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REMARK ON DIFFERENTIABILITY OF SOLUTIONS OF FREE BOUNDARY PROBLEMS DESCRIBING WATER ADSORPTION

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Abstract. In our recent work [1] we discussed free boundary problems with boundary and initial functions depending on some parameter and obtained differentiability of solutions with respect to the parameter. This paper is its sequel and to give more precise estimate for the derivatives by applying the classical theory for weak solutions of differential equations of parabolic type. This result will be applicable to analysis of our two-scale model describing moisture transport appearing in concrete carbonation process.

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1 Introduction and main result

In our previous work [1] we considered the following free boundary problem $\{(1.1)\text{-}(1.6)\}$ denoted by $\mathrm{P}(\lambda)$ for $\lambda \in (0,1)$. We note that the boundary and initial functions of $\mathrm{P}(\lambda)$ depend on the parameter $\lambda \in (0,1)$. For $\lambda \in (0,1)$ $\mathrm{P}(\lambda)$ is to find a curve $x = s(t)$ on $[0,T]$ and a function u on the set $Q_s(T) := \{(t,x) : 0 < t < T,\ s(t) < x < L\}$, $T > 0$, such that

$$\rho_v u_t - k u_{xx} = 0 \text{ in } Q_s(T), \tag{1.1}$$
$$u(t,L) = h(\lambda,t) \text{ for } t \in (0,T), \tag{1.2}$$
$$k u_x(t,s(t)) = (\rho_w - \rho_v u(t,s(t))) s_t(t) \text{ for } t \in (0,T), \tag{1.3}$$
$$s_t(t) = a(u(t,s(t)) - \varphi(s(t))) \text{ for } t \in (0,T), \tag{1.4}$$
$$s(0) = s_0(\lambda), \tag{1.5}$$
$$u(0,x) = u_0(\lambda,x) \text{ for } x \in [s_0(\lambda), L], \tag{1.6}$$

where ρ_v, ρ_w, k and a are given positive constants and φ is a given continuous function on $\mathbb{R}$. Also, for each $\lambda \in (0,1)$, $h(\lambda) = h(\lambda,\cdot)$ is a given function on $(0,T)$, $s_0(\lambda) \in [0,L)$ is a constant and $u_0(\lambda,\cdot)$ is a given function on $[s_0(\lambda), L]$.

The problem $\mathrm{P}(\lambda)$ was originally proposed by Sato-Aiki-Murase-Shirakawa [4, 11] as a mathematical model describing water adsorption in concrete carbonation process and studied in [3, 2, 5, 9, 8]. Its physical interpretation is mentioned in these papers. Moreover, in case λ is varying over the interval $(0,1)$, we can obtain a solution $(s(\lambda), u(\lambda))$ of $\mathrm{P}(\lambda)$ for each $\lambda \in (0,1)$. Accordingly, by regarding s (resp. u) as a function of λ and t (resp. λ, t and x) we established continuity, measurability and differentiability of s and u with respect to λ in [6, 7, 1]. In these results, the example of λ is mentioned.

The aim of this paper is to establish an estimate for maximum of derivatives of solutions to $\mathrm{P}(\lambda)$ with respect to λ under some suitable conditions for h, s_0 and u_0, when λ varies in $(0,1)$.

This paper is organized as follows: In Section 2 we state our main theorem concerned with the maximum of derivatives of s and u with respect to $\lambda \in (0,1)$. We already proved that the derivative of u is a weak solution of a linear parabolic problem so that by applying a classical theory for differential equations of parabolic type we can obtain the estimate for the maximum. In Section 3 we recall our previous results and some basic properties concerned with the theory for weak solutions. Finally, a proof of the theorem will be given.

2 A main result

First, in order to recall our previous results, we give a list of assumptions as follows:

(A1) T, k and a are positive constants.

(A2) $\varphi \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$, $\varphi = 0$ on $(-\infty, 0]$, $\varphi \le 1$ on $\mathbb{R}$, $\varphi' > 0$ on $(0,L]$. Also, we denote by $\hat{\varphi}$ the primitive function of φ with $\hat{\varphi}(0) = 0$ and put $C_\varphi = |\varphi'|_{L^\infty(\mathbb{R})} + |\varphi''|_{L^\infty(\mathbb{R})}$.

(A3) $h(:= h(\lambda,t)) \in L^\infty(0,1; W^{1,2}(0,T))$ with $0 \le h(\lambda,\cdot) \le h^*$ on $(0,T)$ for any $\lambda \in (0,1)$, where h^* is a positive constant satisfying $h^* < \varphi(L)$.

(A4) Two positive constants ρ_w and ρ_v satisfy

$$\rho_w > 2\rho_v, \quad \rho_w \geq \rho_v(C_\varphi + 2), \quad 9aL\rho_v^2 \leq k\rho_w.$$

(A5) $s_0 \in L^2(0,1)$ and $0 \leq s_0(\lambda) \leq L - \ell_0$ for any $\lambda \in (0,1)$, where ℓ_0 is a positive constant. Also, u_0 is defined on the set $\{(\lambda, x) | \lambda \in (0,1), x \in (s_0(\lambda), L)\}$ and for each $\lambda \in (0,1)$ $u_0(\lambda) \in W^{1,2}(s_0(\lambda), L)$ such that

$$|u_0(\lambda)|_{W^{1,2}(s_0(\lambda),L)} \leq C_0, u_0(\lambda, L) = h(0,\lambda) \text{ and } 0 \leq u_0(\lambda) \leq 1 \text{ on } [s_0(\lambda), L] \text{ for } \lambda \in (0,1),$$

where C_0 is a positive constant.

Here, we define a solution of P(λ) for each $\lambda \in (0,1)$.

Definition 2.1. For $T > 0$ and $\lambda \in (0,1)$, let s and u be functions on $[0,T]$ and $Q_s(T)$, respectively. We call that a pair (s,u) is a solution of P(λ) on $[0,T]$ if the conditions (S1)-(S6) hold:

(S1) $s \in W^{1,\infty}(0,T)$, $0 \leq s < L$ on $[0,T]$, $u \in L^\infty(Q_s(T))$, u_t, $u_{xx} \in L^2(Q_s(T))$ and $|u_x(\cdot)|_{L^2(s(\cdot),L)} \in L^\infty(0,T)$.

(S2) $\rho_v u_t - k u_{xx} = 0$ a.e. on $Q_s(T)$.

(S3) $u(t,L) = h(\lambda,t)$ for a.e. $t \in [0,T]$.

(S4) $ku_x(t,s(t)) = (\rho_w - \rho_v u(t,s(t)))\, s_t(t)$ for a.e. $t \in [0,T]$.

(S5) $s_t(t) = a(u(t,s(t)) - \varphi(s(t)))$ for a.e. $t \in [0,T]$.

(S6) $s(0) = s_0(\lambda)$, $u(0,x) = u_0(\lambda,x)$ for $x \in [s_0(\lambda), L]$.

Theorem 2.1 is concerned with the existence and the uniqueness of a solution of P(λ) on $[0,T]$.

Theorem 2.1. *(cf. [2, Theorem 4.1]) Assume (A1)-(A5). Then, for any $\lambda \in (0,1)$ there exists a unique solution (s,u) of P(λ) on $[0,T]$ such that*

$$0 \leq u \leq 1 \text{ a.e. on } Q_s(T), 0 \leq s \leq s_* \text{ on } [0,T] \text{ and } |s_t| \leq a \text{ a.e. on } [0,T], \tag{2.1}$$

where s_* is a positive constant satisfying $s_* < L$ which does not depend on λ.

From Theorem 2.1 we can denote a solution of P(λ) by $(s(\lambda,\cdot), u(\lambda,\cdot,\cdot))$ for each $\lambda \in (0,1)$. For simplicity, we sometimes write $s(\lambda) = s(\lambda,\cdot)$ and $u(\lambda) = u(\lambda,\cdot,\cdot)$. Moreover, in order to give a statement on differentiability we introduce the following notation and change of variables

$$\tilde{u}(\lambda,t,y) = u(\lambda,t,\sigma(\lambda,y)) \text{ for } y \in (0,T) \times [0,1], \tag{2.2}$$

where $\sigma(\lambda,y) = (1-y)s(\lambda,t) + yL$. By using (2.2), P($\lambda$) becomes the following problem on the cylindrical domain $Q(T) := (0,T) \times (0,1)$:

$$\rho_v \tilde{u}_t(\lambda) - \frac{k}{(L - s(\lambda))^2} \tilde{u}_{yy}(\lambda) = \frac{\rho_v(1-y)s_t(\lambda)}{L - s(\lambda)} \tilde{u}_y(\lambda) \text{ on } Q(T), \tag{2.3}$$

$$\tilde{u}(\lambda, t, 1) = h(\lambda, t) \text{ for } t \in [0, T], \tag{2.4}$$

$$\frac{k}{L - s(\lambda, t)} \tilde{u}_y(\lambda, t, 0) = (\rho_w - \rho_v \tilde{u}(\lambda, t, 0)) s_t(\lambda, t) \text{ for } t \in [0, T], \tag{2.5}$$

$$s_t(\lambda, t) = a(\tilde{u}(\lambda, t, 0) - \varphi(s(\lambda, t))) \text{ for } t \in [0, T], \tag{2.6}$$

$$s(\lambda, 0) = s_0(\lambda), \tilde{u}(\lambda, 0, y) = u(0, \sigma(\lambda, y)) =: \tilde{u}_0(\lambda, y) \text{ on } [0, 1]. \tag{2.7}$$

Remark 2.1. For $T > 0$ and $\lambda \in (0,1)$, let $s(\lambda)$ and $u(\lambda)$ be functions on $[0, T]$ and $Q_{s(\lambda)}(T)$, respectively and define a function $\tilde{u}(\lambda)$ on $Q(T)$ by (2.2). The pair $(s(\lambda), u(\lambda))$ is a solution of P(λ) if and only if (S'1) and (S'2) hold:

(S'1) $s(\lambda) \in W^{1,\infty}(0, T), 0 \le s(\lambda) < L$ a.e. on $[0, T]$, $\tilde{u}(\lambda) \in W^{1,2}(0, T; L^2(0, 1)) \cap L^\infty(0, T; H^1(0, 1)) \cap L^\infty(Q(T)) \cap L^2(0, T; H^2(0, 1))$.

(S'2) (2.3)-(2.7) hold.

Hence, if (A1) – (A5) hold, then for any $\lambda \in (0, 1)$ it holds that $0 \le \tilde{u}(\lambda) \le 1$ a.e. $Q(T)$ and $0 \le s(\lambda) \le s_*$ on $[0, T]$, where s_* is defined in Theorem 2.1.

The next theorem guarantees differentiability of s and u with respect to λ.

Theorem 2.2. *(cf. [1, Theorem 2]) Assume (A1) – (A5) hold. Let $\lambda \in (0, 1)$ and $(s(\lambda), u(\lambda))$ be a solution of P(λ) on $[0, T]$ and $\tilde{u}(\lambda)$ be a function decided from $u(\lambda)$ by (2.2). If h, s_0 and u_0 satisfy*

$$\frac{\partial h}{\partial \lambda} \in L^2(0, 1; W^{1,2}(0, T)), \tag{2.8}$$

and

$$\frac{\partial s_0}{\partial \lambda} \in L^2(0, 1), u_0 \in L^\infty(0, 1; W^{1,2}(0, L)), \frac{\partial u_0}{\partial \lambda} \in L^2(0, 1; L^2(0, L)), \tag{2.9}$$

then for a.e. $\lambda \in (0, 1)$ it holds

$$\frac{\partial s}{\partial \lambda}(\lambda) \in W^{1,2}(0, T), \quad \frac{\partial \tilde{u}}{\partial \lambda}(\lambda) \in C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)).$$

Moreover, there exists a positive constant C which depends on ρ_v, ρ_w, k, a, L, C_φ, ℓ_0 and C_0 such that

$$\begin{aligned}
&\left|\frac{\partial s}{\partial \lambda}(\lambda)\right|_{W^{1,2}(0,T)} + \left|\frac{\partial \tilde{u}}{\partial \lambda}(\lambda)\right|_{C([0,T];L^2(0,1))} + \left|\frac{\partial \tilde{u}}{\partial \lambda}(\lambda)\right|_{L^2(0,T;H^1(0,1))} \\
\le & C(\left|\frac{\partial s_0}{\partial \lambda}(\lambda)\right| + \left|\frac{\partial u_0}{\partial \lambda}(\lambda)\right|_{L^2(s_0(\lambda),L)} + \left|\frac{\partial h}{\partial \lambda}(\lambda)\right|_{W^{1,2}(0,T)}).
\end{aligned}$$

Now, we provide a main result of this paper.

Theorem 2.3. *Assume (A1) – (A5), (2.8) and (2.9). If u_0 and h satisfy*

$$\frac{\partial h}{\partial \lambda} \in L^\infty((0, 1) \times (0, T)), \tag{2.10}$$

and

$$\frac{\partial s_0}{\partial \lambda} \in L^\infty(0, 1), u_0 \in W^{1,\infty}((0, 1) \times (0, L)), \tag{2.11}$$

then it holds that

$$\frac{\partial \tilde{u}}{\partial \lambda} \in L^\infty((0,1)\times Q(T)), \frac{\partial^2 s}{\partial \lambda \partial t} \in L^\infty((0,1)\times(0,T)). \tag{2.12}$$

Here, $\left|\frac{\partial \tilde{u}}{\partial \lambda}\right|_{L^\infty((0,1)\times Q(T))}$ *and* $\left|\frac{\partial^2 s}{\partial \lambda \partial t}\right|_{L^\infty((0,1)\times(0,T))}$ *is estimated by a positive constant depending on data appearing in the assumption.*

3 Known results

First, in order to recall a result concerned with derivatives of $\tilde{u}$ with respect to λ we introduce the following problem $\hat{\mathrm{P}}(\lambda) = \{(3.1)-(3.5)\}$:

$$\begin{aligned}
&\rho_v \hat{u}_t(\lambda) - \frac{k}{(L-s(\lambda))^2}\hat{u}_{yy}(\lambda) \\
&\quad = \frac{2k\hat{s}(\lambda)}{(L-s(\lambda))^3}\tilde{u}_{yy}(\lambda) + \frac{\rho_v(1-y)\tilde{u}_y(\lambda)}{L-s(\lambda)}\hat{s}_t(\lambda) + \frac{\rho_v(1-y)s_t(\lambda)}{L-s(\lambda)}\hat{u}_y(\lambda) \\
&\quad + \frac{\rho_v(1-y)s_t(\lambda)}{(L-s(\lambda))^2}\tilde{u}_y(\lambda)\hat{s}(\lambda) \text{ in } Q(T), && (3.1)\\
&\hat{u}(\lambda,t,1) = h_\lambda(\lambda,t) \text{ for a.e. } t\in[0,T], && (3.2)\\
&\frac{k}{L-s(\lambda,t)}\hat{u}_y(\lambda,t,0) = -\frac{k}{(L-s(\lambda,t))^2}\hat{s}(\lambda,t)\tilde{u}_y(\lambda,t,0) + \rho_w\hat{s}_t(\lambda,t) \\
&\qquad\qquad - \rho_v s_t(\lambda,t)\hat{u}(\lambda,t,0) - \rho_v\tilde{u}(\lambda,t,0)\hat{s}_t(t) \text{ for a.e. } t\in[0,T], && (3.3)\\
&\hat{s}_t(\lambda,t) = a(\hat{u}(\lambda,t,0) - \varphi'(s(\lambda,t))\hat{s}(\lambda,t)) \text{ for a.e. } t\in[0,T], \hat{s}(\lambda,0) = s_{0\lambda}(\lambda), && (3.4)\\
&\hat{u}(\lambda,0,y) = u_{0\lambda}(\lambda,\sigma(\lambda,y)) + (1-y)s_{0\lambda}(\lambda,y)u_{0x}(\lambda,\sigma(\lambda,y)) \text{ for a.e. } y\in(0,1). && (3.5)
\end{aligned}$$

In the system above we used following notation:

$$\frac{\partial s_0(\lambda)}{\partial \lambda} := s_{0\lambda}(\lambda),\ \frac{\partial u_0(\lambda)}{\partial \lambda} := u_{0\lambda}(\lambda),\ \text{and}\ \frac{\partial h(\lambda,\cdot)}{\partial \lambda} := h_\lambda(\lambda,\cdot).$$

Next, we define a weak solution of $\hat{\mathrm{P}}(\lambda)$ on $[0,T]$ as follows:

Definition 3.1. *For $T>0$ and $\lambda\in(0,1)$ let $\hat{s}(\lambda)$ and $\hat{u}(\lambda)$ be functions on $[0,T]$ and $Q(T)$, respectively. We call that the pair $(\hat{s}(\lambda),\hat{u}(\lambda))$ is a weak solution of $\hat{P}(\lambda)$ on $[0,T]$ if the following conditions hold:*

(W1) $\hat{s}(\lambda)\in W^{1,2}(0,T)$ and $\hat{u}(\lambda)\in C([0,T];L^2(0,1))\cap L^2(0,T;H^1(0,1))$.

(W2) For any $\eta\in W^{1,2}(0,T;L^2(0,1))\cap L^2(0,T;H^1(0,1))$ with $\eta(T)=0$ and $\eta(t,1)=0$ for $t\in[0,T]$, it holds that

$$\begin{aligned}
&-\int_0^T\int_0^1 \rho_v\hat{u}(\lambda)\eta_t dydt + \int_0^T\int_0^1 \frac{k}{(L-s(\lambda))^2}\hat{u}_y(\lambda)\eta_y dydt \\
&-\int_0^T \frac{k}{(L-s(\lambda))^2}\hat{s}(\lambda)\tilde{u}_y(\lambda,\cdot,0)\eta(\cdot,0)dt + \int_0^T \rho_w\hat{s}_t(\lambda)\eta(\cdot,0)dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T \rho_v s_t(\lambda)\hat{u}(\lambda,\cdot,0)\eta(\cdot,0)dt - \int_0^T \rho_v \tilde{u}(\lambda,\cdot,0)\hat{s}_t(\lambda)\eta(\cdot,0)dt \\
&= \int_0^1 \rho_v \hat{u}_0(\lambda)\eta(0)dy + \int_0^T\int_0^1 \frac{2k\hat{s}(\lambda)}{(L-s(\lambda))^3}\tilde{u}_{yy}(\lambda)\eta dydt \\
&+ \int_0^T\int_0^1 \frac{\rho_v(1-y)\tilde{u}_y(\lambda)}{L-s(\lambda)}\hat{s}_t(\lambda)\eta dydt + \int_0^T\int_0^1 \frac{\rho_v(1-y)s_t(\lambda)}{L-s(\lambda)}\hat{u}_y(\lambda)\eta dydt \\
&+ \int_0^T\int_0^1 \frac{\rho_v(1-y)s_t(\lambda)}{(L-s(\lambda))^2}\tilde{u}_y(\lambda)\hat{s}(\lambda)\eta dydt,
\end{aligned}
$$

where $\hat{u}_0(\lambda,y) = u_{0\lambda}(\lambda,\sigma(\lambda,y)) + u_{0x}(\lambda,\sigma(\lambda,y))(1-y)s_{0\lambda}(\lambda,y)$ *for a.e.* $y \in (0,1)$.

(W3) (3.2) and (3.4) hold.

We have already proved:

Lemma 3.1. *(cf. [1, Theorem 2]) Under the same assumption as in Theorem 2.2 the pair* $(s_\lambda(\lambda),\tilde{u}_\lambda(\lambda))$ *is a weak solution of* $\hat{P}(\lambda)$ *for a.e.* $\lambda \in (0,1)$.

We note that (3.1) is a linear differential equation of parabolic type. Accordingly, our main theorem, Theorem 2.3, is a direct consequence of [10, Theorem 7.1 in Chapter 3]. However, since our boundary condition is different from that of [10, Theorem 7.1 in Chapter 3], we provide its complete proof for reader's convenient.

Here, we introduce function spaces and show a useful inequality as a lemma. For simplicity, we put $H = L^2(0,1)$, $X = \{z \in W^{1,2}(0,1) | z(1) = 0\}$, and $V(T) = L^\infty(0,T;H) \cap L^2(0,T;X)$ for any $T > 0$. Also, we denote their norms as follows;

$$
\begin{aligned}
&|z|_H = (\int_0^1 |z|^2 dy)^{1/2} \text{ for } z \in H, |z|_X = |z_y|_H \text{ for } z \in X, \\
&|z|_{V(T)} = |z|_{L^\infty(0,T;H)} + |z_y|_{L^2(Q(T))} \text{ for } z \in V(T).
\end{aligned}
$$

Lemma 3.2. *([10, (3.8) in Chapter 2]) The following inequality holds.*

$$
|z|_{L^q(Q(T))} \le |z|_{V(T)} \textit{ for } z \in V(T) \textit{ and } q = 5, 6. \tag{3.6}
$$

Proof. Easily, we obtain

$$
\begin{aligned}
\int_{Q(T)} |z|^6 dydt &\le \int_{Q(T)} z^2 (\int_0^1 |\frac{\partial}{\partial y}(z^2)| dy)^2 dydt \\
&\le \int_0^T\int_0^1 |z|^2 |z|_H^2 |z_y|_H^2 dydt \\
&\le |z|_{V(T)}^6 \text{ for } z \in V(T).
\end{aligned}
$$

Similarly, we can obtain (3.6) for $q = 5$. □

At the end of this section we show a property of weak solutions.

Lemma 3.3. *([10, (7.6) in Chapter 3]) For any $\lambda \in (0,1)$ let $(\hat{s},\hat{u})$ be a weak solution of $\hat{P}(\lambda)$. If $j \geq \max\{|h_\lambda(\lambda)|_{L^\infty(0,T)}, |u_{0\lambda}(\lambda)|_{L^\infty(0,L)} + |s_{0\lambda}(\lambda)||u_{0x}(\lambda)|_{L^\infty(0,L)}\} =: m_0$, then it holds:*

$$
\begin{aligned}
&\frac{\rho_v}{2}\int_0^1 |[\hat{u}(t_1)-j]^+|^2 dy + \int_0^{t_1}\int_0^1 \frac{k}{(L-s)^2}\hat{u}_y[\hat{u}-j]^+_y dydt \\
&- \int_0^{t_1} \frac{k}{(L-s)^2}\hat{s}\tilde{u}_y(\cdot,0)[\hat{u}(\cdot,0)-j]^+ dt + \int_0^{t_1} \rho_w \hat{s}_t[\hat{u}(\cdot,0)-j]^+ dt \\
&- \int_0^{t_1} \rho_v s_t \hat{u}(\cdot,0)[\hat{u}(\cdot,0)-j]^+ dt - \int_0^{t_1} \rho_v \tilde{u}(\cdot,0)\hat{s}_t[\hat{u}(\cdot,0)-j]^+ dt \\
= &\int_0^{t_1}\int_0^1 \frac{2k\hat{s}}{(L-s)^3}\tilde{u}_{yy}[\hat{u}-j]^+ dydt \qquad (3.7)\\
&+ \int_0^{t_1}\int_0^1 \frac{\rho_v(1-y)\tilde{u}_y}{L-s}\hat{s}_t[\hat{u}-j]^+ dydt + \int_0^{t_1}\int_0^1 \frac{\rho_v(1-y)s_t}{L-s}\hat{u}_y[\hat{u}-j]^+ dydt \\
&+ \int_0^{t_1}\int_0^1 \frac{\rho_v(1-y)s_t}{(L-s)^2}\tilde{u}_y\hat{s}[\hat{u}-j]^+ dydt \quad \text{for } 0 \leq t_1 \leq T,
\end{aligned}
$$

where $[r]^+ = \max\{r,0\}$ for $r \in \mathbb{R}$.

4 Estimate for maximum of the derivatives

In this section we suppose that the assumption of Theorem 2.3 always holds and shall prove it in a similar way to that of [10, Theorem 7.1 in Chapter 3]. The proof is rather long we divide it to several lemmas.

First, let (s,u) be a solution of P(λ), $\tilde{u}$ be a function defined by (2.2), $\hat{s} = \frac{\partial s}{\partial \lambda}(\lambda)$ and $\hat{u} = \frac{\partial \tilde{u}}{\partial \lambda}(\lambda)$, and denote the right hand side of (3.1) by $\sum_{i=1}^{4} f_i$.

Lemma 4.1. *For each $i = 1,2,3,4$, $f_i \in L^2(0,T;H)$.*

Proof. First, by (2.1) it is easy to see that

$$
\begin{aligned}
&|f_1(t)|_H \leq \frac{2k}{(L-s_*)^3}|\hat{s}(t)||\tilde{u}_{yy}(t)|_H, |f_2(t)|_H \leq \frac{\rho_v}{L-s_*}|\hat{s}_t(t)||\tilde{u}_y(t)|_H, \\
&|f_3(t)|_H \leq \frac{a\rho_v}{L-s_*}|\hat{u}_y(t)|_H, |f_4(t)|_H \leq \frac{a\rho_v}{(L-s_*)^2}|\hat{s}(t)||\tilde{u}_y(t)|_H \quad \text{for a.e. } t \in [0,T].
\end{aligned}
$$

Here, (W1) and (S'1) imply that $f_i \in L^2(0,T;H)$ for $i = 1,2,3,4$. □

Next, we write the third term and the sum of fourth to sixth terms in the left hand side of (3.7) as $I_0(t_1)$ and $S_0(t_1)$, respectively. Also, we put

$$A_j(t) = \{y \in (0,1) | \hat{u}(t,y) \geq j\} \text{ for } t \in [0,T] \text{ and } j \in \mathbb{R}.$$

Lemma 4.2. *There exists a positive constant C_1 independent of j and t_1 such that*

$$I_0(t_1)+S_0(t_1) \geq -\frac{k}{2L^2}\int_0^{t_1} |[\hat{u}(t)-j]^+_y|_H^2 dt - j^2 C_1 \int_0^{t_1} |A_j(t)| dt \text{ for } 0 \leq t_1 \leq T, j \geq m_0+1.$$

Proof. Let $j \geq m_0 + 1$. First, by (2.5), (A4) and (2.1) we see that

$$\begin{aligned} |I_0(t_1)| &= |\int_0^{t_1} \frac{\hat{s}(t)}{L - s(t)} [\hat{u}(t,0) - j]^+ (\rho_w - \rho_v \tilde{u}(t,0)) s_t(t) dt| \\ &\leq |\int_0^{t_1} \frac{a\rho_w |\hat{s}|_{L^\infty(0,T)}}{L - s_*} [\hat{u}(t,0) - j]^+ dt| \\ &\leq jK_1 \int_0^{t_1} [\hat{u}(t,0) - j]^+ dt \quad \text{ for } 0 \leq t_1 \leq T, \end{aligned} \tag{4.1}$$

where $K_1 = \dfrac{a\rho_w |\hat{s}|_{L^\infty(0,T)}}{L - s_*}$.

Next, by (3.4) we observe that

$$\begin{aligned} S_0(t_1) &= \int_0^{t_1} (\rho_w \hat{s}_t(t) - \rho_v s_t(t) \hat{u}(t,0) - \rho_v \tilde{u}(t,0) \hat{s}_t(t)) [\hat{u}(t,0) - j]^+ dt \\ &= \int_0^{t_1} (a(\rho_w - \rho_v \tilde{u}(t,0)) - \rho_v s_t(t)) \hat{u}(t,0) [\hat{u}(t,0) - j]^+ dt \\ &\quad - \int_0^{t_1} a(\rho_w - \rho_v \tilde{u}(t,0)) \varphi'(s(t)) \hat{s}(t) [\hat{u}(t,0) - j]^+ dt \\ &=: I_1(t_1) + I_2(t_1) \quad \text{ for } 0 \leq t_1 \leq T. \end{aligned} \tag{4.2}$$

Here, by putting $\rho_0 = \rho_w - 2\rho_v > 0$ we note that

$$\begin{aligned} I_1(t_1) &= \int_0^{t_1} (a(\rho_w - \rho_v \tilde{u}(t,0)) - \rho_v s_t(t)) (|[\hat{u}(t,0) - j]^+|^2 + j[\hat{u}(t,0) - j]^+) dt \\ &\geq \int_0^{t_1} a\rho_0 (|[\hat{u}(t,0) - j]^+|^2 + j[\hat{u}(t,0) - j]^+) dt \quad \text{ for } 0 \leq t_1 \leq T, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} |I_2(t_1)| &\leq \int_0^{t_1} a(\rho_w - \rho_v \tilde{u}(t,0)) |\varphi'(s(t))| |\hat{s}(t)| [\hat{u}(t,0) - j]^+ dt \\ &\leq \int_0^{t_1} a\rho_w C_\varphi |\hat{s}|_{L^\infty(0,T)} [\hat{u}(t,0) - j]^+ dt \quad \text{ for } 0 \leq t_1 \leq T. \end{aligned} \tag{4.4}$$

From (4.2) $\sim$ (4.4) it follows that

$$\begin{aligned} & I_0(t_1) + S_0(t_1) \\ &\geq -j(K_1 + K_2) \int_0^{t_1} [\hat{u}(t,0) - j]^+ dt \\ &\geq -j(K_1 + K_2) \int_0^{t_1} \int_0^1 |[\hat{u}(t,y) - j]_y^+| dy dt \\ &\geq -\frac{j^2 (K_1 + K_2)^2 L^2}{2k} \int_0^{t_1} |A_j(t)| dt - \frac{k}{2L^2} \int_0^{t_1} |[\hat{u}(t,y) - j]_y^+|_H^2 dt \quad \text{ for } 0 \leq t_1 \leq T, \end{aligned} \tag{4.5}$$

where $K_2 = a\rho_w C_\varphi |\hat{s}|_{L^\infty(0,T)}$. Thus, we have proved this lemma. □

For simplicity we put $f = \sum_{i=1}^4 f_i$.

Lemma 4.3. *There exists a positive constant C_2 independent of j and t_1 such that*

$$
\begin{aligned}
&|[\hat{u} - j]^+|_{V(t_1)}^2 \\
\le & C_2 |f|_{L^2(0,T;H)} t_1^{1/10} |[\hat{u} - j]^+|_{V(t_1)}^2 + C_2 j^2 (\int_0^{t_1} |A_j(t)|dt)^{2/5} \text{ for } 0 \le t_1 \le T, j \ge m_0 + 1.
\end{aligned}
\tag{4.6}
$$

Proof. Let $j \ge m_0 + 1$. Thanks to Lemmas 3.3 and 4.2 we have

$$
\begin{aligned}
&\frac{\rho_v}{2} \int_0^1 |[\hat{u}(t_1) - j]^+|^2 dy + \frac{k}{2L^2} \int_0^{t_1} |[\hat{u}(t) - j]_y^+|_H^2 dt \\
\le & j^2 C_1 \int_0^{t_1} |A_j(t)|dt + \int_0^{t_1} \int_0^1 f(t)[\hat{u}(t) - j]^+ dydt \quad \text{for } 0 \le t_1 \le T.
\end{aligned}
$$

By elementary calculations we see that

$$
\begin{aligned}
&\int_0^{t_1} \int_0^1 f(t)[\hat{u}(t) - j]^+ dydt \\
\le & (\int_0^{t_1} \int_0^1 |f(t)|^{5/3} dydt)^{3/5} (\int_0^{t_1} \int_0^1 |[\hat{u}(t) - j]^+|^{5/2} dydt)^{2/5} \\
\le & (\int_0^{t_1} \int_0^1 |f(t)|^{5/3} dydt)^{3/5} (\int_0^{t_1} \int_{A_j(t)} (|[\hat{u}(t) - j]^+|^5 + j^5) dydt)^{2/5} \\
\le & |f|_{L^2(0,T;H)} t_1^{1/10} \left((\int_{Q(t_1)} |[\hat{u}(t) - j]^+|^5 dydt)^{2/5} + j^2 (\int_0^{t_1} |A_j(t)|dt)^{2/5} \right) \text{ for } 0 \le t_1 \le T.
\end{aligned}
$$

From these inequalities and Lemma 3.2 we obtain

$$
\begin{aligned}
&\frac{\rho_v}{2} \int_0^1 |[\hat{u}(t_1) - j]^+|^2 dy + \frac{k}{2L^2} \int_0^{t_1} |[\hat{u}(t) - j]_y^+|_H^2 dt \\
\le & j^2 (C_1 t_1^{3/5} + |f|_{L^2(0,T;H)} t_1^{1/10}) (\int_0^{t_1} |A_j(t)|dt)^{2/5} + |f|_{L^2(0,T;H)} t_1^{1/10} (\int_{Q(t_1)} |[\hat{u}(t) - j]^+|^5 dydt)^{2/5} \\
\le & j^2 K_2 (\int_0^{t_1} |A_j(t)|dt)^{2/5} + |f|_{L^2(0,T;H)} t_1^{1/10} |[\hat{u}(t) - j]^+|_{V(t_1)}^2 \text{ for } 0 \le t_1 \le T,
\end{aligned}
$$

where $K_2 = C_1 T^{3/5} + |f|_{L^2(0,T;H)} T^{1/10}$.

Here, by putting $\mu_0 = \min\{\frac{\rho_v}{2}, \frac{k}{2L^2}\}$ and $C_2 = (1 + K_2)/\mu_0$, we get (4.6). □

Now, we give a proof of Theorem 2.3.

Proof of Theorem 2.3. First, we choose $T_1 > 0$ such that

$$
C_2 |f|_{L^2(0,T;H)} T_1^{1/10} \le \frac{1}{2}.
$$

By Lemma 4.3 it is clear that

$$|[\hat{u}-j]^+|_{V(T_1)} \le C_3 j(\int_0^{T_1} |A_j(t)|dt)^{1/5} \text{ for } j \ge m_0+1,$$

where $C_3 = \sqrt{2C_2}$.

Let $m_1 \ge m_0+1$ and $j_q = (2-2^{-q})m_1$ for $q = 0, 1, 2, \cdots$. Accordingly, by Lemma 3.2 we see that

$$\begin{aligned}(j_{q+1}-j_q)(\int_0^{T_1} |A_{j_{q+1}}(t)|dt)^{1/6} \le &(\int_{Q(T_1)} |[\hat{u}(t)-j_q]^+|^6 dydt)^{1/6} \\ \le &|[\hat{u}(t)-j_q]^+|_{V(T_1)} \\ \le &C_3 j_q(\int_0^{T_1} |A_{j_q}(t)|dt)^{1/5} \text{ for } q = 0, 1, 2, \cdots, \end{aligned} \tag{4.7}$$

and

$$(\int_0^{T_1} |A_{j_{q+1}}(t)|dt)^{1/6} \le 4C_3 \cdot 2^q (\int_0^{T_1} |A_{j_q}(t)|dt)^{1/5} \text{ for } q = 0, 1, 2, \cdots.$$

Here, we define a sequence $\{\alpha_q\}$ by $\alpha_q = (\int_0^{T_1} |A_{j_q}(t)|dt)^{1/6}$ for $q = 0, 1, 2, \cdots$. Easily, we obtain

$$\alpha_{q+1} \le 4C_3 \cdot 2^q \alpha_q^{1+1/5} \text{ for } q = 0, 1, 2, \cdots.$$

Similarly to (4.7), we have

$$(m_1-(m_0+1))(\int_0^{T_1} |A_{m_1}(t)|dt)^{1/6} \le C_3(m_0+1)(\int_0^{T_1} |A_{m_0+1}(t)|dt)^{1/5} \le C_3(m_0+1)T_1^{1/5},$$

and

$$\alpha_0 \le \frac{C_3(m_0+1)}{m_1-m_0-1} T_1^{1/5}.$$

For a positive number N we put $m_1 = N(m_0+1)$ and obtain

$$\alpha_0 \le \frac{C_3}{N-1} T_1^{1/5}, \alpha_{q+1} \le 4C_3 \cdot 2^q \alpha_q^{1+1/5} \text{ for } q = 0, 1, 2, \cdots.$$

By applying [10, Lemma 5.6 in Chapter 2] we infer that if $\alpha_0 \le (4C_3)^{-5} 2^{-25}$, then $\alpha_q \to 0$ as $q \to \infty$. Hence, by taking $N \ge 1 + C_3 T_1^{1/5} (4C_3)^5 2^{25}$, we get $\int_0^{T_1} |A_{2m_1}|dt = 0$, namely, $\hat{u} \le 2m_1$ a.e. on $Q(T_1)$. Since the choice of T_1 is independent of m_0, by repeating the argument above we can get an estimate for the maximum of $\hat{u}$ on $Q(T)$. Moreover, we can get the same estimate for $-\hat{u}$. Thus, we conclude that $\hat{u} \in L^\infty(Q(T))$.

Finally, (3.4) guarantees that $\hat{s} \in W^{1,\infty}(0,T)$. □

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