v}(t)) > 0$. Now, from $d_0 = J(\boldsymbol{v}_0) < d$, we obtain that $\delta_0 := d - d_0 > 0$ and

$$J\big(\boldsymbol{v}(t)\big) \le d - \delta_0, \quad J\big(\boldsymbol{v}_{\varepsilon_k}(t)\big) \le d - \delta_0$$

for all $t \in [0, T_{\max})$. Thus, from (4.8) we have

$$\begin{aligned}
\varphi_1\big(\boldsymbol{v}_{\varepsilon_k}(t)\big) - \varphi_2\big(\boldsymbol{v}_{\varepsilon_k}(t)\big) &\le d - \delta_0 \\
&= \frac{\alpha p_* - 1}{2}\left(\frac{2}{\alpha p_* + 1}\right)^{(\alpha p_* + 1)/(\alpha p_* - 1)} C^{-2/(\alpha p_* - 1)} - \delta_0.
\end{aligned}$$

Now, $\boldsymbol{v}_{\varepsilon_k}(t) \in \mathcal{W}$, using (4.6) we have

$$\begin{aligned}
\frac{\alpha p_* - 1}{2}\varphi_2\big(\boldsymbol{v}_{\varepsilon_k}(t)\big) &= \frac{\alpha p_* + 1}{2}\varphi_2\big(\boldsymbol{v}_{\varepsilon_k}(t)\big) - \varphi_2\big(\boldsymbol{v}_{\varepsilon_k}(t)\big) \\
&< \varphi_1\big(\boldsymbol{v}_{\varepsilon_k}(t)\big) - \varphi_2\big(\boldsymbol{v}_{\varepsilon_k}(t)\big) \\
&= \frac{\alpha p_* - 1}{2}\left(\frac{2}{\alpha p_* + 1}\right)^{(\alpha p_* + 1)/(\alpha p_* - 1)} C^{-2/(\alpha p_* - 1)} - \delta_0 \\
&\le \frac{\alpha p_* - 1}{2}\left(\frac{2}{\alpha p_* + 1}\right)^{(\alpha p_* + 1)/(\alpha p_* - 1)} \left(\frac{\varphi_1\big(\boldsymbol{v}(t)\big)^{(\alpha p_* + 1)/2}}{\varphi_2\big(\boldsymbol{v}(t)\big)}\right)^{2/(\alpha p_* - 1)} - \delta_0.
\end{aligned}$$

Let $k \to \infty$ in the above; using (4.15) we deduce that

$$\varphi_2\big(\boldsymbol{v}(t)\big) \le \left(\frac{2}{\alpha p_* + 1}\right)^{(\alpha p_* + 1)/(\alpha p_* - 1)} \left(\frac{\varphi_1\big(\boldsymbol{v}(t)\big)^{(\alpha p_* + 1)/2}}{\varphi_2\big(\boldsymbol{v}(t)\big)}\right)^{2/(\alpha p_* - 1)} - \frac{2}{\alpha p_* - 1}\delta_0,$$

that is,

$$(\alpha p_* + 1)\varphi_2\big(\boldsymbol{v}(t)\big) < 2\varphi_1\big(\boldsymbol{v}(t)\big)$$

for all $t \in [0, T_{\max})$. □

Next, we prove Theorem 2.2. Let $(a, b) \in \{(1, 0), (0, 1)\}$, $(\lambda, \mu) = (1, 0)$.

Proof. Assume $1/5 < m < 1$, $1 < p < 5m$. In this case, the estimate (4.5) is replaced by

$$\begin{aligned}
\frac{d}{dt}Y\big(\boldsymbol{v}(t)\big) &= -2\varphi_1\big(\boldsymbol{v}(t)\big) + (\alpha p + 1)\varphi_2\big(\boldsymbol{v}(t)\big) \\
&= -2\varphi_1\big(\boldsymbol{v}(t)\big) + \int_\Omega v^{\alpha p + 1}(t)dx
\end{aligned}$$

for a.a. $t \in (0, \infty)$. Now, there exists a positive constant $C_\mathrm{S} > 0$ such that

$$\begin{aligned}\left(\int_\Omega z^{\alpha p+1} dx\right)^{1/(\alpha p+1)} &= |z|_{L^{\alpha p+1}(\Omega)} \\ &\le C_\mathrm{S} |z|_V\end{aligned}$$

for all $z \in V$ because $\alpha p + 1 < 6$. Thus, there exists a positive constant $C > 0$ such that

$$\varphi_2(\boldsymbol{z}) \le C \varphi_1(\boldsymbol{z})^{(\alpha p+1)/2}.$$

Furthermore, there exists $\varepsilon_1 \in (0,1)$ such that

$$(1-\varepsilon_1) 2\varphi_1\big(\boldsymbol{v}(t)\big) \ge (\alpha p + 1)\varphi_2\big(\boldsymbol{v}(t)\big)$$

for all $t \in [0, \infty)$ replaced with (4.7). Therefore, the proof is completely identical to the proof of Theorem 2.1. The proof of invariance of $\mathcal{W}$ is also the same; indeed, convergence (4.15) holds since $\alpha p_* + 1 = \alpha p + 1 < 6$ from the additional assumption $1 < p < 5m$. $\square$

Appendix

We use the same settings as in the previous sections.

Lemma A.1. *Let $\alpha \ge 1$. Then*

$$\begin{gathered}\frac{4\alpha}{(\alpha+1)^2}\big(r^{(\alpha+1)/2} - s^{(\alpha+1)/2}\big)^2 \le (r^\alpha - s^\alpha)(r - s), \\ |r^\alpha - s^\alpha| \le \frac{2\alpha}{\alpha+1} \max\{r, s\}^{(\alpha-1)/2} \big|r^{(\alpha+1)/2} - s^{(\alpha+1)/2}\big|, \\ |r - s| \le \big|r^{(\alpha+1)/2} - s^{(\alpha+1)/2}\big|^{2/(\alpha+1)}\end{gathered}$$

for all $r, s \ge 0$.

Proof. If $s = 0$, then we can prove that all inequalities hold. Therefore, it is sufficient to prove that

$$\begin{aligned}F(x) &:= (x^\alpha - 1)(x-1) - \frac{4\alpha}{(\alpha+1)^2}\big(x^{(\alpha+1)/2} - 1\big)^2, \\ G(x) &:= \frac{2\alpha}{\alpha+1} x^{(\alpha-1)/2}\big(x^{(\alpha+1)/2} - 1\big) - (x^\alpha - 1), \\ H(x) &:= \big(x^{(\alpha+1)/2} - 1\big) - (x-1)^{(\alpha+1)/2}\end{aligned}$$

are positive for $x \ge 1$. Firstly, from the basic calculation, we see that $G(1) = 0$ and

$$G'(x) = \frac{\alpha(\alpha-1)}{\alpha+1} x^{(\alpha-3)/2}\big(x^{(\alpha+1)/2} - 1\big) \ge 0,$$

these imply that $G(x) \geq 0$ for all $x \geq 1$. Secondly, we put $\ell := (\alpha+1)/2$; then

$$\begin{aligned}
F(x) &= (x^{2\ell-1}-1)(x-1) - \frac{2\ell-1}{\ell^2}(x^\ell-1)^2 \\
&= \left(\frac{\ell-1}{\ell}\right)^2 (x^\ell-1)^2 - x(x^{\ell-1}-1)^2 \\
&= \left(\frac{\ell-1}{\ell}(x^\ell-1) + x^{1/2}(x^{\ell-1}-1)\right)\left(\frac{\ell-1}{\ell}(x^\ell-1) - x^{1/2}(x^{\ell-1}-1)\right) \\
&=: \left(\frac{\ell-1}{\ell}(x^\ell-1) + x^{1/2}(x^{\ell-1}-1)\right) F_1(x),
\end{aligned}$$

where the multiplier of F_1 is positive. Moreover, we can prove that $F_1(x) \geq 0$ for all $x \geq 1$ just as we could for $G(x)$. This means that $F(x) \geq 0$ for all $x \geq 1$. Finally, $H(1) = 0$ and

$$H'(x) = \ell x^{\ell-1} - \ell(x-1)^{\ell-1} \geq 0$$

for all $x \geq 1$ since $\ell x^{\ell-1}$ is monotone increasing for $x \geq 1$. This implies that $H(x) \geq 0$ for all $x \geq 1$. As a remark, a similar inequality for $H(x)$ can be obtained from the Tartar inequality. □

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