

DOUBLY NONLINEAR EVOLUTION INCLUSIONS OF TIME-DEPENDENT SUBDIFFERENTIALS –QUASI-VARIATIONAL APPROACH–

Dedicated to the memory of Professor Isamu Fukuda

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Abstract. This paper is concerned with doubly nonlinear evolution inclusions governed by time-dependent subdifferentials with the unknown-dependent constraints of the form:

$$\partial_* \psi^t(u; u'(t)) + \partial_* \varphi^t(u; u(t)) + g(u; t, u(t)) \ni f(t) \text{ in } V^*, \quad u(0) = u_0 \text{ in } V,$$

where V is a uniformly convex Banach space with its dual space V^* ; $\partial_* \psi^t(v; z)$ and $\partial_* \varphi^t(v; z)$ are subdifferentials from V into V^* of convex functions $z \rightarrow \psi^t(v; z)$ and $z \rightarrow \varphi^t(v; z)$ on V , v being an unknown parameter; $g(v; t, z)$ is a perturbation term depending on the parameter v . The main objective of this paper is to establish an existence result of the above problem which includes the so-called quasi-variational structure, specifying the classes of functions $\psi^t(v; z)$ and $\varphi^t(v; z)$.

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1 Introduction

This paper is concerned with doubly nonlinear quasi-variational evolution inclusions governed by time-dependent subdifferentials.

Throughout this paper, let H be a real Hilbert space and V be a uniformly convex Banach space such that V is dense in H and the injection from V into H is compact, supposing also that the dual space V^* of V is uniformly convex; the norms in V and V^* are denoted by $|\cdot|_V$ and $|\cdot|_{V^*}$, respectively, and the duality pairing between V and V^* is denoted by $\langle \cdot, \cdot \rangle$; similarly the norm and the inner product in H are denoted by $|\cdot|_H$ and $(\cdot, \cdot)_H$, respectively. In this case, identifying H with its dual, we have

$$V \hookrightarrow H \hookrightarrow V^* \quad (\text{with dense and compact embeddings});$$

note that

$$\langle u, v \rangle = (u, v)_H, \quad \text{if } u \in H \text{ and } v \in V.$$

We now recall the usual notation and definitions of subdifferentials of convex functions. Let $\phi : V \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper (i.e., not identically equal to infinity), lower semi-continuous (l.s.c.), and convex function. Then, the effective domain $D(\phi)$ is defined by

$$D(\phi) := \{z \in V; \phi(z) < \infty\}.$$

The subdifferential $\partial_*\phi : V \rightarrow V^*$ of ϕ is a possibly multi-valued operator from V into V^* , and is defined by $z^* \in \partial_*\phi(z)$ if and only if

$$z \in D(\phi) \quad \text{and} \quad \langle z^*, y - z \rangle \leq \phi(y) - \phi(z) \quad \text{for all } y \in V.$$

Its graph is the set $G(\partial_*\phi) := \{[z, z^*] \in V \times V^* ; z^* \in \partial_*\phi(z)\}$, which is often identified with $\partial_*\phi$, namely, $z^* \in \partial_*\phi(z)$ is denoted by $[z, z^*] \in \partial_*\phi$. For various properties and related notions of a proper, l.s.c., convex function ϕ and its subdifferential $\partial_*\phi$, we refer to the monographs by V. Barbu [5, 6]. In particular, for those in Hilbert spaces, we refer to the monographs by H. Brézis [7].

We consider the following doubly quasi-variational evolution inclusions governed by time-dependent subdifferentials in the Banach space V^* :

$$\begin{cases} \partial_*\psi^t(u; u'(t)) + \partial_*\varphi^t(u; u(t)) + g(u; t, u(t)) \ni f(t) & \text{in } V^* \text{ for a.a. } t \in (0, T), \\ u(0) = u_0 & \text{in } V, \end{cases} \quad (1.1)$$

where $0 < T < \infty$, $u' = \frac{du}{dt}$ in V , f is a given V^* -valued function, and $u_0 \in V$ is a given initial datum.

For each parameter v and $t \in [0, T]$, $\psi^t(v; z)$ is a proper, l.s.c., convex function in $z \in V$, $\varphi^t(v; z)$ is a non-negative, continuous convex function in $z \in V$, and $g(v; t, z)$ is a single-valued Lipschitz operator from V into V^* (see Section 2 for their precise definitions). Note that (t, v) is a parameter that determines the convex functions $\psi^t(v; \cdot)$, $\varphi^t(v; \cdot)$, and the perturbation $g(v; t, \cdot)$.

The main objective of this paper is to establish an abstract result on the existence of solutions to (1.1) under some additional assumptions. Also, we show that the solutions to (1.1) is not unique, in general, by giving an example for non-uniqueness.

Similar types of doubly nonlinear evolution inclusions have been discussed from various motivations by many mathematicians, for instance, G. Akagi [1], T. Arai [2], M. Aso–M. Frémond–N. Kenmochi [3], P. Colli [8], P. Colli–A. Visintin [9], O. Grange–F. Mignot [10], and T. Senba [16]. Most of them treated the case

$$\partial\psi^t(u'(t)) + \partial\varphi(u(t)) \ni f(t) \quad \text{in } H \quad \text{for a.a. } t \in (0, T) \quad (1.2)$$

and it should be noticed that two subdifferentials of (1.2) are independent of v and there is no perturbation term g . In such a case the energy inequality is easily obtained.

Subsequently, in the works of N. Kenmochi–K. Shirakawa–N. Yamazaki [12, 13], a more general approach was proposed to treat

$$\partial_*\psi^t(u'(t)) + \partial_*\varphi^t(u; u(t)) + g(t, u(t)) \ni f(t) \quad \text{in } V^* \quad \text{for a.a. } t \in (0, T).$$

The main idea of this approach is based on the treatment for quasi-variational evolution inequalities, and in this paper we shall establish the solvability of the final goal (1.1).

In the application of our abstract result this development enables us to deal with doubly nonlinear evolution inclusions for the quasi-linear partial differential inequalities of parabolic type with gradient constraints for the time-derivative. For instance, consider

$$\varphi^t(v; z) := \frac{1}{2} \int_{\Omega} a(x, t, v(x, t)) |\nabla z(x)|^2 dx, \quad z \in H_0^1(\Omega),$$

$$\psi^t(v; z) := \frac{\tau_0}{2} \int_{\Omega} |z(x)|^2 dx + I_{K(v; t)}(z), \quad z \in H_0^1(\Omega)$$

with

$$K(v; t) := \{z \in H_0^1(\Omega) \mid |\nabla z| \leq \gamma(v(\cdot, t)), \text{ a.e. on } \Omega\},$$

where Ω is a bounded domain in $\mathbb{R}^N$, $a(x, t, v)$ is a positive, bounded, and continuous function on $\Omega \times [0, T] \times \mathbb{R}$, γ is a positive continuous function on $\mathbb{R}$, and $I_{K(v; t)}$ is the indicator function of $K(v; t)$. In such a case,

$$\partial_*\varphi^t(v; z) = -\operatorname{div}(a(\cdot, t, v(\cdot, t))\nabla z),$$

and

$$\partial_*\psi^t(v; z) = \tau_0 z + \partial_* I_{K(v; t)}(z).$$

Therefore, for a solution u of (1.1) the first term of (1.1) requires

$$\left| \nabla \left(\frac{\partial u}{\partial t} \right) \right| \leq \gamma(u), \quad \text{a.e. on } Q := \Omega \times (0, T).$$

This paper is organized as follows. In Section 2, we state the abstract result of the existence of solutions to (1.1) for each $f \in L^2(0, T; V^*)$ and $u_0 \in V$. In Section 3, we give a proof of the existence of solutions to (1.1), which is the main result of this paper. In the final Section 4, we apply our general results to some model problems: quasi-variational inequalities with time-dependent constraints for quasi-linear parabolic inequalities and for Navier–Stokes type.

2 Solvability of (1.1)

Prior to formulate the main result of this paper, we recall a notion of convergence for convex functions, developed by U. Mosco [14, 15] and H. Attouch [4].

Let ϕ, ϕ_n ($n \in \mathbb{N}$) be proper, l.s.c., and convex functions on V . Then, we say that ϕ_n converges to ϕ on V in the sense of U. Mosco [14] as $n \rightarrow \infty$ if the following two conditions are satisfied:

- (i) for any subsequence $\{\phi_{n_k}\}_{k \in \mathbb{N}} \subset \{\phi_n\}_{n \in \mathbb{N}}$, if $z_k \rightarrow z$ weakly in V as $k \rightarrow \infty$, then

$$\liminf_{k \rightarrow \infty} \phi_{n_k}(z_k) \geq \phi(z);$$

- (ii) for any $z \in D(\phi)$, there is a sequence $\{z_n\}_{n \in \mathbb{N}}$ in V such that

$$z_n \rightarrow z \text{ in } V \text{ as } n \rightarrow \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \phi_n(z_n) = \phi(z).$$

For some important properties of the Mosco convergence of convex functions, we refer to the monographs by H. Attouch [4] and N. Kenmochi [11]. Especially the following ones are often used. Let ϕ, ϕ_n ($n \in \mathbb{N}$) be proper, l.s.c., and convex functions on V . Assume ϕ_n converges to ϕ on V in the sense of Mosco as $n \rightarrow \infty$. Then:

(Fact 1) $z_n^* \in \partial_* \phi_n(z_n)$, $z_n \rightarrow z$ weakly in V , $z_n^* \rightarrow z^*$ weakly in V^* , and $\langle z_n^*, z_n \rangle \rightarrow \langle z^*, z \rangle$ (as $n \rightarrow \infty$), then $z^* \in \partial_* \phi(z)$.

(Fact 2) (Graph convergence) If $z^* \in \partial_* \phi(z)$, then there are sequences $\{z_n\}_{n \in \mathbb{N}} \subset V$ and $\{z_n^*\}_{n \in \mathbb{N}} \subset V^*$ such that

$$z_n^* \in \partial_* \phi_n(z_n), \quad z_n \rightarrow z \text{ in } V, \quad \text{and} \quad z_n^* \rightarrow z^* \text{ in } V^*.$$

These results will be often used in the proofs of our statements.

2.1 Statement of the existence result

We begin with the precise formulation of our problem. For this purpose, we use a time-dependent proper, l.s.c., and convex function $\psi_0^t(\cdot)$ on V such that there are positive constants C_1 and C_2 satisfying

$$\psi_0^t(z) \geq C_1 |z|_V^2 - C_2, \quad \forall z \in V, \quad \forall t \in [0, T], \quad 0 < T < \infty, \quad (2.1)$$

and $t \rightarrow \psi_0^t(\cdot)$ is continuous on V in the sense of Mosco.

Our doubly nonlinear evolution inclusions are formulated in terms of two functionals $\psi^t(v; z)$, $\varphi^t(v; z)$, and a mapping $g(v; t, z)$ together with a prescribed initial-value u_0 and a forcing term f , as follows:

$$\begin{cases} \partial_* \psi^t(u; u'(t)) + \partial_* \varphi^t(u; u(t)) + g(u; t, u(t)) \ni f(t) & \text{in } V^*, \text{ a.a. } t \in (0, T), \\ u(0) = u_0 & \text{in } V, \end{cases} \quad (2.2)$$

and their solutions are constructed in the class

$$D_0 := \left\{ v \in W^{1,2}(0, T; V) \mid \int_0^T \psi_0^t(v'(\tau)) d\tau < \infty, v(0) = u_0 \right\}, \quad (2.3)$$

depending on the functional ψ_0^t and the initial-value u_0 .

We suppose that $D_0 \neq \emptyset$ and the following assumptions are fulfilled:

(Assumption (ψ))

The functional $\psi^t(v; z)$ is defined for each $(t, v, z) \in [0, T] \times D_0 \times V$ so that $\psi^t(v; z)$ is proper, l.s.c., and convex in $z \in V$ for any $t \in [0, T]$ and any $v \in D_0$, and

$$\psi^t(v_1; z) = \psi^t(v_2; z), \quad \forall z \in V, \text{ if } v_1 = v_2 \text{ on } [0, t],$$

for $v_i \in D_0$, $i = 1, 2$. Furthermore, assume:

($\psi 1$) If $t_n \in [0, T]$, $v_n \in D_0$, $\sup_{n \in \mathbb{N}} \int_0^T \psi_0^t(v_n'(t)) dt < \infty$, $t_n \rightarrow t$, and $v_n \rightarrow v$ in $C([0, T]; H)$ (hence $v \in D_0$ and $v \in W^{1,2}(0, T; V)$ by (2.1)) as $n \rightarrow \infty$, then

$$\psi^{t_n}(v_n; \cdot) \rightarrow \psi^t(v; \cdot) \text{ on } V \text{ in the sense of Mosco as } n \rightarrow \infty.$$

($\psi 2$) $D(\psi^t(v; \cdot)) \subset D(\psi_0^t)$ for all $v \in D_0$ and $t \in [0, T]$, and

$$\psi^t(v; z) \geq \psi_0^t(z), \quad \forall t \in [0, T], \quad \forall v \in D_0, \quad \forall z \in D(\psi^t(v; \cdot)).$$

($\psi 3$) $\partial_* \psi^t(v; 0) \ni 0$ for all $t \in [0, T]$ and $v \in D_0$, and there is a non-negative function $c_\psi(\cdot) \in L^1(0, T)$ such that

$$\psi^t(v; 0) \leq c_\psi(t), \quad \forall t \in [0, T], \quad \forall v \in D_0.$$

(Assumption (ϕ))

Let $\varphi^t : [0, T] \times D_0 \times V \rightarrow \mathbb{R}$ be a function such that $\varphi^t(v; z)$ is non-negative, finite, continuous, and convex in $z \in V$ for any $t \in [0, T]$ and any $v \in D_0$, and

$$\varphi^t(v_1; z) = \varphi^t(v_2; z), \quad \forall z \in V, \text{ if } v_1 = v_2 \text{ on } [0, t],$$

for $v_i \in D_0$, $i = 1, 2$. Besides, assume the following:

($\phi 1$) The subdifferential $\partial_* \varphi^t(v; z)$ of $\varphi^t(v; z)$ with respect to $z \in V$ is linear and bounded from $D(\partial_* \varphi^t(v; \cdot)) = V$ into V^* for each $t \in [0, T]$ and $v \in D_0$, and there is a positive constant C_3 such that

$$|\partial_* \varphi^t(v; z)|_{V^*} \leq C_3 |z|_V, \quad \forall z \in V, \quad \forall t \in [0, T], \quad \forall v \in D_0.$$

($\phi 2$) If $\{v_n\}_{n \in \mathbb{N}} \subset D_0$, $\sup_{n \in \mathbb{N}} \int_0^T \psi_0^t(v'_n(t)) dt < \infty$, $v \in D_0$, and $v_n \rightarrow v$ in $C([0, T]; H)$ (as $n \rightarrow \infty$), then

$$\varphi^t(v_n; \cdot) \rightarrow \varphi^t(v; \cdot) \quad \text{on } V \text{ in the sense of Mosco, } \forall t \in [0, T].$$

($\phi 3$) $\varphi^0(v; 0) = 0$ for all $v \in D_0$. Moreover, there is a positive constant C_4 such that

$$\varphi^0(v; z) \geq C_4 |z|_V^2, \quad \forall z \in V, \quad \forall v \in D_0.$$

($\phi 4$) There is a function $\alpha \in W^{1,1}(0, T)$ such that

$$|\varphi^t(v; z) - \varphi^s(v; z)| \leq |\alpha(t) - \alpha(s)| \varphi^s(v; z), \quad \forall z \in V, \quad \forall v \in D_0, \quad \forall s, t \in [0, T].$$

(Assumption (g))

Let $g := g(v; t, z)$ be a single-valued operator from $D_0 \times [0, T] \times V$ into V^* such that $g(v; t, z)$ is strongly measurable in $t \in [0, T]$ for each $v \in D_0$ and $z \in V$, and

$$g(v_1; t, z) = g(v_2; t, z), \quad \forall z \in V, \quad \text{if } v_1, v_2 \in D_0 \text{ and } v_1 = v_2 \text{ on } [0, t], \quad \forall t \in [0, T].$$

Moreover, assume:

(g1) Let $\{v_n\}_{n \in \mathbb{N}}$ be a sequence in D_0 such that $\sup_{n \in \mathbb{N}} \int_0^T \psi_0^t(v'_n(t)) dt < \infty$ and $v_n \rightarrow v$ in $C([0, T]; H)$ (as $n \rightarrow \infty$), and $\{z_n\}_{n \in \mathbb{N}}$ be a sequence in V such that $z_n \rightarrow z$ weakly in V . Then,

$$g(v_n; t, z_n) \rightarrow g(v; t, z) \quad \text{in } V^*, \quad \forall t \in [0, T].$$

(g2) $g(v; \cdot, 0) \in L^2(0, T; V^*)$ for any $v \in D_0$, and $g(v; t, \cdot)$ is uniformly Lipschitz from V into V^* , i.e., there is a constant $L_g > 0$ such that

$$|g(v; t, z_1) - g(v; t, z_2)|_{V^*} \leq L_g |z_1 - z_2|_V, \quad \forall z_i \in V \quad (i = 1, 2), \quad \forall v \in D_0, \quad \forall t \in [0, T].$$

(g3) There is a non-negative function $g_0 \in L^2(0, T)$ such that

$$|g(v; t, 0)|_{V^*} \leq g_0(t), \quad \text{a.a. } t \in (0, T), \quad \forall v \in D_0.$$

Next, we give the definition of solutions to evolution inclusion (2.2).

Definition 2.1. Given data $f \in L^2(0, T; V^*)$ and $u_0 \in V$, a function $u : [0, T] \rightarrow V$ is called a solution to $CP(\psi^t, \varphi^t, g; f, u_0)$ or $CP(f, u_0)$ or simply CP , if and only if the following conditions are fulfilled:

(i) $u \in W^{1,2}(0, T; V)$.

(ii) There exists a function $\xi \in L^2(0, T; V^*)$ such that

$$\xi(t) \in \partial_* \psi^t(u; u'(t)) \quad \text{in } V^*, \quad \text{a.a. } t \in (0, T),$$

and

$$\xi(t) + \partial_* \varphi^t(u; u(t)) + g(u; t, u(t)) = f(t) \quad \text{in } V^*, \quad \text{a.a. } t \in (0, T).$$

(iii) $u(0) = u_0$ in V .

Now, we are ready to state our main claim for evolution inclusion $CP(f, u_0)$.

Theorem 2.1. *Assume that $u_0 \in V$ and assumptions (ψ) , (ϕ) , and (g) are fulfilled. Let f be any function in $L^2(0, T; V^*)$. Then, $CP(f, u_0)$ admits at least one solution u . Moreover, there exists a constant $N_1 > 0$, independent of f and u_0 , such that*

$$\int_0^T \psi^t(u; u'(t)) dt + \sup_{t \in [0, T]} \varphi^t(u; u(t)) \leq N_1 \left(|u_0|_V^2 + |f|_{L^2(0, T; V^*)}^2 + 1 \right) \quad (2.4)$$

for any solution u to $CP(f, u_0)$.

The solvability for $CP(f, u_0)$ will be performed in several steps. Our approach is first to consider an auxiliary approximate problem of the form for each fixed $\varepsilon \in (0, 1]$ and $v \in D_0$:

$$\varepsilon F_0 u'_{\varepsilon, v}(t) + \partial_* \psi^t(v; u'_{\varepsilon, v}(t)) + \partial_* \varphi^t(v; u_{\varepsilon, v}(t)) + g(v; t, u_{\varepsilon, v}(t)) \ni f(t) \quad \text{in } V^*, \quad t > 0, \quad (2.5)$$

subject to initial condition $u_{\varepsilon, v}(0) = u_0$ in V , where F_0 is a linear continuous maximal monotone mapping from $D(F_0) := V$ into V^* ; such a mapping F_0 always exists as will be remarked in subsection 2.4. In the first step we shall prove the existence of a unique solution $u_{\varepsilon, v}$ to (2.5) for every $\varepsilon \in (0, 1]$ and $v \in D_0$. In the second step the convergence of approximate solution $u_{\varepsilon, v}$ will be discussed. For each $\varepsilon \in (0, 1]$ and $v \in D_0$, the solution $u_{\varepsilon, v}$ of (2.5) is denoted by $S_\varepsilon(v)$, namely $u_{\varepsilon, v} := S_\varepsilon(v)$, and in the third step we shall find a fixed point of S_ε in D_0 , $u_{\varepsilon, v} = S_\varepsilon(u_{\varepsilon, v})$, which is denoted simply by u_ε . It is a solution to

$$\varepsilon F_0(u'_\varepsilon(t)) + \partial_* \psi^t(u_\varepsilon; u'_\varepsilon(t)) + \partial_* \varphi^t(u_\varepsilon; u_\varepsilon(t)) + g(u_\varepsilon; t, u_\varepsilon(t)) \ni f(t), \quad u_\varepsilon(0) = u_0, \quad (2.6)$$

which is denoted by $CP_\varepsilon(f, u_0)$. In the final step we shall show that the solution u_ε converges in $C([0, T]; V)$ as $\varepsilon \rightarrow 0$ and the limit u is a solution of $CP(f, u_0)$.

2.2 Some lemmas

In this subsection we give some lemmas derived directly from assumptions (ψ) and (ϕ) .

Lemma 2.1. *Suppose that assumption (ϕ) is satisfied. Then, the following inequality holds: for all $t \in [0, T]$, $z \in V$, and $v \in D_0$,*

$$\frac{C_4}{|\alpha'|_{L^1(0, T)} + 1} |z|_V^2 \leq \varphi^t(v; z) \leq \langle \partial_* \varphi^t(v; z), z \rangle \leq C_3 |z|_V^2. \quad (2.7)$$

Proof. Let $v \in D_0$ and fix it. Since $\varphi^0(v; 0) = 0$ by $(\phi 3)$, it follows from $(\phi 1)$ that

$$\varphi^0(v; z) = \varphi^0(v; z) - \varphi^0(v; 0) \leq \langle \partial_* \varphi^0(v; z), z \rangle \leq |\partial_* \varphi^0(v; z)|_{V^*} |z|_V \leq C_3 |z|_V^2,$$

hence

$$C_4 |z|_V^2 \leq \varphi^0(v; z) \leq C_3 |z|_V^2, \quad \forall z \in V. \quad (2.8)$$

Note from $(\phi 4)$ with $s = 0$ that

$$|\varphi^t(v; z) - \varphi^0(v; z)| \leq |\alpha(t) - \alpha(0)| \varphi^0(v; z) \leq \int_0^t |\alpha'(\tau)| d\tau \cdot \varphi^0(v; z),$$

$$\forall t \in [0, T], \quad \forall z \in V,$$

whence

$$\varphi^t(v; z) \leq (|\alpha'|_{L^1(0,T)} + 1) \varphi^0(v; z), \quad \forall t \in [0, T], \quad \forall z \in V. \quad (2.9)$$

From (2.9) we see that $\varphi^t(v; 0) = 0$ for all $t \in [0, T]$ and by $(\phi 1)$,

$$\varphi^t(v; z) \leq \langle \partial_* \varphi^t(v; z), z \rangle \leq C_3 |z|_V^2, \quad \forall z \in V. \quad (2.10)$$

Similarly,

$$\varphi^0(v; z) \leq (|\alpha'|_{L^1(0,T)} + 1) \varphi^t(v; z), \quad \forall t \in [0, T], \quad \forall z \in V. \quad (2.11)$$

Hence, we infer from (2.11) with (2.8) that

$$\varphi^t(v; z) \geq \frac{1}{|\alpha'|_{L^1(0,T)} + 1} \varphi^0(v; z) \geq \frac{C_4}{|\alpha'|_{L^1(0,T)} + 1} |z|_V^2, \quad \forall t \in [0, T], \quad \forall z \in V. \quad (2.12)$$

Thus, we conclude from (2.10) with (2.12) that (2.7) holds. $\square$

Lemma 2.2. *Suppose that assumption (ϕ) is satisfied. Let $\{t_n\}_{n \in \mathbb{N}} \subset [0, T]$ and $\{v_n\}_{n \in \mathbb{N}} \subset D_0$ such that*

$$t_n \rightarrow t, \quad v_n \rightarrow v \text{ in } C([0, T]; H) \text{ (as } n \rightarrow \infty), \quad \sup_{n \in \mathbb{N}} \int_0^T \psi_0^\tau(v_n'(\tau)) d\tau < \infty.$$

Then:

(i) $\varphi^{t_n}(v_n; \cdot) \rightarrow \varphi^t(v; \cdot)$ on V in the sense of Mosco as $n \rightarrow \infty$.

(ii) For any sequence $\{z_n\}_{n \in \mathbb{N}}$ in V such that $z_n \rightarrow z$ in V , we have

$$\partial_* \varphi^{t_n}(v_n; z_n) \rightarrow \partial_* \varphi^t(v; z) \text{ in } V^* \text{ as } n \rightarrow \infty. \quad (2.13)$$

Proof. Let $\{t_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$ be the sequences in the statement of this lemma. We shall prove the following two properties of $\varphi^{t_n}(v_n; \cdot)$ (the definition of Mosco convergence):

(M1) $\varphi^t(v; z) \leq \liminf_{n \rightarrow \infty} \varphi^{t_n}(v_n; \bar{z}_n)$ for any sequence $\{\bar{z}_n\}_{n \in \mathbb{N}}$ with $\bar{z}_n \rightarrow z$ weakly in V .

(M2) For each $z \in V$ there is a sequence $\{\tilde{z}_n\}_{n \in \mathbb{N}}$ in V such that $\tilde{z}_n \rightarrow z$ in V and $\varphi^{t_n}(v_n; \tilde{z}_n) \rightarrow \varphi^t(v; z)$ as $n \rightarrow \infty$.

Let $\{\bar{z}_n\}_{n \in \mathbb{N}}$ be the same as in (M1). First we note from $(\phi 2)$ that

$$\varphi^t(v; z) \leq \liminf_{n \rightarrow \infty} \varphi^t(v_n; \bar{z}_n). \quad (2.14)$$

Since $\varphi^t(v_n; \bar{z}_n)$ is bounded by Lemma 2.1, we infer from $(\phi 4)$ that

$$|\varphi^{t_n}(v_n; \bar{z}_n) - \varphi^t(v_n; \bar{z}_n)| \leq |\alpha(t_n) - \alpha(t)| \varphi^t(v_n; \bar{z}_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, we observe from (2.14) that

$$\liminf_{n \rightarrow \infty} \varphi^{t_n}(v_n; \bar{z}_n) \geq \liminf_{n \rightarrow \infty} \{ \varphi^t(v_n; \bar{z}_n) - |\alpha(t_n) - \alpha(t)| \varphi^t(v_n; \bar{z}_n) \} \geq \varphi^t(v; z).$$

Hence, (M1) was shown.

Next, to prove (M2), let z be any element in V . Since $\varphi^t(v_n; \cdot)$ converges to $\varphi^t(v; \cdot)$ in the sense of Mosco by $(\phi 2)$, there is a sequence $\{\tilde{z}_n\}_{n \in \mathbb{N}}$ in V such that $\tilde{z}_n \rightarrow z$ in V and $\varphi^t(v_n; \tilde{z}_n) \rightarrow \varphi^t(v; z)$ as $n \rightarrow \infty$. For this sequence $\{\tilde{z}_n\}_{n \in \mathbb{N}}$, we observe from $(\phi 4)$ and Lemma 2.1 that

$$\begin{aligned} & |\varphi^{t_n}(v_n; \tilde{z}_n) - \varphi^t(v; z)| \\ & \leq |\varphi^{t_n}(v_n; \tilde{z}_n) - \varphi^t(v_n; \tilde{z}_n)| + |\varphi^t(v_n; \tilde{z}_n) - \varphi^t(v; z)| \\ & \leq |\alpha(t_n) - \alpha(t)| \varphi^t(v_n; \tilde{z}_n) + |\varphi^t(v_n; \tilde{z}_n) - \varphi^t(v; z)| \\ & \rightarrow 0. \end{aligned}$$

Hence, (M2) is obtained. Thus, $\varphi^{t_n}(v_n; \cdot) \rightarrow \varphi^t(v; \cdot)$ on V in the sense of Mosco as $n \rightarrow \infty$.

Finally, we show (2.13). To this end, let $\{z_n\}_{n \in \mathbb{N}}$ be the sequence as in (ii). In addition, note that the assertion (i) is equivalent to the fact that $\partial_* \varphi^{t_n}(v_n; \cdot)$ converges to $\partial_* \varphi^t(v; \cdot)$ in the graph sense (cf. (Fact 2) or [4, 11]). Accordingly, there is a sequence $\{\tilde{z}_n\}_{n \in \mathbb{N}}$ in V such that

$$\tilde{z}_n \rightarrow z \text{ in } V, \quad \partial_* \varphi^{t_n}(v_n; \tilde{z}_n) \rightarrow \partial_* \varphi^t(v; z) \text{ in } V^* \quad \text{as } n \rightarrow \infty.$$

From the above convergences and $(\phi 1)$, we conclude that

$$\begin{aligned} & |\partial_* \varphi^{t_n}(v_n; z_n) - \partial_* \varphi^t(v; z)|_{V^*} \\ & \leq |\partial_* \varphi^{t_n}(v_n; z_n) - \partial_* \varphi^{t_n}(v_n; \tilde{z}_n)|_{V^*} + |\partial_* \varphi^{t_n}(v_n; \tilde{z}_n) - \partial_* \varphi^t(v; z)|_{V^*} \\ & \leq C_3 |z_n - \tilde{z}_n|_V + |\partial_* \varphi^{t_n}(v_n; \tilde{z}_n) - \partial_* \varphi^t(v; z)|_{V^*} \\ & \rightarrow 0. \end{aligned}$$

Thus (2.13) holds. □

Lemma 2.3. *Suppose that assumption (ϕ) is satisfied. Let $v \in D_0$ and $w \in W^{1,1}(0, T; V)$. Then,*

$$\frac{d}{dt} \varphi^t(v; w(t)) - \langle \partial_* \varphi^t(v; w(t)), w'(t) \rangle \leq |\alpha'(t)| \varphi^t(v; w(t)), \quad \text{a.a. } t \in (0, T). \quad (2.15)$$

Proof. By $(\phi 4)$, $\varphi^t(v; w(t))$ is absolutely continuous on $[0, T]$. Hence,

$$\begin{aligned} & \varphi^t(v; w(t)) - \varphi^s(v; w(s)) - \langle \partial_* \varphi^t(v; w(t)), w(t) - w(s) \rangle \\ & \leq \varphi^t(v; w(s)) - \varphi^s(v; w(s)) \\ & \leq |\alpha(t) - \alpha(s)| \varphi^s(v; w(s)), \quad \forall s, t \in [0, T], \quad s < t. \end{aligned}$$

Dividing the above inequalities by $t - s$ and letting $s \uparrow t$, we get (2.15). $\square$

Now, for each $v \in D_0$ and $t \in [0, T]$ we define a functional $\tilde{\Psi}^t(v; \cdot)$ on $L^2(0, t; V)$ by

$$\tilde{\Psi}^t(v; w) := \int_0^t e^{\int_0^s |\alpha'(\tau)| d\tau} \psi^s(v; w(s)) ds, \quad \forall w \in L^2(0, t; V). \quad (2.16)$$

Clearly, $\tilde{\Psi}^t(v; \cdot)$ is proper, l.s.c., and convex on $L^2(0, T; V)$ for every $v \in D_0$ and $t \in [0, T]$.

Lemma 2.4. *Suppose that assumption (ψ) and (ϕ) are satisfied, and let $\{v_n\}_{n \in \mathbb{N}} \subset D_0$ such that*

$$\sup_{n \in \mathbb{N}} \int_0^T \psi_0^t(v_n'(t)) dt < \infty, \quad v_n \rightarrow v \text{ in } C([0, T]; H) \text{ as } n \rightarrow \infty.$$

Then

$$\tilde{\Psi}^t(v_n; \cdot) \rightarrow \tilde{\Psi}^t(v; \cdot) \text{ on } L^2(0, t; V) \text{ in sense of Mosco as } n \rightarrow \infty \quad (2.17)$$

for every $t \in [0, T]$.

Proof. Setting

$$\tilde{\psi}^s(v_n; z) := e^{\int_0^s |\alpha'(\tau)| d\tau} \psi^s(v_n; z), \quad \forall z \in V, \quad \forall s \in [0, T],$$

we easily observe from $(\psi 1)$ that $\tilde{\psi}^s(v_n; \cdot) \rightarrow \tilde{\psi}^s(v; \cdot)$ on V in sense of Mosco as $n \rightarrow \infty$ for all $s \in [0, T]$. It is also easy to see from the definition of Moreau-Yosida approximation that for their Moreau-Yosida approximations $\tilde{\psi}_\lambda^s(v_n; \cdot)$ and $\tilde{\Psi}_\lambda^t(v_n; \cdot)$, $\lambda > 0$, we have

$$\tilde{\Psi}_\lambda^t(v_n; w) = \int_0^t \tilde{\psi}_\lambda^s(v_n; w(s)) ds, \quad \forall w \in L^2(0, t; V). \quad (2.18)$$

By the Lebesgue dominated convergence theorem and the general theory on the Mosco convergence (cf. [4, Theorem 3.26]), for all $w \in L^2(0, t; V)$, all $\lambda > 0$, and all $t \in [0, T]$, the right hand side of (2.18) converges to $\int_0^t \tilde{\psi}_\lambda^s(v; w(s)) ds$, so that $\tilde{\Psi}_\lambda^t(v_n; w) \rightarrow \tilde{\Psi}_\lambda^t(v; w)$ as $n \rightarrow \infty$. This implies that (2.17) holds. $\square$

2.3 Convergence result

Given $u_0 \in V$, $v \in D_0$, and $f \in L^2(0, T; V^*)$, we denote by $CP_v(f, u_0)$ the problem to find a function $u \in W^{1,2}(0, T; V)$ with $\xi \in L^2(0, T; V^*)$ satisfying that

$$\begin{cases} \xi(t) \in \partial_* \psi^t(v; u'(t)) \text{ in } V^*, \text{ a.a. } t \in (0, T), \\ \xi(t) + \partial_* \varphi^t(v; u(t)) + g(v; t, u(t)) = f(t) \text{ in } V^*, \text{ a.a. } t \in (0, T), \\ u(0) = u_0 \text{ in } V. \end{cases} \quad (2.19)$$

Then, in this subsection, we establish the convergence result of solutions to $CP_v(f, u_0)$ with respect to the data v and f .

Note that the existence solutions to $CP_v(f, u_0)$ can be proved applying the abstract theory established in [12, Theorems 1 and 2]. Therefore, we omit the detailed proof of the following Proposition 2.1.

Proposition 2.1 (cf. [12, Theorem 1]). *Suppose that $u_0 \in V$ and assumptions (ψ) , (ϕ) , and (g) are satisfied. Then, for each $v \in D_0$ and $f \in L^2(0, T; V^*)$, there exists at least one solution u to $CP_v(f, u_0)$.*

We discuss the uniqueness of solutions to $CP_v(f, u_0)$ in the next Section 3, more precisely, Corollary 3.1.

We here derive the energy inequality to $CP_v(f, u_0)$.

Lemma 2.5 (Energy inequality). *For any fixed $v \in D_0$, any solution u to $CP_v := CP_v(f, u_0)$ satisfies the bound: there exists a constant $N_1 > 0$, independent of $v \in D_0$ and $f \in L^2(0, T; V^*)$, such that*

$$\int_0^T \psi^t(v; u'(t)) dt + \sup_{t \in [0, T]} \varphi^t(v; u(t)) \leq N_1(|u_0|_V^2 + |f|_{L^2(0, T; V^*)}^2 + 1), \quad (2.20)$$

where N_1 is the same constant as in (2.4) of Theorem 2.1.

Proof. We multiply the equation

$$\xi(t) + \partial_* \varphi^t(v; u(t)) + g(v; t, u(t)) = f(t) \text{ in } V^*$$

by $u'(t)$, where $\xi(t) \in \partial_* \psi^t(v; u'(t))$, a.a. $t \in (0, T)$. Then

$$\begin{aligned} \langle \xi(t), u'(t) \rangle + \langle \partial_* \varphi^t(v; u(t)), u'(t) \rangle + \langle g(v; t, u(t)), u'(t) \rangle \\ = \langle f(t), u'(t) \rangle, \quad \text{a.a. } t \in (0, T). \end{aligned} \quad (2.21)$$

It follows from $(\psi 3)$ and (2.15) of Lemma 2.3 that:

$$\langle \xi(t), u'(t) \rangle \geq \psi^t(v; u'(t)) - \psi^t(v; 0) \geq \psi^t(v; u'(t)) - c_\psi(t), \quad (2.22)$$

$$\langle \partial_* \varphi^t(v; u(t)), u'(t) \rangle \geq \frac{d}{dt} \varphi^t(v; u(t)) - |\alpha'(t)| \varphi^t(v; u(t)). \quad (2.23)$$

Also, from $(\psi 2)$, (2.7) of Lemma 2.1, $(g 2)$, and Schwarz's inequality, we observe that

$$\begin{aligned} |\langle g(v; t, u(t)), u'(t) \rangle| &\leq |g(v; t, u(t))|_{V^*} |u'(t)|_V \\ &\leq \frac{C_1}{4} |u'(t)|_V^2 + \frac{1}{C_1} |g(v; t, u(t))|_{V^*}^2 \\ &\leq \frac{1}{4} \psi^t(v; u'(t)) + \frac{C_2}{4} + \frac{1}{C_1} (|g(v; t, 0)|_{V^*} + L_g |u(t)|_V)^2 \\ &\leq \frac{1}{4} \psi^t(v; u'(t)) + \frac{C_2}{4} + \frac{2|g(v; t, 0)|_{V^*}^2}{C_1} + \frac{2L_g^2(|\alpha'|_{L^1(0, T)} + 1)}{C_1 C_4} \varphi^t(v; u(t)) \end{aligned} \quad (2.24)$$

and

$$|\langle f(t), u'(t) \rangle| \leq \frac{C_1}{4} |u'(t)|_V^2 + \frac{1}{C_1} |f(t)|_{V^*}^2 \leq \frac{1}{4} \psi^t(v; u'(t)) + \frac{C_2}{4} + \frac{1}{C_1} |f(t)|_{V^*}^2. \quad (2.25)$$

Thus, by (2.22)~(2.25), $(\psi 3)$, and $(g 3)$, it follows from (2.21) that

$$\begin{aligned} & \frac{1}{2} \psi^t(v; u'(t)) + \frac{d}{dt} \varphi^t(v; u(t)) \\ & \leq M_1 (|\alpha'(t)| + 1) \varphi^t(v; u(t)) + M_2 (|f(t)|_{V^*}^2 + c_\psi(t) + g_0^2(t) + 1), \\ & \text{a.a. } t \in (0, T), \end{aligned} \quad (2.26)$$

where $M_1 > 0$ and $M_2 > 0$ are constants independent of $v \in D_0$ and $f \in L^2(0, T; V^*)$. Applying Gronwall's inequality to (2.26), we get (2.20). $\square$

We prove a convergence result of solutions to $CP_v(f, u_0)$ with respect to the data v and f .

Proposition 2.2. *Let $u_0 \in V$, $\{v_n\}_{n \in \mathbb{N}}$ and $\{f_n\}_{n \in \mathbb{N}}$ be any sequences in D_0 and $L^2(0, T; V^*)$, respectively, such that*

$$\begin{cases} \sup_{n \in \mathbb{N}} \int_0^T \psi_0^t(v'_n(t)) dt < \infty, & v_n \rightarrow v \text{ in } C([0, T]; H), \\ f_n \rightarrow f \text{ in } L^2(0, T; V^*) & (\text{as } n \rightarrow \infty), \end{cases} \quad (2.27)$$

and let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of solutions to $CP_{v_n}(f_n, u_0)$. Then, any weak cluster point u of $\{u_n\}_{n \in \mathbb{N}}$ in $L^2(0, T; H)$ is a solution to the limit problem $CP_v(f, u_0)$.

Proof. We first note by (2.27) that

$$\int_0^T \psi_0^t(v'(t)) dt \leq \sup_{n \in \mathbb{N}} \int_0^T \psi_0^t(v'_n(t)) dt < \infty,$$

so that $v \in D_0$ and $v_n \rightarrow v$ weakly in $W^{1,2}(0, T; V)$.

From (2.1), $(\psi 2)$, (2.7) of Lemma 2.1, (2.20) of Lemma 2.5, and the Ascoli–Arzelà theorem, it follows that there exist a subsequence of $\{u_n\}_{n \in \mathbb{N}}$, referred to as $\{u_n\}_{n \in \mathbb{N}}$ again, and a function $u \in W^{1,2}(0, T; V)$ such that

$$u_n \rightarrow u \text{ in } C([0, T]; H) \text{ and weakly in } W^{1,2}(0, T; V) \quad (2.28)$$

and

$$u_n(t) \rightarrow u(t) \text{ weakly in } V, \quad \forall t \in [0, T] \quad (2.29)$$

as $n \rightarrow \infty$. Clearly, $u(0) = u_0$ in V .

We next show that $u_n \rightarrow u$ in $C([0, T]; V)$. For simplicity we use the notation

$$\begin{aligned} \tilde{\Psi}^t(w; z) &:= \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} (\psi^s(w; z(s)) + C_2) ds, \\ &\forall w \in D_0, \forall z \in L^2(0, t; V), \forall t \in [0, T]. \end{aligned}$$

By Lemma 2.4,

$$\tilde{\Psi}^t(v_n; \cdot) \rightarrow \tilde{\Psi}^t(v; \cdot) \text{ on } L^2(0, t; V) \text{ in the sense of Mosco, } \forall t \in [0, T]$$

as $n \rightarrow \infty$. Therefore, it follows from (2.28) that

$$\tilde{\Psi}^t(v; u') \leq \liminf_{n \rightarrow \infty} \tilde{\Psi}^t(v_n; u'_n) < \infty,$$

namely,

$$\begin{aligned} & \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} (\psi^s(v; u'(s)) + C_2) ds \\ & \leq \liminf_{n \rightarrow \infty} \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} (\psi^s(v_n; u'_n(s)) + C_2) ds \\ & \leq N'_1 < \infty, \quad \forall t \in [0, T], \end{aligned} \quad (2.30)$$

for some positive constant N'_1 , whence $u' \in D(\tilde{\Psi}^t(v; \cdot))$ for all $t \in [0, T]$, and $u'(s) \in D(\psi^s(v; \cdot))$ for a.a. $s \in (0, T)$. By the Mosco convergence of $\tilde{\Psi}^t(v_n; \cdot)$ for each $t \in [0, T]$ we can find a sequence $\{z_n\}_{n \in \mathbb{N}} \subset L^2(0, t; V)$ such that

$$z_n \rightarrow u' \text{ in } L^2(0, t; V) \text{ and } \tilde{\Psi}^t(v_n; z_n) \rightarrow \tilde{\Psi}^t(v; u'). \quad (2.31)$$

Setting

$$w_n(s) := \int_0^s z_n(\tau) d\tau + u_0 \text{ in } V, \quad \forall s \in [0, t], \quad (2.32)$$

we observe that $z_n(s) \in D(\psi^s(v_n; \cdot))$ for a.a. $s \in (0, t)$, and

$$w'_n(s) = z_n(s) \text{ in } V, \text{ a.a. } s \in (0, t),$$

hence, $w'_n(s) \in D(\psi^s(v_n; \cdot))$ for a.a. $s \in (0, t)$. Also, we infer from (2.27), (2.31), and (2.32) that

$$w_n(s) \rightarrow \int_0^s u'(\tau) d\tau + u_0 (= u(s)) \text{ in } V, \quad \forall s \in [0, t], \quad (2.33)$$

and

$$w'_n(= z_n) \rightarrow u' \text{ in } L^2(0, t; V), \quad \forall t \in [0, T]. \quad (2.34)$$

As u_n is a solution to $CP_{v_n}(f_n, u_0)$, there exists a function ξ_n in $L^2(0, T; V^*)$ such that

$$\xi_n(t) \in \partial_* \psi^t(v_n; u'_n(t)) \text{ in } V^*, \text{ a.a. } t \in (0, T)$$

and

$$\xi_n(t) + \partial_* \varphi^t(v_n; u_n(t)) + g(v_n; t, u_n(t)) = f_n(t) \text{ in } V^*, \text{ a.a. } t \in (0, T).$$

Here, we multiply this equation by $u'_n(s) - w'_n(s)$ to get:

$$\begin{aligned} & \langle \xi_n(s), u'_n(s) - w'_n(s) \rangle + \langle \partial_* \varphi^s(v_n; u_n(s)), u'_n(s) - w'_n(s) \rangle \\ & \quad + \langle g(v_n; s, u_n(s)), u'_n(s) - w'_n(s) \rangle \\ & = \langle f_n(s), u'_n(s) - w'_n(s) \rangle, \quad \text{a.a. } s \in (0, t). \end{aligned} \quad (2.35)$$

Also, we note that

$$\langle \xi_n(s), u'_n(s) - w'_n(s) \rangle \geq \psi^s(v_n; u'_n(s)) - \psi^s(v_n; w'_n(s)), \quad \text{a.a. } s \in (0, t), \quad (2.36)$$

and by $(\phi 1)$ and Lemma 2.3 that

$$\begin{aligned} & \langle \partial_* \varphi^s(v_n; u_n(s)), u'_n(s) - w'_n(s) \rangle \\ &= \langle \partial_* \varphi^s(v_n; u_n(s) - w_n(s)), u'_n(s) - w'_n(s) \rangle + \langle \partial_* \varphi^s(v_n; w_n(s)), u'_n(s) - w'_n(s) \rangle \\ &\geq \frac{d}{ds} \varphi^s(v_n; u_n(s) - w_n(s)) - |\alpha'(s)| \varphi^s(v_n; u_n(s) - w_n(s)) \\ &\quad + \langle \partial_* \varphi^s(v_n; w_n(s)), u'_n(s) - w'_n(s) \rangle, \quad \text{a.a. } s \in (0, t). \end{aligned} \quad (2.37)$$

Therefore, from (2.35)~(2.37) we obtain that:

$$\begin{aligned} & \frac{d}{ds} \varphi^s(v_n; u_n(s) - w_n(s)) \\ &\leq |\alpha'(s)| \varphi^s(v_n; u_n(s) - w_n(s)) + L_n(s) + \psi^s(v_n; w'_n(s)) - \psi^s(v_n; u'_n(s)), \\ &\quad \text{a.a. } s \in (0, t), \end{aligned} \quad (2.38)$$

where $L_n(\cdot)$ is a function defined by:

$$L_n(s) := \langle f_n(s) - \partial_* \varphi^s(v_n; w_n(s)) - g(v_n; s, u_n(s)), u'_n(s) - w'_n(s) \rangle, \quad \text{a.a. } s \in (0, t).$$

Multiplying (2.38) by $e^{-\int_0^s |\alpha'(\tau)| d\tau}$ and integrating in time, we use $\varphi^0(v_n; 0) = 0$ (cf. $(\phi 3)$) to obtain:

$$\begin{aligned} & e^{-\int_0^t |\alpha'(\tau)| d\tau} \varphi^t(v_n; u_n(t) - w_n(t)) \\ &\leq \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} L_n(s) ds + \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} (\psi^s(v_n; w'_n(s)) + C_2) ds \\ &\quad - \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} (\psi^s(v_n; u'_n(s)) + C_2) ds \end{aligned} \quad (2.39)$$

for all $t \in [0, T]$. Here we note from $(\psi 2)$ with (2.1), (2.30), and (2.31) that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left\{ \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} (\psi^s(v_n; w'_n(s)) + C_2) ds \right. \\ &\quad \left. - \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} (\psi^s(v_n; u'_n(s)) + C_2) ds \right\} \\ &\leq \limsup_{n \rightarrow \infty} \left\{ \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} (\psi^s(v_n; w'_n(s)) + C_2) ds \right. \\ &\quad \left. - \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} (\psi^s(v; u'(s)) + C_2) ds \right\} \\ &\quad + \limsup_{n \rightarrow \infty} \left\{ \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} (\psi^s(v; u'(s)) + C_2) ds \right. \\ &\quad \left. - \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} (\psi^s(v_n; u'_n(s)) + C_2) ds \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \tilde{\Psi}^t(v_n; w'_n) - \tilde{\Psi}^t(v; u') \right\} - \liminf_{n \rightarrow \infty} \left\{ -\tilde{\Psi}^t(v; u') + \tilde{\Psi}^t(v_n; u'_n) \right\} \\ &\leq 0. \end{aligned} \quad (2.40)$$

Additionally, note from (2.27), (2.33), and Lemma 2.2(ii) that

$$\partial_* \varphi^s(v_n; w_n(s)) \rightarrow \partial_* \varphi^s(v; u(s)) \text{ in } V^*, \forall s \in [0, t]$$

Therefore, we infer from (2.33), and the Lebesgue dominated convergence theorem that

$$\partial_* \varphi^{(\cdot)}(v_n; w_n) \rightarrow \partial_* \varphi^{(\cdot)}(v; u) \text{ in } L^2(0, t; V^*) \text{ for all } t \in [0, T] \text{ as } n \rightarrow \infty. \quad (2.41)$$

Thus, it concludes from (2.28), (2.34), and (2.41) that

$$\begin{aligned} & \lim_{n \rightarrow 0} \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} \langle \partial_* \varphi^s(v_n; w_n(s)), u'_n(s) - w'_n(s) \rangle ds \\ &= \lim_{n \rightarrow 0} \int_0^t \langle e^{-\int_0^s |\alpha'(\tau)| d\tau} \partial_* \varphi^s(v_n; w_n(s)), u'_n(s) - w'_n(s) \rangle ds \\ &= 0. \end{aligned} \quad (2.42)$$

Similarly, we observe from assumption (g), (2.29), and the Lebesgue dominated convergence theorem that

$$g(v_n; \cdot, u_n) \rightarrow g(v; \cdot, u) \text{ in } L^2(0, t; V^*), \quad \forall t \in [0, T],$$

and therefore

$$\begin{aligned} & \lim_{n \rightarrow 0} \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} \langle g(v_n; s, u_n(s)), u'_n(s) - w'_n(s) \rangle ds \\ &= \lim_{n \rightarrow 0} \int_0^t \langle e^{-\int_0^s |\alpha'(\tau)| d\tau} g(v_n; s, u_n(s)), u'_n(s) - w'_n(s) \rangle ds \\ &= 0. \end{aligned} \quad (2.43)$$

Additionally,

$$\begin{aligned} & \lim_{n \rightarrow 0} \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} \langle f_n(s), u'_n(s) - w'_n(s) \rangle ds \\ &= \lim_{n \rightarrow 0} \int_0^t \langle e^{-\int_0^s |\alpha'(\tau)| d\tau} f_n(s), u'_n(s) - w'_n(s) \rangle ds \\ &= 0. \end{aligned} \quad (2.44)$$

Thus, it follows that

$$\lim_{n \rightarrow \infty} \int_0^t e^{-\int_0^s |\alpha'(\tau)| d\tau} L_n(s) ds = 0, \quad \forall t \in [0, T]. \quad (2.45)$$

We conclude from (2.39) with (2.40)~(2.45) that

$$\limsup_{n \rightarrow \infty} e^{-\int_0^t |\alpha'(\tau)| d\tau} \varphi^t(v_n; u_n(t) - w_n(t)) \leq 0 \quad \text{uniformly in } t \in [0, T]. \quad (2.46)$$

Hence,

$$\limsup_{n \rightarrow \infty} \varphi^t(v_n; u_n(t) - w_n(t)) \leq 0 \quad \text{uniformly in } t \in [0, T],$$

which implies that

$$u_n(t) - w_n(t) \rightarrow 0 \text{ in } V \text{ uniformly in } t \in [0, T] \text{ as } n \rightarrow \infty.$$

This implies from (2.29) and (2.33) that

$$u_n(t) \rightarrow u(t) \text{ uniformly in } V \text{ on } [0, T] \text{ as } n \rightarrow \infty;$$

thus, we conclude that

$$u_n \rightarrow u \text{ in } C([0, T]; V). \quad (2.47)$$

Finally we show that u is a solution to $CP_v(f, u_0)$. We first note from (2.47) and Lemma 2.2(ii) that

$$\partial_* \varphi^t(v_n; u_n(t)) \rightarrow \partial_* \varphi^t(v; u(t)) \text{ in } V^*, \quad \forall t \in [0, T].$$

Therefore, by $(\phi 1)$ with the Lebesgue dominated convergence theorem,

$$\partial_* \varphi^{(\cdot)}(v_n; u_n) \rightarrow \partial_* \varphi^{(\cdot)}(v; u) \text{ in } L^2(0, T; V^*), \quad (2.48)$$

so that

$$\begin{aligned} \partial_* \psi^{(\cdot)}(v_n; u'_n) \ni \xi_n &:= f_n - \partial_* \varphi^{(\cdot)}(v_n; u_n) - g(v_n; \cdot, u_n) \\ &\rightarrow f - \partial_* \varphi^{(\cdot)}(v; u) - g(v; \cdot, u) =: \xi \text{ in } L^2(0, T; V^*), \quad \forall t \in [0, T] \end{aligned}$$

as $n \rightarrow \infty$. As to the limit function ξ , we have by the basic property the Mosco convergence (cf. (Fact 1) with $(\psi 1)$) that $\xi(t) \in \partial_* \psi^t(v; u'(t))$ for a.a. $t \in (0, T)$. Hence,

$$\xi(t) + \partial_* \varphi^t(v; u(t)) + g(v; t, u(t)) = f(t) \text{ in } V^*, \text{ a.a. } t \in (0, T).$$

Since $u(0) = u_0$ in V , thus, u is a solution to $CP_v(f, u_0)$. $\square$

2.4 Approximate problems for $CP_v(f, u_0)$

In this subsection we discuss the approximate problem for $CP_v(f, u_0)$ under the same assumptions as Theorem 2.1, and its convergence.

We begin with the formulation of auxiliary approximate problems for $CP_v(f, u_0)$.

Let us choose a function $v_0 \in D_0$ and consider the continuous convex function $\varphi^0(v_0; \cdot)$ on V and its subdifferential $F_0 := \partial_* \varphi^0(v_0; \cdot)$. Then, on account of (2.7) of Lemma 2.1, we observe that $F_0 : D(F_0) = V \rightarrow V^*$ satisfies

$$c_0 |z|_V^2 \leq \langle F_0 z, z \rangle \leq c'_0 |z|_V^2, \quad \forall z \in V, \quad (2.49)$$

with $c_0 := \frac{C_4}{|\alpha'|_{L^1(0, T)} + 1}$ and $c'_0 = C_3$, so that F_0 is linear, coercive, continuous, and single-valued maximal monotone from V into V^* .

Now, for each $v \in D_0$ and $\varepsilon \in (0, 1]$, we consider the following doubly nonlinear evolution inclusion, referred to as $CP_{\varepsilon, v} := CP_{\varepsilon, v}(f, u_0)$:

$$\varepsilon F_0 u'_{\varepsilon, v}(t) + \xi_{\varepsilon, v}(t) + \partial_* \varphi^t(v; u_{\varepsilon, v}(t)) + g(v; t, u_{\varepsilon, v}(t)) \ni f(t) \text{ in } V^*, \text{ a.a. } t \in (0, T), \quad (2.50a)$$

$$\xi_{\varepsilon,v}(t) \in \partial_* \psi^t(v; u'_{\varepsilon,v}(t)) \text{ in } V^*, \text{ a.a. } t \in (0, T), \quad (2.50b)$$

$$u_{\varepsilon,v}(0) = u_0 \text{ in } V. \quad (2.50c)$$

We now prove an existence result of solutions to $CP_{\varepsilon,v}$.

Lemma 2.6. *Assume that $u_0 \in V$ and assumptions (ψ) , (ϕ) , and (g) are fulfilled. Let f be any function in $L^2(0, T; V^*)$. Then, for every $\varepsilon > 0$ and $v \in D_0$, there is a unique function $u_{\varepsilon,v} \in W^{1,2}(0, T; V)$ with $\xi_{\varepsilon,v} \in L^2(0, T; V)$ such that (2.50):= $\{(2.50a), (2.50b), (2.50c)\}$ holds. Such a function $u_{\varepsilon,v}$ is called a solution to $CP_{\varepsilon,v}$.*

Proof. For the construction of the solution to $CP_{\varepsilon,v}$ we apply the general theory of ordinary differential equations. To this end let us introduce the following mappings for each fixed $v \in D_0$ and $\varepsilon > 0$:

$$\mathcal{B}(t)z^* := (\varepsilon F_0 + \partial_* \psi^t(v; \cdot))^{-1} z^*, \quad \forall z^* \in V^*, \quad \forall t \in [0, T]$$

and

$$\mathcal{F}(t)z := f(t) - \partial_* \varphi^t(v; z) - g(v; t, z), \quad \forall z \in V, \quad \forall t \in [0, T].$$

We show that $\mathcal{B}(t)$ is Lipschitz continuous from V^* into V for each $t \in [0, T]$. In fact, setting $z_i = \mathcal{B}(t)z_i^*$, $i = 1, 2$, we observe that $z_i^* = \varepsilon F_0 z_i + z_{i,*}$ for some $z_{i,*} \in \partial_* \psi^t(v; z_i)$. By (2.49) and the monotonicity of $\partial_* \psi^t(v; \cdot)$, we have

$$\begin{aligned} \langle z_1^* - z_2^*, z_1 - z_2 \rangle &= \langle \varepsilon F_0 z_1 + z_{1,*} - \varepsilon F_0 z_2 - z_{2,*}, z_1 - z_2 \rangle \\ &\geq \varepsilon \langle F_0(z_1 - z_2), z_1 - z_2 \rangle \\ &\geq \varepsilon c_0 |z_1 - z_2|_V^2. \end{aligned}$$

Hence,

$$|\mathcal{B}(t)z_1^* - \mathcal{B}(t)z_2^*|_V = |z_1 - z_2|_V \leq \frac{1}{\varepsilon c_0} |z_1^* - z_2^*|_{V^*} \quad (2.51)$$

and $\mathcal{B}(t) : V^* \rightarrow V$ is Lipschitz for each $t \in [0, T]$. In particular, since $F_0 0 = 0$ and $\partial_* \psi^t(v; 0) \ni 0$, it follows that $0 = \varepsilon F_0 0 + 0$. This shows that $\mathcal{B}(t)0 = 0$ and by (2.51)

$$|\mathcal{B}(t)z^*|_V \leq \frac{1}{\varepsilon c_0} |z^*|_{V^*},$$

thus, the function $t \rightarrow \mathcal{B}(t)z^*$ is bounded in V for each $z^* \in V^*$.

Next, we fix any $z^* \in V^*$ to show that $t \in [0, T] \rightarrow \mathcal{B}(t)z^* \in V$ is continuous. Put $z^t := \mathcal{B}(t)z^*$, hence, $\varepsilon F_0 z^t + \partial_* \psi^t(v; z^t) \ni z^*$. Let $\{s_n\}_{n \in \mathbb{N}} \subset [0, T]$ with $s_n \rightarrow t$ (as $n \rightarrow \infty$). Note that $z^{s_n} \in D(\partial_* \psi^{s_n}(v; \cdot))$ and

$$z^* = \varepsilon F_0 z^{s_n} + z_*^{s_n} \text{ for some } z_*^{s_n} \in \partial_* \psi^{s_n}(v; z^{s_n}).$$

Also, we observe from $(\psi 1)$ that $\partial_* \psi^{s_n}(v; \cdot)$ converges to $\partial_* \psi^t(v; \cdot)$ in the graph sense as $n \rightarrow \infty$. Therefore, for $[z^t, z^* - \varepsilon F_0 z^t] \in G(\partial_* \psi^t(v; \cdot))$, there exists a sequence $\{[z_n, z_n^*]\}_{n \in \mathbb{N}} \subset V \times V^*$ such that $[z_n, z_n^*] \in G(\partial_* \psi^{s_n}(v; \cdot))$ in $V \times V^*$,

$$z_n \rightarrow z^t \text{ in } V \text{ and } z_n^* \rightarrow z^* - \varepsilon F_0 z^t \text{ in } V^* \text{ as } n \rightarrow \infty. \quad (2.52)$$

Since F_0 is Lipschitz continuous from V into V^* (cf. $(\phi 1)$), we observe from (2.52)

$$z_n^* + \varepsilon F_0 z_n \rightarrow z^* - \varepsilon F_0 z^t + \varepsilon F_0 z^t = z^* \quad \text{in } V^* \quad \text{as } n \rightarrow \infty.$$

Hence, we infer from the monotonicity of $\partial_* \psi^{s_n}(v; \cdot)$ that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \langle z^* - z_n^* - \varepsilon F_0 z_n, z^{s_n} - z_n \rangle \\ &= \lim_{n \rightarrow \infty} \langle \varepsilon F_0 z^{s_n} + z_*^{s_n} - z_n^* - \varepsilon F_0 z_n, z^{s_n} - z_n \rangle \\ &\geq \limsup_{n \rightarrow \infty} \varepsilon \langle F_0 z^{s_n} - F_0 z_n, z^{s_n} - z_n \rangle \\ &\geq \varepsilon c_0 \limsup_{n \rightarrow \infty} |z^{s_n} - z_n|_V^2, \end{aligned}$$

which implies that

$$z^{s_n} = \mathcal{B}(s_n)z^* \rightarrow z^t = \mathcal{B}(t)z^* \quad \text{in } V \quad \text{as } s_n \rightarrow t.$$

Thus, the operator $\mathcal{B}(t)z^*$ is continuous in V with respect to $t \in [0, T]$ for all $z^* \in V^*$. Similarly, since $f \in L^2(0, T; V^*)$, it follows from assumptions (ϕ) and (g) that the operator $\mathcal{F}(t)z : [0, T] \times V \rightarrow V^*$ is Lipschitz in $z \in V$ for a.a. $t \in (0, T)$ and integrable in $t \in [0, T]$ for each $z \in V$ and $f \in L^2(0, T; V^*)$.

From the above observation and assumption (g) we see that the composition mapping $\mathcal{B}(t) \circ \mathcal{F}(t)z$ is integrable on $(0, T)$ in V for each $z \in V$ and Lipschitz continuous in $z \in V$ for a.a. $t \in (0, T)$. Therefore, by virtue of the general theory of ordinary differential equations, the problem

$$u'(t) = \mathcal{B}(t) \circ \mathcal{F}(t)u(t) \quad \text{in } V, \quad t \in (0, T), \quad u(0) = u_0 \quad \text{in } V,$$

possesses a unique solution $u \in W^{1,2}(0, T; V)$. Denoting u by $u_{\varepsilon, v}$ we see that $u_{\varepsilon, v}$ satisfies

$$\varepsilon F_0 u'_{\varepsilon, v}(t) + \xi_{\varepsilon, v}(t) + \partial_* \varphi^t(v; u_{\varepsilon, v}(t)) + g(v; t, u_{\varepsilon, v}(t)) = f(t) \quad \text{in } V^*, \quad \text{a.a. } t \in (0, T),$$

where $\xi_{\varepsilon, v} \in L^2(0, T; V^*)$ and

$$\xi_{\varepsilon, v}(t) \in \partial_* \psi^t(v; u'_{\varepsilon, v}(t)) \quad \text{in } V^*, \quad \text{a.a. } t \in (0, T).$$

Thus $u_{\varepsilon, v}$ is a unique solution of $CP_{\varepsilon, v}$. □

We here give the energy inequality to $CP_{\varepsilon, v}$.

Lemma 2.7. *For any fixed $v \in D_0$ and $\varepsilon > 0$, the solution $u_{\varepsilon, v}$ to $CP_{\varepsilon, v}$ satisfies the bound: there exists a constant $N_1 > 0$, independent of $\varepsilon > 0$, $v \in D_0$, and $f \in L^2(0, T; V^*)$, such that*

$$\begin{aligned} &\varepsilon C_4 |u'_{\varepsilon, v}|_{L^2(0, T; V)}^2 + \int_0^T \psi^t(v; u'_{\varepsilon, v}(t)) dt + \sup_{t \in [0, T]} \varphi^t(v; u_{\varepsilon, v}(t)) \\ &\leq N_1 (|u_0|_V^2 + |f|_{L^2(0, T; V^*)}^2 + 1), \end{aligned} \tag{2.53}$$

where C_4 is the same constant as in $(\phi 3)$ as well as N_1 in (2.20) of Lemma 2.5.

Proof. For each $\varepsilon > 0$ we put

$$\psi_\varepsilon^t(v; z) := \varepsilon \varphi^0(v_0; z) + \psi^t(v; z), \quad \forall z \in V. \quad (2.54)$$

As easily checked, the family $\{\psi_\varepsilon^t\}$ satisfies assumption (ψ) with ψ^t replaced by ψ_ε^t . In addition, from (2.54) and the general theory of maximal monotone operators of the sub-differential type, we observe that $CP_{\varepsilon, v}$ can be regarded as CP_v with ψ^t replaced by ψ_ε^t . Therefore, applying the energy estimate obtained by Lemma 2.5, we have

$$\int_0^T \psi_\varepsilon^t(v; u'_{\varepsilon, v}(t)) dt + \sup_{t \in [0, T]} \varphi^t(v; u_{\varepsilon, v}(t)) \leq N_1(|u_0|_V^2 + |f|_{L^2(0, T; V^*)}^2 + 1).$$

Since $\varphi^0(v_0; z) \geq C_4|z|_V^2$ for all $z \in V$ by $(\phi 3)$, estimate (2.53) is immediately obtained from the above inequality. $\square$

Fixing $\varepsilon > 0$ and $f \in L^2(0, T; V^*)$, we denote by $S_\varepsilon(v)$ the solution to $CP_{\varepsilon, v}$ for each $v \in D_0$. It is clear that any fixed point of $S_\varepsilon(\cdot)$ is a solution u_ε to (2.6).

We now prove an existence result of solutions to an approximate equation (2.6) for $CP(f, u_0)$.

Proposition 2.3. *Suppose that $u_0 \in V$ and assumptions (ψ) , (ϕ) , and (g) are fulfilled. Let $\varepsilon > 0$ and $f \in L^2(0, T; V^*)$. Then there are functions $u_\varepsilon \in W^{1,2}(0, T; V)$ and $\xi_\varepsilon \in L^2(0, T; V^*)$ such that*

$$\varepsilon F_0 u'_\varepsilon(t) + \xi_\varepsilon(t) + \partial_* \varphi^t(u_\varepsilon; u_\varepsilon(t)) + g(u_\varepsilon; t, u_\varepsilon(t)) = f(t) \text{ in } V^*, \text{ a.a. } t \in (0, T), \quad (2.55a)$$

$$\xi_\varepsilon(t) \in \partial_* \psi^t(u_\varepsilon; u'_\varepsilon(t)) \text{ in } V^*, \text{ a.a. } t \in (0, T), \quad (2.55b)$$

$$u_\varepsilon(0) = u_0 \text{ in } V. \quad (2.55c)$$

Moreover, we have the following uniform estimate

$$\begin{aligned} & \varepsilon C_4 |u'_\varepsilon|_{L^2(0, T; V)}^2 + \int_0^T \psi^t(u_\varepsilon; u'_\varepsilon(t)) dt + \sup_{t \in [0, T]} \varphi^t(u_\varepsilon; u_\varepsilon(t)) \\ & \leq N_1(|u_0|_V^2 + |f|_{L^2(0, T; V^*)}^2 + 1). \end{aligned} \quad (2.56)$$

Proof. For the existence of solution to (2.55) ($= \{(2.55a), (2.55b), (2.55c)\}$) it is enough to find a fixed point of the mapping S_ε in D_0 . To this end, with the bound (2.56), we put

$$\tilde{N}_1 := N_1(|u_0|_V^2 + |f|_{L^2(0, T; V^*)}^2 + 1)$$

and

$$X(\tilde{N}_1) := \left\{ v \in W^{1,2}(0, T; V) \mid v(0) = u_0 \text{ in } V, \int_0^T \psi_0^t(v'(t)) dt \leq \tilde{N}_1 \right\};$$

note that $X(\tilde{N}_1) \subset D_0$. In addition, note from Lemmas 2.6 and 2.7 that $S_\varepsilon(v) \in X(\tilde{N}_1)$ for any $v \in X(\tilde{N}_1)$. Clearly, $X(\tilde{N}_1)$ is non-empty, compact, and convex in $C([0, T]; H)$ as well as bounded and closed in $W^{1,2}(0, T; V)$.

We next show that S_ε is continuous in $X(\tilde{N}_1)$ with respect to the topology of $C([0, T]; H)$. To this end, let $\{v_n\}_{n \in \mathbb{N}}$ be a sequence in $X(\tilde{N}_1)$ such that $v_n \rightarrow v$ in $C([0, T]; H)$; note from Lemma 2.7 and the definition of $X(\tilde{N}_1)$ that $v_n \rightarrow v$ weakly in $W^{1,2}(0, T; V)$ and $\{S_\varepsilon(v_n)\}_{n \in \mathbb{N}} \subset X(\tilde{N}_1)$. Now, put $u_{\varepsilon, n} := S_\varepsilon(v_n)$. Then, note from (2.54) that CP_{ε, v_n} can be regarded as CP_{v_n} with ψ^t replaced by ψ_ε^t . Therefore, by Proposition 2.2 for CP_{v_n} ($=CP_{\varepsilon, v_n}$) and by extracting a subsequence from $\{u_{\varepsilon, n}\}_{n \in \mathbb{N}}$ if necessary, referred to as $\{u_{\varepsilon, n}\}_{n \in \mathbb{N}}$ again, we observe that $u_{\varepsilon, n} \rightarrow \tilde{u}$ in $C([0, T]; V)$ as $n \rightarrow \infty$ for some function $\tilde{u} \in W^{1,2}(0, T; V)$, and $\tilde{u}$ is a solution to $CP_{\varepsilon, v}$, namely $\tilde{u} = S_\varepsilon(v)$. From the uniqueness of solution to $CP_{\varepsilon, v}$, we conclude that $S_\varepsilon(v_n) \rightarrow S_\varepsilon(v)$ in $C([0, T]; V)$ without extracting any subsequence of $\{S_\varepsilon(v_n)\}_{n \in \mathbb{N}}$. This shows that S_ε is continuous in $X(\tilde{N}_1)$ with respect to the topology of $C([0, T]; H)$.

Therefore, by the Schauder fixed point theorem, S_ε has at least one fixed point u_ε in $X(\tilde{N}_1)$, $u_\varepsilon = S_\varepsilon(u_\varepsilon)$. This is a solution to (2.55). Besides, the uniform estimate (2.56) is immediately obtained by Lemma 2.7. $\square$

3 Proof of Theorem 2.1 and comments of uniqueness

In this section, we give the proof of Theorem 2.1 and remark on uniqueness of solutions to $CP(f, u_0)$. To this end, system (2.55) is denoted by $CP_\varepsilon(\psi^t, \varphi^t, g; f, u_0) := CP(\psi_\varepsilon^t, \varphi^t, g; f, u_0)$.

We first prove Theorem 2.1 by letting $\varepsilon \rightarrow 0$ in $CP_\varepsilon(\psi^t, \varphi^t, g; f, u_0)$.

Proof of Theorem 2.1. Let u_ε be a solution to $CP_\varepsilon(\psi^t, \varphi^t, g; f, u_0)$. Note from Proposition 2.3 that the uniform estimate (2.56) holds and u_ε satisfies that

$$\varepsilon F_0 u'_\varepsilon(t) + \xi_\varepsilon(t) + \partial_* \varphi^t(u_\varepsilon; u_\varepsilon(t)) + g(u_\varepsilon; t, u_\varepsilon(t)) = f(t) \quad \text{in } V^*, \text{ a.a. } t \in (0, T),$$

namely

$$\xi_\varepsilon(t) + \partial_* \varphi^t(u_\varepsilon; u_\varepsilon(t)) + g(u_\varepsilon; t, u_\varepsilon(t)) = f(t) - \varepsilon F_0 u'_\varepsilon(t) \quad \text{in } V^*, \text{ a.a. } t \in (0, T), \quad (3.1)$$

with

$$\xi_\varepsilon \in L^2(0, T; V^*), \quad \xi_\varepsilon(t) \in \partial_* \psi^t(u_\varepsilon; u'_\varepsilon(t)) \text{ in } V^*, \text{ a.a. } t \in (0, T), \quad u_\varepsilon(0) = u_0 \text{ in } V.$$

From the uniform estimate (2.56) and the Ascoli–Arzelà theorem, we can find a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ with $\varepsilon_n \downarrow 0$ (as $n \rightarrow \infty$) and a function $u \in W^{1,2}(0, T; V)$ such that

$$u_{\varepsilon_n} \rightarrow u \quad \text{in } C([0, T]; H) \text{ and weakly in } W^{1,2}(0, T; V),$$

$$u_{\varepsilon_n}(t) \rightarrow u(t) \text{ weakly in } V, \quad \forall t \in [0, T]$$

as $n \rightarrow \infty$. Since $\varepsilon_n F_0 u'_{\varepsilon_n} \rightarrow 0$ in $L^2(0, T; V^*)$ (as $n \rightarrow \infty$), it follows from Proposition 2.2 that the limit u is a solution of $CP_u(f, u_0)$, namely u is a solution to $CP(f, u_0)$, satisfying (2.4). By Lemma 2.5, any solution of $CP(f, u_0)$ satisfies (2.4). Thus, the proof of Theorem 2.1 is complete. $\square$

In general, $CP(f, u_0)$ has multiple solutions as the next simple example shows.

Example 3.1. Let $\Omega = (0, 1)$, and set $V = H^1(\Omega)$ and $H = L^2(\Omega)$. As usual, set $Q := \Omega \times (0, T)$, and let ρ be a prescribed obstacle function in $C(\overline{Q})$ such that

$$1 \leq \rho(x, t) \leq \rho^*, \quad \forall (x, t) \in \overline{Q}, \quad (3.2)$$

where ρ^* is a positive constant.

Now, for each $t \in [0, T]$, define a closed convex subset $K(t)$ of V by

$$K(t) := \{z \in V ; |z(x)| \leq \rho(x, t), |z_x(x)| \leq \rho(x, t), \text{ a.a. } x \in \Omega\}. \quad (3.3)$$

Then, we consider the following variational inclusion:

$$\begin{cases} u_t(t) \in K(t) & \text{for a.a. } t \in (0, T), \\ \int_{\Omega} u_x(x, t)(u_{xt}(x, t) - w_x(x))dx \leq 0, & \forall w \in K(t), \text{ a.a. } t \in (0, T), \\ u(x, 0) = 0, & x \in \Omega. \end{cases} \quad (3.4)$$

For each $t \in [0, T]$, as the time-dependent functional $\psi^t(v; \cdot)$ we consider

$$\psi^t(z) := I_{K(t)}(z) = \begin{cases} 0, & \text{if } z \in K(t), \\ \infty, & \text{otherwise,} \end{cases}$$

and as the functional $\varphi^t(v; z)$ we choose

$$\varphi^t(z) := \frac{1}{2} \int_{\Omega} |z_x(x)|^2 dx + \frac{1}{2} \int_{\Omega} |z(x)|^2 dx, \quad \forall t \in [0, T], \quad \forall z \in V;$$

in the present case, D_0 is formally given by (2.3) with $\psi_0^t = I_{K(t)}$, namely

$$D_0 = \{v \in W^{1,2}(0, T; V) \mid v'(t) \in K(t) \text{ for a.a. } t \in [0, T], v(0) = 0 \text{ in } V\},$$

but actually this set is not used, since ψ^t and φ^t are independent of $v \in D_0$. Now we have:

(i) $z^* \in \partial_* \psi^t(z)$ if and only if

$$z^* \in V^*, \quad z \in K(t), \quad \text{and} \quad \langle z^*, w - z \rangle \leq 0 \text{ for all } w \in K(t),$$

$$(ii) \quad \langle \partial_* \varphi^t(z), w \rangle = \int_{\Omega} z_x(x) w_x(x) dx + \int_{\Omega} z(x) w(x) dx \quad \text{for all } z, w \in V,$$

for all $t \in [0, T]$.

Also, we observe that problem (3.4) is written as $CP(\psi^t, \varphi^t, g; 0, 0)$ with $g(v; t, z) = -z$. It is easy to check assumptions (ψ) , (ϕ) , and (g) . Therefore, by Theorem 2.1, problem (3.4) has at least one solution $u \in W^{1,2}(0, T; V)$.

Moreover, note that for each constant $c \in (0, 1)$ the function u^c defined by

$$u^c(x, t) := c(1 - e^{-t}) \quad \text{for all } (x, t) \in Q$$

is a solution to (3.4) on $[0, T]$. Indeed, we observe that

$$u_t^c(x, t) = ce^{-t}, \quad u_x^c(x, t) = 0, \quad u_{xt}^c(x, t) = 0$$

for all $(x, t) \in Q$. Therefore, by (3.3),

$$u_t^c(t) \in K(t), \quad \forall t \in (0, T).$$

Hence, we easily observe that for each $c \in (0, 1)$ the function u^c satisfies (3.4). Thus, $\{u^c\}_{c \in (0, 1)}$ provides with an infinite family of solutions to (3.4) on $[0, T]$.

As is seen from the above counterexample, the uniqueness of solutions to doubly nonlinear evolution inclusions is not expected, in general. However, in a restricted class for ψ^t and φ^t we have the uniqueness of solution.

Proposition 3.1. *Suppose that ψ^t and g are independent of $v \in D_0$, namely $\psi^t(v; z) = \psi^t(z)$ and $g(v; t, z) = g(t, z)$ for all $v \in D_0$ and $z \in V$; in this case $\psi_0^t := \psi^t$ on V . In addition, assume that ψ^t is uniformly monotone for any $t \in [0, T]$, namely, there exists a positive constant $C_5 > 0$ such that*

$$\begin{aligned} \langle \xi_1 - \xi_2, z_1 - z_2 \rangle &\geq C_5 |z_1 - z_2|_V^2, \\ \forall z_i &\in D(\partial_* \psi^t), \quad \xi_i \in \partial_* \psi^t(z_i) \quad (i = 1, 2), \quad \forall t \in [0, T]. \end{aligned} \quad (3.5)$$

Furthermore, suppose that $u_0 \in V$ and assumptions (ψ) , (ϕ) , and (g) are fulfilled. Also, assume that $\partial_* \varphi^t(v; \cdot)$ is Lipschitz in $v \in D_0$, more precisely, there exists a positive constant $C_6 > 0$ such that

$$\begin{aligned} |\partial_* \varphi^t(v_1; z) - \partial_* \varphi^t(v_2; z)|_{V^*} &\leq C_6 |v_1(t) - v_2(t)|_V (1 + |z|_V), \\ \forall v_i &\in D_0 \quad (i = 1, 2), \quad \forall z \in D_0, \quad \forall t \in [0, T]. \end{aligned} \quad (3.6)$$

Let f be any function in $L^2(0, T; V^*)$. Then, $CP(\psi^t, \varphi^t, g; f, u_0)$ has at most one solution.

Proof. Let u_i , $i = 1, 2$, be two solutions of $CP(\psi^t, \varphi^t, g; f, u_0)$. Then, by Theorem 2.1, we have $u_i \in W^{1,2}(0, T; V)$ and $u_i \in D_0$ for $i = 1, 2$. Subtract the evolution inclusion for $i = 2$ from the one for $i = 1$, and multiply the resultant by $u'_1 - u'_2$ to get

$$\begin{aligned} \langle \xi_1(t) - \xi_2(t), u'_1(t) - u'_2(t) \rangle &+ \langle \partial_* \varphi^t(u_1; u_1(t)) - \partial_* \varphi^t(u_2; u_2(t)), u'_1(t) - u'_2(t) \rangle \\ &+ \langle g(t, u_1(t)) - g(t, u_2(t)), u'_1(t) - u'_2(t) \rangle = 0 \quad \text{for a.a. } t \in (0, T), \end{aligned} \quad (3.7)$$

where $\xi_i(t) \in \partial_* \psi^t(u'_i(t))$ for a.a. $t \in (0, T)$ ($i = 1, 2$). From (3.5) we observe that

$$\langle \xi_1(t) - \xi_2(t), u'_1(t) - u'_2(t) \rangle \geq C_5 |u'_1(t) - u'_2(t)|_V^2 \quad \text{for a.a. } t \in (0, T), \quad (3.8)$$

and by Lemma 2.3 and (3.6) that

$$\begin{aligned} &\langle \partial_* \varphi^t(u_1; u_1(t)) - \partial_* \varphi^t(u_2; u_2(t)), u'_1(t) - u'_2(t) \rangle \\ &= \langle \partial_* \varphi^t(u_1; u_1(t)) - \partial_* \varphi^t(u_2; u_1(t)), u'_1(t) - u'_2(t) \rangle \\ &\quad + \langle \partial_* \varphi^t(u_2; u_1(t)) - \partial_* \varphi^t(u_2; u_2(t)), u'_1(t) - u'_2(t) \rangle \\ &\geq -|\partial_* \varphi^t(u_1; u_1(t)) - \partial_* \varphi^t(u_2; u_1(t))|_{V^*} |u'_1(t) - u'_2(t)|_V \\ &\quad + \frac{d}{dt} \varphi^t(u_2; u_1(t) - u_2(t)) - |\alpha'(t)| \varphi^t(u_2; u_1(t) - u_2(t)) \\ &\geq -C_6 |u_1(t) - u_2(t)|_V (1 + |u_1(t)|_V) |u'_1(t) - u'_2(t)|_V \\ &\quad + \frac{d}{dt} \varphi^t(u_2; u_1(t) - u_2(t)) - |\alpha'(t)| \varphi^t(u_2; u_1(t) - u_2(t)) \end{aligned} \quad (3.9)$$

for a.a. $t \in (0, T)$. Therefore, we observe from (3.7)~(3.9) and (g2) with the help of the Schwarz inequality that

$$\begin{aligned}
& C_5 |u'_1(t) - u'_2(t)|_V^2 + \frac{d}{dt} \varphi^t(u_2; u_1(t) - u_2(t)) \\
& \leq |\alpha'(t)| \varphi^t(u_2; u_1(t) - u_2(t)) + C_6 |u_1(t) - u_2(t)|_V (1 + |u_1(t)|_V) |u'_1(t) - u'_2(t)|_V \\
& \quad + |g(t, u_1(t)) - g(t, u_2(t))|_{V^*} |u'_1(t) - u'_2(t)|_V \\
& \leq |\alpha'(t)| \varphi^t(u_2; u_1(t) - u_2(t)) + \frac{C_6^2}{C_5} |u_1(t) - u_2(t)|_V^2 (1 + |u_1(t)|_V)^2 + \frac{C_5}{4} |u'_1(t) - u'_2(t)|_V^2 \\
& \quad + \frac{1}{C_5} |g(t, u_1(t)) - g(t, u_2(t))|_{V^*}^2 + \frac{C_5}{4} |u'_1(t) - u'_2(t)|_V^2 \\
& \leq |\alpha'(t)| \varphi^t(u_2; u_1(t) - u_2(t)) + \frac{C_6^2}{C_5} |u_1(t) - u_2(t)|_V^2 (1 + |u_1(t)|_V)^2 \\
& \quad + \frac{L_g^2}{C_5} |u_1(t) - u_2(t)|_V^2 + \frac{C_5}{2} |u'_1(t) - u'_2(t)|_V^2
\end{aligned}$$

for a.a. $t \in (0, T)$. From the above inequality with (2.7) of Lemma 2.1 we infer that

$$\begin{aligned}
& \frac{C_5}{2} |u'_1(t) - u'_2(t)|_V^2 + \frac{d}{dt} \varphi^t(u_2; u_1(t) - u_2(t)) \\
& \leq K_1 (|\alpha'(t)| + |u_1(t)|_V^2 + 1) \varphi^t(u_2; u_1(t) - u_2(t)) \quad \text{for a.a. } t \in (0, T)
\end{aligned} \tag{3.10}$$

for some constant $K_1 > 0$ being independent of u_i ($i = 1, 2$). Hence, applying the Gronwall inequality to (3.10), we conclude that $u_1(t) - u_2(t) = 0$ in V for all $t \in [0, T]$. Thus the proof of Proposition 3.1 has been completed. $\square$

By arguments similar to that as in Proposition 3.1, we can obtain the following uniqueness result of solutions to $CP_v(f, u_0)$.

Corollary 3.1. *Suppose that $u_0 \in V$ and assumptions (ψ) , (ϕ) , and (g) are fulfilled. In addition, assume that $\psi^t(v; \cdot)$ is uniformly monotone for any $t \in [0, T]$ and $v \in D_0$, namely, there exists a positive constant $C_5 > 0$ such that*

$$\begin{aligned}
& \langle \xi_1 - \xi_2, z_1 - z_2 \rangle \geq C_5 |z_1 - z_2|_V^2, \\
& \forall z_i \in D(\partial_* \psi^t(v; \cdot)), \quad \xi_i \in \partial_* \psi^t(v; z_i), \quad i = 1, 2, \quad \forall v \in D_0, \quad \forall t \in [0, T].
\end{aligned}$$

Let f be any function in $L^2(0, T; V^*)$. Then, problem $CP_v(f, u_0)$, namely

$$\begin{cases} \partial_* \psi^t(v; u'(t)) + \partial_* \varphi^t(v; u(t)) + g(v; t, u(t)) \ni f(t) & \text{in } V^*, \quad \text{a.a. } t \in (0, T), \\ u(0) = u_0 & \text{in } V, \end{cases}$$

admits a unique solution u on $[0, T]$.

Remark 3.1. Under the following assumptions:

$$\begin{aligned} \langle \xi_1 - \xi_2, z_1 - z_2 \rangle &\geq C_5 |z_1 - z_2|_H^2, \\ \forall z_i &\in D(\partial_* \psi^t(v; \cdot)), \quad \xi_i \in \partial_* \psi^t(v; z_i), \quad i = 1, 2, \quad \forall v \in D_0, \quad \forall t \in [0, T], \end{aligned}$$

and $g(v; t, \cdot)$ is uniformly Lipschitz from V into H in the sense that

$$|g(v; t, z_1) - g(v; t, z_2)|_H \leq L'_g |z_1 - z_2|_H, \quad \forall z_i \in V, \quad i = 1, 2, \quad \forall v \in D_0, \quad \forall t \in [0, T]$$

for a positive constant L'_g . Then, Proposition 3.1 guarantees that for any fixed $v \in D_0$, problem $CP_v(f, u_0)$ admits a unique solution u on $[0, T]$. Because, by a modification of the computation in the proof of Proposition 3.1, we have for any two solutions u_1 and u_2

$$\begin{aligned} &\frac{C_5}{2} |u'_1(t) - u'_2(t)|_H^2 + \frac{d}{dt} \varphi^t(v; u_1(t) - u_2(t)) \\ &\leq K'_1 (|\alpha'(t)| + 1) \varphi^t(v; u_1(t) - u_2(t)), \quad \text{a.a. } t \in (0, T) \end{aligned}$$

for a certain constant $K'_1 > 0$ being independent of u_i ($i = 1, 2$). Therefore, it results by the Gronwall inequality that $u_1(t) - u_2(t) = 0$ in V for all $t \in [0, T]$.

4 Applications

A sort of quasi-variational structure is found in our class of doubly nonlinear evolution inclusions, more concretely in the v -dependence of $\psi^t(v; \cdot)$ and $\varphi^t(v; \cdot)$. In this section we shall deal with some doubly nonlinear quasi-variational evolution inclusions as applications of our abstract result.

4.1 Doubly nonlinear quasi-variational evolution inclusions

Let Ω be a bounded domain in $\mathbb{R}^N$ ($1 \leq N < \infty$) with a smooth boundary $\Gamma := \partial\Omega$, $Q := \Omega \times (0, T)$, and $\Sigma := \Gamma \times (0, T)$ for $0 < T < \infty$, and put

$$V := H_0^1(\Omega), \quad H := L^2(\Omega), \quad V^* := H^{-1}(\Omega);$$

we employ $|z|_V := |\nabla z|_H$ as the norm of V ; $\langle \cdot, \cdot \rangle$ stands for the duality between V^* and V . In this subsection we treat a quasi-variational inequality with gradient constraint for the time derivatives.

Let ρ be a prescribed obstacle function such that

$$\begin{aligned} \rho &:= \rho(x, t, r) \in C(\overline{Q} \times \mathbb{R}), \\ 0 &< \rho_* \leq \rho(x, t, r) \leq \rho^*, \quad \forall (x, t, r) \in \overline{Q} \times \mathbb{R}, \\ |\rho(x, t_1, r_1) - \rho(x, t_2, r_2)| &\leq L_\rho (|t_1 - t_2| + |r_1 - r_2|), \\ \forall t_i &\in [0, T], \quad \forall r_i \in \mathbb{R}, \quad i = 1, 2, \quad \forall x \in \overline{\Omega}, \end{aligned} \tag{4.1}$$

where ρ_* , ρ^* , and L_ρ are positive constants.

(Application 1)

For each $t \in [0, T]$ and $v \in C([0, T]; H)$, we define a convex constraint set $K(v; t)$ in V by

$$K(v; t) := \{z \in V ; |\nabla z(x)| \leq \rho(x, t, v(x, t)), \text{ a.a. } x \in \Omega\}, \quad \forall t \in [0, T], \quad (4.2)$$

and a convex subset K_0 of V by

$$K_0 := \{z \in V ; |\nabla z(x)| \leq \rho^*, \text{ a.a. } x \in \Omega\}. \quad (4.3)$$

Now consider the following quasi-variational inequality with time-dependent gradient constraint:

$$u'(t) := \frac{\partial u(\cdot, t)}{\partial t} \in K(u; t), \text{ a.a. } t \in (0, T), \quad u(\cdot, 0) = u_0 \text{ in } V, \quad (4.4a)$$

$$\begin{aligned} & \tau_0 \int_{\Omega} u'(x, t)(u'(x, t) - z(x))dx \\ & \quad + \int_{\Omega} a(x, t, u(x, t)) \nabla u(x, t) \cdot \nabla (u'(x, t) - z(x))dx \\ & \quad + \int_{\Omega} g(x, t, u(x, t))(u'(x, t) - z(x))dx \\ & \leq \int_{\Omega} f(x, t)(u'(x, t) - z(x))dx, \quad \forall z \in K(u; t), \text{ a.a. } t \in (0, T), \end{aligned} \quad (4.4b)$$

where $\tau_0 \geq 0$ is a constant, $g(\cdot, \cdot, \cdot)$ is a Lipschitz continuous function on $\overline{Q} \times \mathbb{R}$, i.e.,

$$\begin{aligned} |g(x_1, t_1, r_1) - g(x_2, t_2, r_2)| & \leq L_g(|x_1 - x_2| + |t_1 - t_2| + |r_1 - r_2|), \\ \forall (x_i, t_i) & \in \overline{Q}, \forall r_i \in \mathbb{R}, i = 1, 2, \end{aligned} \quad (4.5)$$

with a positive constant L_g , f is a function in $L^2(0, T; H)$, u_0 is an initial datum given in V , and $a(\cdot, \cdot, \cdot)$ is a prescribed function in $C(\overline{Q} \times \mathbb{R})$ such that

$$\begin{aligned} a_* & \leq a(x, t, r) \leq a^*, \quad \forall (x, t) \in \overline{Q}, \forall r \in \mathbb{R}, \\ |a(x, t_1, r_1) - a(x, t_2, r_2)| & \leq L_a(|t_1 - t_2| + |r_1 - r_2|), \\ \forall t_i & \in [0, T], \forall r_i \in \mathbb{R}, i = 1, 2, \forall x \in \overline{\Omega}, \end{aligned} \quad (4.6)$$

where a_* , a^* , and L_a are positive constants.

A function $u : [0, T] \rightarrow V$ is called a solution to (4.4):= {(4.4a), (4.4b)}, if $u \in W^{1,2}(0, T; V)$ and all of the properties required in (4.4) are fulfilled. In order to reformulate problem (4.4) as the form $CP(\psi^t, \varphi^t, g; f, u_0)$, the functionals $\psi_0^t(\cdot)$, $\psi^t(v; \cdot)$, $\varphi^t(v; \cdot)$ are set up as follows:

$$\psi_0^t(z) := I_{K_0}(z), \quad \forall z \in V, \forall t \in [0, T], \quad (4.7)$$

and we define

$$D_0 = \{v \in W^{1,2}(0, T; V) \mid v'(t) \in K_0, \text{ a.a. } t \in (0, T), v(0) = u_0 \text{ in } V\}, \quad (4.8)$$

$$\psi^t(v; z) := \frac{\tau_0}{2} \int_{\Omega} |z(x)|^2 dx + I_{K(v;t)}(z), \quad \forall z \in V, \quad \forall t \in [0, T], \quad \forall v \in D_0, \quad (4.9)$$

$$\varphi^t(v; z) := \frac{1}{2} \int_{\Omega} a(x, t, v(x, t)) |\nabla z(x)|^2 dx, \quad \forall z \in V, \quad \forall t \in [0, T], \quad \forall v \in D_0. \quad (4.10)$$

It is easy to see from the definition of subdifferential that

- (i) Let $v \in D_0$ and $t \in [0, T]$. Then $z^* \in \partial_* \psi^t(v; z)$ if and only if $z^* \in V^*$, $z \in K(v; t)$, and

$$\tau_0 \int_{\Omega} z(x)(z(x) - w(x)) dx + \langle -z^*, z - w \rangle \leq 0, \quad \forall w \in K(v; t).$$

- (ii) Let $v \in D_0$ and $t \in [0, T]$. Then $\partial_* \varphi^t(v; \cdot)$ is singlevalued, linear, and bounded from V into V^* and

$$\langle \partial_* \varphi^t(v; z), w \rangle = \int_{\Omega} a(x, t, v(x, t)) \nabla z(x) \cdot \nabla w(x) dx, \quad \forall w \in V.$$

In addition, for each $v \in D_0$ and $t \in [0, T]$, we define $g(v; t, \cdot) : V \rightarrow V^*$ by

$$\langle g(v; t, z), w \rangle := \int_{\Omega} g(x, t, z(x)) w(x) dx, \quad \forall z, w \in V. \quad (4.11)$$

From the above characterizations (i) and (ii) of subdifferentials of $\psi^t(v; \cdot)$ and $\varphi^t(v; \cdot)$ it follows that problem (4.4) is reformulated as $CP(\psi^t, \varphi^t, g; f, u_0)$.

Lemma 4.1. *Let D_0 be the set defined by (4.8). Then, we have:*

- (i) *There is a positive constant k_0 such that*

$$|v(x, t) - v(x, 0) - v(y, s) + v(y, 0)| \leq k_0(|x - y| + |t - s|),$$

$$\forall v \in D_0, \quad \forall x, y \in \overline{\Omega}, \quad \forall s, t \in [0, T].$$

- (ii) *If $\{v_n\}_{n \in \mathbb{N}} \subset D_0$ and $v_n \rightarrow v$ in $C([0, T]; H)$ (as $n \rightarrow \infty$), then $v_n - v \rightarrow 0$ in $C(\overline{Q})$.*

Proof. For any $v \in D_0$, we infer from assumption (4.3) that $v'(x, t)$ is Lipschitz in $x \in \overline{\Omega}$ and is bounded in Q , since $v'(x, t) = 0$ for $x \in \Gamma$. Therefore, with $M := |v'|_{L^\infty(Q)}$, we have

$$|v(x, t) - v(x, s)| \leq M|t - s|, \quad \forall x \in \overline{\Omega}, \quad \forall t, s \in [0, T].$$

Also, $|\nabla(v(x, t) - v(x, 0))| = |\int_0^t \nabla v'(x, \tau) d\tau| \leq T\rho^*$, a.e. on Ω , which implies that $\bar{v}(x, t) := v(x, t) - v(x, 0)$ is Lipschitz in $x \in \Omega$ uniformly in $t \in [0, T]$. This shows that

$$|\bar{v}(x, t) - \bar{v}(y, s)| \leq k_0(|x - y| + |t - s|), \quad \forall x, y \in \overline{\Omega}, \quad \forall t, s \in [0, T],$$

for some positive constant k_0 . Thus, we have (i).

Next, assume that $v_n \in D_0$ and $v_n \rightarrow v$ in $C([0, T]; H)$ as $n \rightarrow \infty$. Then, by (i) and the Ascoli-Arzelà theorem, there is a subsequence $\{\bar{v}_{n_k}\}_{k \in \mathbb{N}}$ of $\{\bar{v}_n := v_n - v_n(\cdot, 0)\}_{n \in \mathbb{N}}$

which uniformly converges to $\bar{v} := v - v(\cdot, 0)$ on $\bar{Q}$. The limit is determined uniquely by v , so that the whole sequence $\{\bar{v}_n\}_{n \in \mathbb{N}}$ converges to $\bar{v}$ uniformly on $\bar{Q}$. Note from (4.8) that $v_n(x, 0) = v(x, 0) = u_0(x)$, a.a. $x \in \Omega$, hence,

$$|v_n - v|_{C(\bar{Q})} = |\bar{v}_n - \bar{v}|_{C(\bar{Q})} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, (ii) holds. $\square$

Lemma 4.2. *Let ψ^t and φ^t be the functionals defined by (4.9) and (4.10), respectively. Then, assumptions (ψ) and (ϕ) are fulfilled.*

Proof. Regarding assumption (ψ) , its $(\psi 2)$ and $(\psi 3)$ are trivially satisfied. Therefore, we have only to check $(\psi 1)$. We now assume that $t_n \in [0, T]$, $t_n \rightarrow t$, $v_n \in D_0$, and $v_n \rightarrow v$ in $C([0, T]; H)$ (as $n \rightarrow \infty$). Then, we show the first condition of Mosco convergence (cf. (M1)). To this end, we assume that $z_n \in K(v_n; t_n) \subset K_0$ with $z_n \rightarrow z$ weakly in V (as $n \rightarrow \infty$). Then, by the Mazur theorem, there is a convex combinations $z^{(k)}$ of the form

$$z^{(k)} := \sum_{j=1}^{N_k} c_j^{(k)} z_{n_j(k)}, \quad n_j(k) \geq k, \quad c_j^{(k)} \geq 0, \quad \sum_{j=1}^{N_k} c_j^{(k)} = 1, \quad k = 1, 2, \dots,$$

such that $z^{(k)} \rightarrow z$ in V and $\nabla z^{(k)}(x) \rightarrow \nabla z(x)$, a.e. on Ω as $k \rightarrow \infty$. In this case,

$$|\nabla z^{(k)}(x)| \leq \sum_{j=1}^{N_k} c_j^{(k)} |\nabla z_{n_j(k)}(x)| \leq \sum_{j=1}^{N_k} c_j^{(k)} \rho(x, t_{n_j(k)}, v_{n_j(k)}(x, t_{n_j(k)})). \quad (4.12)$$

Passing to the limit as $k \rightarrow \infty$ and noting the last term of (4.12) converges to $\rho(x, t, v(x, t))$ uniformly on Ω by Lemma 4.1(ii), we have that

$$|\nabla z(x)| \leq \rho(x, t, v(x, t)), \quad \text{a.a. } x \in \Omega,$$

which implies that $z \in K(v; t)$. Therefore, since $z_n \rightarrow z$ in H as $n \rightarrow \infty$, we have

$$\psi^t(v; z) = \frac{\tau_0}{2} \int_{\Omega} |z(x)|^2 dx = \liminf_{n \rightarrow \infty} \frac{\tau_0}{2} \int_{\Omega} |z_n(x)|^2 dx = \liminf_{n \rightarrow \infty} \psi^{t_n}(v_n; z_n).$$

Hence, the first condition of Mosco convergence holds.

Next, we show the second condition of Mosco convergence (cf. (M2)). To this end, let z be any element in $K(v; t)$ and put

$$z_n(x) := \left(1 - \frac{1}{\rho_*} |\rho(\cdot, t_n, v_n(\cdot, t_n)) - \rho(\cdot, t, v(\cdot, t))|_{L^\infty(\Omega)} \right) z(x);$$

note from (4.1) that $\rho_* > 0$ and z_n is well defined. Then, we observe that $z_n \rightarrow z$ in V and $z_n \in K(v_n; t_n)$ for all large n as well as $\psi^{t_n}(v_n; z_n) \rightarrow \psi^t(v; z)$ (as $n \rightarrow \infty$). Therefore, the second condition of Mosco convergence holds. Thus, $\psi^{t_n}(v_n; \cdot) \rightarrow \psi^t(v; \cdot)$ on V in the sense of Mosco and $(\psi 1)$ holds.

As to assumption (ϕ) , it is easy to check $(\phi 1) \sim (\phi 3)$. We check $(\phi 4)$ as follows. Let z be any element in V and $v \in D_0$. Then, using (4.6) and Lemma 4.1(i),

$$\begin{aligned} |\varphi^t(v; z) - \varphi^s(v; z)| &\leq \frac{1}{2} \int_{\Omega} |a(x, t, v(x, t)) - a(x, s, v(x, s))| |\nabla z(x)|^2 dx \\ &\leq L_a(|t - s| + M|t - s|) \frac{1}{2} \int_{\Omega} |\nabla z(x)|^2 dx \\ &\leq C_{\phi} |t - s| \varphi^s(v; z), \end{aligned}$$

for some positive constant C_{ϕ} ; hence $(\phi 4)$ holds with $\alpha(t) := C_{\phi} t$. $\square$

Proposition 4.1. *Assume that (4.1), (4.5), and (4.6) are fulfilled for functions $\rho(x, t, r)$, $g(x, t, r)$, and $a(x, t, r)$. Let $\tau_0 \geq 0$, $f \in L^2(0, T; H)$, and $u_0 \in V$. Then, problem (4.4) = $\{(4.4a), (4.4b)\}$ admits at least one solution u in $W^{1,2}(0, T; V)$.*

Proof. On account of Lemmas 4.1 and 4.2, $\{\psi^t(v; \cdot)\}$ and $\{\varphi^t(v; \cdot)\}$ given by (4.2), (4.9), and (4.10) fulfill assumptions (ψ) , (ϕ) , and (g) is trivially verified. Hence all the assumptions of Theorem 2.1 are fulfilled. Therefore,

$$\partial_* \psi^t(u; u'(t)) + \partial_* \varphi^t(u; u(t)) + g(u; t, u(t)) \ni f(t) \text{ in } V^*, \quad u(0) = u_0 \text{ in } V,$$

admits a solution u in $W^{1,2}(0, T; V)$. By the characterization (i) and (ii) of $\partial_* \psi^t(u; \cdot)$ and $\partial_* \varphi^t(u; \cdot)$, and the definition of $g(u; t, \cdot)$, u is a solution to (4.4). $\square$

(Application 2)

Next we are going to consider a doubly nonlinear quasi-variational inequality with non-local obstacle function.

We here define an operator $\mathcal{L} : L^2(0, T; H) \rightarrow L^2(0, T; H)$ by

$$\mathcal{L}(v; x, t) := \int_0^t \int_{\Omega} \ell(x, t, y, \tau, v(y, \tau)) dy d\tau + \delta_0, \quad \forall (x, t) \in Q, \quad \forall v \in L^2(0, T; H),$$

with a given function $\ell \in C(\overline{Q} \times \overline{Q} \times \mathbb{R})$ and a positive constant δ_0 satisfying that:

$$\begin{cases} 0 \leq \ell(x, t, y, \tau, r) \leq \ell^*, & \forall (x, t, y, \tau, r) \in \overline{Q} \times \overline{Q} \times \mathbb{R}, \\ |\ell(x, t_1, y, \tau, r_1) - \ell(x, t_2, y, \tau, r_2)| \leq L_{\ell} (|t_1 - t_2| + |r_1 - r_2|), \\ \forall t_i \in [0, T], \quad \forall r_i \in \mathbb{R}, \quad i = 1, 2, \quad \forall (x, y, \tau) \in \overline{\Omega} \times \overline{Q}, \end{cases}$$

where ℓ^* and L_{ℓ} are positive constants. With the functional $\mathcal{L}(v; x, t)$ we define a constraint set $K(v; t)$ by

$$K(v; t) := \{z \in V \mid |\nabla z(x)| \leq \mathcal{L}(v; x, t), \text{ a.a. } x \in \Omega\}, \quad \forall t \in [0, T]. \quad (4.13)$$

Now, we consider the following quasi-variational inequality:

$$u'(t) \in K(u; t), \text{ a.a. } t \in (0, T), \quad u(x, 0) = u_0(x), \quad x \in \Omega, \quad (4.14a)$$

$$\begin{aligned}
& \tau_0 \int_{\Omega} u'(x, t)(u'(x, t) - z(x))dx \\
& + \int_{\Omega} a(x, t, u(x, t)) \nabla u(x, t) \cdot \nabla (u'(x, t) - z(x))dx \\
& + \int_{\Omega} g(x, t, u(x, t))(u'(x, t) - z(x))dx \\
& \leq \int_{\Omega} f(x, t)(u'(x, t) - z(x))dx, \quad \forall z \in K(u; t), \text{ a.a. } t \in (0, T),
\end{aligned} \tag{4.14b}$$

where $\tau_0 \geq 0$, $a(\cdot, \cdot, \cdot)$, $g(\cdot, \cdot, \cdot)$, f , and u_0 are the same as in problem (4.4). Similarly define functionals $\psi^t(v; \cdot)$ and $\varphi^t(v; \cdot)$ by (4.9), (4.13), and (4.10). In addition, define the (t, v) -dependent operator $g(v; t, z)$ by (4.11). Then we observe that all the assumptions of Theorem 2.1 are fulfilled, and hence the above problem (4.14):= {(4.14a), (4.14b)} has at least one solution u in $W^{1,2}(0, T; V)$.

4.2 Doubly nonlinear evolution inclusions of Navier–Stokes type

We begin our study with the solenoidal function space formulated on a smooth bounded domain Ω in $\mathbb{R}^N$, $N = 2$ or 3 , with usual notation $Q := \Omega \times (0, T)$, $0 < T < \infty$, $\Gamma := \partial\Omega$, and $\Sigma := \Gamma \times (0, T)$;

$$V := H_0^1(\Omega), \quad H := L^2(\Omega).$$

Now, we define:

$$\mathcal{D}_{\sigma} := \{z = (z^{(1)}, z^{(2)}, \dots, z^{(N)}) \in C_0^{\infty}(\Omega)^N \mid \operatorname{div} z = 0 \text{ in } \Omega\},$$

and put

$$\mathbf{H}_{\sigma} := \text{the closure of } \mathcal{D}_{\sigma} \text{ in } H^N, \quad \mathbf{V}_{\sigma} := \text{the closure of } \mathcal{D}_{\sigma} \text{ in } V^N,$$

with usual norms

$$|z|_{\mathbf{H}_{\sigma}} := \left\{ \sum_{k=1}^N |z^{(k)}|_H^2 \right\}^{\frac{1}{2}}, \quad |z|_{\mathbf{V}_{\sigma}} := \left\{ \sum_{k=1}^N |\nabla z^{(k)}|_H^2 \right\}^{\frac{1}{2}}.$$

In addition, we denote the dual space of $\mathbf{V}_{\sigma}$ by $\mathbf{V}_{\sigma}^*$. For simplicity we denote the inner product in $\mathbf{H}_{\sigma}$ by $(\cdot, \cdot)_{\sigma}$ and the duality pairing between $\mathbf{V}_{\sigma}^*$ and $\mathbf{V}_{\sigma}$ by $\langle \cdot, \cdot \rangle_{\sigma}$.

Let ρ be a prescribed obstacle function in $C(\overline{Q} \times \mathbb{R}^N)$ such that

$$0 < \rho_* \leq \rho(x, t, \mathbf{r}) \leq \rho^*, \quad \forall (x, t, \mathbf{r}) \in \overline{Q} \times \mathbb{R}^N, \tag{4.15}$$

where ρ_* and ρ^* are positive constants, and $\mathbf{K}_0$ is a closed convex subset of $\mathbf{V}_{\sigma}$ given by

$$\mathbf{K}_0 := \{z \in \mathbf{V}_{\sigma} \mid |\nabla z(x)| \leq \rho^*, \text{ a.e. on } \Omega\},$$

and put

$$\mathbf{D}_0 := \{v \in W^{1,2}(0, T; \mathbf{V}_{\sigma}) \mid v'(t) \in \mathbf{K}_0, \text{ a.a. } t \in (0, T), \quad v(0) = u_0 \text{ in } \mathbf{V}_{\sigma}\}, \tag{4.16}$$

where $\mathbf{u}_0$ is a given initial datum in $\mathbf{V}_\sigma$ (cf. (4.21a)).

Also, for each $\mathbf{v} \in \mathbf{D}_0$ and $t \in [0, T]$, define

$$\mathbf{K}(\mathbf{v}; t) := \{\mathbf{z} \in \mathbf{V}_\sigma \mid |\nabla \mathbf{z}(x)| \leq \rho(x, t, \mathbf{v}(x, t)), \text{ a.e. on } \Omega\}, \quad \forall t \in [0, T]. \quad (4.17)$$

Now, for each $\mathbf{v} \in \mathbf{D}_0$ and $t \in [0, T]$, we define functionals $\psi_0^t(\cdot)$, $\psi^t(\mathbf{v}; \cdot)$, and $\varphi^t(\mathbf{v}; \cdot)$ on $\mathbf{V}_\sigma$ by

$$\psi_0^t(\mathbf{z}) := I_{\mathbf{K}_0}(\mathbf{z}), \quad \forall \mathbf{z} \in \mathbf{V}_\sigma, \quad \forall t \in [0, T],$$

$$\psi^t(\mathbf{v}; \mathbf{z}) := \frac{\tau_0}{2} \int_{\Omega} |\mathbf{z}(x)|^2 dx + I_{\mathbf{K}(\mathbf{v}; t)}(\mathbf{z}), \quad \forall \mathbf{z} \in \mathbf{V}_\sigma, \quad \forall t \in [0, T], \quad (4.18)$$

$$\varphi^t(\mathbf{v}; \mathbf{z}) := \frac{1}{2} \int_{\Omega} a(x, t, \mathbf{v}(x, t)) |\nabla \mathbf{z}(x)|^2 dx, \quad \forall \mathbf{z} \in \mathbf{V}_\sigma, \quad \forall t \in [0, T], \quad (4.19)$$

and $\mathbf{g}(\mathbf{v}; t, \cdot) : \mathbf{V}_\sigma \rightarrow \mathbf{V}_\sigma^*$ by

$$\begin{aligned} \langle \mathbf{g}(\mathbf{v}; t, \mathbf{w}), \mathbf{z} \rangle_\sigma &:= \int_{\Omega} (\mathbf{v}(x, t) \cdot \nabla) \mathbf{w}(x) \cdot \mathbf{z}(x) dx \\ &= \sum_{i,j=1}^N \int_{\Omega} v^{(i)}(x, t) \frac{\partial w^{(j)}(x)}{\partial x_i} z^{(j)}(x) dx, \\ \forall \mathbf{w} &= (w^{(1)}, w^{(2)}, \dots, w^{(N)}), \quad \mathbf{z} = (z^{(1)}, z^{(2)}, \dots, z^{(N)}) \in \mathbf{V}_\sigma. \end{aligned} \quad (4.20)$$

Now we propose the following quasi-variational inequality of Navier–Stokes type with gradient constraint for time-derivative:

$$\mathbf{u}'(t) := \frac{\partial \mathbf{u}(x, t)}{\partial t} \in \mathbf{K}(\mathbf{u}; t), \quad \text{a.a. } t \in (0, T), \quad \mathbf{u}(\cdot, 0) = \mathbf{u}_0 \text{ in } \mathbf{V}_\sigma, \quad (4.21a)$$

$$\begin{aligned} &\tau_0 \int_{\Omega} \mathbf{u}'(x, t) \cdot (\mathbf{u}'(x, t) - \mathbf{z}(x)) dx \\ &\quad + \int_{\Omega} a(x, t, \mathbf{u}(x, t)) \nabla \mathbf{u}(x, t) \cdot \nabla (\mathbf{u}'(x, t) - \mathbf{z}(x)) dx \\ &\quad + \langle \mathbf{g}(\mathbf{u}; t, \mathbf{u}(t)), \mathbf{u}'(t) - \mathbf{z} \rangle_\sigma \end{aligned} \quad (4.21b)$$

$$\leq \int_{\Omega} \mathbf{f}(x, t) \cdot (\mathbf{u}'(x, t) - \mathbf{z}(x)) dx, \quad \forall \mathbf{z} \in \mathbf{K}(\mathbf{u}; t), \quad \text{a.a. } t \in (0, T),$$

where $\tau_0 \geq 0$ is a constant, $\mathbf{f}$ is a function in $L^2(0, T; \mathbf{H}_\sigma)$, $\mathbf{u}_0$ is an initial datum in $\mathbf{V}_\sigma$, and $a(\cdot, \cdot, \cdot)$ is a function in $C(\overline{Q} \times \mathbb{R}^N)$ such that

$$\begin{aligned} a_* &\leq a(x, t, \mathbf{r}) \leq a^*, \quad \forall (x, t) \in \overline{Q}, \quad \forall \mathbf{r} \in \mathbb{R}^N, \\ |a(x, t_1, \mathbf{r}_1) - a(x, t_2, \mathbf{r}_2)| &\leq L_a(|t_1 - t_2| + |\mathbf{r}_1 - \mathbf{r}_2|), \\ \forall t_i &\in [0, T], \quad \forall \mathbf{r}_i \in \mathbb{R}^N, \quad i = 1, 2, \quad \forall x \in \overline{\Omega}, \end{aligned} \quad (4.22)$$

where a_* , a^* , and L_a are positive constants.

Proposition 4.2. Let $\rho := \rho(x, t, \mathbf{r})$ and $a := a(x, t, \mathbf{r})$ be functions for which (4.15) and (4.22) hold, and let $\mathbf{f} \in L^2(0, T; \mathbf{H}_\sigma)$ and $\mathbf{u}_0 \in \mathbf{V}_\sigma$. Then, problem (4.21) := $\{(4.21a), (4.21b)\}$ admits at least one solution $\mathbf{u}$ in $W^{1,2}(0, T; \mathbf{V}_\sigma)$.

In order to show the existence of a solution to (4.21) we apply Theorem 2.1. To this end we check the assumptions (ψ) , (ϕ) , and (g) with $g = \mathbf{g}$.

Lemma 4.3. Let ψ^t , φ^t , and $\mathbf{g}$ be given by (4.17)~(4.20). Then assumptions (ψ) , (ϕ) , and (g) are fulfilled.

Proof. With regard to assumptions (ψ) and (ϕ) , we can verify them by using Lemmas 4.1 and 4.2 extensively to the vectorial case, too. We are going to check (g) .

We now show $(g1)$. To this end, let $\{\mathbf{v}_n\}_{n \in \mathbb{N}} \subset \mathbf{D}_0$ such that $\mathbf{v}_n \rightarrow \mathbf{v}$ in $C([0, T]; \mathbf{H}_\sigma)$ (as $n \rightarrow \infty$). Then, from Lemma 4.1 it follows that

$$\mathbf{v}_n - \mathbf{v} \rightarrow 0 \text{ in } C(\overline{Q}) \quad \text{and} \quad \mathbf{v}_n \rightarrow \mathbf{v} \text{ weakly in } W^{1,2}(0, T; \mathbf{V}_\sigma) \quad \text{as } n \rightarrow \infty. \quad (4.23)$$

Given $\{\mathbf{w}_n\}_{n \in \mathbb{N}} \subset \mathbf{V}_\sigma$ satisfying that $\mathbf{w}_n \rightarrow \mathbf{w}$ weakly in $\mathbf{V}_\sigma$, and $\mathbf{z} \in \mathbf{V}_\sigma$, we observe that

$$\begin{aligned} & \langle \mathbf{g}(\mathbf{v}_n; t, \mathbf{w}_n) - \mathbf{g}(\mathbf{v}; t, \mathbf{w}), \mathbf{z} \rangle_\sigma \\ &= \sum_{i,j=1}^N \int_{\Omega} v_n^{(i)}(x, t) \frac{\partial w_n^{(j)}(x)}{\partial x_i} z^{(j)}(x) dx - \sum_{i,j=1}^N \int_{\Omega} v^{(i)}(x, t) \frac{\partial w^{(j)}(x)}{\partial x_i} z^{(j)}(x) dx \\ &= \sum_{i,j=1}^N \int_{\Omega} (v_n^{(i)}(x, t) - v^{(i)}(x, t)) \frac{\partial w_n^{(j)}(x)}{\partial x_i} z^{(j)}(x) dx \\ & \quad + \sum_{i,j=1}^N \int_{\Omega} v^{(i)}(x, t) \frac{\partial (w_n^{(j)}(x) - w^{(j)}(x))}{\partial x_i} z^{(j)}(x) dx, \end{aligned}$$

so that

$$\begin{aligned} & |\langle \mathbf{g}(\mathbf{v}_n; t, \mathbf{w}_n) - \mathbf{g}(\mathbf{v}; t, \mathbf{w}), \mathbf{z} \rangle_\sigma| \\ &\leq k_1 |\mathbf{v}_n - \mathbf{v}|_{C(\overline{Q})^N} |\mathbf{w}_n|_{\mathbf{V}_\sigma} |\mathbf{z}|_{\mathbf{V}_\sigma} \\ & \quad + \sup_{\mathbf{z} \in \mathbf{V}_\sigma, |\mathbf{z}|_{\mathbf{V}_\sigma} \leq 1} \sum_{i,j=1}^N \left| \int_{\Omega} v^{(i)}(x, t) \frac{\partial (w_n^{(j)}(x) - w^{(j)}(x))}{\partial x_i} z^{(j)}(x) dx \right| \end{aligned}$$

for some positive constant k_1 independent of $\{\mathbf{v}_n\}_{n \in \mathbb{N}} \subset \mathbf{D}_0$, $\mathbf{v} \in \mathbf{D}_0$, $\{\mathbf{w}_n\}_{n \in \mathbb{N}} \subset \mathbf{V}_\sigma$, and $\mathbf{z} \in \mathbf{V}_\sigma$. Since $Y := \{v^{(i)}(t)z^{(j)} \mid |\mathbf{z}|_{\mathbf{V}_\sigma} \leq 1\}$ is compact in H , for every $i, j = 1, 2, \dots, N$, the last term of the above inequality converges to 0 uniformly on Y as $n \rightarrow \infty$. Therefore,

$$|\mathbf{g}(\mathbf{v}_n; t, \mathbf{w}_n) - \mathbf{g}(\mathbf{v}; t, \mathbf{w})|_{\mathbf{V}_\sigma^*} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, we have $(g1)$.

We next show $(g2)$. In a similar calculation to the above, we get

$$|\langle \mathbf{g}(\mathbf{v}; t, \mathbf{w}_1) - \mathbf{g}(\mathbf{v}; t, \mathbf{w}_2), \mathbf{z} \rangle_\sigma| \leq k_2 |\mathbf{v}|_{L^\infty(0, T; L^4(\Omega)^N)} |\mathbf{w}_1 - \mathbf{w}_2|_{\mathbf{V}_\sigma} |\mathbf{z}|_{\mathbf{V}_\sigma}, \quad \forall \mathbf{z} \in \mathbf{V}_\sigma,$$

where k_2 is a positive constant independent of $\mathbf{v} \in \mathbf{D}_0$ and $\mathbf{w}_i \in \mathbf{V}_\sigma$ ($i = 1, 2$). This implies that

$$|\mathbf{g}(\mathbf{v}; t, \mathbf{w}_1) - \mathbf{g}(\mathbf{v}; t, \mathbf{w}_2)|_{\mathbf{V}_\sigma^*} \leq k_4 |\mathbf{w}_1 - \mathbf{w}_2|_{\mathbf{V}_\sigma}, \quad \text{with } k_4 := k_3 \sup_{\mathbf{v} \in \mathbf{D}_0} |\mathbf{v}|_{L^\infty(0, T; \mathbf{V}_\sigma)},$$

where k_3 is some positive constant, dependent on k_2 and the constant of the embedding $V \hookrightarrow L^4(\Omega)$. Thus, (g2) holds, too.

(g3) is trivially satisfied. Therefore, assumption (g) is fulfilled. Thus, the proof of Lemma 4.3 is complete. $\square$

Proof of Proposition 4.2. From (4.18) and (4.19) of the definitions of ψ^t and φ^t we see the following characterizations of their subdifferentials:

- (i) Let $\mathbf{v} \in \mathbf{D}_0$ and $t \in [0, T]$. Then $\mathbf{z}^* \in \partial_* \psi^t(\mathbf{v}; \mathbf{z})$ if and only if $\mathbf{z}^* \in \mathbf{V}_\sigma^*$, $\mathbf{z} \in \mathbf{K}(\mathbf{v}; t)$ and

$$\tau_0 \int_{\Omega} \mathbf{z}(x) \cdot (\mathbf{z}(x) - \mathbf{w}(x)) dx + \langle -\mathbf{z}^*, \mathbf{z} - \mathbf{w} \rangle_\sigma \leq 0, \quad \forall \mathbf{w} \in \mathbf{K}(\mathbf{v}; t).$$

- (ii) Let $\mathbf{v} \in \mathbf{D}_0$ and $t \in [0, T]$. Then $\partial_* \varphi^t(\mathbf{v}; \cdot)$ is singlevalued, linear, and bounded from $\mathbf{V}_\sigma$ into $\mathbf{V}_\sigma^*$ and

$$\langle \partial_* \varphi^t(\mathbf{v}; \mathbf{z}), \mathbf{w} \rangle_\sigma = \int_{\Omega} a(x, t, \mathbf{v}(x, t)) \nabla \mathbf{z}(x) \cdot \nabla \mathbf{w}(x) dx, \quad \forall \mathbf{z}, \mathbf{w} \in \mathbf{V}_\sigma.$$

These characterizations show that the quasi-variational inequality (4.21) is described as

$$\begin{cases} \partial_* \psi^t(\mathbf{u}; \mathbf{u}'(t)) + \partial_* \varphi^t(\mathbf{u}; \mathbf{u}(t)) + \mathbf{g}(\mathbf{u}; t, \mathbf{u}(t)) \ni \mathbf{f}(t) & \text{in } \mathbf{V}_\sigma^*, \text{ a.a. } t \in (0, T), \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \mathbf{V}_\sigma. \end{cases}$$

Therefore, on account of Theorem 2.1, we obtain Proposition 4.2. $\square$

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