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# VISCOUS CAHN–HILLIARD EQUATION WITH DIRICHLET BOUNDARY CONDITION

Keiichiro Kagawa
Major in Pure and Applied Physics, Graduate School of Advanced Science and Engineering,
Waseda University, 3-4-1 Okubo Shinjuku-ku, Tokyo, 169-8555, JAPAN
(E-mail: heiankyo615@fuji.waseda.jp)

and

Mitsuharu Ôtani
Department of Applied Physics, School of Science and Engineering,
Waseda University, 3-4-1 Okubo Shinjuku-ku, Tokyo, 169-8555, JAPAN
(E-mail: otani@waseda.jp)

**Abstract.** In this paper, we show the existence of solutions of the initial value problem for the viscous Cahn–Hilliard equation with the homogeneous Dirichlet boundary condition. In the previous researches, the existence of strong solutions is shown for the case where the Sobolev subcritical condition is imposed for the nonlinear term. We exclude this restriction by decomposing the nonlinear term into the difference between a maximal monotone term and a perturbation term, and prove the existence of the strong solutions to the initial boundary value problem of the viscous Cahn–Hilliard equation. Furthermore, some smoothing effect of the solutions is also discussed. Our proof relies on the abstract theory of the evolution equation governed by the subdifferential operator.

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# 1 Introduction

The Cahn–Hilliard equation was derived in 1958 to describe the phase separation phenomena in alloys by [3]. Since the 1980s, the initial boundary value problem for the Cahn–Hilliard equation has been extensively investigated. In 1986, Elliott and Songmu [10] considered the initial value problem with the homogeneous Neumann boundary condition in one dimensional bounded domains. They adopted a third order polynomial as the nonlinear term. At that time, the homogeneous Neumann boundary condition was commonly imposed from the physical requirement that the sum of order parameters inside the domain should be conserved. In 1996, Gurtin [11] proposed a model which takes into account internal microforces, where some new viscosity term caused by internal microforces appears in the Cahn–Hilliard equation, which is called the viscous Cahn-Hilliard equation.

In this paper, we consider the existence of solutions of the initial boundary value problem (P) of the viscous Cahn–Hilliard equation with the homogeneous Dirichlet boundary condition:

$$\begin{cases} \gamma\partial_t u = \Delta\left[\varphi(u) - \alpha\Delta u + \beta\partial_t u - h\right], & (t,x) \in (0,T)\times\Omega, \\ u = \varphi(u) - \alpha\Delta u + \beta\partial_t u - h = 0, & (t,x) \in (0,T)\times\partial\Omega, \\ u|_{t=0} = u_0, & x \in \Omega, \end{cases} \tag{P}$$

where $0 < T < \infty$; $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}$) with sufficiently smooth boundary $\partial\Omega$; $u : [0,T]\times\Omega \to \mathbb{R}$ is an unknown function; $\varphi(u)$ is a given nonlinear function, and $h$ is a given external force term; and $\alpha$, $\beta$, and $\gamma$ are given positive parameters. The Cahn–Hilliard equation is nothing but (P) with $\beta = 0$. In the phase separation in a binary alloy, the unknown function $u$ represents the order parameter, which corresponds to the ratio of the concentrations of two metals. In physics, the derivative of the double well potential, $\frac{1}{4}(u-1)^2(u+1)^2$, which causes phase separation, is often adopted as the nonlinear term, i.e., $\varphi(u) = u^3 - u$. The term $\varphi(u) - \alpha\Delta u + \beta\partial_t u - h$ in the right-hand side of the equation is frequently referred as $v$. Then the equation can be written as:

$$\begin{cases} \gamma\partial_t u = \Delta v, \\ v = \varphi(u) - \alpha\Delta u + \beta\partial_t u - h. \end{cases}$$

Initial value problems for the Cahn–Hilliard equation with Dirichlet boundary conditions have been considered since the 2000s. In 2002, Efendiev, Gajewski and Zelik [8] studied the existence of the solutions for the Dirichlet boundary value problem. They considered the unknown function as a $k$-th vector valued function and assumed that $\varphi(u) \in C^1(\mathbb{R}^k;\mathbb{R}^k)$, $\varphi'(u)$, and $\varphi(u)u$ are bounded below by a negative constant. In 2004, Efendiev, Miranville and Zelik [9] also consider the same problem with additional assumptions on $\varphi$, i.e., the boundedness of the nonlinear term and $\varphi(u) \in C^2(\mathbb{R}^k;\mathbb{R}^k)$. They proved the well-posedness of the problem and also considered the limit of the exponential attractors for the viscous Cahn–Hilliard equation, to that for the Cahn–Hilliard equation. Recently, in 2014, Bui, Smarrazzo and Tesei [13] proved the existence of a strong solution under the condition that the nonlinear term $\varphi(u)$ satisfies $\varphi(u)u \geq 0$ for all $u \in \mathbb{R}$ and the Sobolev subcritical growth condition. We note that there are a few other researches for the initial value problem with the Dirichlet boundary condition [6, 7].

In this paper, we decompose the nonlinear term $\varphi(u)$ into the sum of a maximal monotone operator $f : \mathbb{R} \to \mathbb{R}$ and a perturbation term $g : \mathbb{R} \to \mathbb{R}$ (a maximal monotone function or a locally Lipschitz continuous function), i.e., $\varphi = f - g$, and investigate the existence of the strong solutions of (P). Approximating the monotone function and the locally Lipschitz perturbation by globally Lipschitz continuous functions, we show the existence of the solutions of approximate problems by applying the abstract theory of the evolution equation. Furthermore, we derive a priori estimates of solutions of approximate equations independent of approximation parameters, and obtain the solution of the original problem (P) by taking the limit of the approximation parameters. In order to obtain a priori estimates, we impose an assumption such that $g$ is dominated by $f$ (see (2.1) and (2.2)). As for the convergence of solutions of approximate equations, we rely on the compactness argument.

By our method, we can exclude the Sobolev subcritical growth condition on $\varphi$ in [13], and do not need the boundedness of $\varphi(u)u$ or the derivative of $\varphi(u)$ which are assumed in [8, 9]. Furthermore, our structure condition on $\varphi$ can cover the double well potential commonly used in physics, which does not satisfy conditions imposed in [13]. In addition, the logarithmic potentials and the nonsmooth potentials fall within the framework of our treatment.

In section 2, we fix some notations and state main results. Theorem 2.2 claims the existence of the strong solutions for the initial data belonging to $H^1_0(\Omega)$ and the effective domain of $f$. Theorem 2.3 is concerned with the smoothing effect. In section 3 and 4, we give proofs for Theorem 2.2 and Theorem 2.3 respectively. In appendix, we consider the uniqueness of the solution, and the existence of solutions for the homogeneous Neumann boundary value problem.

# 2 Main results

In order to state our main results, we fix some notations. Let $L^2(\Omega)$ be the Hilbert space with inner product $(u, v)_2 := \int_\Omega u(x)v(x)dx$, and norm $|u|_2^2 := (u, u)_2$. Let $H^1_0(\Omega) = \{u \in L^2(\Omega); |\nabla u| \in L^2(\Omega), u|_{\partial\Omega} = 0\}$ with norm $|u|_{H^1_0(\Omega)} = |\nabla u|_{(L^2)^N}$ and denote its dual space by $H^{-1}(\Omega)$ and the duality paring by ${}_{H^1_0} < \cdot, \cdot >_{H^{-1}}$. Then $-\Delta$ gives the duality mapping from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

In this paper, we assume the following structure condition (A$\varphi$-I) or (A$\varphi$-II) on $\varphi(\cdot)$.

(A$\varphi$-I) $\varphi$ can be decomposed into the difference between $f$ and $g$, i.e., $\varphi = f - g$ such that $0 \in f(0)$, $0 \in g(0)$, and followings (i) - (iii) are satisfied.

(i) There is a proper, lower semicontinuous, convex functional $\hat{f} : \mathbb{R} \to (-\infty, \infty]$ satisfying $\partial\hat{f}(r) = f(r)$ for any $r \in \mathbb{R}$ and $\hat{f}(0) = 0$, where $\partial\hat{f}$ denotes the subdifferential of $\hat{f}$. Then $f$ is a (possibly multivalued) maximal monotone graph in $\mathbb{R} \times \mathbb{R}$.

(ii) There is a proper, lower semicontinuous, convex functional $\hat{g} : \mathbb{R} \to (-\infty, \infty]$ satisfying $\partial\hat{g}(r) = g(r)$ for any $r \in \mathbb{R}$ and $\hat{g}(0) = 0$, where $\partial\hat{g}$ denotes the subdifferential of $\hat{g}$. Then $g$ is a (possibly multivalued) maximal monotone graph in $\mathbb{R} \times \mathbb{R}$.

(iii) There exist $k \in [0,1)$ and $K \in [0,\infty)$ such that

$$|||g(r)||| \le k|f^\circ(r)| + K \quad \forall r \in D(f) := \{r \in \mathbb{R}^1 | f(r) \ne \emptyset\}. \tag{2.1}$$

Here, $f^\circ(r)$ denotes the minimal section of $f(r)$ and $|||g(r)||| := \sup\{|b|; b \in g(r)\}$.

(A$\varphi$-II) $\varphi$ can be decomposed into the difference between $f$ and $g$, i.e., $\varphi = f - g$ such that $0 \in f(0)$, and followings (i) - (iii) are satisfied.

(i) (Same as (A$\varphi$-I)-(i)) There is a proper, lower semicontinuous, convex functional $\hat{f} : \mathbb{R} \to (-\infty, \infty]$ satisfying $\partial \hat{f}(r) = f(r)$ for any $r \in \mathbb{R}$ and $\hat{f}(0) = 0$, where $\partial \hat{f}$ denotes the subdifferential of $\hat{f}$. Then $f$ is a (possibly multivalued) maximal monotone graph in $\mathbb{R} \times \mathbb{R}$.

(ii) $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying $g(0) = 0$.

(iii) There exist $k \in [0,1)$ and $K \in [0,\infty)$ such that

$$|g(r)| \le k|f^\circ(r)| + K \quad \forall r \in D(f). \tag{2.2}$$

Here we collect some fundamental properties concerning maximal monotone operators and subdifferential operators (see Brézis [2] and Barbu [1]). Let $H$ be a real Hilbert space with inner product $(\cdot,\cdot)_H = (\cdot,\cdot)$ and norm $||\cdot||_H$. Let $A : H \to 2^H$ be a maximal monotone operator. The minimal section $A^\circ x$ of $Ax$ is the unique element of $Ax$ satisfying $||A^\circ x||_H = \inf\{||y||_H; y \in Ax\}$ for all $x \in D(A) = \{z; Az \ne \emptyset\}$. The resolvent $J_\lambda^A$ and the Yosida approximation $A_\lambda$ of $A$ are defined by

$$J_\lambda^A := (I + \lambda A)^{-1} \text{ and } A_\lambda := \frac{1}{\lambda}(I - J_\lambda^A) \quad \forall \lambda > 0 \tag{2.3}$$

respectively. Then $A_\lambda$ is monotone and Lipschitz continuous and we have

$$J_\lambda^A x \to x \text{ as } \lambda \to 0 \text{ for all } x \in H,$$
$$A_\lambda(x) \in A(J_\lambda^A x) \text{ for all } x \in H.$$

Recall that the graph of any maximal monotone operator is demiclosed, i.e., is closed in $H \times H_w$, where $H_w$ denotes $H$ endowed with the weak topology.

Let $\Phi(H)$ be the set of all lower semicontinuous convex functions $\varphi : H \to (-\infty, +\infty]$ such that its effective domain $D(\varphi)$ defined by

$$D(\varphi) := \{x \in H; \varphi(x) < \infty\} \tag{2.4}$$

is nonempty. For each $\varphi \in \Phi(H)$, the subdifferential operator $\partial\varphi$ of $\varphi$ is defined by

$$\partial\varphi(x) := \{g \in H; \varphi(w) - \varphi(x) \ge (g, w - x)_H \quad \forall w \in H\}. \tag{2.5}$$

Then $\partial\varphi$ becomes a (possibly multivalued) maximal monotone operator with domain

$$D(\partial\varphi) := \{x \in H; \partial\varphi(x) \ne \emptyset\} \subset D(\varphi). \tag{2.6}$$

The Moreau–Yosida regularization $\varphi_\lambda$ of $\varphi \in \Phi(H)$ is given by

$$\varphi_\lambda(x) := \inf\{\varphi(y) + \frac{1}{2\lambda}||x-y||_H^2; y \in H\}. \tag{2.7}$$

Then $\varphi_\lambda$ becomes a convex Fréchet differentiable, so $\varphi_\lambda \in \Phi(H)$ and it holds that

$$\begin{aligned}
\varphi_\lambda(x) &= \frac{1}{2\lambda}||x - J_\lambda^{\partial\varphi}x||_H^2 + \varphi(J_\lambda^{\partial\varphi}x) \\
&= \frac{\lambda}{2}||(\partial\varphi)_\lambda(x)||_H^2 + \varphi(J_\lambda^{\partial\varphi}x) \\
&\leq \varphi(x), \\
\partial(\varphi_\lambda) &= (\partial\varphi)_\lambda.
\end{aligned}$$

Now we introduce realizations $\hat{F}(\cdot)$ and $\bar{F}(\cdot)$ of $\hat{f}$ in $L^2(\Omega)$ and $\mathcal{H} = L^2(0,T;L^2(\Omega))$ respectively by

$$\hat{F}(u) := \begin{cases} \displaystyle\int_\Omega \hat{f}(u(x))dx & \text{if } \hat{f}(u(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise}, \end{cases} \tag{2.8}$$

$$\bar{F}(u) := \begin{cases} \displaystyle\int_0^T \hat{F}(u(s))ds & \text{if } \hat{F}(u(\cdot)) \in L^1(0,T;L^2(\Omega)), \\ +\infty & \text{otherwise}. \end{cases} \tag{2.9}$$

Then $\hat{F} \in \Phi(L^2(\Omega))$ and $\bar{F} \in \Phi(\mathcal{H})$, furthermore we have

$$\begin{gathered}
\partial\hat{F}(u)(x) = (\partial\hat{f})(u(x)) = f(u(x)) \text{ a.e. } x \in \Omega, \\
\partial\bar{F}(u)(t,x) = (\partial\hat{f})(u(t,x)) = f(u(t,x)) \text{ a.e. } (t,x) \in (0,T)\times\Omega, \\
\hat{F}_\lambda(u) = \int_\Omega \hat{f}_\lambda(u(x))dx, \\
\partial\hat{F}_\lambda(u)(x) = (\partial\hat{f})_\lambda(u(x)) = f_\lambda(u(x)) \text{ a.e. } x \in \Omega, \\
\bar{F}_\lambda(u) = \int_0^T \hat{F}_\lambda(u(s))ds, \\
\partial\bar{F}_\lambda(u)(t,x) = (\partial\hat{f})_\lambda(u(t,x)) = f_\lambda(u(t,x)) \text{ a.e. } (t,x) \in (0,T)\times\Omega.
\end{gathered}$$

For the case where $g$ is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$, the above properties hold true with $\hat{F}$ and $\bar{F}$ replaced by $\hat{G}$ and $\bar{G}$ defined by (2.8) and (2.9) with $\hat{f}$ and $\hat{F}$ replaced by $\hat{g}$ and $\hat{G}$ respectively. If $f$ and $g$ satisfy (A$\varphi$-I), then by (2.1) we easily get

$$0 \leq \hat{G}_\mu(u) \leq \hat{G}(u) \leq k\hat{F}(u) + K|u|_{L^1(\Omega)} \quad \forall u \in D(\hat{F})\ \forall\mu > 0, \tag{2.10}$$

$$0 \leq (g_\mu(u), u)_2 \leq (g_0(u), u)_2 \leq k(f_0(u), u)_2 + K|u|_{L^1(\Omega)} \quad \forall u \in D(\partial\hat{F})\ \forall\mu > 0, \tag{2.11}$$

where $g_0(u)$ and $f_0(u)$ are arbitrary sections of $g(u)$ and $f(u)$ respectively.

In this paper, we are concerned with solutions of (P) in the following sense.

**Definition 2.1.** We say $u \in C([0,T]; L^2(\Omega))$ is a solution of (P) if and only if there exist sections $f_0(t,x) \in f(u(t,x))$ and $g_0(t,x) \in g(u(t,x))$ such that

$$\begin{cases} \gamma\partial_t u(t,x) = \Delta v(t,x) & \text{a.e. } (t,x) \in (0,T) \times \Omega, \\ v(t,x) = f_0(t,x) - g_0(t,x) - \alpha\Delta u(t,x) + \beta\partial_t u(t,x) - h(t,x) & \text{a.e. } (t,x) \in (0,T) \times \Omega, \\ u(t,x)|_{t=0} = u_0(x) & \text{a.e. } x \in \Omega. \end{cases} \tag{2.12}$$

Here for the case where $g(\cdot)$ is single valued such as in (A$\varphi$-II), $g_0(t,x)$ coincides with $g(u(t,x))$.

Then our first main result is stated as follows.

**Theorem 2.2.** (Existence) *Assume that* (A$\varphi$-I) *or* (A$\varphi$-II) *is satisfied. Let* $u_0 \in H_0^1(\Omega) \cap D(\hat{F})$ *and* $h \in L^2(0,T; L^2(\Omega))$*, then there exists a solution* $u$ *of* (P) *satisfying*

$$\begin{aligned} &u \in C([0,T]; H_0^1(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)), \\ &f_0, g_0 \in L^2(0,T; L^2(\Omega)), \\ &v \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)), \end{aligned} \tag{2.13}$$

*where* $f_0$*,* $g_0$ *and* $v$ *are functions appearing in (2.12).*

Furthermore we can derive a result of smoothing effect, more precisely, (P) admits a solution when $u_0$ belongs to $H_F := \overline{D(\hat{F})}^{L^2(\Omega)}$, the closure of $D(\hat{F})$ in $L^2(\Omega)$.

**Theorem 2.3.** (Smoothing effect) *Assume that* (A$\varphi$-I) *or* (A$\varphi$-II) *is satisfied. Furthermore assume that there exist possitive constants* $C_0$ *and* $\delta \in (0,1)$ *such that*

$$|||f(r)||| \leq C_0(\hat{f}(r)^{1-\delta} + 1) \quad r \in D(\hat{f}). \tag{A$f$}$$

*Let* $u_0 \in H_F$ *and* $h \in L^2(0,T; L^2(\Omega))$*, then there exists a solution of* (P) *satisfying*

$$\begin{aligned} &u \in C([0,T]; L^2(\Omega)), \\ &\sqrt{t}\Delta u(t), \sqrt{t}\partial_t u(t), \sqrt{t} f_0(u(t)), \sqrt{t} g_0(u(t)) \in L^2(0,T; L^2(\Omega)), \\ &\sqrt{t} v(t) \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)). \end{aligned} \tag{2.14}$$

*where* $f_0$*,* $g_0$ *and* $v$ *are functions appearing in (2.12).*

Here we remark that $H_F = L^2(\Omega)$ if $D(f) = \mathbb{R}$.

The uniqueness of the solution for the problem (P) holds when the perturbation term $g$ is globally Lipschitz continuous function. The uniqueness of the solution is discussed in appendix A.

We give some remarks on conditions imposed on the nonlinear terms.

**Remark 2.4.** Specific examples for the maximal monotone functions $f(u)$ satisfying (A$\varphi$-I)-(i) or (A$\varphi$-II)-(i) are given by the followings:

- $f(u) = C|u|^{p-2}u$, $C \geq 0$, $2 < p < \infty$, which satisfies (A$f$);
- $f(u) = e^u - 1$, which does not satisfy (A$f$);

- Heaviside function, that is,

$$f(u) = \begin{cases} 0 & \text{if } u < 0, \\ [0,1] & \text{if } u = 0, \\ 1 & \text{if } u > 0, \end{cases} \tag{2.15}$$

which satisfies (A$f$);

- $f(u) = \ln(\frac{1+u}{1-u})$, which is the subdifferential of a logarithmic potential $\hat{f}(u)$.

$$\hat{f}(u) = \begin{cases} (1+u)\ln(1+u) + (1-u)\ln(1-u) & \text{if } u \in (-1,1), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.16}$$

It does not satisfy (A$f$) and $D(f) = (-1,1) \subsetneq \mathbb{R}$;

- $f(u) = \partial I_{[-1,1]}(u)$ which is the subdifferential of the indicator function $I_{[-1,1]}(u)$ given by

$$I_{[-1,1]}(u) = \begin{cases} 0 & \text{if } u \in [-1,1], \\ +\infty & \text{otherwise.} \end{cases} \tag{2.17}$$

It does not satisfy (A$f$) and $D(f) = [-1,1] \subsetneq \mathbb{R}$.

**Remark 2.5.** Specific examples of the nonlinear terms $\varphi(u)$ satisfying (A$\varphi$-I) or (A$\varphi$-II) are given by the following:

- $\varphi(u) = C_1|u|^{p-2}u - C_2|u|^{q-2}u, \quad C_1, C_2 \geq 0,\ 2 < q < p < \infty,$
- $\varphi(u) = (e^u - 1) - \frac{1}{2}ue^{\frac{1}{2}u},$
- $\varphi(u) = \ln(\frac{1+u}{1-u}) - C_2|u|^{q-2}u, \quad C_2 \geq 0,\ 2 \leq q < \infty.$
- $\varphi(u) = |u|^{p-2}u(\varepsilon + \sin u)$, $p > 2$, $\varepsilon > 0$. For this case, we can take $f(u) = (1+\varepsilon)|u|^{p-2}u$ and $g(u) = |u|^{p-2}u(\sin u - 1)$.

# 3 Proof of Theorem 2.2

First we assume (A$\varphi$-I) and prepare some Lemmas for the $L^2(\Omega)$-inner product between $u$ and $v$; and $f_0(u) \in f(u)$ and $-\Delta u$.

**Lemma 3.1.** ($L^2$-inner product between $u$ and $v$) *Let $t \in (0,T]$ and assume that $u(t) \in L^2(\Omega)$ and $v(t) \in H_0^1(\Omega)$ satisfy the equation $\gamma\partial_t u(t) = \Delta v(t)$ in $H^{-1}(\Omega)$. Then we obtain*

$$(v(t), u(t))_2 = -\frac{\gamma}{2}\frac{d}{dt}|u(t)|_{H^{-1}}^2. \tag{3.1}$$

*Proof.* Noting that $v(t) \in H_0^1(\Omega)$ and $u(t) \in L^2(\Omega) \subset H^{-1}(\Omega)$, we have

$$(v(t), u(t))_2 = {}_{H_0^1}\langle v(t), u(t) \rangle_{H^{-1}}. \tag{3.2}$$

Recall that $-\Delta$ is a bijection and the duality mapping from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. If $\Lambda$ denotes the inverse of $-\Delta$, $v(t) = -\gamma\Lambda\partial_t u(t)$ holds. Then from the definition of the duality mapping, we get

$$\begin{aligned} {}_{H_0^1} < v(t), u(t) >_{H^{-1}} &= -\gamma_{H_0^1} < \Lambda\partial_t u(t), u(t) >_{H^{-1}} \\ &= -\gamma(\partial_t u(t), u(t))_{H^{-1}} \\ &= -\frac{\gamma}{2}\frac{d}{dt}|u(t)|_{H^{-1}}^2. \end{aligned} \tag{3.3}$$

□

**Lemma 3.2.** ($L^2$-inner product between $f(u)$ and $-\Delta u$) *If $f$ satisfies* (A$\varphi$-I)-(i) *or* (A$\varphi$-II)-(i)*, the following relation holds for any $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f_0(u) \in \partial\hat{F}(u)$.*

$$(f_0(u), -\Delta u)_2 \geq 0. \tag{3.4}$$

*Proof.* From Proposition 2.17 and Theorem 4.4 in [2], $-\Delta + \partial\hat{F}$ becomes a maximal monotone operator and if we define

$$\hat{\phi}(u) := \begin{cases} \frac{1}{2}|\nabla u|_{(L^2(\Omega))^N}^2 + \hat{F}(u) & \text{if } u \in H_0^1(\Omega) \cap D(\hat{F}), \\ +\infty & \text{otherwise}, \end{cases} \tag{3.5}$$

then $\hat{\phi} \in \Phi(L^2(\Omega))$ and $-\Delta+\partial\hat{F} = \partial\hat{\phi}$. For any $f_0 := f_0(u) \in \partial\hat{F}(u)$, put $z := -\Delta u + f_0 \in \partial\hat{\phi}(u)$. Then we get

$$|z|_2^2 = |-\Delta u|_2^2 + |f_0|_2^2 + 2(-\Delta u, f_0)_2. \tag{3.6}$$

Since $u \mapsto -\Delta u + \partial\hat{F}_\lambda(u)$ is maximal monotone in $L^2(\Omega)$, there exists $u_\lambda \in D(-\Delta)$ satisfying the following equation:

$$u_\lambda - \Delta u_\lambda + \partial\hat{F}_\lambda(u_\lambda) = u + z. \tag{3.7}$$

Here $\hat{F}_\lambda$ is the Yosida regularization of $\hat{F}$ with $\lambda > 0$. Multiplying (3.7) by $u_\lambda$ and using the Schwarz inequality and the Young inequality, we obtain

$$\begin{aligned} |u_\lambda|_2^2 + |\nabla u_\lambda|_2^2 + (\partial\hat{F}_\lambda(u_\lambda), u_\lambda)_2 &\leq |u_\lambda|_2|u+z|_2 \\ &\leq |u|_2^2 + |z|_2^2 + \frac{1}{2}|u_\lambda|_2^2, \end{aligned} \tag{3.8}$$

which gives

$$\frac{1}{2}|u_\lambda|_2^2 + |\nabla u_\lambda|_2^2 + (\partial\hat{F}_\lambda(u_\lambda), u_\lambda)_2 \leq |u|_2^2 + |z|_2^2. \tag{3.9}$$

From (3.7), we also obtain

$$\begin{aligned} |u+z-u_\lambda|_2^2 &= (-\Delta u_\lambda + \partial\hat{F}_\lambda(u_\lambda), -\Delta u_\lambda + \partial\hat{F}_\lambda(u_\lambda))_2 \\ &= |-\Delta u_\lambda|_2^2 + |\partial\hat{F}_\lambda(u_\lambda)|_2^2 + 2(-\Delta u_\lambda, \partial\hat{F}_\lambda(u_\lambda))_2 \\ &= |-\Delta u_\lambda|_2^2 + |\partial\hat{F}_\lambda(u_\lambda)|_2^2 + 2\int_\Omega (\partial\hat{F}_\lambda)'(u_\lambda)|\nabla u_\lambda|^2 dx. \end{aligned} \tag{3.10}$$

Since $\partial\hat{F}_\lambda$ is monotone and Lipschitz continuous, the last term of the right-hand side of (3.10) is non-negative. Hence, using Schwarz's and Young's inequalities and (3.9), we obtain

$$|-\Delta u_\lambda|_2^2 + |\partial\hat{F}_\lambda(u_\lambda)|_2^2 \le |u+z-u_\lambda|_2^2 \le 8(|u|_2^2+|z|_2^2). \tag{3.11}$$

Next taking the difference between (3.7) with $\lambda > 0$ and (3.7) with $\lambda = \mu > 0$, and multiplying $u_\lambda - u_\mu$, we have

$$\begin{aligned} 0 &= ((u_\lambda - u_\mu) - \Delta(u_\lambda - u_\mu) + (\partial\hat{F}_\lambda(u_\lambda) - \partial\hat{F}_\mu(u_\mu)), u_\lambda - u_\mu)_2 \\ &= |u_\lambda - u_\mu|_2^2 + |\nabla(u_\lambda - u_\mu)|_2^2 + (\partial\hat{F}_\lambda(u_\lambda) - \partial\hat{F}_\mu(u_\mu), u_\lambda - u_\mu)_2. \end{aligned} \tag{3.12}$$

Using Kōmura's trick, we get

$$\begin{aligned} |u_\lambda - u_\mu|_2^2 + |\nabla(u_\lambda - u_\mu)|_2^2 &= -(\partial\hat{F}_\lambda(u_\lambda) - \partial\hat{F}_\mu(u_\mu), u_\lambda - u_\mu)_2 \\ &\le 2(\lambda+\mu)(|\partial\hat{F}_\lambda(u_\lambda)|_2^2 + |\partial\hat{F}_\mu(u_\mu)|_2^2). \end{aligned} \tag{3.13}$$

Therefore by (3.11), $(u_\lambda)_\lambda$ forms a Cauchy sequence in $H_0^1(\Omega)$ and there exists a subsequence $(u_{\lambda_k})_k$ satisfying

$$\begin{gathered} u_{\lambda_k} \to \tilde{u} \text{ strongly in } H_0^1(\Omega), \\ \Delta u_{\lambda_k} \rightharpoonup \Delta\tilde{u} \text{ weakly in } L^2(\Omega), \\ \partial\hat{F}_{\lambda_k}(u_{\lambda_k}) \rightharpoonup \tilde{f} \in \partial\hat{F}(\tilde{u}) \text{ weakly in } L^2(\Omega), \end{gathered}$$

as $\lambda_k \to 0$. From (3.7), we derive

$$\begin{aligned} \tilde{u} - \Delta\tilde{u} + \tilde{f} &= u + z \\ &= u - \Delta u + f_0. \end{aligned} \tag{3.14}$$

Then multiplying the difference between the left-hand side and the right-hand side of (3.14) by $u - \tilde{u}$, we have

$$|u-\tilde{u}|_2^2 + |\nabla(u-\tilde{u})|_2^2 + (f_0 - \tilde{f}, u - \tilde{u})_2 = 0, \tag{3.15}$$

whence follows $u = \tilde{u}$ and (3.14) implies $f_0 = \tilde{f}$. Hence, letting $\lambda = \lambda_k \to 0$ in (3.11), we obtain

$$|-\Delta u|_2^2 + |f_0|_2^2 \le |z|_2^2. \tag{3.16}$$

Then (3.6) and (3.16) imply (3.4). □

## 3.1 Approximation of the problem

In this subsection, we introduce the following approximate problems $(\mathrm{P})_{\lambda,\mu}$ for (P).

$$\begin{cases} \gamma\partial_t u(t) = \Delta v(t), & (*)_{\lambda,\mu} \\ v(t) = f_\lambda(u(t)) - g_\mu(u(t)) - \alpha\Delta u(t) + \beta\partial_t u(t) - h(t), & (**)_{\lambda,\mu} \\ u|_{t=0} = u_0, \end{cases} \tag{P$_{\lambda,\mu}$}$$

where $f_\lambda(\cdot)$ and $g_\mu(\cdot)$ are Yosida approximations of $f(\cdot)$ and $g(\cdot)$ with $\lambda$, $\mu > 0$ respectively. In order to show the solvability of $(\mathrm{P})_{\lambda,\mu}$, we reduce $(\mathrm{P})_{\lambda,\mu}$ to an abstract evolution equation in $H = L^2(\Omega)$. To this end, we put

$$\psi(u) = \begin{cases} \dfrac{\alpha}{2\gamma}\displaystyle\int_\Omega \left(|A_{\beta'}u|^2 + \beta'|\nabla A_{\beta'}u|^2\right) dx & \text{if } u \in D(\psi) = H_0^1(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega)\backslash H_0^1(\Omega), \end{cases} \tag{3.17}$$

Here $\beta' = \beta/\gamma$ and $A_{\beta'}$ denotes the Yosida approximation of $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, i.e., $A_{\beta'} = (I - J_{\beta'}^A)/\beta' = A(J_{\beta'}^A)$, $J_{\beta'}^A = (I + \beta' A)^{-1}$. Then we easily find that

$$\begin{aligned} \psi(u+\phi) - \psi(u) &= \frac{\alpha}{\gamma}\{(A_{\beta'}u, A_{\beta'}\phi)_2 + \beta'(AA_{\beta'}u, A_{\beta'}\phi)_2\} + \mathcal{O}(||\phi||_{H^1}^2) \\ &= \frac{\alpha}{\gamma}\left\{(A_{\beta'}u, A_{\beta'}\phi)_2 + (AA_{\beta'}u, \phi)_2 - (AA_{\beta'}u, J_{\beta'}^A\phi)_2\right\} + \mathcal{O}(||\phi||_{H^1}^2) \\ &= \frac{\alpha}{\gamma}(AA_{\beta'}u, \phi)_2 + \mathcal{O}(||\phi||_{H^1}^2) \quad \forall \phi \in H_0^1(\Omega). \end{aligned} \tag{3.18}$$

Hence we conclude that

$$\partial\psi(u) = \frac{\alpha}{\gamma}AA_{\beta'}u = \frac{\alpha}{\gamma}A_{\beta'}Au \quad \forall u \in D(\partial\psi) = D(A). \tag{3.19}$$

On the other hand, by $(*)_{\lambda,\mu}$ and $(**)_{\lambda,\mu}$, we get

$$\begin{aligned} \gamma\partial_t u &= -A(f_\lambda(u) - g_\mu(u) + \alpha Au + \beta\partial_t u - h), \\ \gamma\left(I + \frac{\beta}{\gamma}A\right)\partial_t u &= -A(f_\lambda(u) - g_\mu(u) + \alpha Au - h), \end{aligned}$$

Hence we have

$$\begin{aligned} \partial_t u &= \frac{1}{\gamma}J_{\beta'}^A\left[-A(f_\lambda(u) - g_\mu(u)) - \alpha A^2 u + Ah\right] \\ &= -\frac{\alpha}{\gamma}A_{\beta'}Au - \frac{1}{\gamma}A_{\beta'}(f_\lambda(u) - g_\mu(u)) + \frac{1}{\gamma}A_{\beta'}h. \end{aligned} \tag{3.20}$$

Here we put

$$B(u) = \frac{1}{\gamma}A_{\beta'}(\partial\hat{F}_\lambda(u) - \partial\hat{G}_\mu(u)) \text{ and } \mathcal{H}(t) = \frac{1}{\gamma}A_{\beta'}h(t). \tag{3.21}$$

Then $(\mathrm{P})_{\lambda,\mu}$ is reduced to the following abstract evolution equation (E) in $H = L^2(\Omega)$.

$$\begin{cases} \dfrac{d}{dt}u(t) + \partial\psi(u(t)) + B(u(t)) = \mathcal{H}(t), \\ u(0) = u_0. \end{cases} \tag{E}$$

Therefore, since $B$ is Lipschitz continuous from $L^2(\Omega)$ into $L^2(\Omega)$ and $\mathcal{H}(t) \in L^2(0,T;L^2(\Omega))$ for any $h \in L^2(0,T;L^2(\Omega))$, the standard result (see, e.g., Brézis [2]) assures that for any $u_0 \in D(\psi) = H_0^1(\Omega)$, (E) admits a unique global solution $u(t) \in C([0,T];H_0^1(\Omega))$ satisfying

$$\frac{d}{dt}u(t), \partial\psi(u(t)) \in L^2(0,T;L^2(\Omega)). \tag{3.22}$$

## 3.2 A priori estimates independent of the parameter $\lambda$

For the time being, we fix $\mu > 0$ and denote $u$ and $v$ in $(*)_{\lambda,\mu}$ and $(**)_{\lambda,\mu}$ by $u_\lambda$ and $v_\lambda$ respectively. In what follows, $C$ denotes a general constant independent of $\lambda$.

**Lemma 3.3.** *There exists a constant $C$ independent of $\lambda$ such that*

$$|u_\lambda(t)|_2^2 + \int_0^t |g_\mu(u_\lambda(s))|_2^2 ds \le C$$

*for any $t \in (0, T]$.*

*Proof.* Multiplying $(**)_{\lambda,\mu}$ by $u_\lambda(s)$ $(s \in (0, T))$, we get

$$\begin{aligned}(v_\lambda(s), u_\lambda(s))_2 = (f_\lambda(u_\lambda(s)), u_\lambda(s))_2 - (g_\mu(u_\lambda(s)), u_\lambda(s))_2 \\ + \alpha|\nabla u_\lambda(s)|_2^2 + \frac{\beta}{2}\frac{d}{ds}|u_\lambda(s)|_2^2 - (h(s), u_\lambda(s))_2.\end{aligned} \tag{3.23}$$

From Lemma 3.1, it follows that

$$\begin{aligned}(f_\lambda(u_\lambda(s)), u_\lambda(s))_2 + \alpha|\nabla u_\lambda(s)|_2^2 + \frac{\beta}{2}\frac{d}{ds}|u_\lambda(s)|_2^2 + \frac{\gamma}{2}\frac{d}{ds}|u_\lambda(s)|_{H^{-1}}^2 \\ = (g_\mu(u_\lambda(s)), u_\lambda(s))_2 + (h(s), u_\lambda(s))_2.\end{aligned} \tag{3.24}$$

Since $g_\mu$ is Lipschitz continuous with Lipschitz constant $L_\mu$, from Schwarz's and Young's inequalities, we have

$$\begin{aligned}(f_\lambda(u_\lambda(s)), u_\lambda(s))_2 + \alpha|\nabla u_\lambda(s)|_2^2 + \frac{\beta}{2}\frac{d}{ds}|u_\lambda(s)|_2^2 + \frac{\gamma}{2}\frac{d}{ds}|u_\lambda(s)|_{H^{-1}}^2 \\ \le \frac{1}{2}|h(s)|_2^2 + \left(L_\mu + \frac{1}{2}\right)|u_\lambda(s)|_2^2.\end{aligned} \tag{3.25}$$

Then integrating this with respect to $s$ over $(0, t)$ $(t \in (0, T])$, we obtain

$$\begin{aligned}&\int_0^t (f_\lambda(u_\lambda(s)), u_\lambda(s))_2 ds + \alpha \int_0^t |\nabla u_\lambda(s)|_2^2 ds + \frac{\beta}{2}|u_\lambda(t)|_2^2 + \frac{\gamma}{2}|u_\lambda(t)|_{H^{-1}}^2 \\ &\le \frac{\beta}{2}|u_0|_2^2 + \frac{\gamma}{2}|u_0|_{H^{-1}}^2 + \frac{1}{2}\int_0^t |h(s)|_2^2 ds + \left(L_\mu + \frac{1}{2}\right)\int_0^t |u_\lambda(s)|_2^2 ds \\ &\le C + \left(L_\mu + \frac{1}{2}\right)\int_0^t |u_\lambda(s)|_2^2 ds.\end{aligned} \tag{3.26}$$

From Gronwall's lemma, we obtain the following estimates:

$$\int_0^t (f_\lambda(u_\lambda(s)), u_\lambda(s))_2 ds + \alpha \int_0^t |\nabla u_\lambda(s)|_2^2 ds + \frac{\beta}{2}|u_\lambda(t)|_2^2 + \frac{\gamma}{2}|u_\lambda(t)|_{H^{-1}}^2 \le C. \tag{3.27}$$

Therefore we also get

$$\begin{aligned}\int_0^t |g_\mu(u_\lambda(s))|_2^2 ds &\le L_\mu^2 \int_0^t |u_\lambda(s)|_2^2 ds \\ &\le C.\end{aligned} \tag{3.28}$$

$\square$

**Lemma 3.4.** *There exists a constant $C$ independent of $\lambda$ such that*

$$|\nabla u_\lambda(t)|_2^2 + \int_0^t |\partial_s u_\lambda(s)|_2^2 ds + \int_0^t |\nabla v_\lambda(s)|_2^2 ds + \int_0^t |v_\lambda(s)|_2^2 ds + \int_0^t |\Delta v_\lambda(s)|_2^2 ds \le C.$$

*for any $t \in (0, T]$.*

*Proof.* Multiplying $(**)_{\lambda,\mu}$ by $\partial_s u_\lambda(s)$ $(s \in (0,T))$, we get

$$\begin{aligned}(v_\lambda(s), \partial_s u_\lambda(s))_2 &= (f_\lambda(u_\lambda(s)), \partial_s u_\lambda(s))_2 - (g_\mu(u_\lambda(s)), \partial_s u_\lambda(s))_2 \\ &\quad + \frac{\alpha}{2}\frac{d}{ds}|\nabla u_\lambda(s)|_2^2 + \beta|\partial_s u_\lambda(s)|_2^2 - (h(s), \partial_s u_\lambda(s))_2 \\ &= \frac{d}{ds}\hat{F}_\lambda(u_\lambda(s)) - (g_\mu(u_\lambda(s)), \partial_s u_\lambda(s))_2 \\ &\quad + \frac{\alpha}{2}\frac{d}{ds}|\nabla u_\lambda(s)|_2^2 + \beta|\partial_s u_\lambda(s)|_2^2 - (h(s), \partial_s u_\lambda(s))_2. \end{aligned} \tag{3.29}$$

From $(*)_{\lambda,\mu}$, we have

$$\begin{aligned}(v_\lambda(s), \partial_s u_\lambda(s))_2 &= (v_\lambda(s), \frac{1}{\gamma}\Delta v_\lambda(s))_2 \\ &= -\frac{1}{\gamma}|\nabla v_\lambda(s)|_2^2.\end{aligned}$$

Then we obtain

$$\begin{aligned}\frac{d}{ds}\hat{F}_\lambda(u_\lambda(s)) + \frac{\alpha}{2}\frac{d}{ds}|\nabla u_\lambda(s)|_2^2 + \beta|\partial_s u_\lambda(s)|_2^2 + \frac{1}{\gamma}|\nabla v_\lambda(s)|_2^2 \\ = (g_\mu(u_\lambda(s)), \partial_s u_\lambda(s))_2 + (h(s), \partial_s u_\lambda(s))_2.\end{aligned} \tag{3.30}$$

Integrating (3.30) on $[0,t]$ $(t \in (0,T])$, from Schwarz's and Young's inequalities, we get

$$\begin{aligned}&\hat{F}_\lambda(u_\lambda(t)) + \frac{\alpha}{2}|\nabla u_\lambda(t)|_2^2 + \beta\int_0^t |\partial_s u_\lambda(s)|_2^2 ds + \frac{1}{\gamma}\int_0^t |\nabla v_\lambda(s)|_2^2 ds \\ &= \hat{F}_\lambda(u_0) + \frac{\alpha}{2}|\nabla u_0|_2^2 + \int_0^t (g_\mu(u_\lambda(s)), \partial_s u_\lambda(s))_2 ds + \int_0^t (h(s), \partial_s u_\lambda(s))_2 ds \\ &\le \hat{F}(u_0) + \frac{\alpha}{2}|\nabla u_0|_2^2 + \frac{1}{\beta}\int_0^t |g_\mu(u_\lambda(s))|_2^2 ds + \frac{1}{\beta}\int_0^t |h(s)|_2^2 ds + \frac{\beta}{2}\int_0^t |\partial_s u_\lambda(s)|_2^2 ds.\end{aligned} \tag{3.31}$$

Since we already obtain the estimate of $g_\mu(u_\lambda(\cdot))$ in Lemma 3.3, we get

$$\hat{F}_\lambda(u_\lambda(t)) + \frac{\alpha}{2}|\nabla u_\lambda(t)|_2^2 + \frac{\beta}{2}\int_0^t |\partial_s u_\lambda(s)|_2^2 ds + \frac{1}{\gamma}\int_0^t |\nabla v_\lambda(s)|_2^2 ds \le C. \tag{3.32}$$

Hence, we derive the following a priori estimates from the Poincaré inequality and $(*)_{\lambda,\mu}$:

$$\begin{aligned}\int_0^t |v_\lambda(s)|_2^2 ds &\le C\int_0^t |\nabla v_\lambda(s)|_2^2 ds \\ &\le C,\end{aligned} \tag{3.33}$$

$$\int_0^t |\Delta v_\lambda(s)|_2^2 ds \leq \gamma^2 \int_0^t |\partial_s u_\lambda(s)|_2^2 ds$$
$$\leq C. \tag{3.34}$$

□

**Lemma 3.5.** *There exists a constant $C$ independent of $\lambda$ such that*

$$\int_0^t |f_\lambda(u_\lambda(s))|_2^2 ds + \int_0^t |\Delta u_\lambda(s)|_2^2 ds \leq C$$

*for any $t \in (0, T]$.*

*Proof.* From $(**)_{\lambda,\mu}$, we have $f_\lambda(u_\lambda(s)) - \alpha\Delta u_\lambda(s) = v_\lambda(s) + g_\mu(u_\lambda(s)) - \beta\partial_s u_\lambda(s) - h(s)$ for $s \in (0, T)$. By virtue of Lemmas 3.3 and 3.4, the right-hand side of the above equation is bounded in $L^2(0, t; L^2(\Omega))$. Hence we get

$$\begin{aligned}
&\int_0^t |f_\lambda(u_\lambda(s)) - \alpha\Delta u_\lambda(s)|_2^2 ds \\
&= \int_0^t |f_\lambda(u_\lambda(s))|_2^2 ds - 2\alpha \int_0^t (f_\lambda(u_\lambda(s)), \Delta u_\lambda(s))_2 ds + \alpha^2 \int_0^t |\Delta u_\lambda(s)|_2^2 ds \\
&\leq C.
\end{aligned} \tag{3.35}$$

Here noting that

$$\int_0^t (f_\lambda(u_\lambda(s)), \Delta u_\lambda(s))_2 ds = -\int_0^t \int_\Omega f'_\lambda(u_\lambda(s)) |\nabla u_\lambda(s)|^2 dx ds \leq 0, \tag{3.36}$$

we obtain

$$\int_0^t |f_\lambda(u_\lambda(s))|_2^2 ds + \alpha^2 \int_0^t |\Delta u_\lambda(s)|_2^2 ds \leq C. \tag{3.37}$$

□

## 3.3 Passage to the limit as $\lambda \to 0$

In this subsection, by letting $\lambda \to 0$ in $(\mathrm{P})_{\lambda,\mu}$, we show that (P) with $g(\cdot)$ replaced by $g_\mu(\cdot)$ admits a solution $u_\mu$ in the following sense.

**Lemma 3.6.** *For $u_0 \in H_0^1(\Omega) \cap D(\hat{F})$ and $h \in L^2(0, T; L^2(\Omega))$, there exists $u_\mu$ satisfying the following:*

1. *There is a section $f_{0,\mu}(u_\mu) \in \partial\bar{F}(u_\mu)$ and $u_\mu$ satisfying the following regularities:*

$$u_\mu \in C([0, T]; L^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)),$$

$$f_{0,\mu}(u_\mu), g_\mu(u_\mu) \in L^2(0, T; L^2(\Omega)),$$
$$v_\mu \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)).$$

*2. $u_\mu$ satisfies the equations in the following sense:*

$$\begin{cases} \gamma\partial_t u_\mu = \Delta v_\mu & \text{in } L^2(0,T;L^2(\Omega)), \quad (*)_\mu \\ v_\mu = f_{0,\mu}(u_\mu) - g_\mu(u_\mu) - \alpha\Delta u_\mu + \beta\partial_t u_\mu - h & \text{in } L^2(0,T;L^2(\Omega)). \quad (**)_\mu \end{cases}$$

*Proof.* Since $|u_\lambda(t)|_{H_0^1(\Omega)} \leq C$ for any $t \in (0,T]$ by Lemma 3.4, and the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the sequence $(u_\lambda(t))_\lambda$ is relatively compact in $L^2(\Omega)$. Moreover, since we have

$$\begin{aligned} |u_\lambda(t) - u_\lambda(s)|_2 &\leq \left| \int_s^t |\partial_\tau u_\lambda(\tau)|_2 d\tau \right| \\ &\leq |t-s|^{\frac{1}{2}} \left( \int_0^T |\partial_\tau u_\lambda(\tau)|_2^2 \, d\tau \right)^{\frac{1}{2}} \end{aligned} \tag{3.38}$$

for any $t, s \in [0,T]$ and Lemma 3.4 assures that $\partial_\tau u_\lambda(\tau)$ is bounded in $L^2(0,T;L^2(\Omega))$, the sequence $(u_\lambda(t))_\lambda$ is equicontinuous in $L^2(\Omega)$. Hence by Ascoli's theorem, there exists a subsequence $(u_{\lambda_k})_k$ which strongly converges to $u_\mu$ in $C([0,T];L^2(\Omega))$, that is,

$$u_{\lambda_k} \to u_\mu \text{ strongly in } C([0,T];L^2(\Omega)) \text{ as } k \to \infty \ (\lambda_k \to 0). \tag{3.39}$$

Let $J_\lambda^{\partial\bar{F}} = (I + \lambda\partial\bar{F})^{-1}$ be the resolvent of $\partial\bar{F}$, then we have

$$\begin{aligned} |J_{\lambda_k}^{\partial\bar{F}} u_{\lambda_k} - u_\mu|_{L^2(0,T;L^2(\Omega))} &\leq |J_{\lambda_k}^{\partial\bar{F}} u_{\lambda_k} - u_{\lambda_k}|_{L^2(0,T;L^2(\Omega))} + |u_{\lambda_k} - u_\mu|_{L^2(0,T;L^2(\Omega))} \\ &\leq \lambda_k |f_{\lambda_k}(u_{\lambda_k})|_{L^2(0,T;L^2(\Omega))} + |u_{\lambda_k} - u_\mu|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

Then by Lemma 3.5, we get

$$J_{\lambda_k}^{\partial\bar{F}} u_{\lambda_k} \to u_\mu \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ as } k \to \infty, \tag{3.40}$$

$$f_{\lambda_k}(u_{\lambda_k}) \rightharpoonup f_0 \text{ weakly in } L^2(0,T;L^2(\Omega)) \text{ as } k \to \infty. \tag{3.41}$$

Since $f_{\lambda_k}(u_{\lambda_k}) \in \partial\bar{F}(J_{\lambda_k}^{\partial\bar{F}} u_{\lambda_k})$, the demiclosedness of $\partial\bar{F}$ assures

$$f_0 =: f_{0,\mu}(u_\mu) \in \partial\bar{F}(u_\mu). \tag{3.42}$$

Since $\partial_t$ and $\Delta$ are weakly closed, from a priori estimates of $\partial_t u_\lambda$, $\Delta u_\lambda$, $v_\lambda$ and $\Delta v_\lambda$ in Lemma 3.4 and 3.5, we easily see

$$\Delta u_{\lambda_k} \rightharpoonup \Delta u_\mu \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{3.43}$$

$$\partial_t u_{\lambda_k} \rightharpoonup \partial_t u_\mu \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{3.44}$$

$$v_{\lambda_k} \rightharpoonup v_\mu \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{3.45}$$

$$\Delta v_{\lambda_k} \rightharpoonup \Delta v_\mu \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{3.46}$$

as $k \to \infty$. Since $g_\mu$ is Lipschitz continuous, from a priori estimate of $g_\mu(u_\lambda)$ in Lemma 3.5, we get

$$g_\mu(u_{\lambda_k}) \to g_\mu(u_\mu) \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ as } k \to \infty. \tag{3.47}$$

Thus it is shown that $(u_\mu, v_\mu)$ satisfies $(*)_\mu$ and $(**)_\mu$. $\square$

Since $\int_0^T |\nabla u_\mu(t)|_2^2 dt$, $\int_0^T |\Delta u_\mu(t)|_2^2 dt$ and $\int_0^T |\partial_t u_\mu(t)|_2^2 dt$ are bounded because of the lower semicontinuity of norm with respect to the weak convergence, we get

$$\begin{aligned}\int_0^T \left| \frac{d}{dt} |\nabla u_\mu(t)|_2^2 \right| dt &= 2 \int_0^T |(\partial_t u_\mu(t), \Delta u_\mu(t))_2|\, dt \\ &\le 2 \int_0^T |\partial_t u_\mu(t)|_2^2 dt \int_0^T |\Delta u_\mu(t)|_2^2 dt \\ &\le C. \end{aligned} \tag{3.48}$$

Hence $|\nabla u_\mu(t)|_2^2$ is absolutely continuous on $[0,T]$ and we get

$$|\nabla u_\mu(0)|_2^2 = |\nabla u_0|_2^2. \tag{3.49}$$

Also, since $\int_0^T \hat{F}(u_\mu(t)) dt$, $\int_0^T |f_{0,\mu}(u_\mu(t))|_2^2 dt$ and $\int_0^T |\partial_t u_\mu(t)|_2^2 dt$ are bounded,

$$\begin{aligned}\int_0^T \left| \frac{d}{dt} \hat{F}(u_\mu(t)) \right| dt &= \int_0^T |(\partial_t u_\mu(t), f_{0,\mu}(u_\mu(t)))_2|\, dt \\ &\le C. \end{aligned} \tag{3.50}$$

That is, $\hat{F}(u_\mu(t))$ is absolutely continuous on $[0,T]$. Then we obtain

$$\hat{F}(u_\mu(0)) = \hat{F}(u_0). \tag{3.51}$$

## 3.4 A priori estimates independent of the parameter $\mu$

In this subsection, we establish some a priori estimates for $u_\mu$ and $v_\mu$ independent of $\mu$. In what follows, $C'$ denotes a general constant independent of $\mu$.

**Lemma 3.7.** *There exists a constant $C'$ independent of $\mu$ such that*

$$|u_\mu(t)|_2^2 \le C'$$

*for any $t \in (0,T]$.*

*Proof.* Multiplying $(**)_\mu$ by $u_\mu(s)$, we get by the same argument in the proof of Lemma 3.3 (see (3.24)),

$$\begin{aligned}&(f_{0,\mu}(u_\mu(s)), u_\mu(s))_2 - (g_\mu(u_\mu(s)), u_\mu(s))_2 \\ &\quad + \alpha |\nabla u_\mu(s)|_2^2 + \frac{\beta}{2} \frac{d}{ds} |u_\mu(s)|_2^2 + \frac{\gamma}{2} \frac{d}{ds} |u_\mu(s)|_{H^{-1}}^2 = (h(s), u_\mu(s))_2. \end{aligned} \tag{3.52}$$

From (2.11), Schwarz's and Young's inequalities, we get

$$\begin{aligned}&(1-k)(f_{0,\mu}(u_\mu(s)), u_\mu(s))_2 + \alpha |\nabla u_\mu(s)|_2^2 + \frac{\beta}{2} \frac{d}{ds} |u_\mu(s)|_2^2 + \frac{\gamma}{2} \frac{d}{ds} |u_\mu(s)|_{H^{-1}}^2 \\ &\le \frac{1}{2} |h(s)|_2^2 + \frac{1}{2} |u_\mu(s)|_2^2 + K |u_\mu(s)|_{L^1(\Omega)} \\ &\le \frac{1}{2} |h(s)|_2^2 + \frac{K^2 |\Omega|}{2} + |u_\mu(s)|_2^2. \end{aligned} \tag{3.53}$$

Then integrating this over $s \in [0,t]$ $(t \in (0,T])$, we have

$$\begin{aligned}(1-k)\int_0^t (f_{0,\mu}(u_\mu(s)),u_\mu(s))_2 ds + \alpha\int_0^t |\nabla u_\mu(s)|_2^2 ds + \frac{\beta}{2}|u_\mu(t)|_2^2 + \frac{\gamma}{2}|u_\mu(t)|_{H^{-1}}^2 \\ \leq \frac{\beta}{2}|u_0|_2^2 + \frac{\gamma}{2}|u_0|_{H^{-1}}^2 + \frac{1}{2}\int_0^T |h(s)|_2^2 ds + \frac{K^2|\Omega|T}{2} + \int_0^t |u_\mu(s)|_2^2 ds \\ \leq C' + \int_0^t |u_\mu(s)|_2^2 ds. \end{aligned} \tag{3.54}$$

From Gronwall's lemma, we obtain

$$\begin{aligned}(1-k)\int_0^t (f_{0,\mu}(u_\mu(s)),u_\mu(s))_2 ds + \alpha\int_0^t |\nabla u_\mu(s)|_2^2 ds \\ + \frac{\beta}{2}|u_\mu(t)|_2^2 + \frac{\gamma}{2}|u_\mu(t)|_{H^{-1}}^2 \leq C'. \end{aligned} \tag{3.55}$$

□

**Lemma 3.8.** *There exists a constant $C'$ independent of $\mu$ such that*

$$|\nabla u_\mu(t)|_2^2 + \int_0^t |\partial_s u_\mu(s)|_2^2 ds + \int_0^t |\nabla v_\mu(s)|_2^2 ds + \int_0^t |v_\mu(s)|_2^2 ds + \int_0^t |\Delta v_\mu(s)|_2^2 ds \leq C'.$$

*for any $t \in (0,T]$.*

*Proof.* Multiplying $(**)_\mu$ by $\partial_s u_\mu(s)$ $(s \in (0,T))$, we get (see (3.29))

$$\begin{aligned}(v_\mu(s),\partial_s u_\mu(s))_2 = \frac{d}{ds}\hat{F}(u_\mu(s)) - \frac{d}{ds}\hat{G}_\mu(u_\mu(s)) + \frac{\alpha}{2}\frac{d}{ds}|\nabla u_\mu(s)|_2^2 \\ + \beta|\partial_s u_\mu(s)|_2^2 - (h(s),\partial_s u_\mu(s))_2. \end{aligned} \tag{3.56}$$

From $(*)_\mu$, we obtain

$$\begin{aligned}\frac{d}{ds}\hat{F}(u_\mu(s)) - \frac{d}{ds}\hat{G}_\mu(u_\mu(s)) + \frac{\alpha}{2}\frac{d}{ds}|\nabla u_\mu(s)|_2^2 + \beta|\partial_s u_\mu(s)|_2^2 + \frac{1}{\gamma}|\nabla v_\mu(s)|_2^2 \\ = (h(s),\partial_s u_\mu(s))_2. \end{aligned} \tag{3.57}$$

Then integrating this on $(0,t)$ $(t \in (0,T])$, using Schwarz's and Young's inequalities, we get

$$\begin{aligned}\hat{F}(u_\mu(t)) - \hat{G}_\mu(u_\mu(t)) + \frac{\alpha}{2}|\nabla u_\mu(t)|_2^2 + \beta\int_0^t |\partial_s u_\mu(s)|_2^2 ds + \frac{1}{\gamma}\int_0^t |\nabla v_\mu(s)|_2^2 ds \\ \leq \hat{F}(u_0) - \hat{G}_\mu(u_0) + \frac{\alpha}{2}|\nabla u_0|_2^2 + \frac{1}{2\beta}\int_0^t |h(s)|_2^2 ds + \frac{\beta}{2}\int_0^t |\partial_s u_\mu(s)|_2^2 ds. \end{aligned} \tag{3.58}$$

From (2.10) and $\hat{G}_\mu(u_\mu(t)) \geq 0$, we have

$$\begin{aligned}(1-k)\hat{F}(u_\mu(t)) + \frac{\alpha}{2}|\nabla u_\mu(t)|_2^2 + \frac{\beta}{2}\int_0^t |\partial_s u_\mu(s)|_2^2 ds + \frac{1}{\gamma}\int_0^t |\nabla v_\mu(s)|_2^2 ds \\ \leq \hat{F}(u_0) + \frac{\alpha}{2}|\nabla u_0|_2^2 + \frac{1}{2\beta}\int_0^t |h(s)|_2^2 ds + K|u_\mu(t)|_{L^1(\Omega)}. \end{aligned} \tag{3.59}$$

Noting that we can also obtain the $L^1$-estimate of $u_\mu(t)$ from Lemma 3.7, we get

$$(1-k)\hat{F}(u_\mu(t)) + \frac{\alpha}{2}|\nabla u_\mu(t)|_2^2 + \frac{\beta}{2}\int_0^t |\partial_s u_\mu(s)|_2^2 ds + \frac{1}{\gamma}\int_0^t |\nabla v_\mu(s)|_2^2 ds \le C'. \tag{3.60}$$

Hence from the Poincaré inequality and $(*)_\mu$, we also obtain the following estimates:

$$\int_0^t |v_\mu(s)|_2^2 ds \le C', \tag{3.61}$$

$$\int_0^t |\Delta v_\mu(s)|_2^2 ds \le C'. \tag{3.62}$$

□

**Lemma 3.9.** *There exists a constant $C'$ independent of $\mu$ such that*

$$\int_0^t |f_{0,\mu}(u_\mu(s))|_2^2 ds + \int_0^t |g_\mu(u_\mu(s))|_2^2 ds + \int_0^t |\Delta u_\mu(s)|_2^2 ds \le C'.$$

*for any $t \in (0,T]$.*

*Proof.* Let $\bar{k} := \frac{1-k}{2}$. From $(**)_\mu$, we have

$$\bar{k} f_{0,\mu}(u_\mu(s)) + (1-\bar{k}) f_{0,\mu}(u_\mu(s)) - g_\mu(u_\mu(s)) - \alpha\Delta u_\mu(s) = v_\mu(s) - \beta\partial_s u_\mu(s) + h(s)$$

for $s \in (0,T)$. Since Lemma 3.8 assures the $L^2(0,t;L^2(\Omega))$-boundedness of the right-hand side of this identity, the left-hand side of this identity is bounded in $L^2(0,t;L^2(\Omega))$, i.e., we have

$$\begin{aligned}
&\bar{k}^2 \int_0^t |f_{0,\mu}(u_\mu(s))|_2^2 ds \\
&\quad + 2\bar{k}\int_0^t (f_{0,\mu}(u_\mu(s)), (1-\bar{k}) f_{0,\mu}(u_\mu(s)) - g_\mu(u_\mu(s)) - \alpha\Delta u_\mu(s))_2 ds \\
&\quad + \int_0^t |(1-\bar{k}) f_{0,\mu}(u_\mu(s)) - g_\mu(u_\mu(s)) - \alpha\Delta u_\mu(s)|_2^2 ds \le C'.
\end{aligned} \tag{3.63}$$

In order to estimate the cross-term, we apply Lemma 3.2 to get

$$\begin{aligned}
&(f_{0,\mu}(u_\mu), (1-\bar{k}) f_{0,\mu}(u_\mu) - g_\mu(u_\mu) - \alpha\Delta u_\mu(s))_2 \\
&\qquad \ge (1-\bar{k})|f_{0,\mu}(u_\mu)|_2^2 - (f_{0,\mu}(u_\mu), g_\mu(u_\mu))_2.
\end{aligned} \tag{3.64}$$

Applying condition (A$\varphi$-I)-(iii) to the second term of the right-hand side, we have

$$\begin{aligned}
-(f_{0,\mu}(u_\mu), g_\mu(u_\mu))_2 &\ge -\int_\Omega |f_{0,\mu}(u_\mu)||g_\mu(u_\mu)| dx \\
&\ge -\int_\Omega k|f_{0,\mu}(u_\mu)|^2 + K|f_{0,\mu}(u_\mu)| dx \\
&\ge -k|f_{0,\mu}(u_\mu)|_2^2 - \frac{1-k-\bar{k}}{2}|f_{0,\mu}(u_\mu)|_2^2 - \frac{K^2|\Omega|}{2(1-k-\bar{k})}.
\end{aligned} \tag{3.65}$$

Then we obtain

$$
\begin{aligned}
&(f_{0,\mu}(u_\mu), (1-\bar{k})f_{0,\mu}(u_\mu) - g_\mu(u_\mu) - \alpha\Delta u_\mu(s))_2 \\
&\geq \frac{1-k-\bar{k}}{2}|f_{0,\mu}(u_\mu(s))|_2^2 - \frac{K^2|\Omega|}{2(1-k-\bar{k})} \\
&= \frac{1-k}{4}|f_{0,\mu}(u_\mu(s))|_2^2 - \frac{K^2|\Omega|}{1-k} \qquad (3.66)
\end{aligned}
$$

Hence (3.63) with $\bar{k} = \frac{1-k}{2}$ gives

$$
\frac{(1-k)^2}{2}\int_0^t |f_{0,\mu}(u_\mu(s))|_2^2 ds \leq C'. \qquad (3.67)
$$

Furthermore by (A$\varphi$-I)-(iii), we get

$$
\begin{aligned}
\int_0^t |g_\mu(u_\mu(s))|_2^2 ds &\leq C' \int_0^t |f_{0,\mu}(u_\mu(s))|_2^2 ds + C' \\
&\leq C'. \qquad (3.68)
\end{aligned}
$$

Thus in view of Lemma 3.8, (3.67) and (3.68), we can derive the following estimate for $\Delta u_\mu$ from $(**)_\mu$:

$$
\alpha^2 \int_0^t |\Delta u_\mu(s)|_2^2 ds \leq C'. \qquad (3.69)
$$

□

## 3.5 Passage to the limit as $\mu \to 0$

In this subsection, we discuss the convergence of $(u_\mu)_\mu$ as $\mu \to 0$. In subsection 3.4, we obtained the same estimates as those given in subsection 3.2. So by virtue of these a priori estimates, we can extract a subsequence $\{\mu_k\}$ of $\{\mu\}$ such that

$$
u_{\mu_k} \to u \text{ strongly in } C([0,T]; L^2(\Omega)), \qquad (3.70)
$$

$$
\Delta u_{\mu_k} \rightharpoonup \Delta u \text{ weakly in } L^2(0,T; L^2(\Omega)), \qquad (3.71)
$$

$$
\partial_t u_{\mu_k} \rightharpoonup \partial_t u \text{ weakly in } L^2(0,T; L^2(\Omega)), \qquad (3.72)
$$

$$
\partial \bar{F}(u_{\mu_k}) \ni f_{0,\mu_k}(u_{\mu_k}) \rightharpoonup f_0(u) \in \partial \bar{F}(u) \text{ weakly in } L^2(0,T; L^2(\Omega)), \qquad (3.73)
$$

$$
\partial \bar{G}(J_{\mu_k}^{\partial \bar{G}} u_{\mu_k}) \ni g_{\mu_k}(u_{\mu_k}) \rightharpoonup g_0(u) \in \partial \bar{G}(u) \text{ weakly in } L^2(0,T; L^2(\Omega)), \qquad (3.74)
$$

$$
v_{\mu_k} \rightharpoonup v \text{ weakly in } L^2(0,T; L^2(\Omega)), \qquad (3.75)
$$

$$
\Delta v_{\mu_k} \rightharpoonup \Delta v \text{ weakly in } L^2(0,T; L^2(\Omega)), \qquad (3.76)
$$

as $k \to \infty$ ($\mu_k \to 0$), where $v = f_0(u) - g_0(u) - \alpha\Delta u + \beta\partial_t u - h$.

Therefore we can easily see that the limit $u$ gives a solution of (P), i.e., $u$ satisfies the following:

1. There exist $f_0(u) \in f(u)$ and $g_0(u) \in g(u)$, and $u$ satisfying the following regularities:

$$
\begin{aligned}
&u \in C([0,T]; L^2(\Omega)) \cap H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega));\\
&f_0(u), g_0(u) \in L^2(0,T; L^2(\Omega));\\
&v \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)).
\end{aligned}
\tag{3.77}
$$

2. $u$ satisfies the equations in the following sense:

$$
\begin{cases}
\gamma \partial_t u = \Delta v, & \text{in } L^2(0,T; L^2(\Omega)), \quad (*)\\
v = f_0(u) - g_0(u) - \alpha \Delta u + \beta \partial_t u - h, & \text{in } L^2(0,T; L^2(\Omega)), \quad (**)\\
u|_{t=0} = u_0, & \text{in } L^2(\Omega).
\end{cases}
\tag{3.78}
$$

Since it is shown that $\hat{F}(u(t))$ and $|\nabla u(t)|_2^2$ are absolutely continuous on $[0,T]$ by the same argument in subsection 3.3, the followings hold:

$$
|\nabla u(0)|_2^2 = |\nabla u_0|_2^2, \tag{3.79}
$$

$$
\hat{F}(u(0)) = \hat{F}(u_0). \tag{3.80}
$$

Therefore $u \in C([0,T]; H_0^1(\Omega))$ and the initial condition holds in $H_0^1(\Omega) \cap D(\hat{F})$.

## 3.6 In the case of (A$\varphi$-II)

*Proof.* Here we give a proof of Theorem 2.2, when we assume (A$\varphi$-II), We prove Theorem 2.2 by almost the same arguments as for the case (A$\varphi$-I). However we need a couple of modifications. More precisely, $\hat{g}(u)$ introduced for the case of (A$\varphi$-I) should be replaced by the primitive function of $g$, i.e., $\hat{g}(u) := \int_0^u g(z)dz$, and instead of Yosida approximation $g_\mu$ of $g$, we introduce the following cut-off function $g_\mu$ defined by

$$
g_\mu(u) = \begin{cases}
g(\mu^{-1}) & \text{if } u > \mu^{-1},\\
g(u) & \text{if } |u| \le \mu^{-1},\\
g(-\mu^{-1}) & \text{if } u < -\mu^{-1}.
\end{cases}
\tag{3.81}
$$

Since $g$ is assumed to be locally Lipschitz continuous, $g_\mu$ becomes a globally Lipschitz continuous function with Lipschitz constant $L_\mu$. We again define $\hat{G}(u) = \int_\Omega \int_0^u g(z)dzdx$ and $\hat{G}_\mu(u) = \int_\Omega \int_0^u g_\mu(z)dzdx$, then relations (2.10) and (2.11) also hold.

we can repeat exactly the same arguments as before except the verification for the convergence of $g_\mu(u_\mu)$ to $g(u)$. In order to discuss the convergence of $g_\mu(u_\mu)$, we introduce the following cut-off function $\chi_\mu$:

$$
\chi_\mu(u) := \begin{cases}
\mu^{-1} & \text{if } u > \mu^{-1},\\
u & \text{if } |u| \le \mu^{-1},\\
-\mu^{-1} & \text{if } u < -\mu^{-1}.
\end{cases}
\tag{3.82}
$$

Then, since by (3.70) $u_{\mu_k}(t,x) \to u(t,x)$ a.e. $(t,x) \in (0,T) \times \Omega$, we can easily get

$$
\chi_{\mu_k}(u_{\mu_k}(t,x)) \to u(t,x) \text{ a.e. } (t,x) \in (0,T) \times \Omega \text{ as } k \to \infty \ (\mu_k \to 0). \tag{3.83}
$$

Here Egorov's theorem assures that, for any $\varepsilon > 0$, there exists a closed set $A_\varepsilon \subset (0,T) \times \Omega =: Q$ such that $|Q/A_\varepsilon| < \varepsilon$ and $\chi_{\mu_k}(u_{\mu_k}(t,x))$ converges to $u(t,x)$ uniformly in $A_\varepsilon$. From a priori estimate for $\int_0^T |g_\mu(u_\mu(t))|_2^2 dt$ in Lemma 3.9, $g_\mu(u_\mu) = g(\chi_\mu(u_\mu))$ converges weakly to some function $\bar{g}$ in $L^2(0,T;L^2(\Omega))$. Then for any test function $\rho \in C_0^\infty(Q)$, we obtain

$$\begin{aligned}
&\int_0^T (g(u) - \bar{g}, \rho)_2 dt \\
&\quad = \int_Q \{g(u(t,x)) - g(\chi_{\mu_k}(u_{\mu_k}(t,x)))\}\rho dxdt \\
&\qquad + \int_Q \{g(\chi_{\mu_k}(u_{\mu_k}(t,x))) - \bar{g}\}\rho dxdt \\
&\quad = \int_{Q/A_\varepsilon} \{g(u(t,x)) - g(\chi_{\mu_k}(u_{\mu_k}(t,x)))\}\rho dxdt \\
&\qquad + \int_{A_\varepsilon} \{g(u(t,x)) - g(\chi_{\mu_k}(u_{\mu_k}(t,x)))\}\rho dxdt \\
&\qquad + \int_Q \{g_{\mu_k}(u_{\mu_k}(t,x)) - \bar{g}\}\rho dxdt. 
\end{aligned} \tag{3.84}$$

Considering the first term of the right-hand side of (3.84), since $g_\mu(u_\mu) = g(\chi_\mu(u_\mu))$ and $g(u)$ are bounded in $L^2(Q)$, we obtain

$$\begin{aligned}
&\int_{Q/A_\varepsilon} \{g(u(t,x)) - g(\chi_{\mu_k}(u_{\mu_k}(t,x)))\}\rho dxdt \\
&\quad \le \int_{Q/A_\varepsilon} \{|g(u(t,x))| + |g(\chi_{\mu_k}(u_{\mu_k}(t,x)))|\}\rho dxdt \\
&\quad \le |\rho|_{L^\infty(Q)} \int_{Q/A_\varepsilon} |g(u(t,x))| + |g(\chi_{\mu_k}(u_{\mu_k}(t,x)))| dxdt \\
&\quad \le |\rho|_{L^\infty(Q)} \left\{ \int_0^T |g(u)|_2^2 dt + \int_0^T |g(\chi_{\mu_k}(u_{\mu_k}))|_2^2 dt \right\}^{\frac{1}{2}} |Q/A_\varepsilon|^{\frac{1}{2}} \\
&\quad \le C' \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{3.85}$$

Since $|\chi_{\mu_k}(u_{\mu_k}(t,x))| \le C$ and $|u(t,x)| \le C$ on $A_\varepsilon$ and $g(\cdot)$ is assumed to be locally Lipschitz continuous, there exists a constant $L_C$ such that

$$\begin{aligned}
&\int_{A_\varepsilon} \{g(u(t,x)) - g(\chi_{\mu_k}(u_{\mu_k}(t,x)))\}\rho dxdt \\
&\quad \le L_C \int_{A_\varepsilon} |u(t,x) - \chi_{\mu_k}(u_{\mu_k}(t,x))||\rho| dxdt \\
&\quad \to 0 \text{ as } k \to \infty.
\end{aligned} \tag{3.86}$$

From the weak convergence of $g_{\mu_k}(u_{\mu_k}) = g(\chi_{\mu_k}(u_{\mu_k}))$, the third term of the right-hand side of (3.84) converges to 0. Thus we conclude $\bar{g} = g(u)$ and

$$g_{\mu_k}(u_{\mu_k}) = g(\chi_{\mu_k}(u_{\mu_k})) \rightharpoonup g(u) \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{3.87}$$

which completes the proof. $\square$

# 4 Proof of Theorem 2.3

For any $u_0 \in H_F = \overline{D(\hat{F})}^{L^2(\Omega)}$, we take an approximate sequence $\{u_{0n}\} \subset H_0^1(\Omega) \cap D(\hat{F})$ satisfying $u_{0n} \to u_0$ strongly in $L^2(\Omega)$ and $|u_{0n}|_2 \le 2|u_0|_2$. Let $u_n$ be the solutions of (P) given in Theorem 2.2 for the initial data $u_{0n}$, i.e., there exist sections $f_{0,n}(u_n) \in \partial\bar{F}(u_n)$, $g_{0,n}(u_n) \in \partial\bar{G}(u_n)$, and $u_n$ satisfies the following equations:

$$\begin{cases} \gamma\partial_t u_n = \Delta v_n, & (*)_n \\ v_n = f_{0,n}(u_n) - g_{0,n}(u_n) - \alpha\Delta u_n + \beta\partial_t u_n - h, & (**)_n \\ u_n|_{t=0} = u_{0n}. \end{cases}$$

Then we discuss below the convergence of $u_n$.

## 4.1 A priori estimates independent of $n$

In this subsection, we establish the following a priori estimates for $u_n$ independent of $n$. In what follows, $C''$ denotes a general constant independent of the approximation parameter $n$.

**Lemma 4.1.** *There exists a constant $C''$ independent of $n$ such that*

$$|u_n(t)|_2^2 + \int_0^t |\nabla u_n(s)|_2^2 ds + \int_0^t (f_{0,n}(u_n(s)), u_n(s))_2 ds + \int_0^t \hat{F}(u_n(s)) ds \le C'',$$

*for any $t \in [0, T]$.*

*Proof.* In parallel with (3.55), we now get

$$\frac{\gamma}{2}|u_n(t)|_{H^{-1}}^2 + \frac{\beta}{2}|u_n(t)|_2^2 + \alpha\int_0^t |\nabla u_n(s)|_2^2 ds + (1-k)\int_0^t (f_{0,n}(u_n(s)), u_n(s))_2 ds \le C''. \quad (4.1)$$

Since $\hat{F}(0) = 0$ and $f_{0,n}(u_n) \in \partial\hat{F}(u_n)$, the definition of subdifferential yields

$$\begin{aligned} \int_0^t \hat{F}(u_n(s)) ds &\le \int_0^t (f_{0,n}(u_n(s)), u_n(s))_2 ds \\ &\le C''. \end{aligned} \quad (4.2)$$

□

**Lemma 4.2.** *There exists a constant $C''$ independent of $n$ such that*

$$t|\nabla u_n(t)|_2^2 + \int_0^t s|\partial_s u_n(s)|_2^2 ds + \int_0^t s|\nabla v_n(s)|_2^2 ds + \int_0^t s|v_n(s)|_2^2 ds + \int_0^t s|\Delta v_n(s)|_2^2 ds \le C'',$$

*for any $t \in [0, T]$.*

*Proof.* Multiplying $(**)_n$ by $s\partial_s u_n(s)$ for $s \in (0, T]$, we have

$$\begin{aligned} &s\frac{d}{ds}\hat{F}(u_n(s)) - s\frac{d}{ds}\hat{G}(u_n(s)) + \frac{\alpha s}{2}\frac{d}{ds}|\nabla u_n(s)|_2^2 \\ &\quad + \beta s|\partial_s u_n(s)|_2^2 + \frac{s}{\gamma}|\nabla v_n(s)|_2^2 = s(h(s), \partial_s u_n(s))_2. \end{aligned} \quad (4.3)$$

Then integrating this on $[0,t]$, using Schwarz's and Young's inequalities, we get

$$\begin{aligned}
&\int_0^t s\frac{d}{ds}\hat{F}(u_n(s))ds - \int_0^t s\frac{d}{ds}\hat{G}(u_n(s))ds + \frac{\alpha}{2}\int_0^t s\frac{d}{ds}|\nabla u_n(s)|_2^2 ds \\
&\quad + \beta\int_0^t s|\partial_s u_n(s)|_2^2 ds + \frac{1}{\gamma}\int_0^t s|\nabla v_n(s)|_2^2 ds \\
&= \int_0^t s(h(s), \partial_s u_n(s))_2 ds \\
&\leq \frac{T}{2\beta}\int_0^t |h(s)|_2^2 ds + \frac{\beta}{2}\int_0^t s|\partial_s u_n(s)|_2^2 ds. \qquad (4.4)
\end{aligned}$$

Now we consider the first three terms of the left-hand side. By virtue of Lemma 4.1 and (2.10), we have

$$\begin{aligned}
&\int_0^t s\frac{d}{ds}\hat{F}(u_n(s))ds - \int_0^t s\frac{d}{ds}\hat{G}(u_n(s))ds + \frac{\alpha}{2}\int_0^t s\frac{d}{ds}|\nabla u_n(s)|_2^2 ds \\
&= \int_0^t \frac{d}{ds}\left[s\hat{F}(u_n(s)) - s\hat{G}(u_n(s)) + \frac{\alpha s}{2}|\nabla u_n(s)|_2^2\right] ds \\
&\quad - \int_0^t \left[\hat{F}(u_n(s)) - \hat{G}(u_n(s)) + \frac{\alpha}{2}|\nabla u_n(s)|_2^2 ds\right] ds \\
&\geq (1-k)t\hat{F}(u_n(t)) - Kt|u_n(t)|_{L^1(\Omega)} + \frac{\alpha t}{2}|\nabla u_n(t)|_2^2 \\
&\quad - \int_0^t \hat{F}(u_n(s))ds - \frac{\alpha}{2}\int_0^t |\nabla u_n(s)|_2^2 ds \\
&\geq (1-k)t\hat{F}(u_n(t)) + \frac{\alpha t}{2}|\nabla u_n(t)|_2^2 - C''. \qquad (4.5)
\end{aligned}$$

Then by (4.4), we obtain

$$(1-k)t\hat{F}(u_n(t)) + \frac{\alpha t}{2}|\nabla u_n(t)|_2^2 + \frac{\beta}{2}\int_0^t s|\partial_s u_n(s)|_2^2 ds + \frac{1}{\gamma}\int_0^t s|\nabla v_n(s)|_2^2 ds \leq C''. \quad (4.6)$$

Hence Poincaré's inequality and $(*)_n$ yield

$$\int_0^t s|v_n(s)|_2^2 ds \leq C'', \qquad (4.7)$$

$$\int_0^t s|\Delta v_n(s)|_2^2 ds \leq C''. \qquad (4.8)$$

□

**Lemma 4.3.** *There exists a constant $C''$ independent of $n$ such that*

$$\int_0^t s|f_{0,n}(u_n(s))|_2^2 ds + \int_0^t s|g_{0,n}(u_n(s))|_2^2 ds + \int_0^t s|\Delta u_n(s)|_2^2 ds \leq C'',$$

*for any $t \in [0,T]$*

*Proof.* This estimate follows from much the same arguments as in the proof of Lemma 3.9. □

## 4.2 Passage to the limit as $n \to \infty$

We now discuss the convergence of $u_n$. We first consider the convergence in $(0, T]$. Take any $\varepsilon \in (0, T]$, then Lemma 4.2 assures that $u_n$ is bounded in $C([\varepsilon, T]; H_0^1(\Omega))$ and $W^{1,2}([\varepsilon, T]; L^2(\Omega))$. Hence by Ascoli's theorem, there exists a subsequence $(u_{n_k})_k$ converging to $u$ strongly in $C([\varepsilon, T]; L^2(\Omega))$, i.e.,

$$u_{n_k} \to u \text{ strongly in } C([\varepsilon, T]; L^2(\Omega)) \text{ as } k \to \infty \ (n_k \to \infty). \tag{4.9}$$

Therefore $u \in C((0, T]; L^2(\Omega))$. By virtue of estimates in Lemmas 4.1, 4.2 and 4.3, there exists a subsequence of $(u_{n_k})_k$ again denoted by $(u_{n_k})_k$ such that

$$\sqrt{t}\partial_t u_{n_k}(t) \rightharpoonup \sqrt{t}\partial_t u(t) \text{ weakly in } L^2(0, T; L^2(\Omega)), \tag{4.10}$$

$$\sqrt{t}\Delta u_{n_k}(t) \rightharpoonup \sqrt{t}\Delta u(t) \text{ weakly in } L^2(0, T; L^2(\Omega)), \tag{4.11}$$

$$\sqrt{t} f_{0,n_k}(u_{n_k}(t)) \rightharpoonup \sqrt{t} f_0(u(t)) \text{ weakly in } L^2(0, T; L^2(\Omega)), \tag{4.12}$$

$$\sqrt{t} g_{0,n_k}(u_{n_k}(t)) \rightharpoonup \sqrt{t} g_0(u(t)) \text{ weakly in } L^2(0, T; L^2(\Omega)), \tag{4.13}$$

$$\sqrt{t} v_{n_k}(t) \rightharpoonup \sqrt{t} v(t) \text{ weakly in } L^2(0, T; L^2(\Omega)), \tag{4.14}$$

$$\sqrt{t}\Delta v_{n_k}(t) \rightharpoonup \sqrt{t}\Delta v(t) \text{ weakly in } L^2(0, T; L^2(\Omega)), \tag{4.15}$$

as $k \to \infty$, where $v = f_0(u) - g_0(u) - \alpha\Delta u + \beta\partial_t u - h$, $f_0(u) \in \partial\bar{F}(u)$, and $g_0(u) \in \partial\bar{G}(u)$. Hence the limit $u$ satisfies the following equations:

$$\begin{cases} \gamma\partial_t u(t, x) = \Delta v(t, x), & (*) \\ v(t, x) = f_0(u(t, x)) - g_0(u(t, x)) - \alpha\Delta u(t, x) + \beta\partial_t u(t, x) - h(t, x), & (**) \end{cases} \tag{4.16}$$

for a.e. $(t, x) \in (0, T) \times \Omega$.

To complete the proof, it suffices to check $u(t) \to u_0$ strongly in $L^2(\Omega)$ as $t \to 0$. We first check $u(t) \to u_0$ weakly in $L^2(\Omega)$ as $t \to 0$. To this end, we test $(**)_n$ by $\psi \in C_0^\infty(\Omega)$ to get

$$\begin{aligned} &\frac{d}{ds}\left\{\beta(u_n(s), \psi)_2 + \gamma(u_n(s), \psi)_{H^{-1}}\right\} \\ &\quad = (-f_{0,n}(u_n(s)) + g_{0,n}(u_n(s)), \psi)_2 + \alpha(\Delta u_n(s), \psi)_2 + (h(s), \psi)_2, \end{aligned} \tag{4.17}$$

for a.e. $s \in (0, T)$. Integrating (4.17) on $(0, t)$ with $t \in (0, T]$, we obtain

$$\begin{aligned} &\beta(u_n(t) - u_{0n}, \psi)_2 + \gamma(u_n(t) - u_{0n}, \psi)_{H^{-1}} \\ &\quad = \int_0^t (-f_{0,n}(u_n) + g_{0,n}(u_n), \psi)_2 ds + \alpha\int_0^t (\Delta u_n, \psi)_2 ds + \int_0^t (h, \psi)_2 ds. \end{aligned} \tag{4.18}$$

By (A$\varphi$-I)-(iii), we obtain

$$\begin{aligned} \left|\int_0^t (-f_{0,n}(u_n) + g_{0,n}(u_n), \psi)_2 ds\right| &\le \int_0^t \int_\Omega (|f_{0,n}(u_n)||\psi| + k|f^\circ(u_n)||\psi| + K|\psi|)\, dx ds \\ &\le (1 + k)\int_0^t \int_\Omega |f_{0,n}(u_n)||\psi| dx ds + K|\psi|_{L^1(\Omega)} t. \end{aligned} \tag{4.19}$$

Here by virtue of (A$f$) and Lemma 4.1, we have

$$\begin{aligned}\int_0^t\int_\Omega |f_{0,n}(u_n(s,x))|dxds &\le \int_0^t\int_\Omega C_0(\hat{f}(u_n(s,x))^{1-\delta}+1)dxds \\ &\le C_0\left(\int_0^t\int_\Omega \hat{f}(u_n(s,x))dxds\right)^{1-\delta}(|\Omega|t)^\delta + C_0|\Omega|t \\ &\le C_0C''|\Omega|^\delta t^\delta + C_0|\Omega|t. \qquad (4.20)\end{aligned}$$

As for the second term of the right-hand side of (4.18), we get by Lemma 4.1

$$\begin{aligned}\left|\alpha\int_0^t(\Delta u_n(s),\psi)_2 ds\right| &\le \alpha\left(\int_0^t |\nabla u_n(s)|_2^2 ds\right)^{\frac{1}{2}}|\nabla\psi|_2 t^{\frac{1}{2}} \\ &\le \alpha C''^{\frac{1}{2}}|\nabla\psi|_2 t^{\frac{1}{2}}. \qquad (4.21)\end{aligned}$$

Furthermore we have

$$\left|\int_0^t (h(s),\psi)_2 ds\right| \le |\psi|_2|h|_{L^2(0,T;L^2(\Omega))}t^{\frac{1}{2}}. \qquad (4.22)$$

Thus, in view of (4.19), (4.20), (4.21) and (4.22), letting $n\to\infty$ in (4.18), we find that there exists a constant $C_1$ depending on $k$, $C_0$, $C''$, $|\Omega|$, $\delta$, $\alpha$, $|\nabla\psi|_2$, $|\psi|_2$ and $|h|_{L^2(0,T;L^2(\Omega))}$ such that

$$\begin{aligned}|\beta(u(t)-u_0,\psi)_2+\gamma(u(t)-u_0,\psi)_{H^{-1}}| &= |(u(t)-u_0,(\beta+\gamma\Lambda)\psi)_2| \\ &\le C_1(t^\delta+t^{\frac{1}{2}}+t), \quad \forall\psi\in C_0^\infty(\Omega). \qquad (4.23)\end{aligned}$$

Hence $u(t)\to u_0$ in $\mathscr{D}'(\Omega)$ as $t\to\infty$. Since $|u(t)|_2$ is bounded, we easily find that $u(t)$ converges to $u_0$ weakly in $L^2(\Omega)$ and strongly in $H^{-1}(\Omega)$. Therefore we have $\liminf_{t\to 0}|u(t)|_2 \ge |u_0|_2$ and $\lim_{t\to\infty}|u(t)|_{H^{-1}} = |u_0|_{H^{-1}}$. In order to show the strong continuity of $u(t)$ in $L^2(\Omega)$ at $t=+0$, we have only to check $|u_0|_2 \ge \limsup_{t\to 0}|u(t)|_2$. Multiplying $(**)_n$ by $u_n(t)$, we get by Lemma 4.1

$$\begin{aligned}&\frac{d}{ds}\left\{\frac{\gamma}{2}|u_n(s)|^2_{H^{-1}(\Omega)}+\frac{\beta}{2}|u_n(s)|_2^2\right\} \\ &\quad = (-f_{0,n}(u_n(s))+g_{0,n}(u_n(s))+\alpha\Delta u_n(s)+h(s),u_n(s))_2 \\ &\quad \le -(1-k)(f_{0,n}(u_n(s)),u_n(s))_2-\alpha|\nabla u_n(s)|_2^2 \\ &\qquad + |h(s)|_2|u_n(s)|_2+K|u_n(s)|_{L^1(\Omega)} \\ &\quad \le \sqrt{C''}\left(|h(s)|_2+K|\Omega|^{\frac{1}{2}}\right). \qquad (4.24)\end{aligned}$$

Integrating this over $(0,t)$ and letting $n\to\infty$, we get

$$\begin{aligned}&\frac{\gamma}{2}|u(t)|^2_{H^{-1}(\Omega)}-\frac{\gamma}{2}|u_0|^2_{H^{-1}(\Omega)}+\frac{\beta}{2}|u(t)|_2^2-\frac{\beta}{2}|u_0|_2^2 \\ &\quad \le \sqrt{C''}\left(|h|_{L^2(0,T;L^2(\Omega))}t^{\frac{1}{2}}+K|\Omega|^{\frac{1}{2}}t\right). \qquad (4.25)\end{aligned}$$

Hence we obtain $\limsup_{t\to 0}|u(t)|_2^2 \le |u_0|_2^2$. □

# Appendix

## A. Uniqueness

In addition to (A$\varphi$-I)-(i) or (A$\varphi$-II)-(i), assume the following condition (A1), then the uniqueness of the solution for (P) holds true.

(A1) The perturbation term $g$ is a globally Lipschitz continuous in $\mathbb{R}$ with its Lipschitz constant $K_1$.

**Theorem A.1.** (Uniqueness) *Let $\alpha$, $\beta$ and $\gamma > 0$. Assume* (A$\varphi$-I)-(i)*, and* (A1)*. Let $u_0 \in L^2(\Omega)$ and $h \in L^2(0,T;L^2(\Omega))$, then* (P) *admits a unique solution satisfying (2.14).*

**Remark A.2.** *If we assume* (A$\varphi$-II) *and $D(f) \subset [a,b]$ $(-\infty < a < b < \infty)$, then the solution of (P) is unique.*

*Proof of Theorem A.1.* Let $u_i$ $(i = 1,2)$ be solutions of (P) for initial values $u_{0i} \in L^2(\Omega)$ and $h_i \in L^2(0,T;L^2(\Omega))$ $(i = 1,2)$ satisfying (2.14). That is, $u_i$ $(i = 1,2)$ satisfy the following equations:

$$\begin{cases} \gamma\partial_t u_i = \Delta v_i & \text{in } L^2(0,T;L^2(\Omega)), \qquad (*)_i \\ v_i = f_{0,i}(u_i) - g(u_i) - \alpha\Delta u_i + \beta\partial_t u_i - h_i & \text{in } L^2(0,T;L^2(\Omega)), \qquad (**)_i \\ u_i|_{t=0} = u_{0i}, & \text{in } L^2(\Omega). \end{cases}$$

Multiplying $(**)_1 - (**)_2$ by $w(s) := u_1(s) - u_2(s)$, we get, by Lemma 3.1,

$$\begin{aligned} &\frac{\gamma}{2}\frac{d}{ds}|w(s)|^2_{H^{-1}} + \frac{\beta}{2}\frac{d}{ds}|w(s)|_2^2 + \alpha|\nabla w(s)|_2^2 + (f_{0,1}(u_1(s)) - f_{0,2}(u_2(s)), w(s))_2 \\ &\quad = (g(u_1(s)) - g(u_2(s)), w(s))_2 + (h_1(s) - h_2(s), w(s))_2. \end{aligned} \tag{a.1}$$

Then by using (A1) and the monotonicity of $f$, we obtain

$$\frac{\gamma}{2}\frac{d}{ds}|w(s)|^2_{H^{-1}} + \frac{\beta}{2}\frac{d}{ds}|w(s)|_2^2 + \alpha|\nabla w(s)|_2^2 \le \left(K_1 + \frac{1}{2}\right)|w(s)|_2^2 + \frac{1}{2}|h_1(s) - h_2(s)|_2^2. \tag{a.2}$$

The integration of this over $[0,t]$ $(t \in (0,T])$ yields

$$\begin{aligned} &\frac{\gamma}{2}|w(t)|^2_{H^{-1}} + \frac{\beta}{2}|w(t)|_2^2 \\ &\le \frac{\gamma}{2}|w_0|^2_{H^{-1}} + \frac{\beta}{2}|w_0|_2^2 + \left(K_1 + \frac{1}{2}\right)\int_0^t |w(s)|_2^2 ds + \frac{1}{2}\int_0^t |h_1 - h_2|_2^2 ds, \end{aligned} \tag{a.3}$$

where $w_0 := u_{01} - u_{02}$. Hence by Gronwall's lemma, we have

$$\frac{\gamma}{2}|w(t)|^2_{H^{-1}} + \frac{\beta}{2}|w(t)|_2^2 \le \left\{\frac{\gamma}{2}|w_0|^2_{H^{-1}} + \frac{\beta}{2}|w_0|_2^2 + \frac{1}{2}\int_0^T |h_1(s) - h_2(s)|_2^2 ds\right\} e^{\frac{2}{\beta}(K_1 + \frac{1}{2})t}. \tag{a.4}$$

Then $u_{01} = u_{02}$ and $h_1 = h_2$ imply $u_1(t) = u_2(t)$ in $L^2(\Omega)$.

As for the existence of solution $u$ satisfying (2.14), we can repeat the proof of Theorem 2.3 up to (4.16). The verification for the fact that $u(t) \to u_0$ strongly in $L^2(\Omega)$ as $t \to 0$, can be done much easier than in the proof of Theorem 2.3. In fact, applying the above arguments with $u_1$ and $u_2$ replaced by $u_n$ and $u_m$, we obtain

$$\begin{aligned} &\frac{\gamma}{2}|u_n(t) - u_m(t)|^2_{H^{-1}} + \frac{\beta}{2}|u_n(t) - u_m(t)|^2_2 \\ &\quad \leq \left\{ \frac{\gamma}{2}|u_{0n} - u_{0m}|_{H^{-1}} + \frac{\beta}{2}|u_{0n} - u_{0m}|^2_2 \right\} e^{\frac{2}{\beta}(K_1 + \frac{1}{2})t} \quad \forall t \in [0, T]. \end{aligned} \tag{a.5}$$

Hence $\{u_n(t)\}_n$ forms a Cauchy sequence and we find that $\sup_{t\in[0,T]} |u_n(t) - u(t)|_2 \to 0$ as $n \to \infty$. Therefore for any $\eta > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} |u_n(t) - u(t)|_2 + |u_{0n} - u_0|_2 < \eta \quad \forall n \geq N. \tag{a.6}$$

Hence we obtain

$$\begin{aligned} |u(t) - u_0|_2 &\leq |u(t) - u_N(t)|_2 + |u_N(t) - u_{0N}|_2 + |u_{0N} - u_0|_2 \\ &\leq \eta + |u_N(t) - u_{0N}|_2. \end{aligned} \tag{a.7}$$

Then letting $t \to 0$ in (a.7), we get

$$\limsup_{t\to 0} |u(t) - u_0|_2 \leq \eta, \tag{a.8}$$

whence follows $\lim_{t\to 0} |u(t) - u_0|_2 = 0$. □

# B. Neumann boundary value problem

We can prove the existence of the solutions to the following Neumann boundary value problem by arguments similar to those for the Dirichlet boundary value problem.

$$\begin{cases} \gamma\partial_t u = \Delta v, & (t, x) \in (0, T) \times \Omega, \quad (*) \\ v = f_0(u) - g_0(u) - \alpha\Delta u + \beta\partial_t u - h, & (t, x) \in (0, T) \times \Omega, \quad (**) \\ f_0(u) \in f(u), g_0(u) \in g(u), & (t, x) \in (0, T) \times \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & (t, x) \in (0, T) \times \partial\Omega, \\ u|_{t=0} = u_0, & x \in \Omega, \end{cases} \tag{NBVP}$$

where $\partial_\nu$ represents the outward normal derivative on $\partial\Omega$.

**Theorem B.1.** *We assume condition* (A$\varphi$-I) *or* (A$\varphi$-II)*, and the following* (A2).

(A2) *There exists* $K_2 \in (0, \infty)$ *satisfying,*

$$|||f(u)|||_1 \leq K_2(\hat{F}(u) + 1) \quad \forall u \in D(\hat{F}),$$

*where* $|||f(u)|||_1 := \sup\{|b|_{L^1(\Omega)}; b \in \partial\hat{F}(u)\}$.

*Let* $u_0 \in H^1(\Omega) \cap D(\hat{F})$ *and* $h \in L^2(0,T;L^2(\Omega))$, *then there exists a solution* $u$ *of* (NBVP) *satisfying*

$$\begin{aligned}
&u \in C([0,T];H^1(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2_N(\Omega)),\\
&f_0, g_0 \in L^2(0,T;L^2(\Omega)),\\
&v \in L^2(0,T;H^2_N(\Omega)),
\end{aligned}$$

*where* $H^2_N(\Omega) := \{z \in H^2(\Omega); \partial_\nu z = 0 \text{ on } \partial\Omega\}$.

**Remark B.2.** Under the Neumann boundary condition, the following conservation law holds.

$$\int_\Omega u(t)dx = \int_\Omega u_0 dx. \tag{b.1}$$

This law can be shown by integrating $(*)$ on $\Omega$.

We can carry out the proof almost the same way as in the proof of Theorem 2.2, except the usage of Poincaré's inequality such as in (3.61) and (4.7). Instead of it, we rely on the Poincaré–Wirtinger inequality. From the second energy estimates, we obtain the following estimates (cf. Lemma 3.8).

$$|\nabla u(t)|_2^2 + \hat{F}(u(t)) + \int_0^t |\partial_s u(s)|_2^2 ds + \int_0^t |\nabla v(s)|_2^2 ds + \int_0^t |\Delta v(s)|_2^2 ds \le C. \tag{b.2}$$

Then from the conservation law (b.1), we also get the estimate of $|u(t)|_2$ for any $t \in (0,T)$. Furthermore from assumption (A2), we can derive an a priori bound for $\int_\Omega |f_0(u(t))|dx$. Integrating $(**)$, from $(*)$ and the boundary conditions, we obtain by (b.1)

$$\begin{aligned}
\int_\Omega v(t)dx &= \int_\Omega f_0(u(t))dx - \int_\Omega g_0(u(t))dx + \beta \int_\Omega \partial_t u(t)dx - \int_\Omega h(t)dx\\
&= \int_\Omega f_0(u(t))dx - \int_\Omega g_0(u(t))dx - \int_\Omega h(t)dx.
\end{aligned} \tag{b.3}$$

Hence by (b.2), we can get the estimate for $\int_0^t \int_\Omega v(s)dxds$. By virtue of the Poincaré–Wirtinger inequality and the estimates of $\int_0^t |\nabla v(s)|_2^2 ds$ and $\int_0^t \int_\Omega v(s)dxds$, we obtain the estimate of $\int_0^t |v(s)|_2^2 ds$. Then from the same arguments as in the proof of Lemma 3.9, we can derive the following estimates.

$$\int_0^t |f_0(u(s))|_2^2 ds + \int_0^t |g_0(u(s))|_2^2 ds + \int_0^t |\Delta u(s)|_2^2 ds \le C. \tag{b.4}$$

Thus the rest of the proof can be done as in the proof of Theorem 2.2.

Furthermore we can derive a result of smoothing effect under the Neumann boundary condition, i.e., (NBVP) admits a solution when $u_0$ belongs to $H_F := \overline{D(\hat{F})}^{L^2(\Omega)}$.

**Theorem B.3.** *Assume* (A$\varphi$-I) *or* (A$\varphi$-II), (A2), *and (Af). Let* $u_0 \in H_F$ *and* $h \in L^2(0,T;L^2(\Omega))$, *then there exists a solution of* (NBVP) *satisfying*

$$\begin{aligned}
&u \in C([0,T];L^2(\Omega)),\\
&\sqrt{t}\Delta u(t), \sqrt{t}\partial_t u(t), \sqrt{t} f_0(u(t)), \sqrt{t} g_0(u(t)) \in L^2(0,T;L^2(\Omega)),\\
&\sqrt{t} v(t) \in L^2(0,T;H^2_N(\Omega)),
\end{aligned}$$

*where* $f_0$, $g_0$ *and* $v$ *are functions appearing in* (NBVP).

We can prove the existence of the solutions of the following auxiliary problem with relaxation term $\eta v$ for $u_0 \in H^1(\Omega) \cap D(\hat{F})$ and $h \in L^2(0,T;L^2(\Omega))$, by modifying the proof of Theorem B.1 slightly:

$$\begin{cases} \gamma\partial_t u = \Delta v - \eta v, & (t,x) \in (0,T) \times \Omega, \\ v = f_0(u) - g_0(u) - \alpha\Delta u + \beta\partial_t u - h, & (t,x) \in (0,T) \times \Omega, \\ f_0(u) \in f(u), g_0(u) \in g(u), & (t,x) \in (0,T) \times \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ u|_{t=0} = u_0 & x \in \Omega. \end{cases} \tag{ANBVP}$$

Noting that $(-\Delta + \eta I)^{-1}$ is a bijection and the duality mapping from $(H^1(\Omega))^*$ to $H^1(\Omega)$, we get the following relation same as in Lemma 3.1:

$$(v(t), u(t))_2 = -\frac{\gamma}{2}\frac{d}{dt}|u(t)|^2_{(H^1(\Omega))^*}. \tag{b.5}$$

Then we can establish a priori estimates independent of $\eta$ by using the Poincaré–Wirtinger inequality instead of the Poincaré inequality. The rest of the proof can be carried out almost the same way as in the proof of Theorem 2.3.

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