

DIFFERENTIABILITY OF A SOLUTION OF A FREE BOUNDARY PROBLEM DESCRIBING WATER ADSORPTION

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Abstract. Sato-Aiki-Murase-Shirakawa [7, 16] proposed a free boundary problem in order to describe water adsorption appearing in moisture transport of concrete carbonation process. The problem is to find a pair of a curve and a function for given boundary and initial functions. In this paper we consider that the boundary and initial functions depend on some parameter. Kumazaki [9, 10] already proved the continuous and measurable properties of solutions with respect to the parameter. The aim of this paper is to establish differentiability of solutions with respect to the parameter as a new property. This result will be applied to research of the multi-scale model to water adsorption in concrete carbonation.

Communicated by Editors; Received May 16, 2020.

This work is supported by JSPS KAKENHI Grant Number JP16K17636, JP19K03572 and JP20K03704

AMS Subject Classification: 35R35 35K61 74F25

Keywords: Free boundary problem, differentiability of solutions, multi-scale problem, concrete carbonation

1 Introduction and main result

A two-scale model for moisture transport in concrete carbonation process was proposed in [6, 12]. The model is a system of a nonlinear diffusion equation in a macro domain $\Omega \subset \mathbb{R}^3$ and free boundary problems in a micro domain $(0, L)$ for a positive constant L . Here, we note that boundary and initial data in the free boundary problem are given as functions of a parameter $\lambda \in \Omega$. In this paper we mainly deal with the free boundary problem having the boundary and initial data which depend on the parameter varying in an open interval $(0, 1)$. Also, a result for the case the parameter varies in Ω is provided in the last section.

Accordingly, we consider the following free boundary problem $\{(1.1)-(1.6)\}$ denoted by $P(\lambda)$ for $\lambda \in (0, 1)$. Unknown functions of $P(\lambda)$ are a curve $x = s(\lambda)(t)$ on $[0, T]$ and a function $u(\lambda)$ on the set $Q_{s(\lambda)}(T) := \{(t, x) : 0 < t < T, s(\lambda)(t) < x < L\}$, $T > 0$, such that

$$\rho_v u_t(\lambda) - k u_{xx}(\lambda) = 0 \text{ for } t \in [0, T] \text{ and } x \in (s(\lambda)(t), L), \quad (1.1)$$

$$u(\lambda)(t, L) = h(\lambda, t) \text{ for } t \in (0, T), \quad (1.2)$$

$$k u_x(\lambda)(t, s(\lambda)(t)) = (\rho_w - \rho_v u(\lambda)(t, s(\lambda)(t))) s_t(\lambda)(t) \text{ for } t \in (0, T), \quad (1.3)$$

$$s_t(\lambda)(t) = a(u(\lambda)(t, s(\lambda)(t)) - \varphi(s(\lambda)(t))) \text{ for } t \in (0, T), \quad (1.4)$$

$$s(\lambda)(0) = s_0(\lambda), \quad (1.5)$$

$$u(\lambda)(0, x) = u_0(\lambda, x) \text{ for } x \in [s_0(\lambda), L], \quad (1.6)$$

where ρ_v , ρ_w , k and a are given positive constants and φ is a given continuous function on $\mathbb{R}$. Also, for each $\lambda \in (0, 1)$, $h = h(\lambda) = h(\lambda, \cdot)$ is a given function on $(0, T)$, $s_0(\lambda)$ is a constant and $u_0(\lambda)$ is a given function on $[s_0(\lambda), L]$.

The problem $\{(1.1) - (1.6)\}$ is originally proposed by Sato-Aiki-Murase-Shirakawa [7, 16] as a mathematical model describing water adsorption in concrete carbonation process, and the two-scale problem containing the free boundary problem is also proposed in [6]. Also, in case λ is fixed, the existence of a solution locally in time and the uniqueness were proved in [16], the global existence of the solution and a result on a large time behavior were obtained in Aiki-Murase [5], and existence of periodic solutions in time was shown in Aiki-Sato [8]. Moreover, in case λ is varying over Ω the problem $P(\lambda)$ was already studied. In this case, continuous and measurable properties of s and u with respect to λ were investigated in Kumazaki [9, 10]. By applying these properties the local existence of solutions in time and the uniqueness were obtained in Kumazaki-Aiki-Sato-Murase [12] and the global existence was discussed in Kumazaki [11]. The aim of this paper is to establish differentiability of solutions of $P(\lambda)$ with respect to λ under some conditions for h , s_0 and u_0 , when λ varies in $(0, 1)$.

In this paper, we proceed in the following way: In Section 2 we state our main theorem concerned with the differentiability of s and u with respect to $\lambda \in (0, 1)$. In the proof of the theorem we shall show convergence of differential quotients of s and u with denominator δ . In Section 3 we give some uniform estimates for the differential quotients with respect to δ and in Section 4 we infer that the quotients are Cauchy sequences. Due to the results obtained in Sections 3 and 4 we prove the convergences and the pair of the limits of the quotients satisfies a new system in Section 5. At the end of this paper we provide a

corollary of our main theorem. The corollary implies some estimates for derivative of s in case the parameter λ takes values in $\Omega \subset \mathbb{R}^3$.

2 Main results

First, in order to recall the existence and the uniqueness result for $P(\lambda)$ we give a list of assumptions as follows:

(A1) T , k and a are positive constants.

(A2) $\varphi \in C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$, $\varphi = 0$ on $(-\infty, 0]$, $\varphi \leq 1$ on $\mathbb{R}$, $\varphi' > 0$ on $(0, L]$. Also, we denote by $\hat{\varphi}$ the primitive function of φ with $\hat{\varphi}(0) = 0$ and put $C_\varphi = |\varphi'|_{L^\infty(\mathbb{R})} + |\varphi''|_{L^\infty(\mathbb{R})}$.

(A3) $h(= h(\lambda, t)) \in L^\infty(0, 1; W^{1,2}(0, T))$ with $0 \leq h(\lambda, \cdot) \leq h^*$ on $(0, T)$ for any $\lambda \in (0, 1)$, where h^* is a positive constant satisfying $h^* < \varphi(L)$.

(A4) Two positive constants ρ_w and ρ_v satisfy

$$\rho_w > 2\rho_v, \quad \rho_w \geq \rho_v(C_\varphi + 2), \quad 9aL\rho_v^2 \leq k\rho_w.$$

(A5) $s_0 \in L^2(0, 1)$ and $0 \leq s_0(\lambda) \leq L - \ell_0$ for any $\lambda \in (0, 1)$, where ℓ_0 is a positive constant, and $u_0(\lambda) \in W^{1,2}(0, L)$ such that $|u_0(\lambda)|_{W^{1,2}(s_0(\lambda), L)} \leq C(u_0)$ for any $\lambda \in (0, 1)$, where $C(u_0)$ is a positive constant, $u_0(\lambda)(L) = h(0, \lambda)$ and $0 \leq u_0(\lambda) \leq 1$ on $[s_0(\lambda), L]$ for $\lambda \in (0, 1)$.

Here, we define a solution of $P(\lambda)$ for $\lambda \in (0, 1)$.

Definition 1. For $T > 0$ and $\lambda \in (0, 1)$ let $s(\lambda)$ and $u(\lambda)$ be functions on $[0, T]$ and $Q_{s(\lambda)}(T)$, respectively. We call that a pair $(s(\lambda), u(\lambda))$ is a solution of $P(\lambda)$ on $[0, T]$ if the conditions (S1)-(S6) hold:

(S1) $s(\lambda) \in W^{1,\infty}(0, T)$, $0 \leq s(\lambda) < L$ on $[0, T]$, $u(\lambda) \in L^\infty(Q_{s(\lambda)}(T))$, $u_t(\lambda)$, $u_{xx}(\lambda) \in L^2(Q_{s(\lambda)}(T))$ and $|u_x(\lambda)(\cdot)|_{L^2(s(\lambda)(\cdot), L)} \in L^\infty(0, T)$.

(S2) $\rho_v u_t(\lambda) - k u_{xx}(\lambda) = 0$ a.e. on $Q_{s(\lambda)}(T)$.

(S3) $u(\lambda)(t, L) = h(\lambda, t)$ for a.e. $t \in [0, T]$.

(S4) $k u_x(\lambda)(t, s(\lambda)(t)) = (\rho_w - \rho_v u(\lambda)(t, s(\lambda)(t))) s_t(\lambda)(t)$ for a.e. $t \in [0, T]$.

(S5) $s_t(\lambda)(t) = a(u(\lambda)(t, s(\lambda)(t)) - \varphi(s(\lambda)(t)))$ for a.e. $t \in [0, T]$.

(S6) $s(\lambda)(0) = s_0(\lambda)$, $u(\lambda)(0, x) = u_0(\lambda, x)$ for $x \in [s_0(\lambda), L]$.

Theorem 1 is concerned with the existence and the uniqueness of a solution of $P(\lambda)$ on $[0, T]$.

Theorem 1. (cf. [5, Theorem 4.1]) Assume (A1)-(A5). Then, for any $\lambda \in (0, 1)$ there exists a unique solution $(s(\lambda), u(\lambda))$ of $P(\lambda)$ on $[0, T]$ such that

$$0 \leq u(\lambda) \leq 1 \text{ a.e. on } Q_{s(\lambda)}(T), 0 \leq s(\lambda) \leq s_* \text{ on } [0, T] \text{ and } |s_t(\lambda)| \leq a \text{ a.e. on } [0, T], \quad (2.1)$$

where s_* is a positive constant satisfying $s_* < L$ which does not depend on λ .

In order to give a statement of our main result we introduce the following notation and change of variables

$$\tilde{u}(\lambda)(t, y) = u(t, (1 - y)s(\lambda)(t) + yL) \text{ for } y \in (0, T) \times [0, 1]. \quad (2.2)$$

Then, we transform $P(\lambda)$ into the following problem on the cylindrical domain $Q(T) := (0, T) \times (0, 1)$:

$$\rho_v \tilde{u}_t(\lambda) - \frac{k}{(L - s(\lambda))^2} \tilde{u}_{yy}(\lambda) = \frac{\rho_v(1 - y)s_t(\lambda)}{L - s(\lambda)} \tilde{u}_y(\lambda) \text{ on } Q(T), \quad (2.3)$$

$$\tilde{u}(\lambda)(t, 1) = h(\lambda, t) \text{ for } t \in [0, T], \quad (2.4)$$

$$\frac{k}{L - s(\lambda)(t)} \tilde{u}_y(\lambda)(t, 0) = (\rho_w - \rho_v \tilde{u}(\lambda)(t, 0))s_t(\lambda)(t) \text{ for } t \in [0, T], \quad (2.5)$$

$$s_t(\lambda)(t) = a(\tilde{u}(\lambda)(t, 0) - \varphi(s(\lambda)(t))) \text{ for } t \in [0, T], \quad (2.6)$$

$$s(\lambda)(0) = s_0(\lambda), \tilde{u}(\lambda)(0, y) = u(0, (1 - y)s_0(\lambda) + yL) =: \tilde{u}_0(\lambda)(y) \text{ on } [0, 1]. \quad (2.7)$$

Remark 1. For $T > 0$ and $\lambda \in (0, 1)$, let $s(\lambda)$ and $u(\lambda)$ be functions on $[0, T]$ and $Q_{s(\lambda)}(T)$, respectively and define a function $\tilde{u}(\lambda)$ on $Q(T)$ by (2.2). The pair $(s(\lambda), u(\lambda))$ is a solution of $P(\lambda)$ if and only if (S'1) and (S'2) hold:

(S'1) $s(\lambda) \in W^{1,\infty}(0, T)$, $0 \leq s(\lambda) < L$ a.e. on $[0, T]$, $\tilde{u}(\lambda) \in W^{1,2}(0, T; L^2(0, 1)) \cap L^\infty(0, T; H^1(0, 1)) \cap L^\infty(Q(T)) \cap L^2(0, T; H^2(0, 1))$.

(S'2) (2.3)-(2.7) hold.

Hence, if (A1) $\sim$ (A5) hold, then for any $\lambda \in (0, 1)$ it holds that $0 \leq \tilde{u}(\lambda) \leq 1$ a.e. $Q(T)$ and $0 \leq s(\lambda) \leq s_*$ on $[0, T]$, where s_* is defined in Theorem 1.

Furthermore, we introduce notation to discuss differentiability of s and u . For $\lambda \in (0, 1)$ let $(s(\lambda), u(\lambda))$ be a solution of $P(\lambda)$ on $[0, T]$ and $\tilde{u}(\lambda)$ be a function decided from $u(\lambda)$ by (2.2). Also, for $\delta \in I(\lambda) := \{\delta' \in \mathbb{R}; 0 < \lambda + \delta' < 1\}$ we define the following differential quotients:

$$\begin{aligned} \Delta_\delta s(\lambda)(t) &:= \frac{s(\lambda + \delta)(t) - s(\lambda)(t)}{\delta}, & \Delta_\delta \tilde{u}(\lambda)(t) &:= \frac{\tilde{u}(\lambda + \delta)(t) - \tilde{u}(\lambda)(t)}{\delta}, \\ \Delta_\delta s_0(\lambda) &:= \frac{s_0(\lambda + \delta) - s_0(\lambda)}{\delta}, & \Delta_\delta \tilde{u}_0(\lambda) &:= \frac{\tilde{u}_0(\lambda + \delta) - \tilde{u}_0(\lambda)}{\delta}, \\ \Delta_\delta h(\lambda, t) &:= \frac{h(\lambda + \delta, t) - h(\lambda, t)}{\delta}. \end{aligned}$$

Also, the symbol $\frac{\partial}{\partial \lambda}$ represents the derivative respect to λ and we put

$$\frac{\partial s_0(\lambda)}{\partial \lambda} := s_{0\lambda}(\lambda), \quad \frac{\partial u_0(\lambda)}{\partial \lambda} := u_{0\lambda}(\lambda), \quad \frac{\partial h(\lambda, \cdot)}{\partial \lambda} := h_\lambda(\lambda, \cdot) \text{ and } \frac{\partial}{\partial t} \left(\frac{\partial h(\lambda, \cdot)}{\partial \lambda} \right) := h_{\lambda t}(\cdot, \lambda).$$

Now, we state our main theorem of this paper.

Theorem 2. Assume (A1) $\sim$ (A5) hold. Let $\lambda \in (0, 1)$ and $(s(\lambda), u(\lambda))$ be a solution of $P(\lambda)$ on $[0, T]$ and $\tilde{u}(\lambda)$ be a function decided from $u(\lambda)$ by (2.2). If h , s_0 and u_0 satisfy

$$\frac{\partial h}{\partial \lambda} \in L^2(0, 1; W^{1,2}(0, T)), \quad (2.8)$$

and

$$\frac{\partial s_0}{\partial \lambda} \in L^2(0, 1), u_0 \in L^\infty(0, 1; W^{1,2}(0, L)), \frac{\partial u_0}{\partial \lambda} \in L^2(0, 1; L^2(0, L)), \quad (2.9)$$

then for a.e. $\lambda \in (0, 1)$ there exists a unique $\hat{s}(\lambda) \in W^{1,2}(0, T)$ and $\hat{u}(\lambda) \in C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1))$ such that (i), (ii) and (iii):
(i) it holds that

$$\begin{aligned} \Delta_\delta s(\lambda) &\rightarrow \hat{s}(\lambda) \text{ in } W^{1,2}(0, T), \\ \Delta_\delta \tilde{u}(\lambda) &\rightarrow \hat{u}(\lambda) \text{ in } C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)) \text{ as } \delta \rightarrow 0. \end{aligned}$$

(ii) There exists a positive constant C which depends on $\rho_v, \rho_w, k, a, L, C_\varphi, \ell_0$ and $C(u_0)$ and is independent of δ such that

$$\begin{aligned} &|\hat{s}(\lambda)|_{W^{1,2}(0,T)} + |\hat{u}(\lambda)|_{C([0,T];L^2(0,1))} + |\hat{u}(\lambda)|_{L^2(0,T;H^1(0,1))} \\ &\leq C(|s_{0\lambda}(\lambda)| + |u_{0\lambda}(\lambda)|_{L^2(s_0(\lambda),L)} + |h_\lambda(\lambda)|_{W^{1,2}(0,T)}). \end{aligned}$$

(iii) $(\hat{s}(\lambda), \hat{u}(\lambda))$ is a unique weak solution of the following problem $\hat{P}(\lambda) = \{(2.10) - (2.14)\}$:

$$\begin{aligned} \rho_v \hat{u}_t(\lambda) - \frac{k}{(L - s(\lambda))^2} \hat{u}_{yy}(\lambda) \\ = \frac{2k\hat{s}(\lambda)}{(L - s(\lambda))^3} \tilde{u}_{yy}(\lambda) + \frac{\rho_v(1-y)\tilde{u}_y(\lambda)}{L - s(\lambda)} \hat{s}_t(\lambda) + \frac{\rho_v(1-y)s_t(\lambda)}{L - s(\lambda)} \hat{u}_y(\lambda) \\ + \frac{\rho_v(1-y)s_t(\lambda)}{(L - s(\lambda))^2} \tilde{u}_y(\lambda) \hat{s}(\lambda) \text{ in } Q(T), \end{aligned} \quad (2.10)$$

$$\hat{u}(\lambda)(t, 1) = h_\lambda(\lambda, t) \text{ for a.e. } t \in [0, T], \quad (2.11)$$

$$\begin{aligned} \frac{k}{L - s(\lambda)(t)} \hat{u}_y(\lambda)(t, 0) = - \frac{k}{(L - s(\lambda)(t))^2} \hat{s}(\lambda)(t) \tilde{u}_y(\lambda)(t, 0) + \rho_w \hat{s}_t(\lambda)(t) \\ - \rho_v s_t(\lambda)(t) \hat{u}(\lambda)(t, 0) - \rho_v \tilde{u}(\lambda)(t, 0) \hat{s}_t(t) \text{ for a.e. } t \in [0, T], \end{aligned} \quad (2.12)$$

$$\hat{s}_t(\lambda)(t) = a(\hat{u}(\lambda)(t, 0) - \varphi'(s(\lambda)(t))\hat{s}(\lambda)(t)) \text{ for a.e. } t \in [0, T], \hat{s}(\lambda)(0) = s_{0\lambda}(\lambda), \quad (2.13)$$

$$\hat{u}(\lambda)(0, y) = u_{0\lambda}(\lambda, \sigma(\lambda, y)) + u_{0x}(\lambda, \sigma(\lambda, y))(1 - y)s_{0\lambda}(\lambda, y) \text{ for a.e. } y \in (0, 1), \quad (2.14)$$

where $\sigma(\lambda, y) = (1 - y)s_0(\lambda) + yL$ for $y \in (0, 1)$.

Here, we define a weak solution of $\hat{P}(\lambda)$ on $[0, T]$ as follows:

Definiton 2. For $T > 0$ and $\lambda \in (0, 1)$ let $\hat{s}(\lambda)$ be a function on $[0, T]$ and $\hat{u}(\lambda)$ be a function on $Q(T)$. We call that the pair $(\hat{s}(\lambda), \hat{u}(\lambda))$ is a weak solution of $\hat{P}(\lambda)$ on $[0, T]$ if the following conditions hold:

(W1) $\hat{s}(\lambda) \in W^{1,2}(0, T)$ and $\hat{u}(\lambda) \in C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1))$.

(W2) For any $\eta \in W^{1,2}(0, T; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1))$ with $\eta(T) = 0$ and $\eta(t, 1) = 0$ for $t \in [0, T]$, it holds that

$$- \int_0^T \int_0^1 \rho_v \hat{u}(\lambda)(t) \eta_t(t) dy dt + \int_0^T \int_0^1 \frac{k}{(L - s(\lambda)(t))^2} \hat{u}_y(\lambda)(t) \eta_y(t) dy dt$$

$$\begin{aligned}
& - \int_0^T \frac{k}{(L - s(\lambda)(t))^2} \hat{s}(\lambda)(t) \tilde{u}_y(\lambda)(t, 0) \eta(0) dt + \int_0^T \rho_w \hat{s}_t(\lambda)(t) \eta(t, 0) dt \\
& - \int_0^T \rho_v s_t(\lambda)(t) \hat{u}(\lambda)(t, 0) \eta(t, 0) dt - \int_0^T \rho_v \tilde{u}(\lambda)(t, 0) \hat{s}_t(\lambda)(t) \eta(t, 0) dt \\
& = \int_0^1 \rho_v \hat{u}_0(\lambda) \eta(0) dy + \int_0^T \int_0^1 \frac{2k \hat{s}(\lambda)(t)}{(L - s(\lambda)(t))^3} \tilde{u}_{yy}(\lambda)(t) \eta(t) dy dt \\
& + \int_0^T \int_0^1 \frac{\rho_v (1 - y) \tilde{u}_y(\lambda)(t)}{L - s(\lambda)(t)} \hat{s}_t(\lambda)(t) \eta(t) dy dt \\
& + \int_0^T \int_0^1 \frac{\rho_v (1 - y) s_t(\lambda)(t)}{L - s(\lambda)(t)} \hat{u}_y(\lambda)(t) \eta(t) dy dt \\
& + \int_0^T \int_0^1 \frac{\rho_v (1 - y) s_t(\lambda)(t)}{(L - s(\lambda)(t))^2} \tilde{u}_y(\lambda)(t) \hat{s}(\lambda)(t) \eta(t) dy dt,
\end{aligned}$$

where $\hat{u}_0(\lambda, y) = u_{0\lambda}(\lambda, \sigma(\lambda, y)) + u_{0x}(\lambda, \sigma(\lambda, y))(1 - y)s_{0\lambda}(\lambda, y)$ for a.e. $y \in (0, 1)$.
(W3) (2.11) and (2.13) hold.

In the proof of Theorem 2 we shall show convergences of $\Delta_\delta s(\lambda)$ and $\Delta_\delta \tilde{u}(\lambda)$ only for $\delta > 0$, since we can prove the convergences similarly when δ is negative.

Remark 2. (1) For φ the condition $\varphi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ is sufficient to show the measurability of solutions in [9, 10]. Here, in order to obtain more better regularities of solutions we impose the stronger condition (A2) in Theorem 2.

(2) In Theorem 2 we say that the properties (i) $\sim$ (iii) hold for a.e. $\lambda \in (0, 1)$. Namely, it is necessary to show existence of a set $N \subset (0, 1)$ such that $\text{meas}(N) = 0$ and (i) $\sim$ (iii) hold on $(0, 1) \setminus N$, where "meas" indicates the Lebesgue measure in $\mathbb{R}$. Choice of the set N will be discussed in Lemma 2 in detail.

3 Uniform estimate with respect to δ

First, we provide a lemma concerned with estimates for the solution $(s(\lambda), u(\lambda))$ of $P(\lambda)$ with respect to λ .

Lemma 1. Let $\lambda \in (0, 1)$ and $(s(\lambda), u(\lambda))$ be a solution of $P(\lambda)$ on $[0, T]$ and $\tilde{u}(\lambda)$ be a function decided from $u(\lambda)$ by (2.2). If (A1) $\sim$ (A5) hold then there exists a positive constant K independent of λ such that

$$|\tilde{u}_t(\lambda)|_{L^2(0,t;L^2(0,1))} + |\tilde{u}_y(\lambda)(t)|_{L^2(0,1)} + |\tilde{u}_{yy}(\lambda)|_{L^2(0,t;L^2(0,1))} \leq K \text{ for } t \in [0, T] \text{ and } \lambda \in (0, 1).$$

Proof. Let $\lambda \in (0, 1)$ and $(s(\lambda), u(\lambda))$ be a solution of $P(\lambda)$ on $[0, T]$. As showed in [5, Theorem 2.4] and [12, Theorem 4.1] already, there exists a positive constant K_1 independent of $\lambda \in (0, 1)$ such that

$$|u_t(\lambda)|_{L^2(Q_{s(\lambda)}(t))} + |u_x(\lambda)|_{L^2(s(\lambda)(t), L)} \leq K_1 \text{ for } t \in [0, T] \text{ and } \lambda \in (0, 1).$$

By changing variable as in (2.2), we observe that

$$\begin{aligned}
& \int_0^t \int_0^1 |\tilde{u}_t(\lambda)(\tau, y)|^2 dy d\tau \\
&= \int_0^t \int_{s(\lambda)(\tau)}^L \left(\frac{1}{L - s(\lambda)(\tau)} \right) \left| u_t(\lambda)(\tau, x) + u_x(\lambda)(\tau, x) \frac{L - x}{L - s(\lambda)(\tau)} s_t(\lambda)(\tau) \right|^2 dx d\tau \\
&\leq 2 \int_0^t \int_{s(\lambda)(\tau)}^L \left(\frac{1}{L - s(\lambda)(\tau)} \right) |u_t(\lambda)(t, x)|^2 dx d\tau \\
&\quad + 2 \int_0^t \int_{s(\lambda)(\tau)}^L \left(\frac{1}{L - s(\lambda)(\tau)} \right) \left| u_x(\lambda)(\tau, x) \frac{L - x}{L - s(\lambda)(\tau)} s_t(\lambda)(\tau) \right|^2 dx d\tau \text{ for } t \in [0, T],
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\int_0^1 |\tilde{u}_y(\lambda)(t)|^2 dy &= \int_{s(\lambda)(t)}^L \frac{1}{L - s(\lambda)(t)} |u_x(\lambda)(t)(L - s(\lambda)(t))|^2 dx dt \\
&= \int_{s(\lambda)(t)}^L (L - s(\lambda)(t)) |u_x(\lambda)(t)|^2 dx dt \text{ for } t \in [0, T].
\end{aligned} \tag{3.2}$$

Therefore, from (2.1), (3.1) and (3.2) we can find a positive constant K_2 satisfying

$$|\tilde{u}_t(\lambda)|_{L^2(0,t;L^2(0,1))} + |\tilde{u}_y(\lambda)(t)|_{L^2(0,1)} \leq K_2 \text{ for } t \in [0, T] \text{ and } \lambda \in (0, 1). \tag{3.3}$$

Also, by (2.3), (3.3) and (2.1) we can infer that

$$|\tilde{u}_{yy}(\lambda)|_{L^2(0,t;L^2(0,1))} \leq K_3 \text{ for } t \in [0, T],$$

where K_3 is a positive constant depending on a , s_* , L and K_2 . Therefore, by putting $K = K_2 + K_3$ we see that Lemma 1 is true. $\square$

Next, we give a lemma concerned with convergence for differential quotients of boundary and initial functions.

Lemma 2. *If (A5), (2.8) and (2.9) hold, then there exists a set $N \subset (0, 1)$ such that $\text{meas}(N) = 0$ and for $\lambda \in (0, 1) \setminus N$ it holds that*

$$\Delta_\delta h(\lambda) \rightarrow h_\lambda(\lambda) \text{ in } W^{1,2}(0, T), \tag{3.4}$$

$$\Delta_\delta s_0(\lambda) \rightarrow s_{0\lambda}(\lambda) \text{ in } \mathbb{R}, \tag{3.5}$$

$$\Delta_\delta \tilde{u}_0(\lambda, \cdot) \rightarrow u_{0\lambda}(\lambda, \sigma(\lambda, \cdot)) + (1 - \cdot) s_{0\lambda}(\lambda) u_{0x}(\lambda, \sigma(\lambda, \cdot)) \text{ in } L^2(0, 1) \text{ as } \delta \rightarrow 0, \tag{3.6}$$

where $\sigma(\lambda, y) = (1 - y)s_0(\lambda) + yL$ for $y \in (0, 1)$. Clearly, the sets $\{\Delta_\delta h(\lambda)\}_{\delta \in I_+(\lambda)}$, $\{\Delta_\delta s_0(\lambda)\}_{\delta \in I_+(\lambda)}$ and $\{\Delta_\delta \tilde{u}_0(\lambda)\}_{\delta \in I_+(\lambda)}$ are bounded in $L^2(0, T)$, $\mathbb{R}$ and $L^2(0, 1)$, respectively, where $I_+(\lambda) := \{\delta > 0 : \delta \in I(\lambda)\}$.

Proof. By (2.8) we see that $h(\lambda, \cdot)$ is differentiable as a function from $(0, 1)$ to $W^{1,2}(0, T)$. Accordingly, (3.4) holds for a.e. $\lambda \in (0, 1)$. Also, (3.5) holds for a.e. $\lambda \in (0, 1)$.

Let $\lambda \in (0, 1)$ and $\delta \in I_+(\lambda)$. Then, we have

$$\begin{aligned} & \Delta_\delta \tilde{u}_0(\lambda, y) \\ &= \frac{1}{\delta}(u_0(\lambda + \delta, \sigma(\lambda + \delta, y)) - u_0(\lambda, \sigma(\lambda + \delta, y))) + \frac{1}{\delta}(u_0(\lambda, \sigma(\lambda + \delta, y)) - u_0(\lambda, \sigma(\lambda, y))) \\ &=: U_1^\delta(\lambda, y) + U_2^\delta(\lambda, y) \quad \text{for } (\lambda, y) \in (0, 1) \times (0, 1). \end{aligned}$$

For simplicity, we put

$$U_1(\lambda, y) := u_{0\lambda}(\lambda, \sigma(\lambda, y)), U_2(\lambda, y) := (1-y)s_{0\lambda}(\lambda)u_{0x}(\lambda, \sigma(\lambda, y)) \text{ for } (\lambda, y) \in (0, 1) \times (0, 1).$$

By using these notation we infer that

$$\begin{aligned} & |U_1^\delta(\lambda, \cdot) - U_1(\lambda, \cdot)|_{L^2(0,1)} \\ & \leq |U_1^\delta(\lambda, \cdot) - u_{0\lambda}(\lambda, \sigma(\lambda + \delta, \cdot))|_{L^2(0,1)} + |u_{0\lambda}(\lambda, \sigma(\lambda + \delta, \cdot)) - U_1(\lambda, \cdot)|_{L^2(0,1)} =: I_1^\delta + I_2^\delta. \end{aligned}$$

Here, by changing variable we see that

$$\begin{aligned} I_1^\delta & \leq \left(\int_0^1 \left| \frac{1}{\delta}(u_0(\lambda + \delta, \sigma(\lambda + \delta, y)) - u_0(\lambda, \sigma(\lambda + \delta, y))) - u_{0\lambda}(\lambda, \sigma(\lambda + \delta, y)) \right|^2 dy \right)^{1/2} \\ & = \frac{1}{\sqrt{L - s_0(\lambda + \delta)}} \left(\int_{s_0(\lambda + \delta)}^L \left| \frac{1}{\delta}(u_0(\lambda + \delta, x) - u_0(\lambda, x)) - u_{0\lambda}(\lambda, x) \right|^2 dx \right)^{1/2}. \end{aligned} \quad (3.7)$$

The assumption (2.9) implies that

$$\left(\int_0^L \left| \frac{1}{\delta}(u_0(\lambda + \delta, x) - u_0(\lambda, x)) - u_{0\lambda}(\lambda, x) \right|^2 dx \right)^{1/2} \rightarrow 0 \text{ as } \delta \rightarrow 0 \text{ for a.e. } \lambda \in (0, 1).$$

Accordingly, it holds that $I_1^\delta \rightarrow 0$ as $\delta \rightarrow 0$. Similarly to (3.7), we have

$$\begin{aligned} I_2^\delta & \leq \left(\int_0^1 |u_{0\lambda}(\lambda, \sigma(\lambda + \delta, y)) - u_{0\lambda}(\lambda, \sigma(\lambda, y))|^2 dy \right)^{1/2} \\ & = \frac{1}{\sqrt{L - s_0(\lambda)}} \left(\int_{s_0(\lambda)}^L \left| u_{0\lambda}(\lambda, x + (\frac{L-x}{L-s_0(\lambda)})(s_0(\lambda + \delta) - s_0(\lambda))) - u_{0\lambda}(\lambda, x) \right|^2 dx \right)^{1/2}. \end{aligned}$$

Since $\int_{s_0(\lambda)}^L |u_{0\lambda}(\lambda, x)|^2 dx < \infty$ for a.e. $\lambda \in (0, 1)$ and $|(\frac{L-x}{L-s_0(\lambda)})(s_0(\lambda + \delta) - s_0(\lambda))| \rightarrow 0$ as $\delta \rightarrow 0$, we obtain that $I_2^\delta \rightarrow 0$ as $\delta \rightarrow 0$.

Obviously, we have

$$U_2^\delta(\lambda, y) = \frac{1}{\delta} \int_\lambda^{\lambda + \delta} (1-y)s_{0\lambda}(r)u_{0x}(\lambda, \sigma(r, y))dr \text{ for } (\lambda, y) \in (0, 1) \times (0, 1),$$

and then,

$$\begin{aligned} & |U_2^\delta(\lambda) - U_2(\lambda)|_{L^2(0,1)} \\ & \leq \left| \frac{1}{\delta} \int_\lambda^{\lambda + \delta} (1-y)(s_{0\lambda}(r) - s_{0\lambda}(\lambda))u_{0x}(\lambda, \sigma(r, y))dr \right|_{L^2(0,1)} \end{aligned}$$

$$+ \left| \frac{1}{\delta} \int_{\lambda}^{\lambda+\delta} (1-y) s_{0\lambda}(\lambda) (u_{0x}(\lambda, \sigma(r, y)) - u_{0x}(\lambda, \sigma(\lambda, y))) dr \right|_{L^2(0,1)} =: I_3^\delta + I_4^\delta.$$

It is easy to see that

$$\begin{aligned} I_3^\delta &\leq \left(\frac{1}{\delta} \int_{\lambda}^{\lambda+\delta} \int_0^1 |u_{0x}(\lambda, \sigma(r, y))|^2 |s_{0\lambda}(r) - s_{0\lambda}(\lambda)|^2 dy dr \right)^{1/2} \\ &\leq \frac{1}{\sqrt{L-\ell_0}} \left(\frac{1}{\delta} \int_{\lambda}^{\lambda+\delta} |s_{0\lambda}(r) - s_{0\lambda}(\lambda)|^2 \int_{s_0(r)}^L |u_{0x}(\lambda, x)|^2 dx dr \right)^{1/2} \\ &\leq \frac{1}{\sqrt{L-\ell_0}} |u_{0x}(\lambda)|_{L^2(0,L)} \left(\frac{1}{\delta} \int_{\lambda}^{\lambda+\delta} |s_{0\lambda}(r) - s_{0\lambda}(\lambda)|^2 dr \right)^{1/2}. \end{aligned}$$

Here, by applying Lebesgue's density theorem it holds that

$$\frac{1}{\delta} \int_{\lambda}^{\lambda+\delta} |s_{0\lambda}(r) - s_{0\lambda}(\lambda)|^2 dr \rightarrow 0 \text{ as } \delta \rightarrow 0 \text{ for a.e. } \lambda \in (0, 1). \quad (3.8)$$

Indeed, for any rational number q the function $r \rightarrow |s_{0\lambda}(r) - q|^2$ is integrable on $(0, 1)$. Then, Lebesgue's density theorem implies that there exists $N_q \subset (0, 1)$ with $\text{meas}(N_q) = 0$ such that

$$\frac{1}{\delta} \int_{\lambda}^{\lambda+\delta} |s_{0\lambda}(r) - q|^2 dr \rightarrow 0 \text{ as } \delta \rightarrow 0 \text{ for } \lambda \in (0, 1) \setminus N_q.$$

Also, the set $E_1 := \{\lambda \in (0, 1) : |s_{0\lambda}(\lambda)| < \infty\}$ satisfies $\text{meas}((0, 1) \setminus E_1) = 0$. Moreover, we put $N = ((0, 1) \setminus E_1) \cup (\cup_{q \in \mathbb{Q}} N_q)$. Then, for any $\lambda \in (0, 1) \setminus N$ and $\varepsilon > 0$ we can take $q \in \mathbb{Q}$ such that $|s_{0\lambda}(\lambda) - q| < \varepsilon$. It clear that

$$\left(\frac{1}{\delta} \int_{\lambda}^{\lambda+\delta} |s_{0\lambda}(r) - s_{0\lambda}(\lambda)|^2 dr \right)^{1/2} \leq \left(\frac{1}{\delta} \int_{\lambda}^{\lambda+\delta} |s_{0\lambda}(r) - q|^2 dr \right)^{1/2} + \varepsilon.$$

This guarantees (3.8) and then, $I_3^\delta \rightarrow 0$ as $\delta \rightarrow 0$. The argument for I_4^δ is similar to that for I_2^δ . Thus, we have proved (3.6). $\square$

Next, by using Lemma 1, we give uniform estimates for the differential quotient $\Delta_\delta s(\lambda)$ and $\Delta_\delta \tilde{u}(\lambda)$ with respect to δ .

Lemma 3. *Assume (A1) $\sim$ (A5), (2.8) and (2.9) and for $\lambda \in (0, 1)$ let $(s(\lambda), u(\lambda))$ be a solution of $P(\lambda)$ on $[0, T]$ and $\tilde{u}(\lambda)$ be a function decided from $u(\lambda)$ by (2.2). Then, there exists a positive constant C which depends on ρ_v , ρ_w , k , a , C_φ , ℓ_0 , L , $C(u_0)$ and $|h|_{L^\infty(0,1;W^{1,2}(0,T))}$, and is independent of δ such that*

$$\begin{aligned} &|\Delta_\delta s(\lambda)|_{W^{1,2}(0,T)} + |\Delta_\delta \tilde{u}(\lambda)|_{C([0,T];L^2(0,1))} + |\Delta_\delta \tilde{u}(\lambda)|_{L^2(0,T;H^1(0,1))} \\ &\leq C(|\Delta_\delta s_0(\lambda)| + |\Delta_\delta \tilde{u}_0(\lambda)|_{L^2(0,1)} + |\Delta_\delta h(\lambda)|_{W^{1,2}(0,T)}) \text{ for a.e. } \lambda \in (0, 1) \text{ and } \delta \in I_+(\lambda). \end{aligned} \quad (3.9)$$

Moreover, the sets $\{\Delta_\delta s(\lambda)\}_{\delta \in I_+(\lambda)}$ and $\{\Delta_\delta \tilde{u}(\lambda)\}_{\delta \in I_+(\lambda)}$ are bounded in $W^{1,2}(0, T)$ and $C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1))$, respectively.

Proof. Let $\lambda \in (0, 1) \setminus N$ and $\delta \in I_+(\lambda)$, where N is the set obtained in Lemma 2. Then, from Remark 1 and (2.4) $\sim$ (2.7) it follows that

$$\begin{aligned} & \rho_v \Delta_\delta \tilde{u}_t(\lambda) - \frac{k}{(L - s(\lambda))^2} \Delta_\delta \tilde{u}_{yy}(\lambda) \\ &= \frac{k(2L - s(\lambda + \delta) - s(\lambda))}{(L - s(\lambda + \delta))^2 (L - s(\lambda))^2} \Delta_\delta s(\lambda) \tilde{u}_{yy}(\lambda + \delta) + \rho_v(1 - y) \frac{\tilde{u}_y(\lambda + \delta)}{L - s(\lambda + \delta)} \Delta_\delta s_t(\lambda) \\ &+ \rho_v(1 - y) \left(\frac{s_t(\lambda)}{L - s(\lambda + \delta)} \Delta_\delta \tilde{u}_y(\lambda) + \frac{s_t(\lambda) \tilde{u}_y(\lambda)}{(L - s(\lambda + \delta))(L - s(\lambda))} \Delta_\delta s(\lambda) \right) \\ &=: \sum_{i=1}^4 F_{i\delta}(\lambda) \text{ in } Q(T), \end{aligned} \quad (3.10)$$

$$\Delta_\delta \tilde{u}(\lambda)(t, 1) = \Delta_\delta h(\lambda, t) \text{ for } t \in [0, T], \quad (3.11)$$

$$\Delta_\delta s_t(\lambda)(t) = a \left(\Delta_\delta \tilde{u}(\lambda)(t, 0) - \frac{\varphi(s(\lambda + \delta)(t)) - \varphi(s(\lambda)(t))}{\delta} \right) \text{ for } t \in [0, T], \quad (3.12)$$

$$\begin{aligned} & \frac{k}{L - s(\lambda)(t)} \Delta_\delta \tilde{u}_y(\lambda)(t, 0) + \frac{k}{(L - s(\lambda + \delta)(t))(L - s(\lambda)(t))} \Delta_\delta s(\lambda)(t) \tilde{u}_y(\lambda + \delta)(t, 0) \\ &= \rho_w \Delta_\delta s_t(\lambda)(t) - \rho_v s_t(\lambda + \delta)(t) \Delta_\delta \tilde{u}(\lambda)(t, 0) - \rho_v \tilde{u}(\lambda)(t, 0) \Delta_\delta s_t(\lambda)(t) \text{ for } t \in [0, T], \end{aligned} \quad (3.13)$$

$$\Delta_\delta s(\lambda)(0) = \Delta_\delta s_0(\lambda), \quad \Delta_\delta \tilde{u}(\lambda)(0, y) = \Delta_\delta \tilde{u}_0(\lambda)(y) \text{ for } y \in (0, 1). \quad (3.14)$$

Moreover, we put $F_\delta(\lambda) := \sum_{i=1}^4 F_{i\delta}(\lambda)$ and $\Delta_\delta \bar{u}(\lambda)(t, y) = \Delta_\delta \tilde{u}(\lambda)(t, y) - \Delta_\delta h(\lambda, t)$ for $\delta \in I_+(\lambda)$, $t \in [0, T]$ and $y \in [0, 1]$. By multiplying (3.10) with $\Delta_\delta \bar{u}(\lambda)(t)$, we obtain

$$\begin{aligned} & \frac{\rho_v}{2} \frac{d}{dt} \int_0^1 |\Delta_\delta \bar{u}(\lambda)(t)|^2 dy - \frac{k}{(L - s(\lambda)(t))^2} \int_0^1 \Delta_\delta \tilde{u}_{yy}(\lambda)(t) \Delta_\delta \bar{u}(\lambda)(t) dy \\ &= \int_0^1 \left(F_\delta(\lambda)(t) - \rho_v \Delta_\delta h_t(\lambda, t) \right) \Delta_\delta \bar{u}(\lambda)(t) dy \quad \text{for a.e. } t \in [0, T]. \end{aligned} \quad (3.15)$$

Using the boundary condition (3.13), we can deal with the second term of the right hand side of (3.15) as follows:

$$\begin{aligned} & - \frac{k}{(L - s(\lambda)(t))^2} \int_0^1 \Delta_\delta \tilde{u}_{yy}(\lambda)(t) \Delta_\delta \bar{u}(\lambda)(t) dy \\ &= \frac{k}{(L - s(\lambda)(t))^2} \int_0^1 |\Delta_\delta \tilde{u}_y(\lambda)(t)|^2 dy + \frac{k}{(L - s(\lambda)(t))^2} \Delta_\delta \tilde{u}_y(\lambda)(t, 0) \Delta_\delta \bar{u}(\lambda)(t, 0) \\ &= \frac{k}{(L - s(\lambda)(t))^2} \int_0^1 |\Delta_\delta \tilde{u}_y(\lambda)(t)|^2 dy \\ &+ \frac{1}{L - s(\lambda)(t)} \Delta_\delta \bar{u}(\lambda)(t, 0) \left(- \frac{\Delta_\delta s(\lambda)(t) \tilde{u}_y(\lambda + \delta)(t, 0)}{(L - s(\lambda + \delta)(t))(L - s(\lambda)(t))} \right) \\ &+ \frac{1}{L - s(\lambda)(t)} \Delta_\delta \bar{u}(\lambda)(t, 0) \left(\rho_w \Delta_\delta s_t(\lambda)(t) - \rho_v s_t(\lambda + \delta)(t) \Delta_\delta \tilde{u}(\lambda)(t, 0) \right) \\ &+ \frac{1}{L - s(\lambda)(t)} \Delta_\delta \bar{u}(\lambda)(t, 0) \left(- \rho_v \tilde{u}(\lambda)(t, 0) \Delta_\delta s_t(\lambda)(t) \right) \end{aligned}$$

$$=: \sum_{i=1}^5 I_{i\delta}(\lambda)(t) \quad \text{for } t \in [0, T].$$

In the following calculations, we use (2.1) and $\Delta_\delta \bar{u}(\lambda)(\cdot, 0) = \int_0^1 \Delta_\delta \tilde{u}_y(\lambda)(\cdot, y) dy$ on $[0, T]$. First, by (2.5) it holds that

$$\begin{aligned} |I_{2\delta}(\lambda)(t)| &= \left| \frac{\Delta_\delta \bar{u}(\lambda)(t, 0)}{(L - s(\lambda)(t))^2} \Delta_\delta s(\lambda)(t) \frac{1}{k} \left((\rho_w - \rho_v \tilde{u}(\lambda + \delta)(t, 0)) s_t(\lambda + \delta)(t) \right) \right| \\ &\leq \frac{(\rho_w + \rho_v)a}{k(L - s(\lambda)(t))^2} |\Delta_\delta \bar{u}(\lambda)(t, 0)| |\Delta_\delta s(\lambda)(t)| \\ &= \frac{(\rho_w + \rho_v)a}{k(L - s(\lambda)(t))^2} \left| \int_0^1 \Delta_\delta \tilde{u}_y(\lambda)(t) dy \right| |\Delta_\delta s(\lambda)(t)| \\ &\leq \frac{k}{2\eta(L - s(\lambda)(t))^2} |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \frac{\eta}{2k} \left(\frac{(\rho_w + \rho_v)a}{k(L - s(\lambda)(t))} \right)^2 |\Delta_\delta s(\lambda)(t)|^2 \\ &\quad \text{for a.e. } t \in [0, T], \end{aligned} \tag{3.16}$$

where η is an arbitrary positive number. Similarly to (3.16), we can estimate $I_{3\delta}(\lambda)$, $I_{4\delta}(\lambda)$ and $I_{5\delta}(\lambda)$ in the following way:

$$\begin{aligned} |I_{3\delta}(\lambda)(t)| &= \frac{\rho_w}{L - s(\lambda)(t)} |\Delta_\delta \bar{u}(\lambda)(t, 0)| |\Delta_\delta s_t(\lambda)(t)| \\ &= \frac{\rho_w}{L - s(\lambda)(t)} \left| \int_0^1 \Delta_\delta \tilde{u}_y(\lambda)(t) dy \right| |\Delta_\delta s_t(\lambda)(t)| \\ &\leq \frac{k}{2\eta(L - s(\lambda)(t))^2} |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \frac{\rho_w^2 \eta}{2k} |\Delta_\delta s_t(\lambda)(t)|^2, \end{aligned} \tag{3.17}$$

$$\begin{aligned} |I_{4\delta}(\lambda)(t)| &= \frac{\rho_v}{L - s(\lambda)(t)} |\Delta_\delta \bar{u}(\lambda)(t, 0) \Delta_\delta \tilde{u}(\lambda)(t, 0)| |s_t(\lambda + \delta)(t)| \\ &\leq \frac{\rho_v a}{L - s(\lambda)(t)} \left(|\Delta_\delta \bar{u}(\lambda)(t, 0)|^2 + \Delta_\delta \bar{u}(\lambda)(t, 0) \Delta_\delta h(t, \lambda) \right) \\ &\leq \frac{\rho_v a}{L - s(\lambda)(t)} \left(\int_0^1 \frac{\partial}{\partial y} |\Delta_\delta \bar{u}(\lambda)(t)|^2 dy + \left| \int_0^1 \Delta_\delta \tilde{u}_y(\lambda)(t) dy \right| |\Delta_\delta h(t, \lambda)| \right) \\ &\leq \frac{\rho_v a}{L - s(\lambda)(t)} \left(2 |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)} |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)} + |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)} |\Delta_\delta h(t, \lambda)| \right) \\ &\leq \frac{k}{2\eta(L - s(\lambda)(t))^2} |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \frac{2\eta(\rho_v a)^2}{k} |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 \\ &\quad + \frac{k}{2\eta(L - s(\lambda)(t))^2} |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \frac{\eta(\rho_v a)^2}{2k} |\Delta_\delta h(t, \lambda)|^2, \end{aligned} \tag{3.18}$$

and

$$|I_{5\delta}(\lambda)(t)| = \frac{\rho_v}{L - s(\lambda)(t)} |\Delta_\delta \bar{u}(\lambda)(t, 0)| |\tilde{u}(\lambda)(t, 0)| |\Delta_\delta s_t(\lambda)(t)|$$

$$\begin{aligned}
&\leq \frac{\rho_v}{L - s(\lambda)(t)} |\Delta_\delta \bar{u}(\lambda)(t, 0)| |\Delta_\delta s_t(\lambda)(t)| \\
&\leq \frac{k}{2\eta(L - s(\lambda)(t))^2} |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \frac{\eta \rho_v^2}{2k} |\Delta_\delta s_t(\lambda)(t)|^2.
\end{aligned} \tag{3.19}$$

Next, we give estimates for some terms in the right hand side of (3.15) as follows:

$$\begin{aligned}
&| \int_0^1 F_{1\delta}(\lambda)(t) \Delta_\delta \bar{u}(\lambda)(t) dy | \\
&\leq \frac{2kL}{(L - s(\lambda + \delta)(t))^2 (L - s(\lambda)(t))^2} |\Delta_\delta s(\lambda)(t)| \int_0^1 |\tilde{u}_{yy}(\lambda + \delta)(t)| |\Delta_\delta \bar{u}(\lambda)(t)| dy \\
&\leq \frac{2kL}{(L - s_*)^4} |\Delta_\delta s(\lambda)(t)| |\tilde{u}_{yy}(\lambda + \delta)(t)|_{L^2(0,1)} |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)} \\
&\leq \frac{\eta}{2} |\Delta_\delta s(\lambda)(t)|^2 |\tilde{u}_{yy}(\lambda + \delta)(t)|_{L^2(0,1)}^2 + \frac{1}{2\eta} \left(\frac{2kL}{(L - s_*)^4} \right)^2 |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
| \int_0^1 F_{2\delta}(\lambda)(t) \Delta_\delta \bar{u}(\lambda)(t) dy | &\leq \frac{\rho_v}{L - s(\lambda + \delta)(t)} |\Delta_\delta s_t(\lambda)(t)| \int_0^1 |\tilde{u}_y(\lambda + \delta)(t)| |\Delta_\delta \bar{u}(\lambda)(t)| dy \\
&\leq \frac{\rho_v}{L - s_*} |\Delta_\delta s_t(\lambda)(t)| |\tilde{u}_y(\lambda + \delta)(t)|_{L^2(0,1)} |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)} \\
&\leq \frac{\eta}{2} |\Delta_\delta s_t(\lambda)(t)|^2 + \frac{1}{2\eta} \left(\frac{\rho_v K}{L - s_*} \right)^2 |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2,
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
&| \int_0^1 F_{3\delta}(\lambda)(t) \Delta_\delta \bar{u}(\lambda)(t) dy | \\
&\leq \frac{\rho_v}{L - s(\lambda + \delta)(t)} |s_t(\lambda)(t)| \int_0^1 |\Delta_\delta \tilde{u}_y(\lambda)(t)| |\Delta_\delta \bar{u}(\lambda)(t)| dy \\
&\leq \frac{\rho_v a}{L - s_*} |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)} |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)} \\
&\leq \frac{k}{2\eta(L - s(\lambda)(t))^2} |\Delta_\delta \tilde{u}_y(\lambda)(t)|^2 + \frac{\eta L^2}{2k} \left(\frac{\rho_v a}{L - s_*} \right)^2 |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2,
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
&| \int_0^1 F_{4\delta}(\lambda)(t) \Delta_\delta \bar{u}(\lambda)(t) dy | \\
&\leq \frac{\rho_v}{(L - s(\lambda + \delta)(t))(L - s(\lambda)(t))} |s_t(\lambda)(t)| |\Delta_\delta s(\lambda)(t)| \int_0^1 |\tilde{u}_y(\lambda)(t)| |\Delta_\delta \bar{u}(\lambda)(t)| dy \\
&\leq \frac{\rho_v a}{(L - s_*)^2} |\Delta_\delta s(\lambda)(t)| |\tilde{u}_y(\lambda)(t)|_{L^2(0,1)} |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)} \\
&\leq \frac{\eta}{2} |\Delta_\delta s(\lambda)(t)|^2 + \frac{1}{2\eta} \left(\frac{\rho_v a K}{(L - s_*)^2} \right)^2 |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2,
\end{aligned} \tag{3.23}$$

where K is the same constant as in Lemma 1. Also, we estimate the last term of the right hand side of (3.15) as follows:

$$\int_0^1 (-\rho_v \Delta_\delta h_t(t, \lambda)) \Delta_\delta \bar{u}(\lambda)(t) dy \leq \frac{\rho_v}{2} |\Delta_\delta h_t(t, \lambda)|^2 + \frac{\rho_v}{2} |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2. \quad (3.24)$$

Combining all estimates (3.16)-(3.24) with (3.15), we deduce that

$$\begin{aligned} & \frac{\rho_v}{2} \frac{d}{dt} \int_0^1 |\Delta_\delta \bar{u}(\lambda)(t)|^2 dy + \frac{k}{(L - s(\lambda)(t))^2} \left(1 - \frac{3}{\eta}\right) |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 \\ & \leq C_1(\eta) \left(1 + |\tilde{u}_{yy}(\lambda + \delta)(t)|_{L^2(0,1)}^2\right) |\Delta_\delta s(\lambda)(t)|^2 + C_2(\eta) |\Delta_\delta s_t(\lambda)(t)|^2 \\ & \quad + C_3(\eta) |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + \frac{\eta(\rho_v a)^2}{2k} |\Delta_\delta h(t, \lambda)|^2 + \frac{\rho_v}{2} |\Delta_\delta h_t(t, \lambda)|^2 \text{ for a.e. } t \in [0, T], \end{aligned} \quad (3.25)$$

where $C_i(\eta)$ ($1 \leq i \leq 3$) is a positive constant depending on η . Here, by using (3.12) and the derivation of (3.18) we have

$$\begin{aligned} & |\Delta_\delta s_t(\lambda)(t)|^2 \\ & \leq 2a^2 \left(|\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2 + \left| \frac{\varphi(s(\lambda + \delta)(t)) - \varphi(s(\lambda)(t))}{\delta} \right|^2 \right) \\ & \leq 2a^2 \left(4|\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)} |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)} + 2|\Delta_\delta h(t, \lambda)|^2 + C_\varphi^2 |\Delta_\delta s(\lambda)(t)|^2 \right) \\ & \leq \frac{k}{2\varepsilon(L - s(\lambda)(t))^2} |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \frac{\varepsilon(8a^2 L)^2}{2k} |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 \\ & \quad + 4a^2 |\Delta_\delta h(t, \lambda)|^2 + 2a^2 C_\varphi^2 |\Delta_\delta s(\lambda)(t)|^2 \quad \text{for } t \in [0, T], \end{aligned} \quad (3.26)$$

where ε is an arbitrary positive number and C_φ is the same constant as in (A2). Similarly, we can get

$$\begin{aligned} & \frac{d}{dt} |\Delta_\delta s(\lambda)(t)|^2 \\ & \leq a \left(|\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2 + \left| \frac{\varphi(s(\lambda + \delta)(t)) - \varphi(s(\lambda)(t))}{\delta} \right|^2 \right) + 2a |\Delta_\delta s(\lambda)(t)|^2 \\ & \leq \frac{k}{2\eta(L - s(\lambda)(t))^2} |\Delta_\delta \tilde{u}_y(\lambda)|_{L^2(0,1)}^2 + \frac{\eta(4aL)^2}{2k} |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + 2a |\Delta_\delta h(t, \lambda)|^2 \\ & \quad + a(C_\varphi^2 + 2) |\Delta_\delta s(\lambda)(t)|^2 \text{ for } t \in [0, T]. \end{aligned} \quad (3.27)$$

Therefore, by (3.25) $\sim$ (3.27) we obtain that

$$\begin{aligned} & \frac{\rho_v}{2} \frac{d}{dt} \int_0^1 |\Delta_\delta \bar{u}(\lambda)(t)|^2 dy + \frac{d}{dt} |\Delta_\delta s(\lambda)(t)|^2 \\ & \quad + \frac{k}{(L - s(\lambda)(t))^2} \left(1 - \frac{7}{2\eta} - \frac{C_2(\eta)}{2\varepsilon}\right) |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \left[a(C_\varphi^2 + 2) + 2a^2C_2(\eta)C_\varphi^2 + C_1(\eta) \left(1 + |\tilde{u}_{yy}(\lambda + \delta)(t)|_{L^2(0,1)}^2 \right) \right] |\Delta_\delta s(\lambda)(t)|^2 \\
&\quad + \left(C_3(\eta) + \frac{\eta(4aL)^2}{2k} + \frac{\varepsilon(8a^2L)^2}{2k} C_2(\eta) \right) |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 \\
&\quad + \left(\frac{\eta(\rho_v a)^2}{2k} + 4a^2C_2(\eta) + 2a \right) |\Delta_\delta h(t, \lambda)|^2 + \frac{\rho_v}{2} |\Delta_\delta h_t(t, \lambda)|^2 \text{ for a.e. } t \in [0, T].
\end{aligned}$$

By choosing a suitable ε after taking a suitable η , we see that there exist positive constants C_4 , C_5 and C_6 such that

$$\begin{aligned}
&\frac{\rho_v}{2} \frac{d}{dt} \int_0^1 |\Delta_\delta \bar{u}(\lambda)(t)|^2 dy + \frac{d}{dt} |\Delta_\delta s(\lambda)(t)|^2 + \frac{k}{2(L - s(\lambda)(t))^2} |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 \\
&\leq C_4 \left(1 + |\tilde{u}_{yy}(\lambda + \delta)(t)|_{L^2(0,1)}^2 \right) |\Delta_\delta s(\lambda)(t)|^2 + C_5 |\Delta_\delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 \\
&\quad + C_6 (|\Delta_\delta h(\lambda, t)|^2 + |\Delta_\delta h_t(\lambda, t)|^2) \text{ for a.e. } t \in [0, T].
\end{aligned} \tag{3.28}$$

Hence, by applying Gronwall's lemma to (3.28) we see that

$$\begin{aligned}
&\frac{\rho_v}{2} \int_0^1 |\Delta_\delta \bar{u}(\lambda)(t)|^2 dy + |\Delta_\delta s(\lambda)(t)|^2 + \frac{k}{2L^2} \int_0^t |\Delta_\delta \tilde{u}_y(\lambda)(\tau)|_{L^2(0,1)}^2 d\tau \\
&\leq \exp(C_4 \int_0^t (1 + |\tilde{u}_{yy}(\tau)|_{L^2(0,1)}^2) d\tau + C_5 \frac{2}{\rho_v} t) \left(\frac{\rho_v}{2} \int_0^1 |\Delta_\delta \bar{u}(\lambda)(0)|^2 dy + |\Delta_\delta s(\lambda)(0)|^2 \right. \\
&\quad \left. + C_6 \int_0^t (|\Delta_\delta h(\lambda, \tau)|^2 + |\Delta_\delta h_t(\lambda, \tau)|^2) d\tau \right) \text{ for } t \in [0, T].
\end{aligned} \tag{3.29}$$

Here, we note that by from Lemma 1 $\tilde{u}_{yy}(\lambda) \in L^2(0, T; L^2(0, 1))$ for any $\lambda \in (0, 1)$.

Also, by integrating (3.26) over $[0, t]$ for $t \in [0, T]$ and using (3.29), we infer that

$$\begin{aligned}
&\int_0^t |\Delta_\delta s_t(\lambda)(\tau)|^2 d\tau \\
&\leq C_7 (|\Delta_\delta \bar{u}(\lambda)(0)|_{L^2(0,1)}^2 + |\Delta_\delta s(\lambda)(0)|^2 + |\Delta_\delta h(\lambda)|_{W^{1,2}(0,t)}^2) \text{ for } t \in [0, T],
\end{aligned}$$

where C_7 is some positive constant independent of δ . Thus, we get (3.9).

Finally, the boundednesses of $\{\Delta_\delta s(\lambda)\}_{\delta \in I_+(\lambda)}$ in $W^{1,2}(0, T)$ and $\{\Delta_\delta \tilde{u}(\lambda)\}_{\delta \in I_+(\lambda)}$ in $C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1))$ are direct consequences of (3.9) and Lemma 2. $\square$

4 Cauchy property of the differential quotients

In Section 3, we have obtained the uniform estimate for $\Delta_\delta s(\lambda)$ and $\Delta_\delta \tilde{u}(\lambda)$ with respect to δ . In this section, by using these uniform estimates we show that for a.e. $\lambda \in (0, 1)$, $\{\Delta_\delta s(\lambda)\}_{\delta \in I_+(\lambda)}$ and $\{\Delta_\delta \tilde{u}(\lambda)\}_{\delta \in I_+(\lambda)}$ are Cauchy sequences in $W^{1,2}(0, T)$ and in $C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1))$, respectively. Here, we note from Lemma 3 that there exists a positive constant $\tilde{C}$ which is independent of δ such that

$$\begin{aligned}
&|\Delta_\delta s(\lambda)|_{W^{1,2}(0,T)} \leq \tilde{C}, \\
&|\Delta_\delta \tilde{u}(\lambda)|_{C([0,T];L^2(0,1))} + |\Delta_\delta \tilde{u}(\lambda)|_{L^2(0,T;H^1(0,1))} \leq \tilde{C} \text{ for } \delta \in I_+(\lambda).
\end{aligned} \tag{4.1}$$

Lemma 4. *Under the same assumption and notation as in Lemma 3 for $\lambda \in (0, 1) \setminus N$, $\{\Delta_\delta s(\lambda)\}_{\delta \in I_+(\lambda)}$ and $\{\Delta_\delta \tilde{u}(\lambda)\}_{\delta \in I_+(\lambda)}$ are Cauchy sequences in $W^{1,2}(0, T)$ and in $C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1))$, respectively.*

Proof. First, for $\lambda \in (0, 1) \setminus N$ and $\delta, \delta' \in I_+(\lambda)$ and $t \in [0, T]$, we put

$$\begin{aligned}\Delta s(\lambda)(t) &= \Delta_\delta s(\lambda)(t) - \Delta_{\delta'} s(\lambda)(t), \quad \Delta \tilde{u}(\lambda)(t) = \Delta_\delta \tilde{u}(\lambda)(t) - \Delta_{\delta'} \tilde{u}(\lambda)(t), \\ \Delta h(t, \lambda) &= \Delta_\delta h(t, \lambda) - \Delta_{\delta'} h(t, \lambda), \\ \Delta \varphi(s(\lambda)(t)) &= \frac{\varphi(s(\lambda + \delta)(t)) - \varphi(s(\lambda)(t))}{\delta} - \frac{\varphi(s(\lambda + \delta')(t)) - \varphi(s(\lambda)(t))}{\delta'}.\end{aligned}$$

Since $(\Delta_\delta s(\lambda), \Delta_\delta \tilde{u}(\lambda))$ and $(\Delta_{\delta'} s(\lambda), \Delta_{\delta'} \tilde{u}(\lambda))$ satisfy (3.10)-(3.14), by using the notation $F_\delta(\lambda)$ defined in the proof of Lemma 3, again, we see that

$$\rho_v(\Delta \tilde{u}_t(\lambda) - \Delta h_t) - \frac{k}{(L - s(\lambda))^2} \Delta \tilde{u}_{yy}(\lambda) = F_\delta(\lambda) - F_{\delta'}(\lambda) - \rho_v \Delta h_t \text{ in } Q(T), \quad (4.2)$$

$$\Delta \tilde{u}(\lambda)(t, 1) = \Delta h(\lambda, t) \text{ for } t \in [0, T], \quad (4.3)$$

$$\Delta s_t(\lambda)(t) = a(\Delta \tilde{u}(\lambda)(t, 0) - \Delta \varphi(s(\lambda)(t))) \text{ for } t \in [0, T], \quad (4.4)$$

$$\begin{aligned}& \frac{k}{L - s(\lambda)(t)} \Delta \tilde{u}_y(\lambda)(t, 0) \\ &= -\frac{k}{L - s(\lambda)(t)} \left(\frac{1}{L - s(\lambda + \delta)(t)} - \frac{1}{L - s(\lambda + \delta')(t)} \right) \Delta_\delta s(\lambda)(t) \tilde{u}_y(\lambda + \delta)(t, 0) \\ & \quad - \frac{k}{(L - s(\lambda + \delta')(t))(L - s(\lambda)(t))} \Delta s(\lambda)(t) \tilde{u}_y(\lambda + \delta)(t, 0) \\ & \quad - \frac{k}{(L - s(\lambda + \delta')(t))(L - s(\lambda)(t))} \Delta_{\delta'} s(\lambda)(t) \left(\tilde{u}_y(\lambda + \delta)(t, 0) - \tilde{u}_y(\lambda + \delta')(t, 0) \right) \\ & \quad + \rho_w \Delta s_t(\lambda)(t) - \rho_v \left(s_t(\lambda + \delta)(t) - s_t(\lambda + \delta')(t) \right) \Delta_\delta \tilde{u}(\lambda)(t, 0) \\ & \quad - \rho_v s_t(\lambda + \delta')(t) \Delta \tilde{u}(\lambda)(t, 0) - \rho_v \tilde{u}(\lambda)(t, 0) \Delta s_t(\lambda)(t) \\ &=: \sum_{i=1}^7 G_{i\delta}(\lambda)(t) \text{ for } t \in [0, T],\end{aligned} \quad (4.5)$$

$$\Delta s(\lambda)(0) = \Delta s_0(\lambda), \Delta \tilde{u}(\lambda)(0, y) = \Delta \tilde{u}_0(\lambda)(y) \text{ for } y \in (0, 1). \quad (4.6)$$

Put $\Delta \bar{u}(\lambda)(t, y) := \Delta \tilde{u}(\lambda)(t, y) - \Delta h(\lambda, t)$ for $t \in [0, T]$ and $y \in [0, 1]$. By elementary calculation to (4.2) we have

$$\begin{aligned}& \frac{\rho_v}{2} \frac{d}{dt} \int_0^1 |\Delta \bar{u}(\lambda)(t)|^2 dy - \frac{k}{(L - s(\lambda)(t))^2} \int_0^1 \Delta \tilde{u}_{yy}(\lambda)(t) \Delta \bar{u}(\lambda)(t) dy \\ &= \int_0^1 \left(F_\delta(\lambda)(t) - F_{\delta'}(\lambda)(t) - \rho_v \Delta h_t(t, \lambda) \right) \Delta \bar{u}(\lambda)(t) dy \text{ for a.e. } t \in [0, T].\end{aligned} \quad (4.7)$$

For the second term in the left hand side of (4.7), from (4.3) it follows that

$$-\frac{k}{(L - s(\lambda)(t))^2} \int_0^1 \Delta \tilde{u}_{yy}(\lambda)(t) \Delta \bar{u}(\lambda)(t) dy$$

$$= \frac{k}{(L - s(\lambda)(t))^2} \int_0^1 |\Delta \tilde{u}_y(\lambda)(t)|^2 dy + \frac{k}{(L - s(\lambda)(t))^2} \Delta \tilde{u}_y(\lambda)(t, 0) \Delta \bar{u}(\lambda)(t, 0).$$

Then, by (4.5) we can rewrite (4.7) as

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 |\Delta \bar{u}(\lambda)(t)|^2 dy + \frac{k}{(L - s(\lambda)(t))^2} \int_0^1 |\Delta \tilde{u}_y(\lambda)(t)|^2 dy \\ &= \frac{1}{L - s(\lambda)} \sum_{i=1}^7 G_{i\delta}(\lambda)(t) \Delta \bar{u}(\lambda)(t, 0) + \int_0^1 \left(F_\delta(\lambda)(t) - F_{\delta'}(\lambda)(t) \right) \Delta \bar{u}(\lambda)(t) dy \\ & \quad - \int_0^1 \rho_v \Delta h_t(t, \lambda) \Delta \bar{u}(\lambda)(t) dy \quad \text{for a.e. } t \in [0, T]. \end{aligned} \quad (4.8)$$

On the first term of the right hand side in (4.8), by using $\Delta \bar{u}(\lambda)(\cdot, 0) = \int_0^1 \Delta \bar{u}_y(\lambda) dy = \int_0^1 \Delta \tilde{u}_y(\lambda) dy$ on $[0, T]$, we observe that

$$\begin{aligned} & \frac{1}{L - s(\lambda)(t)} \sum_{i=1}^7 G_{i\delta}(\lambda)(t) \Delta \bar{u}(\lambda)(t, 0) \\ & \leq \frac{1}{L - s(\lambda)(t)} \left| \sum_{i=1}^7 G_{i\delta}(\lambda)(t) \right| \|\Delta \tilde{u}_y(\lambda)(t)\|_{L^2(0,1)} \\ & \leq \frac{k}{2(L - s(\lambda)(t))^2} \|\Delta \tilde{u}_y(\lambda)(t)\|_{L^2(0,1)}^2 + \frac{1}{2k} \left(\sum_{i=1}^7 |G_{i\delta}(\lambda)(t)| \right)^2. \end{aligned} \quad (4.9)$$

Hence, by combining (4.8) and (4.9), we have

$$\begin{aligned} & \frac{\rho_v}{2} \frac{d}{dt} \int_0^1 |\Delta \bar{u}(\lambda)(t)|^2 dy + \frac{k}{2(L - s(\lambda)(t))^2} \int_0^1 |\Delta \tilde{u}_y(\lambda)(t)|^2 dy \\ & \leq M_1 \sum_{i=1}^7 |G_{i\delta}(\lambda)(t)|^2 + \int_0^1 \left(F_\delta(\lambda)(t) - F_{\delta'}(\lambda)(t) \right) \Delta \bar{u}(\lambda)(t) dy \\ & \quad - \int_0^1 \rho_v \Delta h_t(t, \lambda) \Delta \bar{u}(\lambda)(t) dy \quad \text{for a.e. } t \in [0, T], \end{aligned} \quad (4.10)$$

where M_1 is a positive constant independent of δ and δ' .

Now, we give the estimate for each $|G_{i\delta}(\lambda)|^2$. On account of (2.1) and (2.5) we observe that

$$\begin{aligned} & |G_{1\delta}(\lambda)(t)|^2 \\ &= \left| \left(\frac{1}{L - s(\lambda + \delta)(t)} - \frac{1}{L - s(\lambda + \delta')(t)} \right) \Delta_\delta s(\lambda)(t) \frac{k}{L - s(\lambda)(t)} \tilde{u}_y(\lambda + \delta)(t, 0) \right|^2 \\ &\leq \frac{4((\rho_w + \rho_v)a)^2 L^2}{(L - s_*)^4} |s(\lambda + \delta)(t) - s(\lambda + \delta')(t)|^2 |\Delta_\delta s(\lambda)(t)|^2 \text{ for a.e. } t \in [0, T]. \end{aligned} \quad (4.11)$$

Here, we note that

$$|s(\lambda + \delta) - s(\lambda + \delta')|^2$$

$$=|\Delta_\delta s(\lambda)\delta - \Delta_{\delta'} s(\lambda)\delta'|^2 \leq 2(|\Delta_\delta s(\lambda)|^2|\delta|^2 + |\Delta_{\delta'} s(\lambda)|^2|\delta'|^2) \text{ on } [0, T]. \quad (4.12)$$

Accordingly, by (4.11) with (4.1) and (4.12) it follows that

$$\begin{aligned} |G_{1\delta}(\lambda)(t)|^2 &\leq \frac{8((\rho_w + \rho_v)a)^2 L^2}{(L - s_*)^4} \left(|\Delta_\delta s(\lambda)(t)|^2 |\delta|^2 + |\Delta_{\delta'} s(\lambda)(t)|^2 |\delta'|^2 \right) |\Delta_\delta s(\lambda)(t)|^2 \\ &\leq \frac{8((\rho_w + \rho_v)a)^2 L^2}{(L - s_*)^4} \tilde{C}^4 (|\delta|^2 + |\delta'|^2) \quad \text{for a.e. } t \in [0, T]. \end{aligned} \quad (4.13)$$

Similarly to (4.11), the term $G_{2\delta}(\lambda)$ can be estimated as follows:

$$\begin{aligned} |G_{2\delta}(\lambda)(t)|^2 &= \frac{1}{(L - s(\lambda + \delta')(t))^2} |\Delta s(\lambda)(t)|^2 \left| \frac{k}{L - s(\lambda)(t)} \tilde{u}_y(\lambda + \delta)(t, 0) \right|^2 \\ &\leq \frac{((\rho_w + \rho_v)a)^2 L^2}{(L - s_*)^4} |\Delta s(\lambda)(t)|^2 \quad \text{for a.e. } t \in [0, T]. \end{aligned} \quad (4.14)$$

For $G_{3\delta}(\lambda)$, by using (4.1), we have

$$|G_3(\lambda)(t)|^2 \leq \frac{k\tilde{C}^2}{(L - s_*)^4} |\tilde{u}_y(\lambda + \delta)(t, 0) - \tilde{u}_y(\lambda + \delta')(t, 0)|^2 \text{ for a.e. } t \in [0, T]. \quad (4.15)$$

Here, by (2.5) we observe that

$$\begin{aligned} &|\tilde{u}_y(\lambda + \delta)(t, 0) - \tilde{u}_y(\lambda + \delta')(t, 0)| \\ &= \left| \frac{L - s(\lambda + \delta)(t)}{k} \left((\rho_w - \rho_v \tilde{u}(\lambda + \delta)(t, 0)) s_t(\lambda + \delta)(t) \right) \right. \\ &\quad \left. - \frac{L - s(\lambda + \delta')(t)}{k} \left((\rho_w - \rho_v \tilde{u}(\lambda + \delta')(t, 0)) s_t(\lambda + \delta')(t) \right) \right| \\ &\leq \frac{1}{k} |s(\lambda + \delta)(t) - s(\lambda + \delta')(t)| |(\rho_w - \rho_v \tilde{u}(\lambda + \delta)(t, 0)) s_t(\lambda + \delta)(t)| \\ &\quad + \frac{\rho_w}{k} |L - s(\lambda + \delta')(t)| |s_t(\lambda + \delta)(t) - s_t(\lambda + \delta')(t)| \\ &\quad + \frac{1}{k} |L - s(\lambda + \delta')(t)| \left| \rho_v \left(\tilde{u}(\lambda + \delta)(t, 0) - \tilde{u}(\lambda + \delta')(t, 0) \right) \right| |s_t(\lambda + \delta)(t)| \\ &\quad + \frac{1}{k} |L - s(\lambda + \delta')(t)| |\rho_v \tilde{u}(\lambda + \delta')(t, 0)| |s_t(\lambda + \delta)(t) - s_t(\lambda + \delta')(t)|. \end{aligned} \quad (4.16)$$

Similarly to the derivation of (4.12), it holds that

$$|\tilde{u}(\lambda + \delta)(t, 0) - \tilde{u}(\lambda + \delta')(t, 0)| \leq |\Delta_\delta \tilde{u}(\lambda)(t, 0)| |\delta| + |\Delta_{\delta'} \tilde{u}(\lambda)(t, 0)| |\delta'|, \quad (4.17)$$

$$|s_t(\lambda + \delta) - s_t(\lambda + \delta')| \leq |\Delta_\delta s_t(\lambda)(t)| |\delta| + |\Delta_{\delta'} s_t(\lambda)(t)| |\delta'|. \quad (4.18)$$

Hence, by substituting (4.17) and (4.18) into (4.16) and using (4.1), (4.12) and (2.1) we deduce

$$|\tilde{u}_y(\lambda + \delta)(t, 0) - \tilde{u}_y(\lambda + \delta')(t, 0)|$$

$$\begin{aligned}
&\leq \tilde{C} \frac{(\rho_w + \rho_v)a}{k} (|\delta| + |\delta'|) + \frac{\rho_v a L}{k} \left(|\Delta_\delta \tilde{u}(\lambda)(t, 0)| |\delta| + |\Delta_{\delta'} \tilde{u}(\lambda)(t, 0)| |\delta'| \right) \\
&\quad + \frac{(\rho_w + \rho_v)L}{k} \left(|\Delta_\delta s_t(\lambda)(t)| |\delta| + |\Delta_{\delta'} s_t(\lambda)(t)| |\delta'| \right) \text{ for a.e. } t \in [0, T]. \quad (4.19)
\end{aligned}$$

From (4.15) and (4.19), we can deal with $G_{3\delta}(\lambda)$ as follows:

$$\begin{aligned}
|G_{3\delta}(\lambda)(t)|^2 &\leq M_2 \left(|\delta|^2 + |\delta'|^2 + |\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2 |\delta|^2 + |\Delta_{\delta'} \tilde{u}(\lambda)(t, 0)|^2 |\delta'|^2 \right. \\
&\quad \left. + |\Delta_\delta s_t(\lambda)(t)|^2 |\delta|^2 + |\Delta_{\delta'} s_t(\lambda)(t)|^2 |\delta'|^2 \right) \text{ for a.e. } t \in [0, T], \quad (4.20)
\end{aligned}$$

where M_2 is a positive constant independent of δ and δ' . For $G_{4\delta}(\lambda) \sim G_{7\delta}(\lambda)$, thanks to (4.18) and (2.1) we proceed as

$$|G_{4\delta}(\lambda)(t)|^2 \leq \rho_w^2 |\Delta s_t(\lambda)(t)|^2, \quad (4.21)$$

$$\begin{aligned}
|G_{5\delta}(\lambda)(t)|^2 &\leq \rho_v^2 |s_t(\lambda + \delta)(t) - s_t(\lambda + \delta')(t)|^2 |\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2 \\
&\leq 2\rho_v^2 (|\Delta_\delta s_t(\lambda)|^2 |\delta|^2 + |\Delta_{\delta'} s_t(\lambda)|^2 |\delta'|^2) |\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2, \quad (4.22)
\end{aligned}$$

$$|G_{6\delta}(\lambda)(t)|^2 \leq (\rho_v a)^2 |\Delta \tilde{u}(\lambda)(t, 0)|^2, \quad (4.23)$$

and

$$|G_{7\delta}(\lambda)(t)|^2 \leq \rho_v^2 |\Delta s_t(\lambda)(t)|^2. \quad (4.24)$$

By all estimates (4.13), (4.14), (4.20) $\sim$ (4.24), we see that there exists a positive constant M_3 independent of δ and δ' such that

$$\begin{aligned}
&\sum_{i=1}^7 |G_{i\delta}(\lambda)(t)|^2 \\
&\leq M_3 \left(|\delta|^2 + |\delta'|^2 + |\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2 |\delta|^2 + |\Delta_{\delta'} \tilde{u}(\lambda)(t, 0)|^2 |\delta'|^2 \right. \\
&\quad + (|\Delta_\delta s_t(\lambda)(t)|^2 |\delta|^2 + |\Delta_{\delta'} s_t(\lambda)(t)|^2 |\delta'|^2) (1 + |\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2) \\
&\quad \left. + |\Delta s(\lambda)(t)|^2 + |\Delta s_t(\lambda)(t)|^2 + |\Delta \tilde{u}(\lambda)(t, 0)|^2 \right) \text{ for a.e. } t \in [0, T]. \quad (4.25)
\end{aligned}$$

Next, we consider to estimate the second term in the right hand side of (4.8). First, it holds that

$$\begin{aligned}
&\int_0^1 (F_{1\delta}(\lambda)(t) - F_{1\delta'}(\lambda)(t)) \Delta \tilde{u}(\lambda)(t) dy \\
&= - \frac{k(2L - s(\lambda + \delta)(t) - s(\lambda)(t))}{(L - s(\lambda + \delta)(t))^2 (L - s(\lambda)(t))^2} \Delta_\delta s(\lambda)(t) \int_0^1 \tilde{u}_y(\lambda + \delta)(t) \Delta \tilde{u}_y(\lambda)(t) dy \\
&\quad - \frac{k(2L - s(\lambda + \delta)(t) - s(\lambda)(t))}{(L - s(\lambda + \delta)(t))^2 (L - s(\lambda)(t))^2} \Delta_\delta s(\lambda)(t) \tilde{u}_y(\lambda + \delta)(t, 0) \Delta \tilde{u}_y(\lambda)(t, 0)
\end{aligned}$$

$$\begin{aligned}
& + \frac{k(2L - s(\lambda + \delta')(t) - s(\lambda)(t))}{(L - s(\lambda + \delta')(t))^2(L - s(\lambda)(t))^2} \Delta_{\delta'} s(\lambda)(t) \int_0^1 \tilde{u}_y(\lambda + \delta')(t) \Delta \tilde{u}_y(\lambda)(t) dy \\
& + \frac{k(2L - s(\lambda + \delta')(t) - s(\lambda)(t))}{(L - s(\lambda + \delta')(t))^2(L - s(\lambda)(t))^2} \Delta_{\delta'} s(\lambda)(t) \tilde{u}_y(\lambda + \delta')(t, 0) \Delta \tilde{u}_y(\lambda)(t, 0). \\
& =: \sum_{i=1}^4 J_{i\delta}(t).
\end{aligned} \tag{4.26}$$

It is easy to see that

$$\begin{aligned}
& J_{1\delta} + J_{3\delta} \\
& = - \frac{k(2L - s(\lambda + \delta) - s(\lambda))}{(L - s(\lambda + \delta))^2(L - s(\lambda))^2} \Delta_{\delta} s(\lambda) \int_0^1 \left(\tilde{u}_y(\lambda + \delta) - \tilde{u}_y(\lambda + \delta') \right) \Delta \tilde{u}_y(\lambda) dy \\
& \quad - \left(\frac{k(2L - s(\lambda + \delta) - s(\lambda))}{(L - s(\lambda + \delta))^2(L - s(\lambda))^2} \right) (\Delta_{\delta} s(\lambda) - \Delta_{\delta'} s(\lambda)) \int_0^1 \tilde{u}_y(\lambda + \delta') \Delta \tilde{u}_y(\lambda) dy \\
& \quad - \frac{k}{(L - s(\lambda))^2} \left(\frac{-s(\lambda + \delta) + s(\lambda + \delta')}{(L - s(\lambda + \delta))^2} \right) \Delta_{\delta'} s(\lambda) \int_0^1 \tilde{u}_y(\lambda + \delta') \Delta \tilde{u}_y(\lambda) dy \\
& \quad - \frac{k}{(L - s(\lambda))^2} \left(\frac{(2L - s(\lambda + \delta) - s(\lambda)) \left((L - s(\lambda + \delta'))^2 - (L - s(\lambda + \delta))^2 \right)}{(L - s(\lambda + \delta))^2(L - s(\lambda + \delta'))^2} \right) \Delta_{\delta'} s(\lambda) \\
& \quad \times \int_0^1 \tilde{u}_y(\lambda + \delta') \Delta \tilde{u}_y(\lambda) dy.
\end{aligned}$$

Following (4.12) and $|\tilde{u}_y(\lambda + \delta) - \tilde{u}_y(\lambda + \delta')| \leq |\Delta_{\delta} \tilde{u}_y(\lambda)| |\delta| + |\Delta_{\delta'} \tilde{u}_y(\lambda)| |\delta'|$, we infer that

$$\begin{aligned}
& |J_{1\delta} + J_{3\delta}| \\
& \leq \frac{2kL}{(L - s_*)^4} |\Delta_{\delta} s(\lambda)| \left(|\Delta_{\delta} \tilde{u}_y(\lambda)|_{L^2(0,1)} |\delta| + |\Delta_{\delta'} \tilde{u}_y(\lambda)|_{L^2(0,1)} |\delta'| \right) |\Delta \tilde{u}_y(\lambda)|_{L^2(0,1)} \\
& \quad + \frac{2kL}{(L - s_*)^4} |\Delta s(\lambda)| |\tilde{u}_y(\lambda + \delta')|_{L^2(0,1)} |\Delta \tilde{u}_y(\lambda)|_{L^2(0,1)} \\
& \quad + \frac{k}{(L - s_*)^4} \left(|\Delta_{\delta} s(\lambda)| |\delta| + |\Delta_{\delta'} s(\lambda)| |\delta'| \right) |\Delta_{\delta'} s(\lambda)| |\Delta \tilde{u}_y(\lambda)|_{L^2(0,1)} |\tilde{u}_y(\lambda + \delta')|_{L^2(0,1)} \\
& \quad + \frac{4kL^2}{(L - s_*)^6} \left(|\Delta_{\delta} s(\lambda)| |\delta| + |\Delta_{\delta'} s(\lambda)| |\delta'| \right) |\Delta_{\delta'} s(\lambda)| |\Delta \tilde{u}_y(\lambda)|_{L^2(0,1)} |\tilde{u}_y(\lambda + \delta')|_{L^2(0,1)}.
\end{aligned}$$

Using the fact that $|\tilde{u}_y(\lambda)|_{L^2(0,1)} \leq K$ on $[0, T]$ in Lemma 1 and $|\Delta_{\delta} s(\lambda)|_{C([0, T])} \leq \tilde{C}$ for any $\lambda \in (0, 1)$ in (4.1), we deduce that there exists a positive constant M_4 independent of δ and δ' such that

$$\begin{aligned}
|J_{1\delta} + J_{3\delta}| & \leq \frac{k}{4(L - s(\lambda))^2} |\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 \\
& \quad + M_4 \left(|\Delta_{\delta} \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 |\delta|^2 + |\Delta_{\delta'} \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 |\delta'|^2 + |\Delta s(\lambda)(t)|^2 + |\delta|^2 + |\delta'|^2 \right)
\end{aligned}$$

$$\text{for a.e. } t \in [0, T]. \quad (4.27)$$

On $J_{2\delta} + J_{4\delta}$, it holds that

$$\begin{aligned} & J_{2\delta} + J_{4\delta} \\ &= -\frac{k(2L - s(\lambda + \delta) - s(\lambda))}{(L - s(\lambda + \delta))^2(L - s(\lambda))^2}(\Delta_\delta s(\lambda)\tilde{u}_y(\lambda + \delta)(t, 0) - \Delta_{\delta'} s(\lambda)\tilde{u}_y(\lambda + \delta')(t, 0))\Delta\tilde{u}_y(\lambda)(t, 0) \\ & \quad + \left(\frac{k(2L - s(\lambda + \delta') - s(\lambda))}{(L - s(\lambda + \delta'))^2(L - s(\lambda))^2} - \frac{k(2L - s(\lambda + \delta) - s(\lambda))}{(L - s(\lambda + \delta))^2(L - s(\lambda))^2} \right) \\ & \quad \times \Delta_{\delta'} s(\lambda)\tilde{u}_y(\lambda + \delta')(t, 0)\Delta\tilde{u}_y(\lambda)(t, 0) \\ &= -\frac{k(2L - s(\lambda + \delta) - s(\lambda))}{(L - s(\lambda + \delta))^2(L - s(\lambda))^2} \left(\Delta_\delta s(\lambda) - \Delta_{\delta'} s(\lambda) \right) \tilde{u}_y(\lambda + \delta)(t, 0)\Delta\tilde{u}_y(\lambda)(t, 0) \\ & \quad - \frac{k(2L - s(\lambda + \delta) - s(\lambda))}{(L - s(\lambda + \delta))^2(L - s(\lambda))^2} \Delta_{\delta'} s(\lambda) \left(\tilde{u}_y(\lambda + \delta)(t, 0) - \tilde{u}_y(\lambda + \delta')(t, 0) \right) \Delta\tilde{u}_y(\lambda)(t, 0) \\ & \quad + \left(\frac{k(2L - s(\lambda + \delta') - s(\lambda))}{(L - s(\lambda + \delta'))^2(L - s(\lambda))^2} - \frac{k(2L - s(\lambda + \delta) - s(\lambda))}{(L - s(\lambda + \delta))^2(L - s(\lambda))^2} \right) \\ & \quad \times \Delta_{\delta'} s(\lambda)\tilde{u}_y(\lambda + \delta')(t, 0)\Delta\tilde{u}_y(\lambda)(t, 0). \end{aligned}$$

By (4.12) and (4.17) we get

$$\begin{aligned} & |J_{2\delta} + J_{4\delta}| \\ &\leq \frac{2kL}{(L - s_*)^4} |\Delta s(\lambda)| |\tilde{u}_y(\lambda + \delta)(t, 0)| |\Delta\tilde{u}_y(\lambda)(t, 0)| \\ & \quad + \frac{2kL}{(L - s_*)^4} |\Delta_{\delta'} s(\lambda)| \left(|\Delta_\delta \tilde{u}_y(\lambda)(t, 0)| |\delta| + |\Delta_{\delta'} \tilde{u}_y(\lambda)(t, 0)| |\delta'| \right) |\Delta\tilde{u}_y(\lambda)(t, 0)| \\ & \quad + \frac{k}{(L - s_*)^4} \left(|\Delta_\delta s(\lambda)| |\delta| + |\Delta_{\delta'} s(\lambda)| |\delta'| \right) |\Delta_{\delta'} s(\lambda)| |\tilde{u}_y(\lambda + \delta')(t, 0)| |\Delta\tilde{u}_y(\lambda)(t, 0)| \\ & \quad + \frac{4kL^2}{(L - s_*)^6} \left(|\Delta_\delta s(\lambda)| |\delta| + |\Delta_{\delta'} s(\lambda)| |\delta'| \right) |\Delta_{\delta'} s(\lambda)| |\tilde{u}_y(\lambda + \delta')(t, 0)| |\Delta\tilde{u}_y(\lambda)(t, 0)|. \end{aligned} \quad (4.28)$$

Here, by (2.1) and (2.5) we see that

$$\begin{aligned} |\tilde{u}_y(\lambda + \delta)(t, 0)| &= \frac{L - s(\lambda + \delta)(t)}{k} |(\rho_w - \rho_v \tilde{u}(\lambda + \delta)(t, 0)) s_t(\lambda + \delta)(t)| \\ &\leq \frac{L}{k} (\rho_w + \rho_v) a \quad \text{for a.e. } t \in [0, T]. \end{aligned} \quad (4.29)$$

Due to (4.1) and (4.29) we obtain from (4.28) that

$$\begin{aligned} & |J_{2\delta} + J_{4\delta}| \\ &\leq \frac{2kL}{(L - s_*)^4} \frac{L}{k} (\rho_w + \rho_v) a |\Delta s(\lambda)(t)| |\Delta\tilde{u}_y(\lambda)(t, 0)| \end{aligned}$$

$$\begin{aligned}
& + \frac{2kL\tilde{C}}{(L-s_*)^4} \left(|\Delta_\delta \tilde{u}_y(\lambda)(t,0)| |\delta| + |\Delta_{\delta'} \tilde{u}_y(\lambda)(t,0)| |\delta'| \right) |\Delta \tilde{u}_y(\lambda)(t,0)| \\
& + \frac{k\tilde{C}^2}{(L-s_*)^4} \frac{L}{k} (\rho_w + \rho_v) a \left(|\delta| + |\delta'| \right) |\Delta \tilde{u}_y(\lambda)(t,0)| \\
& + \frac{4kL^2\tilde{C}^2}{(L-s_*)^6} \frac{L}{k} (\rho_w + \rho_v) a \left(|\delta| + |\delta'| \right) |\Delta \tilde{u}_y(\lambda)(t,0)| \text{ for a.e. } t \in [0, T]. \tag{4.30}
\end{aligned}$$

By (4.5), it holds that for $t \in [0, T]$,

$$|\Delta \tilde{u}_y(\lambda)(t,0)| = \frac{L-s(\lambda)(t)}{k} \frac{k}{L-s(\lambda)(t)} |\Delta \tilde{u}_y(\lambda)(t,0)| \leq \frac{L}{k} \sum_{i=1}^7 |G_i(\lambda)(t)|. \tag{4.31}$$

From (4.30), (4.31) and (4.25), we can find a positive constant M_5 which does not depend on δ and δ' such that

$$\begin{aligned}
& |J_{2\delta} + J_{4\delta}| \\
& \leq M_5 \left(|\delta|^2 + |\delta'|^2 + |\Delta_\delta \tilde{u}(\lambda)(t,0)|^2 |\delta|^2 + |\Delta_{\delta'} \tilde{u}(\lambda)(t,0)|^2 |\delta'|^2 \right. \\
& \quad + |\Delta_\delta \tilde{u}_y(\lambda)(t,0)|^2 |\delta|^2 + |\Delta_{\delta'} \tilde{u}_y(\lambda)(t,0)|^2 |\delta'|^2 \\
& \quad + (|\Delta_\delta s_t(\lambda)(t)|^2 |\delta|^2 + |\Delta_{\delta'} s_t(\lambda)(t)|^2 |\delta'|^2) (1 + |\Delta_\delta \tilde{u}(\lambda)(t,0)|^2) \\
& \quad \left. + |\Delta s(\lambda)(t)|^2 + |\Delta s_t(\lambda)(t)|^2 + |\Delta \tilde{u}(\lambda)(t,0)|^2 \right) \text{ for a.e. } t \in [0, T]. \tag{4.32}
\end{aligned}$$

Hence, by substituting (4.27) and (4.32) into (4.26), we infer that

$$\begin{aligned}
& \int_0^1 (F_{1\delta}(\lambda)(t) - F_{1\delta'}(\lambda)(t)) \Delta \bar{u}(\lambda)(t) dy \\
& \leq \frac{k}{4(L-s(\lambda)(t))^2} |\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + M_6 \left(|\delta|^2 + |\delta'|^2 + |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 |\delta|^2 \right. \\
& \quad + |\Delta_{\delta'} \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 |\delta'|^2 + |\Delta_\delta \tilde{u}(\lambda)(t,0)|^2 |\delta|^2 + |\Delta_{\delta'} \tilde{u}(\lambda)(t,0)|^2 |\delta'|^2 \\
& \quad + (|\Delta_\delta s_t(\lambda)(t)|^2 |\delta|^2 + |\Delta_{\delta'} s_t(\lambda)(t)|^2 |\delta'|^2) (1 + |\Delta_\delta \tilde{u}(\lambda)(t,0)|^2) \\
& \quad \left. + |\Delta s(\lambda)(t)|^2 + |\Delta s_t(\lambda)(t)|^2 + |\Delta \tilde{u}(\lambda)(t,0)|^2 \right), \tag{4.33}
\end{aligned}$$

where M_6 is a positive constant depending on M_4 and M_5 .

Next, for the difference $F_{2\delta} - F_{2\delta'}$ it follows that

$$\begin{aligned}
& \int_0^1 (F_{2\delta}(\lambda)(t) - F_{2\delta'}(\lambda)(t)) \Delta \bar{u}(\lambda)(t) dy \\
& = \frac{\rho_v \Delta s_t(\lambda)(t)}{L-s(\lambda+\delta)(t)} \int_0^1 (1-y) \tilde{u}_y(\lambda+\delta)(t) \Delta \bar{u}(\lambda)(t) dy \\
& \quad + \frac{\rho_v (-s(\lambda+\delta')(t) + s(\lambda+\delta)(t)) \Delta_{\delta'} s_t(\lambda)(t)}{(L-s(\lambda+\delta)(t))(L-s(\lambda+\delta')(t))} \int_0^1 (1-y) \tilde{u}_y(\lambda+\delta)(t) \Delta \bar{u}(\lambda)(t) dy
\end{aligned}$$

$$+ \frac{\rho_v \Delta_{\delta'} s_t(\lambda)(t)}{L - s(\lambda + \delta')(t)} \int_0^1 (1 - y)(\tilde{u}_y(\lambda + \delta)(t) - \tilde{u}_y(\lambda + \delta')(t)) \Delta \bar{u}(\lambda)(t) dy. \quad (4.34)$$

On account of the result that $|\tilde{u}_y(\lambda)(t)|_{L^2(0,1)} \leq K$ for $t \in [0, T]$ in Lemma 1, we deduce from (4.34) that

$$\begin{aligned} & \int_0^1 (F_{2\delta}(\lambda)(t) - F_{2\delta'}(\lambda)(t)) \Delta \bar{u}(\lambda)(t) dy \\ & \leq \frac{\rho_v K}{L - s_*} |\Delta s_t(\lambda)(t)| |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)} \\ & \quad + \frac{\rho_v K}{(L - s_*)^2} |s(\lambda + \delta)(t) - s(\lambda + \delta')(t)| |\Delta_{\delta'} s_t(\lambda)(t)| |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)} \\ & \quad + \frac{\rho_v}{L - s_*} |\Delta_{\delta'} s_t(\lambda)(t)| |\tilde{u}_y(\lambda + \delta)(t) - \tilde{u}_y(\lambda + \delta')(t)|_{L^2(0,1)} |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}. \end{aligned} \quad (4.35)$$

In the similar way to the derivation of (4.27), using the result that $|\Delta_{\delta} s(\lambda)|_{C([0,T])} \leq \tilde{C}$ in (4.1) and Young's inequality, we derive from (4.35) that

$$\begin{aligned} & \int_0^1 (F_{2\delta}(\lambda)(t) - F_{2\delta'}(\lambda)(t)) \Delta \bar{u}(\lambda)(t) dy \\ & \leq M_8 \left(|\Delta s_t(\lambda)(t)|^2 + |\delta|^2 + |\delta'|^2 + |\Delta_{\delta} \tilde{u}_y(\lambda)(t)|^2 |\delta|^2 + |\Delta_{\delta'} \tilde{u}_y(\lambda)(t)|^2 |\delta'|^2 \right) \\ & \quad + (1 + |\Delta_{\delta'} s_t(\lambda)(t)|^2) |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 \text{ for a.e. } t \in [0, T], \end{aligned} \quad (4.36)$$

where M_8 is a positive constant independent of δ and δ' .

Also, the difference on $F_{3\delta} - F_{3\delta'}$ can be written as

$$\begin{aligned} & \int_0^1 (F_{3\delta}(\lambda)(t) - F_{3\delta'}(\lambda)(t)) \Delta \bar{u}(\lambda)(t) dy \\ & \leq \frac{\rho_v s_t(\lambda)(t)}{L - s(\lambda + \delta)(t)} \int_0^1 (1 - y) \Delta \tilde{u}_y(\lambda)(t) \Delta \bar{u}(\lambda)(t) dy \\ & \quad + \frac{\rho_v s_t(\lambda)(t)(-s(\lambda + \delta')(t) + s(\lambda + \delta)(t))}{(L - s(\lambda + \delta)(t))(L - s(\lambda + \delta')(t))} \int_0^1 (1 - y) \Delta_{\delta'} \tilde{u}_y(\lambda)(t) \Delta \bar{u}(\lambda)(t) dy. \end{aligned} \quad (4.37)$$

By (2.1), (4.1) and (4.37) we see that

$$\begin{aligned} & \int_0^1 (F_{3\delta}(\lambda)(t) - F_{3\delta'}(\lambda)(t)) \Delta \bar{u}(\lambda)(t) dy \\ & \leq \frac{k}{8(L - s(\lambda)(t))^2} |\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \frac{2L^2}{k} \left(\frac{\rho_v a}{L - s_*} \right)^2 |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 \\ & \quad + \left(\frac{\rho_v a \tilde{C}}{(L - s_*)^2} \right)^2 (|\delta|^2 + |\delta'|^2) |\Delta_{\delta'} \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \frac{1}{2} |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2. \end{aligned} \quad (4.38)$$

Moreover, we give the estimate on the following difference $F_{4\delta} - F_{4\delta'}$:

$$\int_0^1 (F_{4\delta}(\lambda)(t) - F_{4\delta'}(\lambda)(t)) \Delta \bar{u}(\lambda)(t) dy$$

$$\begin{aligned}
&= \frac{\rho_v s_t(\lambda)(t) \Delta s(\lambda)(t)}{(L - s(\lambda + \delta)(t))(L - s(\lambda)(t))} \int_0^1 (1 - y) \tilde{u}_y(\lambda)(t) \Delta \bar{u}(\lambda)(t) dy \\
&\quad + \frac{\rho_v s_t(\lambda)(t)}{L - s(\lambda)(t)} \frac{\Delta_{\delta'} s(\lambda)(t) (-s(\lambda + \delta')(t) + s(\lambda + \delta)(t))}{(L - s(\lambda + \delta)(t))(L - s(\lambda + \delta')(t))} \int_0^1 (1 - y) \tilde{u}_y(\lambda)(t) \Delta \bar{u}(\lambda)(t) dy.
\end{aligned} \tag{4.39}$$

Referring to the derivation of (4.38), we obtain from (4.39) that

$$\begin{aligned}
&\int_0^1 (F_{4\delta}(\lambda)(t) - F_{4\delta'}(\lambda)(t)) \Delta \bar{u}(\lambda)(t) dy \\
&\leq \frac{1}{2} \left(\frac{\rho_v a K}{(L - s_*)^2} \right)^2 |\Delta s(\lambda)(t)|^2 + \left(\frac{\rho_v a K \tilde{C}^2}{(L - s_*)^3} \right)^2 (|\delta|^2 + |\delta'|^2) + |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2.
\end{aligned} \tag{4.40}$$

Combining (4.10), (4.25), (4.33), (4.36), (4.38) and (4.40), we have

$$\begin{aligned}
&\frac{\rho_v}{2} \frac{d}{dt} |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + \frac{k}{8(L - s_*)^2} |\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 \\
&\leq M_9 \left((|\delta|^2 + |\delta'|^2) (1 + |\Delta_{\delta} \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + |\Delta_{\delta'} \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2) + |\Delta_{\delta} \tilde{u}(\lambda)(t, 0)|^2 |\delta|^2 \right. \\
&\quad + |\Delta_{\delta'} \tilde{u}(\lambda)(t, 0)|^2 |\delta'|^2 + (|\Delta_{\delta} s_t(\lambda)(t)|^2 |\delta|^2 + |\Delta_{\delta'} s_t(\lambda)(t)|^2 |\delta'|^2) (1 + |\Delta_{\delta} \tilde{u}(\lambda)(t, 0)|^2) \\
&\quad \left. + |\Delta s(\lambda)(t)|^2 + |\Delta s_t(\lambda)(t)|^2 + |\Delta \tilde{u}(\lambda)(t, 0)|^2 \right) \\
&\quad + M_{10} \left((1 + |\Delta_{\delta'} s_t(\lambda)(t)|^2) |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + |\Delta h_t(\lambda, t)|^2 \right) \text{ for a.e. } t \in [0, T],
\end{aligned} \tag{4.41}$$

where M_9 and M_{10} are positive constants independent of δ and δ' . The estimate for $\Delta \tilde{u}(\lambda)(t, 0)$ can be given as follows:

$$\begin{aligned}
&|\Delta \tilde{u}(\lambda)(t, 0)|^2 \\
&\leq 2 |\Delta \tilde{u}(\lambda)(t, 0) - \Delta h(\lambda, t)|^2 + 2 |\Delta h(\lambda, t)|^2 \\
&\leq 4 |\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)} |\Delta \tilde{u}(\lambda)(t) - \Delta h(\lambda, t)|_{L^2(0,1)} + 2 |\Delta h(\lambda, t)|^2 \\
&\leq \frac{\eta k}{2(L - s_*)^2} |\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \frac{8L^2}{\eta k} |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + 2 |\Delta h(\lambda, t)|^2 \text{ for } t \in [0, T],
\end{aligned} \tag{4.42}$$

where η is an arbitrary positive number. Also, $\Delta s_t(\lambda)$ can be rewritten by

$$\begin{aligned}
&\Delta s_t(\lambda)(t) \\
&= a \Delta \tilde{u}(\lambda)(t, 0) - a \left(\frac{\varphi(s(\lambda + \delta)(t) - \varphi(s(\lambda)(t))}{\delta} - \frac{\varphi(s(\lambda + \delta')(t) - \varphi(s(\lambda)(t))}{\delta'} \right) \\
&= a \Delta \tilde{u}(\lambda)(t, 0) - a \varphi'(s(\lambda)(t) + \xi_{\delta} \delta \Delta_{\delta} s(\lambda)(t)) \Delta_{\delta} s(\lambda)(t) \\
&\quad + a \varphi'(s(\lambda)(t) + \xi_{\delta'} \delta' \Delta_{\delta'} s(\lambda)(t)) \Delta_{\delta'} s(\lambda)(t) \\
&= a \Delta \tilde{u}(\lambda)(t, 0) - a \varphi'(s(\lambda)(t) + \xi_{\delta} \delta \Delta_{\delta} s(\lambda)(t)) \Delta s(\lambda)(t) \\
&\quad - a \left(\varphi'(s(\lambda)(t) + \xi_{\delta} \delta \Delta_{\delta} s(\lambda)(t)) - \varphi'(s(\lambda)(t) + \xi_{\delta'} \delta' \Delta_{\delta'} s(\lambda)(t)) \right) \Delta_{\delta'} s(\lambda)(t),
\end{aligned} \tag{4.43}$$

where $0 < \xi_\delta < 1$ and $0 < \xi_{\delta'} < 1$. With the help of (A2), (4.1) and (4.12), we get

$$\begin{aligned}
& |\Delta s_t(\lambda)(t)| \\
& \leq a|\Delta \tilde{u}(\lambda)(t, 0)| + aC_\varphi|\Delta s(\lambda)(t)| + aC_\varphi \left(|\Delta_\delta s(\lambda)(t)||\delta| + |\Delta_{\delta'} s(\lambda)(t)||\delta'| \right) |\Delta_{\delta'} s(\lambda)(t)| \\
& \leq a|\Delta \tilde{u}(\lambda)(t, 0)| + aC_\varphi|\Delta s(\lambda)(t)| + aC_\varphi \tilde{C}^2(|\delta| + |\delta'|) \quad \text{for a.e. } t \in [0, T].
\end{aligned} \tag{4.44}$$

By (4.42) and (4.44), we have

$$\begin{aligned}
& |\Delta s_t(\lambda)(t)|^2 \\
& \leq 4a^2|\Delta \tilde{u}(\lambda)(t, 0)|^2 + 4(aC_\varphi)^2|\Delta s(\lambda)(t)|^2 + 4(aC_\varphi \tilde{C}^2)^2(|\delta|^2 + |\delta'|^2) \\
& \leq \frac{4a^2\eta k}{2(L - s_*)^2}|\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \frac{32(aL)^2}{\eta k}|\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + 8a^2|\Delta h(t, \lambda)|^2 \\
& \quad + 4(aC_\varphi)^2|\Delta s(\lambda)(t)|^2 + 4(aC_\varphi \tilde{C}^2)^2(|\delta|^2 + |\delta'|^2) \quad \text{for a.e. } t \in [0, T].
\end{aligned} \tag{4.45}$$

Using (4.42), (4.45) and (4.41), we infer that

$$\begin{aligned}
& \frac{\rho_v}{2} \frac{d}{dt} |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + \frac{k}{8(L - s_*)^2} (1 - M_9(4\eta + 16a^2\eta)) |\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 \\
& \leq M_9(1 + 4(aC_\varphi)^2) |\Delta s(\lambda)(t)|^2 \\
& \quad + M_9 \left(1 + 4(aC_\varphi \tilde{C}^2)^2 + |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + |\Delta_{\delta'} \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + |\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2 \right. \\
& \quad \left. + |\Delta_{\delta'} \tilde{u}(\lambda)(t, 0)|^2 + (|\Delta_\delta s_t(\lambda)(t)|^2 + |\Delta_{\delta'} s_t(\lambda)(t)|^2)(1 + |\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2) \right) (|\delta|^2 + |\delta'|^2) \\
& \quad + \left(M_9 \frac{4L^2}{\eta k} (2 + 8a^2) + M_{10}(1 + |\Delta_{\delta'} s_t(\lambda)(t)|^2) \right) |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 \\
& \quad + M_{10} |\Delta h(t, \lambda)|^2 + (2 + 8a^2) M_9 |\Delta h(t, \lambda)|^2 \quad \text{for a.e. } t \in [0, T].
\end{aligned} \tag{4.46}$$

In addition, it holds that for $l = \delta, \delta'$

$$\begin{aligned}
|\Delta_l \tilde{u}(\lambda)(t, 0)|^2 & \leq 2|\Delta_l \tilde{u}(\lambda)(t, 0) - \Delta_l h(t, \lambda)|^2 + 2|\Delta_l h(t, \lambda)|^2 \\
& \leq 2|\Delta_l \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + 2|\Delta_l h(t, \lambda)|^2 \quad \text{for } t \in [0, T],
\end{aligned} \tag{4.47}$$

and by (A2) and (4.1) we have

$$\begin{aligned}
|\Delta_l s_t(\lambda)(t)|^2 & = \left| a\Delta_l \tilde{u}(\lambda)(t, 0) - a \left(\frac{\varphi(s(\lambda + l)(t)) - \varphi(s(\lambda)(t))}{l} \right) \right|^2 \\
& \leq 2a^2|\Delta_l \tilde{u}(\lambda)(t, 0)|^2 + 2(aC_\varphi)^2|\Delta_l s(\lambda)(t)|^2 \\
& \leq 2a^2|\Delta_l \tilde{u}(\lambda)(t, 0)|^2 + 2(aC_\varphi)^2 \tilde{C}^2 \quad \text{for a.e. } t \in [0, T].
\end{aligned} \tag{4.48}$$

Furthermore, by (4.47) and (4.48) we observe that

$$\begin{aligned}
& (|\Delta_\delta s_t(\lambda)(t)|^2 + |\Delta_{\delta'} s_t(\lambda)(t)|^2) |\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2 \\
& \leq (2a^2|\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2 + 2a^2|\Delta_{\delta'} \tilde{u}(\lambda)(t, 0)|^2 + 4(aC_\varphi \tilde{C}^2)^2) |\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2
\end{aligned}$$

$$\begin{aligned} &\leq 3a^2|\Delta_\delta \tilde{u}(\lambda)(t, 0)|^4 + a^2|\Delta_{\delta'} \tilde{u}(\lambda)(t, 0)|^4 \\ &\quad + 8(aC_\varphi \tilde{C})^2(|\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + |\Delta_\delta h(t, \lambda)|^2) \quad \text{for a.e. } t \in [0, T], \end{aligned} \quad (4.49)$$

and that for $l = \delta, \delta'$,

$$\begin{aligned} |\Delta_l \tilde{u}(\lambda)(t, 0)|^4 &\leq 8|\Delta_l \tilde{u}(\lambda)(t, 0) - \Delta_l h(t, \lambda)|^4 + 8|\Delta_l h(t, \lambda)|^4 \\ &\leq 32|\Delta_l \tilde{u}(\lambda)(t) - \Delta_l h(t, \lambda)|_{L^2(0,1)}^2 |\Delta_l \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + 8|\Delta_l h(t, \lambda)|^4. \end{aligned} \quad (4.50)$$

Using (4.1) it follows that $|\Delta_l \tilde{u}(\lambda)(t) - \Delta_l h(t, \lambda)|_{L^2(0,1)}^2 \leq 2(\tilde{C}^2 + |\Delta_l h(t, \lambda)|^2)$ for $t \in [0, T]$ so that (4.49) and (4.50) lead to

$$\begin{aligned} &(|\Delta_\delta s_t(\lambda)(t)|^2 + |\Delta_{\delta'} s_t(\lambda)(t)|^2) |\Delta_\delta \tilde{u}(\lambda)(t, 0)|^2 \\ &\leq 3a^2(64(\tilde{C}^2 + |\Delta_\delta h(t, \lambda)|^2) |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + 8|\Delta_\delta h(t, \lambda)|^4) \\ &\quad + a^2(64(\tilde{C}^2 + |\Delta_{\delta'} h(t, \lambda)|^2) |\Delta_{\delta'} \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + 8|\Delta_{\delta'} h(t, \lambda)|^4) \\ &\quad + 8(aC_\varphi \tilde{C})^2(|\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + |\Delta_\delta h(t, \lambda)|^2) \quad \text{for a.e. } t \in [0, T]. \end{aligned} \quad (4.51)$$

From (4.46), (4.47) and (4.51) it yields that

$$\begin{aligned} &\frac{\rho_v}{2} \frac{d}{dt} |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + \frac{k}{8(L - s_*)^2} (1 - M_9(4\eta + 16a^2\eta)) |\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 \\ &\leq M_9(1 + 4(aC_\varphi)^2) |\Delta s(\lambda)(t)|^2 \\ &\quad + M_{11} \left(1 + (1 + |\Delta_\delta h(t, \lambda)|^2) |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + (1 + |\Delta_{\delta'} h(t, \lambda)|^2) |\Delta_{\delta'} \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 \right. \\ &\quad \left. + |\Delta_\delta h(t, \lambda)|^2 + |\Delta_{\delta'} h(t, \lambda)|^2 + |\Delta_\delta h(t, \lambda)|^4 + |\Delta_{\delta'} h(t, \lambda)|^4 \right) (|\delta|^2 + |\delta'|^2) \\ &\quad + \left(M_9 \frac{4L^2}{\eta k} (2 + 8a^2) + M_{10}(1 + |\Delta_{\delta'} s_t(\lambda)(t)|^2) \right) |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 \\ &\quad + M_{10} |\Delta h_t(t, \lambda)|^2 + (2 + 8a^2) M_9 |\Delta h(t, \lambda)|^2 \quad \text{for a.e. } t \in [0, T], \end{aligned} \quad (4.52)$$

where M_{11} is a positive constant depending on $a, C_\varphi, \tilde{C}$ and M_9 . Recalling (4.42), (4.43) and (4.44), it can be guaranteed that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} |\Delta s(\lambda)(t)|^2 \\ &\leq a^2 |\Delta \tilde{u}(\lambda)(t, 0)|^2 + (aC_\varphi)^2 |\Delta s(\lambda)(t)|^2 + (aC_\varphi \tilde{C}^2)^2 (|\delta|^2 + |\delta'|^2) + |\Delta s(\lambda)(t)|^2 \\ &\leq \frac{a^2 \eta k}{4(L - s_*)^2} |\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \frac{4(aL)^2}{\eta k} |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + a^2 |\Delta h(t, \lambda)|^2 \\ &\quad + (aC_\varphi)^2 |\Delta s(\lambda)(t)|^2 + (aC_\varphi \tilde{C}^2)^2 (|\delta|^2 + |\delta'|^2) + |\Delta s(\lambda)(t)|^2 \quad \text{for a.e. } t \in [0, T]. \end{aligned} \quad (4.53)$$

Adding (4.53) to (4.52) we have

$$\frac{\rho_v}{2} \frac{d}{dt} |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + \frac{1}{2} \frac{d}{dt} |\Delta s(\lambda)(t)|^2$$

$$\begin{aligned}
& + \frac{k}{8(L-s_*)^2} \left(1 - M_9(4\eta + 16a^2\eta) - 2a^2\eta \right) |\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 \\
\leq & \left[M_9(1 + 4(aC_\varphi)^2) + (aC_\varphi)^2 + 1 \right] |\Delta s(\lambda)(t)|^2 \\
& + \left[M_{11} \left(1 + (1 + |\Delta_\delta h(t, \lambda)|^2) |\Delta_\delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + (1 + |\Delta_{\delta'} h(t, \lambda)|^2) |\Delta_{\delta'} \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 \right. \right. \\
& \left. \left. + |\Delta_\delta h(t, \lambda)|^2 + |\Delta_{\delta'} h(t, \lambda)|^2 + |\Delta_\delta h(t, \lambda)|^4 + |\Delta_{\delta'} h(t, \lambda)|^4 \right) + (aC_\varphi \tilde{C}^2)^2 \right] (|\delta|^2 + |\delta'|^2) \\
& + \left(M_9 \frac{4L^2}{\eta k} (2 + 8a^2) + M_{10} (1 + |\Delta_{\delta'} s_t(\lambda)(t)|^2) + \frac{4(aL)^2}{\eta k} \right) |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 \\
& + M_{10} |\Delta h_t(t, \lambda)|^2 + \left[(2 + 8a^2) M_9 + a^2 \right] |\Delta h(t, \lambda)|^2 \quad \text{for a.e. } t \in [0, T]. \tag{4.54}
\end{aligned}$$

For simplicity, in (4.54) we set the coefficients of $|\delta|^2 + |\delta'|^2$ and $|\Delta \bar{u}(\lambda)(t)|^2$ by $f_1(\delta, \delta')(t)$ and $f_2(\delta, \delta')(t)$, respectively. Also, by taking a suitable $\eta > 0$ we can rewrite (4.54) as follows:

$$\begin{aligned}
& \frac{\rho_v}{2} \frac{d}{dt} |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + \frac{1}{2} \frac{d}{dt} |\Delta s(\lambda)(t)|^2 + \frac{k}{16(L-s_*)^2} |\Delta \tilde{u}_y(\lambda)(t)|_{L^2(0,1)}^2 \\
\leq & M_{12} |\Delta s(\lambda)(t)|^2 + f_1(\delta, \delta')(t) (|\delta|^2 + |\delta'|^2) + f_2(\delta, \delta')(t) |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 \\
& + M_{10} |\Delta h_t(t, \lambda)|^2 + M_{13} |\Delta h(t, \lambda)|^2 \quad \text{for a.e. } t \in [0, T], \tag{4.55}
\end{aligned}$$

where M_{12} and M_{13} are positive constants independent of δ and δ' .

By applying Gronwall's lemma to (4.55), we obtain

$$\begin{aligned}
& \frac{\rho_v}{2} |\Delta \bar{u}(\lambda)(t)|_{L^2(0,1)}^2 + \frac{1}{2} |\Delta s(\lambda)(t)|^2 + \frac{k}{16(L-s_*)^2} \int_0^t |\Delta \tilde{u}_y(\lambda)(\tau)|_{L^2(0,1)}^2 d\tau \\
\leq & M_{14} (\exp(\int_0^t f_2(\delta, \delta')(\tau) d\tau) \times \\
& \times \left(|\Delta \tilde{u}_0(\lambda) - \Delta h(0, \lambda)|_{L^2(0,1)}^2 + |\Delta s_0(\lambda)|^2 + (|\delta|^2 + |\delta'|^2) \int_0^t f_1(\delta, \delta')(\tau) d\tau \right. \\
& \left. + \int_0^t (|\Delta h_t(\tau, \lambda)|^2 + |\Delta h(\tau, \lambda)|^2) d\tau \right) \quad \text{for } t \in [0, T], \tag{4.56}
\end{aligned}$$

where M_{14} is a positive constant independent of δ and δ' .

Now, we can show that $\{f_1(\delta, \delta')\}_{\delta, \delta' \in I_+(\lambda)}$ and $\{f_2(\delta, \delta')\}_{\delta, \delta' \in I_+(\lambda)}$ are bounded in $L^1(0, T)$. Indeed, by Lemma 3 the set $\{\Delta_l \tilde{u}_y(\lambda)(t)\}_{l \in I_+(\lambda)}$ is bounded in $L^2(0, T; L^2(0, 1))$ for $l = \delta, \delta'$. Also, we see that

$$\begin{aligned}
|\Delta_l h(\lambda, t)|^2 & \leq 2 \left(\int_0^t |\Delta_l h_t(\lambda, \tau)| d\tau \right)^2 + 2 |\Delta_l h(\lambda, 0)|^2 \\
& \leq 2t \int_0^t |\Delta_l h_t(\lambda, \tau)|^2 d\tau + 2 |\Delta_l h(\lambda, 0)|^2 \quad \text{for } t \in [0, T] \text{ and } l = \delta, \delta'.
\end{aligned}$$

Since it is obvious that $|\Delta_l h(0, \lambda)| \leq C' |\Delta_l h(\lambda)|_{W^{1,2}(0,T)}$ for some positive constant C' , by Lemma 2 the set $\{\Delta_l h(0, \lambda)\}_{l \in I_+(\lambda)}$ is bounded in $\mathbb{R}$ for $l = \delta, \delta'$. Then, $|\Delta_l h(t, \lambda)|^2$ and $|\Delta_l h(t, \lambda)|^4$ are bounded in $\mathbb{R}$ on $[0, T]$ with respect to $l \in I_+(\lambda)$ and $t \in [0, T]$ so that $\{f_1(\delta, \delta')\}_{\delta, \delta' \in I_+(\lambda)}$ is bounded in $L^1(0, T)$. Also, by Lemma 3 $\{\Delta_\delta s_t(\lambda)\}_{\delta \in I_+(\lambda)}$ is bounded in $L^2(0, T)$, and we see that $\{f_2(\delta, \delta')\}_{\delta, \delta' \in I_+(\lambda)}$ is bounded in $L^1(0, T)$.

At the end of this proof, thanks to Lemma 2 it is clear that $\Delta \tilde{u}_0(\lambda) \rightarrow 0$ in $L^2(0, 1)$, $\Delta h(0, \lambda), \Delta s_0(\lambda) \rightarrow 0$ in $\mathbb{R}$ and $\Delta h_t(\cdot, \lambda), \Delta h(\cdot, \lambda) \rightarrow 0$ in $L^2(0, T)$ as $\delta, \delta' \rightarrow 0$. Therefore, by (4.56) we conclude that $\{\Delta_\delta \tilde{u}(\lambda)\}_{\delta \in I(\lambda)}$ is a Cauchy sequence in $C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1))$. Finally, by (4.44) it is easy to see that $\{\Delta_\delta s(\lambda)\}_{\delta \in I(\lambda)}$ is a Cauchy sequence in $W^{1,2}(0, T)$. □

5 Proof of Theorem 2

We prove Theorem 2 in the following steps:

1. (Section 5.1) The differential quotients $\{\Delta_\delta s(\lambda)\}_{\delta \in I_+(\lambda)}$ and $\{\Delta_\delta \tilde{u}(\lambda)\}_{\delta \in I_+(\lambda)}$ converge to some functions $\hat{s}(\lambda)$ and $\hat{u}(\lambda)$, respectively.
2. (Section 5.2) The pair $(\hat{s}(\lambda), \hat{u}(\lambda))$ of limit functions is a weak solution of $\hat{P}(\lambda)$.
3. (Section 5.3) The uniqueness of weak solutions of $\hat{P}(\lambda)$ on $[0, T]$ holds.

In this section the same conditions as in Theorem 2 is assumed and all notation in the previous sections will be used.

5.1 Convergence of differential quotients

Proposition 1. *For $\lambda \in (0, 1) \setminus N$ there exist functions $\hat{s}(\lambda) \in W^{1,2}(0, T)$ and $\hat{u}(\lambda) \in C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1))$ such that*

$$\Delta_\delta \tilde{u}(\lambda) \rightarrow \hat{u}(\lambda) \text{ in } C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)), \quad (5.1)$$

$$\Delta_\delta \tilde{u}(\lambda)(\cdot, 0) \rightarrow \hat{u}(\lambda)(\cdot, 0) \text{ in } L^2(0, T), \quad \Delta_\delta \tilde{u}(\lambda)(\cdot, 1) \rightarrow \hat{u}(\lambda)(\cdot, 1) \text{ in } L^2(0, T), \quad (5.2)$$

$$\Delta_\delta s(\lambda) \rightarrow \hat{s}(\lambda) \text{ in } W^{1,2}(0, T) \text{ as } \delta \rightarrow 0. \quad (5.3)$$

Proof. (5.1) and (5.3) are direct consequences of Lemma 4. Also, it is easy to see that (5.1) implies (5.2). □

In the next lemma we show some convergences of $s(\lambda)$ and $\tilde{u}(\lambda)$.

Lemma 5. *For $\lambda \in (0, 1) \setminus N$ there exists a subsequence $\{\delta_n\}$ of $\{\delta\}$ such that*

$$\tilde{u}_y(\lambda + \delta_n) \rightarrow \tilde{u}_y(\lambda) \text{ weakly-}^* \text{ in } L^\infty(0, T; L^2(0, 1)), \quad (5.4)$$

$$\tilde{u}_{yy}(\lambda + \delta_n) \rightarrow \tilde{u}_{yy}(\lambda) \text{ weakly in } L^2(0, T; L^2(0, 1)), \quad (5.5)$$

$$\tilde{u}_y(\lambda + \delta_n)(\cdot, 0) \rightarrow \tilde{u}_y(\lambda)(\cdot, 0) \text{ weakly in } L^2(0, 1), \quad (5.6)$$

$$s(\lambda + \delta_n) \rightarrow s(\lambda) \text{ in } C([0, T]), \quad (5.7)$$

$$s_t(\lambda + \delta_n) \rightarrow s_t(\lambda) \text{ in } L^2(0, T) \text{ as } n \rightarrow \infty. \quad (5.8)$$

Proof. First, by Lemma 3 $\{\Delta_\delta \tilde{u}(\lambda)\}_{\delta \in I_+(\lambda)}$ is bounded in $L^2(0, T; L^2(0, 1))$. Then, from

$$|\tilde{u}(\lambda + \delta)(t) - \tilde{u}(\lambda)(t)|_{L^2(0,1)} \leq |\Delta_\delta \tilde{u}(\lambda)(t)|_{L^2(0,1)} |\delta| \quad \text{for } t \in [0, T]$$

it follows that

$$\tilde{u}(\lambda + \delta) \rightarrow \tilde{u}(\lambda) \text{ in } L^2(0, T; L^2(0, 1)) \text{ as } \delta \rightarrow 0. \quad (5.9)$$

Also, by Lemma 1 $\{\tilde{u}_y(\lambda + \delta)\}_{\delta \in I_+(\lambda)}$ and $\{\tilde{u}_{yy}(\lambda + \delta)\}_{\delta \in I_+(\lambda)}$ are bounded in $L^2(0, T; L^2(0, 1))$. Accordingly, there exist a subsequence $\{\delta_n\} \subset \{\delta\}$ and $\bar{u}_1(\lambda) \in L^\infty(0, T; L^2(0, 1))$ and $\bar{u}_2(\lambda) \in L^2(0, T; L^2(0, 1))$ such that $\tilde{u}_y(\lambda + \delta_n) \rightarrow \bar{u}_1(\lambda)$ weakly-* in $L^\infty(0, T; L^2(0, 1))$ and $\tilde{u}_{yy}(\lambda + \delta_n) \rightarrow \bar{u}_2(\lambda)$ weakly in $L^2(0, T; L^2(0, 1))$ as $n \rightarrow \infty$. From (5.9), we easily see that $\bar{u}_1(\lambda) = \tilde{u}_y(\lambda)$ in $L^2(0, T; L^2(0, 1))$ and $\bar{u}_2(\lambda) = \tilde{u}_{yy}(\lambda)$ in $L^2(0, T; L^2(0, 1))$, namely, (5.4) and (5.5) hold. Obviously, (5.4) and (5.5) imply (5.6).

Next, since $\{\Delta_{\delta_n} s(\lambda)\}_{n \in \mathbb{N}}$ and $\{\Delta_{\delta_n} s_t(\lambda)\}_{n \in \mathbb{N}}$ are bounded in $C([0, T])$ and in $L^2(0, T)$, respectively, we have

$$\begin{aligned} |s(\lambda + \delta_n)(t) - s(\lambda)(t)| &= |\Delta_{\delta_n} s(\lambda)(t)| |\delta_n| \quad \text{for } t \in [0, T], \\ |s_t(\lambda + \delta_n) - s_t(\lambda)|_{L^2(0, T)} &\leq |\Delta_{\delta_n} s_t(\lambda)|_{L^2(0, T)} |\delta_n| \quad \text{for } t \in [0, T]. \end{aligned}$$

Thus, we get (5.7) and (5.8). □

5.2 Existence of a weak solution of $\hat{\mathbf{P}}(\lambda)$

In this section, we show that the pair $(\hat{s}(\lambda), \hat{u}(\lambda))$ of limit functions satisfies Definition 2.

Proof of Theorem 2(existence). Let fix $\lambda \in (0, 1) \setminus N$. Recalling (3.10)-(3.14), we see that for each $n \in \mathbb{N}$, it holds that

$$\begin{aligned} &\rho_v \Delta_{\delta_n} \tilde{u}_t(\lambda) - \frac{k}{(L - s(\lambda))^2} \Delta_{\delta_n} \tilde{u}_{yy}(\lambda) \\ &= \frac{k(2L - s(\lambda + \delta_n) - s(\lambda))}{(L - s(\lambda + \delta_n))^2 (L - s(\lambda))^2} \Delta_{\delta_n} s(\lambda) \tilde{u}_{yy}(\lambda + \delta_n) + \rho_v(1 - y) \frac{\tilde{u}_y(\lambda + \delta_n)}{L - s(\lambda + \delta_n)} \Delta_{\delta_n} s_t(\lambda) \\ &\quad + \rho_v(1 - y) \left(\frac{s_t(\lambda)}{L - s(\lambda + \delta_n)} \Delta_{\delta_n} \tilde{u}_y(\lambda) + \frac{s_t(\lambda) \tilde{u}_y(\lambda)}{(L - s(\lambda + \delta_n))(L - s(\lambda))} \Delta_{\delta_n} s(\lambda) \right) \text{ in } Q(T), \end{aligned} \quad (5.10)$$

$$\Delta_{\delta_n} \tilde{u}(\lambda)(t, 1) = \Delta_{\delta_n} h(t, \lambda) \text{ for } t \in [0, T], \quad (5.11)$$

$$\Delta_{\delta_n} s_t(\lambda)(t) = a \left(\Delta_{\delta_n} \tilde{u}(\lambda)(t, 0) - \frac{\varphi(s(\lambda + \delta_n)(t)) - \varphi(s(\lambda)(t))}{\delta_n} \right) \text{ for } t \in [0, T], \quad (5.12)$$

$$\begin{aligned} &\frac{k}{L - s(\lambda)(t)} \Delta_{\delta_n} \tilde{u}_y(\lambda)(t, 0) + \frac{k}{(L - s(\lambda + \delta_n)(t))(L - s(\lambda)(t))} \Delta_{\delta_n} s(\lambda)(t) \tilde{u}_y(\lambda + \delta_n)(t, 0) \\ &= \rho_w \Delta_{\delta_n} s_t(\lambda)(t) - \rho_v s_t(\lambda + \delta_n)(t) \Delta_{\delta_n} \tilde{u}(\lambda)(t, 0) - \rho_v \tilde{u}(\lambda)(t, 0) \Delta_{\delta_n} s_t(\lambda)(t) \text{ for } t \in [0, T], \end{aligned} \quad (5.13)$$

$$\Delta_{\delta_n} s(\lambda)(0) = \Delta_{\delta_n} s_0(\lambda), \quad \Delta_{\delta_n} \tilde{u}(\lambda)(0, y) = \Delta_{\delta_n} \tilde{u}_0(\lambda)(y) \text{ for } y \in (0, 1). \quad (5.14)$$

Then, for any $\eta \in W^{1,2}(0, T; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1))$ with $\eta(T) = 0$ and $\eta(t, 1) = 0$ on $[0, T]$, from (5.10) it holds that for $n \in \mathbb{N}$,

$$\begin{aligned}
& - \int_0^T \int_0^1 \rho_v \Delta_{\delta_n} \tilde{u}(\lambda)(t) \eta_t(t) dy dt - \int_0^1 \rho_v \Delta_{\delta_n} \tilde{u}_0(\lambda) \eta(0) dy \\
& + \int_0^T \int_0^1 \frac{k}{(L - s(\lambda)(t))^2} \Delta_{\delta_n} \tilde{u}_y(\lambda)(t) \eta_y(t) dy dt + \int_0^T \frac{k}{L - s(\lambda)(t)} \Delta_{\delta_n} \tilde{u}_y(\lambda)(t, 0) \eta(t, 0) dt \\
& = \int_0^T \int_0^1 \frac{k(2L - s(\lambda + \delta_n)(t) - s(\lambda)(t))}{(L - s(\lambda + \delta_n)(t))^2 (L - s(\lambda)(t))^2} \Delta_{\delta_n} s(\lambda)(t) \tilde{u}_{yy}(\lambda + \delta_n)(t) \eta(t) dy dt \\
& + \int_0^T \int_0^1 \frac{\rho_v(1 - y) \tilde{u}_y(\lambda + \delta_n)(t)}{L - s(\lambda + \delta_n)(t)} \Delta_{\delta_n} s_t(\lambda)(t) \eta(t) dy dt \\
& + \int_0^T \int_0^1 \frac{\rho_v(1 - y) s_t(\lambda)(t)}{L - s(\lambda + \delta_n)(t)} \Delta_{\delta_n} \tilde{u}_y(\lambda)(t) \eta(t) dy dt \\
& + \int_0^T \int_0^1 \frac{\rho_v(1 - y) s_t(\lambda)(t) \tilde{u}_y(\lambda)(t)}{(L - s(\lambda + \delta_n)(t))(L - s(\lambda)(t))} \Delta_{\delta_n} s(\lambda)(t) \eta(t) dy dt. \tag{5.15}
\end{aligned}$$

Now, we proceed the limiting process $n \rightarrow \infty$ in (5.15). First, by the strong convergence (5.1) for $\Delta_{\delta_n} \tilde{u}(\lambda)$ it is easy to see that

$$- \int_0^T \int_0^1 \rho_v \Delta_{\delta_n} \tilde{u}(\lambda)(t) \eta_t(t) dy dt \rightarrow - \int_0^T \int_0^1 \rho_v \hat{u}(\lambda)(t) \eta_t(t) dy dt \text{ as } n \rightarrow \infty, \tag{5.16}$$

and

$$\int_0^T \int_0^1 \frac{k \Delta_{\delta_n} \tilde{u}_y(\lambda)(t)}{(L - s(\lambda)(t))^2} \eta_y(t) dy dt \rightarrow \int_0^T \int_0^1 \frac{k \hat{u}_y(\lambda)(t)}{(L - s(\lambda)(t))^2} \eta_y(t) dy dt \text{ as } n \rightarrow \infty. \tag{5.17}$$

On the third term of the left hand side of (5.15), by (5.13) we have

$$\begin{aligned}
& \int_0^T \frac{k}{L - s(\lambda)(t)} \Delta_{\delta_n} \tilde{u}_y(\lambda)(t, 0) \eta(t, 0) dt \\
& = - \int_0^T \frac{k}{(L - s(\lambda + \delta_n)(t))(L - s(\lambda)(t))} \Delta_{\delta_n} s(\lambda)(t) \tilde{u}_y(\lambda + \delta_n)(t, 0) \eta(t, 0) dt \\
& + \int_0^T \rho_w \Delta_{\delta_n} s_t(\lambda)(t) \eta(0) dt - \int_0^T \rho_v s_t(\lambda + \delta_n)(t) \Delta_{\delta_n} \tilde{u}(\lambda)(t, 0) \eta(t, 0) dt \\
& - \int_0^T \rho_v \tilde{u}(\lambda)(t, 0) \Delta_{\delta_n} s_t(\lambda)(t) \eta(t, 0) dt. \tag{5.18}
\end{aligned}$$

Accordingly, from (5.2), (5.3), (5.6), (5.7) and (5.8), the right hand side of (5.18) converges to the following (5.19) as $n \rightarrow \infty$:

$$\begin{aligned}
& - \int_0^T \frac{k}{(L - s(\lambda)(t))^2} \hat{s}(\lambda)(t) \tilde{u}_y(\lambda)(t, 0) \eta(t, 0) dt + \int_0^T \rho_w \hat{s}_t(\lambda)(t) \eta(t, 0) dt \\
& - \int_0^T \rho_v s_t(\lambda)(t) \hat{u}(\lambda)(t, 0) \eta(t, 0) dt - \int_0^T \rho_v \tilde{u}(\lambda)(t, 0) \hat{s}_t(\lambda)(t) \eta(t, 0) dt. \tag{5.19}
\end{aligned}$$

Next, by recalling the notation of $F_{i\delta}(\lambda)$, $i = 1, 2, 3, 4$, we rewrite the right hand side of (5.15) as follows:

$$\sum_{i=1}^4 \int_0^T \int_0^1 F_{i\delta_n}(\lambda)(t) \eta(t) dy dt$$

Here, we note that $\tilde{u}_y, \tilde{u}_{yy} \in L^\infty(0, 1; L^2(0, T; L^2(0, 1)))$. Also, by Lemma 3, $\{\Delta_{\delta_n} s(\lambda)\}_{n \in \mathbb{N}}$ is bounded in $C([0, T])$, $\{\Delta_{\delta_n} s_t(\lambda)\}_{n \in \mathbb{N}}$ is bounded in $L^2(0, T)$ and $\{\Delta_{\delta_n} \tilde{u}_y(\lambda)\}_{n \in \mathbb{N}}$ is bounded in $L^2(0, T; L^2(0, 1))$. Then, by (5.3), (5.5) and (5.7), we see that

$$F_{1\delta_n}(\lambda) \rightarrow \frac{2k(L - s(\lambda))}{(L - s(\lambda))^4} \hat{s}(\lambda) \tilde{u}_{yy}(\lambda) \text{ weakly in } L^2(0, T; L^2(0, 1)) \text{ as } n \rightarrow \infty. \quad (5.20)$$

Also, from the strong convergences (5.3) and (5.7) and the weakly convergence (5.4), it follows that

$$F_{2\delta_n}(\lambda) \rightarrow \frac{\rho_v(1 - y) \tilde{u}_y(\lambda)}{L - s(\lambda)} \hat{s}_t(\lambda) \text{ weakly in } L^2(0, T; L^2(0, 1)) \text{ as } n \rightarrow \infty. \quad (5.21)$$

Moreover, on account of (5.1), (5.3) and (5.7) we have

$$F_{3\delta_n}(\lambda) \rightarrow \frac{\rho_v(1 - y) s_t(\lambda)}{L - s(\lambda)} \hat{u}_y(\lambda) \text{ weakly in } L^2(0, T; L^2(0, 1)) \text{ as } n \rightarrow \infty, \quad (5.22)$$

$$F_{4\delta_n}(\lambda) \rightarrow \frac{\rho_v(1 - y) s_t(\lambda) \tilde{u}_y(\lambda)}{(L - s(\lambda))^2} \hat{s}(\lambda) \text{ weakly in } L^2(0, T; L^2(0, 1)) \text{ as } n \rightarrow \infty. \quad (5.23)$$

Consequently, by all convergences (5.16), (5.17), (5.19), (5.20)-(5.23) and Lemma 2 we obtain from (5.15) that

$$\begin{aligned} & - \int_0^T \int_0^1 \rho_v \hat{u}(\lambda)(t) \eta_t(t) dy dt + \int_0^T \int_0^1 \frac{k}{(L - s(\lambda)(t))^2} \hat{u}_y(\lambda)(t) \eta_y(t) dy dt \\ & - \int_0^T \frac{k}{(L - s(\lambda)(t))^2} \hat{s}(\lambda)(t) \tilde{u}_y(\lambda)(t, 0) \eta(t, 0) dt + \int_0^T \rho_w \hat{s}_t(\lambda)(t) \eta(t, 0) dt \\ & - \int_0^T \rho_v s_t(\lambda)(t) \hat{u}(\lambda)(t, 0) \eta(t, 0) dt - \int_0^T \rho_v \tilde{u}(\lambda)(t, 0) \hat{s}_t(\lambda)(t) \eta(t, 0) dt \\ & = \int_0^T \int_0^1 \frac{2k \hat{s}(\lambda)(t)}{(L - s(\lambda)(t))^3} \tilde{u}_{yy}(\lambda)(t) \eta(t) dy dt + \int_0^T \int_0^1 \frac{\rho_v(1 - y) \tilde{u}_y(\lambda)(t)}{L - s(\lambda)(t)} \hat{s}_t(\lambda)(t) \eta(t) dy dt \\ & + \int_0^T \int_0^1 \frac{\rho_v(1 - y) s_t(\lambda)(t)}{L - s(\lambda)(t)} \hat{u}_y(\lambda)(t) \eta(t) dy dt \\ & + \int_0^T \int_0^1 \frac{\rho_v(1 - y) s_t(\lambda)(t) \tilde{u}_y(\lambda)(t)}{(L - s(\lambda)(t))^2} \hat{s}(\lambda)(t) \eta(t) dy dt + \rho_v \int_0^1 \hat{u}_0(\lambda) \eta(0) dy. \end{aligned}$$

To accomplish the proof we remain to show (2.11) and (2.13). By Lemma 2, (5.2) and (5.11), we infer that (2.11) holds. Next, since φ' is Lipschitz continuous, we observe that

$$\left| \frac{\varphi(s(\lambda + \delta_n)) - \varphi(s(\lambda))}{\delta_n} - \varphi'(s(\lambda)) \hat{s}(\lambda) \right|$$

$$\begin{aligned}
&= \left| \varphi' \left(s(\lambda) + \xi_n(s(\lambda + \delta_n) - s(\lambda)) \right) \Delta_{\delta_n} s(\lambda) - \varphi'(s(\lambda)) \hat{s}(\lambda) \right| \\
&\leq C_\varphi \left(|s(\lambda + \delta_n) - s(\lambda)| |\Delta_{\delta_n} s(\lambda)| + |\Delta_{\delta_n} s(\lambda) - \hat{s}(\lambda)| \right) \text{ on } [0, T], \tag{5.24}
\end{aligned}$$

where $0 < \xi_n < 1$. From (5.24), (5.3) and (5.7) it is proved that

$$\frac{\varphi(s(\lambda + \delta_n)) - \varphi(s(\lambda))}{\delta_n} \rightarrow \varphi'(s(\lambda)) \hat{s}(\lambda) \text{ in } C([0, T]) \text{ as } n \rightarrow \infty. \tag{5.25}$$

Therefore, with the help of (5.2), (5.3) and (5.25), by letting $n \rightarrow \infty$ in (5.12) we get the first equation in (2.13). Finally, from the convergences in Lemma 2, (5.14) we see the second equation in (2.13). Thus, $(\hat{s}(\lambda), \hat{u}(\lambda))$ is the weak solution of $\hat{P}(\lambda)$. $\square$

5.3 Uniqueness of weak solutions of $\hat{P}(\lambda)$

The aim of this part is to establish uniqueness of solutions to $\hat{P}(\lambda)$ on $[0, T]$.

Proof of Theorem 2(uniqueness). Let $\lambda \in (0, 1)$ and $(\hat{s}_1(\lambda), \hat{u}_1(\lambda))$ and $(\hat{s}_2(\lambda), \hat{u}_2(\lambda))$ be weak solutions of $\hat{P}(\lambda)$ on $[0, T]$. Put $\hat{s}(\lambda) = \hat{s}_1(\lambda) - \hat{s}_2(\lambda)$ and $\hat{u}(\lambda) = \hat{u}_1(\lambda) - \hat{u}_2(\lambda)$. Then, for any $\eta \in W^{1,2}(0, T; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1))$ with $\eta(T) = 0$ and $\eta(t, 1) = 0$, it holds that

$$\begin{aligned}
& - \int_0^T \int_0^1 \rho_v \hat{u}(\lambda)(t) \eta_t(t) dy dt + \int_0^T \int_0^1 \frac{k}{(L - s(\lambda)(t))^2} \hat{u}_y(\lambda)(t) \eta_y(t) dy dt \\
& - \int_0^T \frac{k}{(L - s(\lambda)(t))^2} \hat{s}(\lambda)(t) \tilde{u}_y(\lambda)(t, 0) \eta(t, 0) dt + \int_0^T \rho_w \hat{s}_t(\lambda)(t) \eta(t, 0) dt \\
& - \int_0^T \rho_v s_t(\lambda)(t) \hat{u}(\lambda)(t, 0) \eta(t, 0) dt - \int_0^T \rho_v \tilde{u}(\lambda)(t, 0) \hat{s}_t(\lambda)(t) \eta(t, 0) dt \\
& = \int_0^T \int_0^1 \frac{2k \hat{s}(\lambda)(t)}{(L - s(\lambda)(t))^3} \tilde{u}_{yy}(\lambda)(t) \eta(t) dy dt + \int_0^T \int_0^1 \frac{\rho_v (1 - y) \tilde{u}_y(\lambda)(t)}{L - s(\lambda)(t)} \hat{s}_t(\lambda)(t) \eta(t) dy dt \\
& + \int_0^T \int_0^1 \frac{\rho_v (1 - y) s_t(\lambda)(t)}{L - s(\lambda)(t)} \hat{u}_y(\lambda)(t) \eta(t) dy dt \\
& + \int_0^T \int_0^1 \frac{\rho_v (1 - y) s_t(\lambda)(t) \tilde{u}_y(\lambda)(t)}{(L - s(\lambda)(t))^2} \hat{s}(\lambda)(t) \eta(t) dy dt,
\end{aligned}$$

and

$$\hat{u}(\lambda)(t, 1) = 0 \text{ for a.e. } t \in [0, T], \tag{5.26}$$

$$\hat{s}_t(\lambda) = a(\hat{u}(\lambda)(t, 0) - \varphi'(s(\lambda)) \hat{s}(t)) \text{ for a.e. } t \in [0, T], \tag{5.27}$$

$$\hat{u}(\lambda)(0) = 0, \quad \hat{s}(\lambda)(0) = 0. \tag{5.28}$$

By applying the classical theory for weak solutions of linear parabolic equations, for instance [13, Chapter 3] we obtain

$$\frac{\rho_v}{2} |\hat{u}(\lambda)(t_1)|_{L^2(0,1)}^2 + \int_0^{t_1} \int_0^1 \frac{k}{(L - s(\lambda)(t))^2} |\hat{u}_y(\lambda)(t)|^2 dy dt$$

$$\begin{aligned}
&= \int_0^{t_1} \frac{k}{(L - s(\lambda)(t))^2} \hat{s}(\lambda)(t) \tilde{u}_y(\lambda)(t, 0) \hat{u}(\lambda)(t, 0) dt + \int_0^{t_1} \rho_w \hat{s}_t(\lambda)(t) \hat{u}(\lambda)(t, 0) dt \\
&\quad + \int_0^{t_1} \rho_v s_t(\lambda)(t) |\hat{u}(\lambda)(t, 0)|^2 dt - \int_0^{t_1} \rho_v \tilde{u}(\lambda)(t, 0) \hat{s}_t(\lambda)(t) \hat{u}(\lambda)(t, 0) dt \\
&\quad + \int_0^{t_1} \int_0^1 \frac{2k \hat{s}(\lambda)(t)}{(L - s(\lambda)(t))^3} \tilde{u}_{yy}(\lambda)(t) \hat{u}(\lambda)(t) dy dt \\
&\quad + \int_0^{t_1} \int_0^1 \frac{\rho_v (1 - y) \tilde{u}_y(\lambda)(t)}{L - s(\lambda)(t)} \hat{s}_t(\lambda)(t) \hat{u}(\lambda)(t) dy dt \\
&\quad + \int_0^{t_1} \int_0^1 \frac{\rho_v (1 - y) s_t(\lambda)(t)}{L - s(\lambda)(t)} \hat{u}_y(\lambda)(t) \hat{u}(\lambda)(t) dy dt \\
&\quad + \int_0^{t_1} \int_0^1 \frac{\rho_v (1 - y) s_t(\lambda)(t) \tilde{u}_y(\lambda)(t)}{(L - s(\lambda)(t))^2} \hat{s}(\lambda)(t) \hat{u}(\lambda)(t) dy dt \\
&=: \sum_{i=1}^8 I_i(t_1) \text{ for any } t_1 \in [0, T].
\end{aligned} \tag{5.29}$$

First, by (4.29) and (5.26) it is obvious that $|\tilde{u}_y(\lambda)(t, 0)| \leq \frac{L}{k}(\rho_w + \rho_v)a$ for $t \in [0, T]$ and $\hat{u}(\lambda)(t, 0) = \int_0^1 \hat{u}_y(\lambda)(t) dy$ for $t \in [0, T]$. Then, we see that

$$\begin{aligned}
|I_1(t_1)| &\leq \frac{L(\rho_w + \rho_v)a}{(L - s_*)^2} \int_0^{t_1} |\hat{s}(\lambda)(t)| |\hat{u}(\lambda)(t, 0)| dt \\
&\leq \frac{L(\rho_w + \rho_v)a}{(L - s_*)^2} \int_0^{t_1} |\hat{s}(\lambda)(t)| |\hat{u}_y(\lambda)(t)|_{L^2(0,1)} dt \\
&\leq \frac{\eta}{2} \int_0^{t_1} |\hat{u}_y(\lambda)(t)|_{L^2(0,1)}^2 dt + \frac{1}{2\eta} \left(\frac{L(\rho_w + \rho_v)a}{(L - s_*)^2} \right)^2 \int_0^{t_1} |\hat{s}(\lambda)(t)|^2 dt,
\end{aligned} \tag{5.30}$$

where η is an arbitrary positive number. Similarly to (5.30), by Remark 1 we infer that

$$|I_2(t_1)| \leq \frac{\eta}{2} \int_0^{t_1} |\hat{u}_y(\lambda)(t)|_{L^2(0,1)}^2 dt + \frac{\rho_w^2}{2\eta} \int_0^{t_1} |\hat{s}_t(\lambda)(t)|^2 dt, \tag{5.31}$$

and

$$|I_4(t_1)| \leq \frac{\eta}{2} \int_0^{t_1} |\hat{u}_y(\lambda)(t)|_{L^2(0,1)}^2 dt + \frac{\rho_v^2}{2\eta} \int_0^{t_1} |\hat{s}_t(\lambda)(t)|^2 dt. \tag{5.32}$$

Also, by standard calculations and (2.1) we obtain

$$\begin{aligned}
|I_3(t_1)| &\leq \rho_v a \int_0^{t_1} |\hat{u}(\lambda)(t, 0)|^2 dt \\
&\leq \rho_v a C_e \int_0^{t_1} |\hat{u}(\lambda)(t)|_{L^2(0,1)} |\hat{u}(\lambda)(t)|_{H^1(0,1)} dt \\
&\leq \frac{\eta}{2} \int_0^{t_1} |\hat{u}_y(\lambda)(t)|_{L^2(0,1)}^2 dt + \left(\rho_v a C_e + \frac{(\rho_v a C_e)^2}{2\eta} \right) \int_0^{t_1} |\hat{u}(\lambda)(t)|_{L^2(0,1)}^2 dt,
\end{aligned} \tag{5.33}$$

where C_e is some positive constant. Because of $\tilde{u}_y(\lambda) \in L^\infty(0, T; L^2(0, 1))$, we can deal with the terms $I_i(t_1)$ for $5 \leq i \leq 8$ as follows:

$$|I_5(t_1)| \leq \frac{k}{(L - s_*)^3} \left(\int_0^{t_1} |\hat{u}(\lambda)(t)|_{L^2(0,1)}^2 dt + \int_0^{t_1} |\tilde{u}_{yy}(\lambda)(t)|_{L^2(0,1)}^2 |\hat{s}(\lambda)(t)|^2 dt \right), \quad (5.34)$$

$$\begin{aligned} |I_6(t_1)| &\leq \frac{\rho_v}{L - s_*} \int_0^{t_1} |\hat{s}_t(\lambda)(t)| |\tilde{u}_y(\lambda)(t)|_{L^2(0,1)} |\hat{u}(\lambda)(t)|_{L^2(0,1)} \\ &\leq \frac{\rho_v K}{2(L - s_*)} \left(\int_0^{t_1} |\hat{s}_t(\lambda)(t)|^2 dt + \int_0^{t_1} |\hat{u}(\lambda)(t)|_{L^2(0,1)}^2 dt \right), \end{aligned} \quad (5.35)$$

$$|I_7(t_1)| \leq \frac{\eta}{2} \int_0^{t_1} |\hat{u}_y(\lambda)(t)|_{L^2(0,1)}^2 dt + \frac{1}{2\eta} \left(\frac{\rho_v a}{(L - s_*)} \right)^2 \int_0^{t_1} |\hat{u}(\lambda)(t)|_{L^2(0,1)}^2 dt, \quad (5.36)$$

and

$$|I_8(t_1)| \leq \frac{\rho_v a K}{2(L - s_*)^2} \left(\int_0^{t_1} |\hat{u}(\lambda)(t)|_{L^2(0,1)}^2 dt + \int_0^{t_1} |\hat{s}(\lambda)(t)|^2 dt \right), \quad (5.37)$$

where K is the positive constant defined in Lemma 1. Therefore, by combining (5.29) with (5.30)-(5.37), we obtain that

$$\begin{aligned} &\frac{\rho_v}{2} |\hat{u}(\lambda)(t_1)|_{L^2(0,1)}^2 + \left(\frac{k}{2L^2} - \frac{5}{2}\eta \right) \int_0^{t_1} \int_0^1 |\hat{u}_y(\lambda)(t)|^2 dy dt \\ &\leq \tilde{M}_1 \left(\int_0^{t_1} (1 + |\tilde{u}_{yy}(\lambda)(t)|_{L^2(0,1)}^2) |\hat{s}(\lambda)(t)|^2 dt + \int_0^{t_1} |\hat{s}_t(\lambda)(t)|^2 dt + \int_0^{t_1} |\hat{u}(\lambda)(t)|_{L^2(0,1)}^2 dt \right), \end{aligned} \quad (5.38)$$

where $\tilde{M}_1$ is a positive constant depending on ρ_v , ρ_w , a , s_* , C_e , K , L and η . Here, by recalling (5.27) and (5.33) it holds that

$$\begin{aligned} \frac{d}{dt} |\hat{s}(\lambda)(t)|^2 &\leq 2a |\hat{u}(\lambda)(t, 0)| |\hat{s}(\lambda)(t)| + 2a C_\varphi |\hat{s}(\lambda)(t)|^2 \\ &\leq 2a |\hat{u}_y(\lambda)(t)|_{L^2(0,1)} |\hat{s}(\lambda)(t)| + 2a C_\varphi |\hat{s}(\lambda)(t)|^2 \\ &\leq \frac{\eta}{2} |\hat{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \left(\frac{2a^2}{\eta} + 2a C_\varphi \right) |\hat{s}(\lambda)(t)|^2 \text{ for } t \in [0, T], \end{aligned} \quad (5.39)$$

and

$$\begin{aligned} &|\hat{s}_t(\lambda)(t)|^2 \\ &\leq 2a^2 (|\hat{u}(\lambda)(t, 0)|^2 + C_\varphi^2 |\hat{s}(\lambda)(t)|^2) \\ &\leq 2a^2 C_e |\hat{u}(\lambda)(t)|_{L^2(0,1)} |\hat{u}(\lambda)(t)|_{H^1(0,1)} + 2a^2 C_\varphi^2 |\hat{s}(\lambda)(t)|^2 \\ &\leq \frac{\eta}{2} |\hat{u}_y(\lambda)(t)|_{L^2(0,1)}^2 + \left(\frac{2(a^2 C_e)^2}{\eta} + 2a^2 C_e \right) |\hat{u}(\lambda)(t)|_{L^2(0,1)}^2 + 2a^2 C_\varphi^2 |\hat{s}(\lambda)(t)|^2 \text{ for a.e. } t \in [0, T]. \end{aligned} \quad (5.40)$$

We integrate (5.39) over $[0, t_1]$, substitute (5.40) into (5.38) and add them. Then, by taking a suitable η we have

$$\begin{aligned} & \frac{\rho_v}{2} |\hat{u}(\lambda)(t_1)|_{L^2(0,1)}^2 + |\hat{s}(\lambda)(t_1)|^2 + \frac{k}{4L^2} \int_0^{t_1} \int_0^1 |\hat{u}_y(\lambda)(t)|^2 dy dt \\ & \leq \tilde{M}_2 \left(\int_0^{t_1} (1 + |\tilde{u}_{yy}(\lambda)(t)|_{L^2(0,1)}^2) |\hat{s}(\lambda)(t)|^2 dt + \int_0^{t_1} |\hat{u}(\lambda)(t)|_{L^2(0,1)}^2 dt \right), \end{aligned} \quad (5.41)$$

where $\tilde{M}_2$ is a positive constant. Consequently, by applying Gronwall's lemma to (5.41) with help of (5.28), we conclude that $\hat{s}(\lambda) = 0$ and $\hat{u}(\lambda) = 0$ in $L^2(0, 1)$ on $[0, T]$. Thus, $\hat{P}(\lambda)$ has at most one solution. Hence, Theorem 2 is completely proved. $\square$

6 Application

In this section we apply Theorem 2 to the case the parameter varies in $\Omega \subset \mathbb{R}^3$, that is, the boundary and initial functions depend on $\xi = (\xi_1, \xi_2, \xi_3) \in \Omega$. For this case we establish estimates for the partial derivative of s with respect to ξ_i for $i = 1, 2, 3$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. For $\xi \in \Omega$ we consider the following free boundary problem $P(\xi) := \{(6.1)-(6.6)\}$ is to find a pair of a curve $x = s(\xi)(t)$ on $[0, T]$ and a function $u(\xi)$ on the set $Q_{s(\xi)}(T) := \{(t, x) : 0 < t < T, s(\xi)(t) < x < L\}$, $T > 0$, such that

$$\rho_v u_t(\xi) - k u_{xx}(\xi) = 0 \text{ for } t \in [0, T] \text{ and } x \in (s(\xi)(t), L), \quad (6.1)$$

$$u(\xi)(t, L) = h(t, \xi) \text{ for } t \in (0, T), \quad (6.2)$$

$$k u_x(\xi)(t, s(\xi)(t)) = (\rho_w - \rho_v u(\xi)(t, s(\xi)(t))) s_t(\xi)(t) \text{ for } t \in (0, T), \quad (6.3)$$

$$s_t(\xi)(t) = a(u(\xi)(t, s(\xi)(t)) - \varphi(s(\xi)(t))) \text{ for } t \in (0, T), \quad (6.4)$$

$$s(\xi)(0) = s_0(\xi), \quad (6.5)$$

$$u(\xi)(0, x) = u_0(\xi, x) \text{ for } x \in [s_0(\xi), L], \quad (6.6)$$

where h , s_0 and u_0 are given functions on $Q_\Omega := \Omega \times (0, T)$, Ω and $\Omega(s_0) := \{(\xi, x) : \xi \in \Omega, s_0(\xi) < x < L\}$, respectively.

Here, we provide assumptions for h , s_0 and u_0 .

(A3') $h(= h(t, \xi)) \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$ and $h_t \in L^\infty((0, T) \times \Omega) \cap L^2(0, T; H^1(\Omega))$ with $0 \leq h(t, \xi) \leq h^*$ for $t \in (0, T)$ and $\xi \in \Omega$, where h^* is a positive constant satisfying $h^* < \varphi(L)$.

(A5') $s_0(= s_0(\xi)) \in H^1(\Omega)$ with $0 \leq s_0(\xi) \leq L - \ell_0$ for any $\xi \in \Omega$, where ℓ_0 is a positive constant, and $u_0(= u_0(\xi, x)) \in L^\infty(\Omega; W^{1,2}(0, L))$, $\frac{\partial u_0}{\partial \xi_i} \in L^2(\Omega; L^2(0, L))$ for $i = 1, 2, 3$, $u_0(\xi)(L) = h(0, \xi)$ and $0 \leq u_0(\xi) \leq 1$ on $[s_0(\xi), L]$ for $\xi \in \Omega$.

Here, we give estimates for the partial derivative of s as a corollary of Theorem 2.

Corollary 1. *If (A1), (A2), (A4), (A3') and (A5') hold, then for a.e. $\xi = (\xi_1, \xi_2, \xi_3) \in \Omega$ there exists a solution $(s(\xi), u(\xi))$ of $P(\xi)$ on $[0, T]$ such that s is differentiable with respect*

to ξ_i for each $i = 1, 2, 3$. Moreover, there exists a positive constant C_* such that for each $i = 1, 2, 3$

$$\left| \frac{\partial s}{\partial \xi_i} \right|_{W^{1,2}(0,T)} \leq C_* \left(\left| \frac{\partial s_0}{\partial \xi_i} \right| + \left| \frac{\partial u_0}{\partial \xi_i} \right|_{L^2(s_0(\xi),L)} + \left| \frac{\partial h}{\partial \xi_i} \right|_{W^{1,2}(0,T)} \right) \text{ a.e. on } \Omega, \quad (6.7)$$

$$\int_{\Omega} \left| \frac{\partial s}{\partial \xi_i} \right|_{W^{1,2}(0,T)}^2 d\xi \leq C_* \int_{\Omega} \left(\left| \frac{\partial s_0}{\partial \xi_i} \right|^2 + \left| \frac{\partial u_0}{\partial \xi_i} \right|_{L^2(s_0(\xi),L)}^2 + \left| \frac{\partial h}{\partial \xi_i} \right|_{W^{1,2}(0,T)}^2 \right) d\xi. \quad (6.8)$$

Proof. By (A3') and (A5') for a.e. $\xi = (\xi_1, \xi_2, \xi_3) \in \Omega$ there exists an open interval (a_1, b_1) such that $(a_1, b_1) \times \{\xi_2\} \times \{\xi_3\} \subset \Omega$, and $\hat{h}(\xi_1, t) := h(t, \xi_1, \xi_2, \xi_3)$ and $\hat{u}_0(\xi_1, x) := u_0(\xi_1, \xi_2, \xi_3, x)$ satisfy

$$\begin{aligned} \hat{h} &\in L^\infty(a_1, b_1; W^{1,2}(0, T)), \frac{\partial \hat{h}}{\partial \xi_1} \in L^2(a_1, b_1; L^2(0, T)), \frac{\partial^2 \hat{h}}{\partial \xi_1 \partial t} \in L^2(a_1, b_1; L^2(0, T)), \\ s_0 &\in H^1(a_1, b_1), \hat{u}_0 \in L^\infty(a_1, b_1; W^{1,2}(0, L)), \frac{\partial \hat{u}_0}{\partial \xi_1} \in L^2(a_1, b_1; L^2(0, L)). \end{aligned}$$

Namely, all assumptions of Theorem 2 hold. Hence, Theorem 2 implies (6.7). By integrating it, we get (6.8) for $i = 1$. Similarly, we can prove (6.7) and (6.8) for $i = 2, 3$. $\square$

References

- [1] T. Aiki and K. Kumazaki, Mathematical model for hysteresis phenomenon in moisture transport in concrete carbonation process, *Phys. B*, **407** (2012), 1424–1426.
- [2] T. Aiki and K. Kumazaki, Mathematical modelling of concrete carbonation process with hysteresis effect, *RIMS, Kyoto Univ., sūrikaiseikikenkyūsho, kōkyūroku*, **1792** (2012), 99–107.
- [3] T. Aiki and K. Kumazaki, Uniqueness of solutions to a mathematical model describing moisture transport in concrete materials, *Netw. Heterogeneous media*, **9** (2014), no. 4,
- [4] T. Aiki and K. Kumazaki, Well-posedness of a mathematical model for moisture transport appearing in concrete carbonation process, *Adv. Math. Sci. Appl.*, **21** (2011), 361–381.
- [5] T. Aiki and Y. Murase, On a large time behavior of a solution to a one-dimensional free boundary problem for adsorption phenomena, *J. Math. Anal. Appl.*, **445** (2017), 837–854.
- [6] T. Aiki, Y. Murase, N. Sato and K. Kumazaki, A two scale model for concrete carbonation process in a three dimensional domain, *RIMS, Kyoto Univ., sūrikaiseikikenkyūsho, kōkyūroku*, **1997** (2016), 133–139.
- [7] T. Aiki, Y. Murase, N. Sato and K. Shirakawa, A mathematical model for a hysteresis appearing in adsorption phenomena, *RIMS, Kyoto Univ., sūrikaiseikikenkyūsho, kōkyūroku*, **1856** (2013), 1–11.

- [8] T. Aiki and N. Sato, Existence of periodic solution of one dimensional free boundary problem for adsorption phenomena, *Bulletin of Irkutsk State University, Series Mathematics*, **25** (2018), 3–18.
- [9] K. Kumazaki, Continuous dependence of a solution of a free boundary problem describing adsorption phenomenon for a given data, *Adv. Math. Sci. Appl.*, **25** (2016), no. 1, 283–299.
- [10] K. Kumazaki, Measurability of a solution of a free boundary problem describing adsorption phenomenon, *Adv. Math. Sci. Appl.*, **26** (2017), 19–27.
- [11] K. Kumazaki, Global existence of a solution for a multiscale model describing moisture transport in concrete materials, *Izv. Irkutsk. Gos. Univ. Ser. Mat.*, **28** (2019), 69–84.
- [12] K. Kumazaki, T. Aiki, N. Sato and Y. Murase, Multiscale model for moisture transport with adsorption phenomenon in concrete materials, *Appl. Anal.*, **97**(2018), 41–54.
- [13] O. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'ceva *Linear and Quasi-linear Equations of Parabolic Type*, Transl. Math. Monograph 23, Amer. Math. Soc., Providence R. I., 1968.
- [14] K. Maekawa, R. Chaube, and T. Kishi. *Modeling of concrete carbonation*. Taylor and Francis, 1999.
- [15] K. Maekawa, T. Ishida, and T. Kishi. Multi-scale modeling of concrete performance. *J. Adv. Concrete Technol.*, **1**(2003), 91–126.
- [16] N. Sato, T. Aiki, Y. Murase and K. Shirakawa, A one dimensional free boundary problem for adsorption phenomena, *Netw. Heterogeneous media.*, **9** (2014), no. 4, 655–668.