Advances in Mathematical
Sciences and Applications
Vol.29, No.1 (2020), pp.211-229

GAKKOTOSHO
TOKYO JAPAN

ON AN ABSTRACT TRANSPORT EQUATION AND SOME OF ITS APPLICATIONS IN CELL POPULATION DYNAMICS

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Abstract. This work is devoted to an abstract mathematical model involving the transport equation. We prove that this model is governed by a strongly continuous semigroup and we give some of its applications arising from population dynamics.

Communicated by N. Kenmochi; Received May 6, 2020.
This work is supported by LMCM-RSA.
Key words : Boundary Value Problems, Semigroups, Population Dynamic, Transport Equation.
AMS Subject Classification : 47D06, 92D25, 82D75

1 Introduction

It is well known that the transport equation is involved in many mathematical models describing biological or physical phenomena, for instance: growing cell populations (e.g. cells, bacteria) and transport of particles (e.g. neutrons, photons). The differences between these phenomena are mainly due to their dynamics.

This work deals with an abstract mathematical model involving the first derivative of an unknown density $f(t) = f(t, x, y)$. We suppose that this model is governed by the following partial differential equation

$$\frac{\partial f}{\partial t} = -\mathrm{h}(y)\frac{\partial f}{\partial x} + \mathrm{P}f \qquad (x, y) \in \Omega, \quad t \geqslant 0 \tag{1}$$

and equipped with the following boundary condition

$$f\big(t, \alpha(y), y\big) = \mathrm{K}\Big(f\big(t, \beta(\cdot), \cdot\big)\Big)(y), \qquad y \in (\mathrm{a}, \mathrm{b}), \tag{2}$$

where h, α and β are given functions of the variable y, and K and P are linear operators defined on suitable spaces (see the next section for more details).

The full model (1)–(2) appears as a linear perturbation of the unperturbed model (1)–(2)–(with P $= 0$). We know that if the unperturbed model is well–posed then the full model is well–posed too whenever P is a bounded linear operator. Accordingly, our attention will be focused only on the unperturbed model.

In Section 2, we set suitable assumptions on the functions α, β and h, and we define the spaces and mappings to investigate the unperturbed model (1)–(2)–(with P $= 0$). In particular, in order to give a sense to the boundary condition (2), we prove that the traces, at $\alpha(\cdot)$ and $\beta(\cdot)$, are continuous mappings on suitable Banach spaces (see Lemma 1).

In Section 3, we prove that the unperturbed model (1)–(2)–(with P $= 0$) is governed by a strongly continuous semigroup. We also give its growth inequality. Finally, according to the boundedness of the linear operator P, the well–posedness of the full model (1)–(2) can be inferred as a simple corollary.

Section 4 and Section 5 are devoted to some applications arising from population dynamic. Section 6 is devoted to transport equation in slab geometry. We refer the reader to [7] for the mathematical background used in this work.

In our knowledge, this study is new and has never been investigated.

2 Mathematical Background

We consider, until the end of this work a and b such that

$$-\infty \leqslant \mathrm{a} < \mathrm{b} \leqslant \infty.$$

Let α and β be two continuous functions defined on (a,b) and subject to following assumptions

$$\left(\mathcal{A}^1_{\alpha,\beta}\right) \ : \qquad \beta(y) > \alpha(y) \geqslant 0 \qquad \text{for all} \ \ y \in (\mathrm{a},\mathrm{b})$$

$$\left(\mathcal{A}^2_{\alpha,\beta}\right) \ : \qquad \beta(y) - \alpha(y) \geqslant \mathrm{m} > 0 \qquad \text{for all} \ \ y \in (\mathrm{a},\mathrm{b}).$$

Whenever the assumption $\left(\mathcal{A}^1_{\alpha,\beta}\right)$ is fulfilled, we set

$$\Omega := \Big\{(x,y) \in (0,\infty) \times \mathbb{R} \quad : \quad \alpha(y) < x < \beta(y) \qquad \text{and} \qquad \mathrm{a} < y < \mathrm{b}\Big\}$$

and we consider the following Banach space

$$\mathrm{X}_p := \mathrm{L}^p(\Omega) \ (p \geqslant 1) \quad \text{whose norm is} \quad \left\|\varphi\right\|_p := \left[\int_\Omega |\varphi(x,y)|^p \,\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{p}}.$$

Let h be a continuous function defined on (a,b) and subject to the following assumption

$$(\mathcal{A}_\mathrm{h}) \ : \qquad 0 < \mathrm{h}(y) \leqslant \mathrm{M} < \infty \qquad \text{for all} \ \ y \in (\mathrm{a},\mathrm{b}).$$

Whenever $(\mathcal{A}_\mathrm{h})$ is fulfilled, we consider the following Banach spaces

$$\mathrm{W}_p := \left\{\varphi \in \mathrm{X}_p \quad : \quad \mathrm{h}\frac{\partial\varphi}{\partial x} \in \mathrm{X}_p\right\} \quad \text{with} \quad \left\|\varphi\right\|_{\mathrm{W}_p} := \left[\left\|\varphi\right\|_p + \left\|\mathrm{h}\frac{\partial\varphi}{\partial x}\right\|_p\right]^{\frac{1}{p}}$$

and

$$\mathrm{Y}_p := \mathrm{L}^p\Big((\mathrm{a},\mathrm{b})\,,\mathrm{h}\Big) \quad \text{with} \quad \left\|\psi\right\|_{\mathrm{Y}_p} := \left[\int_\mathrm{a}^\mathrm{b} |\psi(y)|^p \,\mathrm{h}(y)\mathrm{d}y\right]^{\frac{1}{p}}.$$

Now we can give a sense to trace mappings involving in the boundary condition (2) as follows

Lemma 1. *Suppose that assumptions* $\left(\mathcal{A}^1_{\alpha,\beta}\right)$ *and* $\left(\mathcal{A}^2_{\alpha,\beta}\right)$, *and* $(\mathcal{A}_\mathrm{h})$ *hold true. The following trace mappings*

$$\Phi_\alpha\varphi(y) := \varphi\big(\alpha(y),y\big) \qquad \text{and} \qquad \Phi_\beta\varphi(y) := \varphi\big(\beta(y),y\big), \qquad y \in (\mathrm{a},\mathrm{b}),$$

are linear and continuous from W_p $(p \geqslant 1)$ *into* Y_p. *More precisely,*

$$\left\|\Phi_\alpha\varphi\right\|_{\mathrm{Y}_p} \leqslant \max\left\{\left(\frac{\mathrm{M}}{\mathrm{m}} + p - 1\right)^{\frac{1}{p}}, 1\right\}\left\|\varphi\right\|_{\mathrm{W}_p} \tag{3}$$

and

$$\left\|\Phi_\beta\varphi\right\|_{\mathrm{Y}_p} \leqslant \max\left\{\left(\frac{\mathrm{M}}{\mathrm{m}} + p - 1\right)^{\frac{1}{p}}, 1\right\}\left\|\varphi\right\|_{\mathrm{W}_p} \tag{4}$$

for all $\varphi \in \mathrm{W}_p$ $(p \geqslant 1)$.

Proof. Suppose that assumptions $\left(\mathcal{A}^1_{\alpha,\beta}\right)$ and $\left(\mathcal{A}^2_{\alpha,\beta}\right)$, and $(\mathcal{A}_{\mathrm{h}})$ hold true.

STEP 1 (Φ_α is a continuous mapping).

Let $\varphi \in \mathrm{W}_p$ $(p \geqslant 1)$. For almost all $(x, y) \in \Omega$ we can write that

$$\begin{aligned}|\Phi_\alpha \varphi(y)|^p &= |\varphi(x,y)|^p - \int_{\alpha(y)}^{x} \frac{\partial |\varphi|^p}{\partial x}(s,y)\mathrm{d}s \\ &= |\varphi(x,y)|^p - p\int_{\alpha(y)}^{x} (\operatorname{sgn}\varphi)(s,y)\,|\varphi(s,y)|^{p-1}\frac{\partial \varphi}{\partial x}(s,y)\mathrm{d}s \\ &\leqslant |\varphi(x,y)|^p + p\int_{\alpha(y)}^{\beta(y)} |\varphi(s,y)|^{p-1}\left|\frac{\partial \varphi}{\partial x}(s,y)\right|\mathrm{d}s\end{aligned}$$

and therefore

$$\mathrm{h}(y)\,|\Phi_\alpha \varphi(y)|^p \leqslant \mathrm{h}(y)\,|\varphi(x,y)|^p + p\int_{\alpha(y)}^{\beta(y)} |\varphi(s,y)|^{p-1}\left|\mathrm{h}(y)\frac{\partial \varphi}{\partial x}(s,y)\right|\mathrm{d}s. \tag{5}$$

Integrating (5) with respect to x and then with respect to y, we get that

$$\|\Phi_\alpha \varphi\|^p_{\mathrm{Y}_p} \leqslant \frac{\mathrm{M}}{\mathrm{m}}\|\varphi\|^p_p + p\int_\Omega |\varphi(s,y)|^{p-1}\left|\mathrm{h}(y)\frac{\partial \varphi}{\partial x}(s,y)\right|\mathrm{d}s\mathrm{d}y. \tag{6}$$

Now, if $p = 1$ then (6) becomes

$$\|\Phi_\alpha \varphi\|_{\mathrm{Y}_1} \leqslant \frac{\mathrm{M}}{\mathrm{m}}\|\varphi\|_1 + \left\|\mathrm{h}\frac{\partial \varphi}{\partial x}\right\|_1 \leqslant \max\left\{\frac{\mathrm{M}}{\mathrm{m}}, 1\right\}\|\varphi\|_{\mathrm{W}_1}$$

which proves (3)–(with $p = 1$) and by the way Φ_α is a continuous mapping from W_1 into Y_1.

Next. Suppose that $p > 1$ and let $q > 1$ be its conjugate. The Young inequality yields that

$$\begin{aligned}&\int_\Omega |\varphi(s,y)|^{p-1}\left|\mathrm{h}(y)\frac{\partial \varphi}{\partial x}(s,y)\right|\mathrm{d}s\mathrm{d}y \\ &\qquad\leqslant \frac{1}{q}\int_\Omega |\varphi(s,y)|^{q(p-1)}\,\mathrm{d}s\mathrm{d}y + \frac{1}{p}\int_\Omega \left|\mathrm{h}(y)\frac{\partial \varphi}{\partial x}(s,y)\right|^p\mathrm{d}s\mathrm{d}y \\ &\qquad= \frac{p-1}{p}\int_\Omega |\varphi(s,y)|^p\,\mathrm{d}s\mathrm{d}y + \frac{1}{p}\int_\Omega \left|\mathrm{h}(y)\frac{\partial \varphi}{\partial x}(s,y)\right|^p\mathrm{d}s\mathrm{d}y\end{aligned}$$

which can be written as

$$\int_\Omega |\varphi(s,y)|^{p-1}\left|\mathrm{h}(y)\frac{\partial \varphi}{\partial x}(s,y)\right|\mathrm{d}s\mathrm{d}y \leqslant \frac{p-1}{p}\|\varphi\|^p_p + \frac{1}{p}\left\|\mathrm{h}\frac{\partial \varphi}{\partial x}\right\|^p_p. \tag{7}$$

Combining now (6) and (7) we get that

$$\begin{aligned}\|\Phi_\alpha \varphi\|^p_{\mathrm{Y}_p} &\leqslant \left(\frac{\mathrm{M}}{\mathrm{m}} + p - 1\right)\|\varphi\|^p_p + \left\|\mathrm{h}\frac{\partial \varphi}{\partial x}\right\|^p_p \\ &\leqslant \max\left\{\left(\frac{\mathrm{M}}{\mathrm{m}} + p - 1\right), 1\right\}\|\varphi\|^p_{\mathrm{W}_p}\end{aligned}$$

which proves (3)–(with $p > 1$) and by the way Φ_α is a continuous mapping from W_p $(p > 1)$ into Y_p.

Step 2 (Φ_β is a continuous mapping).

Let $\varphi \in \mathrm{W}_p$ $(p \geqslant 1)$. For almost all $(x, y) \in \Omega$ we can write that

$$\begin{aligned} \left|\Phi_\beta\varphi(y)\right|^p &= |\varphi(x,y)|^p + \int_x^{\beta(y)} \frac{\partial\,|\varphi|^p}{\partial x}(s,y)\mathrm{d}s \\ &= |\varphi(x,y)|^p + p\int_x^{\beta(y)} (\operatorname{sgn}\varphi)\,(s,y)\,|\varphi(s,y)|^{p-1}\,\frac{\partial\varphi}{\partial x}(s,y)\mathrm{d}s \\ &\leqslant |\varphi(x,y)|^p + p\int_{\alpha(y)}^{\beta(y)} |\varphi(s,y)|^{p-1}\left|\frac{\partial\varphi}{\partial x}(s,y)\right|\mathrm{d}s \end{aligned}$$

and therefore

$$\mathrm{h}(y)\left|\Phi_\beta\varphi(y)\right|^p \leqslant \mathrm{h}(y)\,|\varphi(x,y)|^p + p\int_{\alpha(y)}^{\beta(y)} |\varphi(s,y)|^{p-1}\left|\mathrm{h}(y)\frac{\partial\varphi}{\partial x}(s,y)\right|\mathrm{d}s. \tag{8}$$

The rest of the proof follows from the step 1 because (5) and (8) have the same right hand side. □

3 The Well Posedness of the Model (1)–(2)

This section is devoted to the well posedness of the full model (1)–(2) in the sense of the semigroup theory. Let then A_K be the unbounded linear operator associated to the unperturbed model (1)–(2)–(with $\mathrm{P} = 0$), that is to say that,

$$\mathrm{A}_\mathrm{K}\varphi := -\mathrm{h}\frac{\partial\varphi}{\partial x} \tag{9}$$

on the domain

$$\mathrm{D}_\mathrm{K} := \left\{\varphi \in \mathrm{W}_p \qquad : \qquad \Phi_\alpha\varphi = \mathrm{K}\Phi_\beta\varphi\right\} \tag{10}$$

where K is a given linear operator acting from Y_p $(p \geqslant 1)$ into itself.

Due to Lemma 1, the unbounded linear operator $(\mathrm{A}_\mathrm{K}, \mathrm{D}_\mathrm{K})$ is well defined whenever K is bounded from Y_p $(p \geqslant 1)$ into itself.

Before we state that $(\mathrm{A}_\mathrm{K}, \mathrm{D}_\mathrm{K})$ is an infinitesimal generator, we need to prove two propositions. The first one can be announced as follows

Proposition 1. *Suppose that assumptions* $\left(\mathcal{A}^1_{\alpha,\beta}\right)$ *and* $\left(\mathcal{A}^2_{\alpha,\beta}\right)$, *and* $(\mathcal{A}_\mathrm{h})$ *hold true. Suppose also that* K *is a bounded linear operator from* Y_p $(p \geqslant 1)$ *into itself. Then the following assertions hold true.*

1. *The resolvent set of* A_K *contains a half line, namely,*

$$\left(\frac{\mathrm{M}}{\mathrm{m}}\ln\mathrm{k}\,,\ \infty\right) \subset \rho\left(\mathrm{A}_\mathrm{K}\right) \tag{11}$$

where

$$\mathrm{k} := \max\left\{1, \|\mathrm{K}\|\right\}. \tag{12}$$

2. A_{K} *is closed and densely defined.*

3. *If* K *is positive, then so is* $(\lambda - \mathrm{A}_{\mathrm{K}})^{-1}$ *for all* $\lambda > \frac{\mathrm{M}}{\mathrm{m}} \ln \mathrm{k}$

Proof. For convenience, we divide the proof in several steps.

Step I (First Auxiliary Operator).

Let $\lambda > 0$ and $g \in \mathrm{X}_p$ $(p \geqslant 1)$. Let L_λ be the following linear operator

$$\mathrm{L}_\lambda g(x,y) = \frac{1}{\mathrm{h}(y)} \int_{\alpha(y)}^{x} e^{-\lambda \frac{(x-z)}{\mathrm{h}(y)}} g(z,y) \mathrm{d}z \qquad (x,y) \in \Omega \tag{13}$$

which is obviously a positive operator. Accordingly,

$$\begin{aligned} \|\mathrm{L}_\lambda g\|_p^p &= \int_\Omega \left| \frac{1}{\mathrm{h}(y)} \int_{\alpha(y)}^{x} e^{-\lambda \frac{(x-z)}{\mathrm{h}(y)}} g(z,y) \mathrm{d}z \right|^p \mathrm{d}x \mathrm{d}y \\ &\leqslant \int_\Omega \left[\frac{1}{\mathrm{h}(y)} \int_{\alpha(y)}^{x} e^{-\lambda \frac{(x-z)}{\mathrm{h}(y)}} |g(z,y)| \, \mathrm{d}z \right]^p \mathrm{d}x \mathrm{d}y \\ &= \|\mathrm{L}_\lambda \, |g|\|_p^p \end{aligned}$$

and therefore

$$\|\mathrm{L}_\lambda g\|_p \leqslant \|\mathrm{L}_\lambda \, |g|\|_p. \tag{14}$$

Next, writing that

$$\|\mathrm{L}_\lambda \, |g|\|_p^p = \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{g}_\lambda(y) \mathrm{d}y \tag{15}$$

where

$$\mathrm{g}_\lambda(y) := \int_{\alpha(y)}^{\beta(y)} \left[e^{-\frac{p\lambda x}{\mathrm{h}(y)}} \right] \left[\frac{1}{\mathrm{h}(y)} \int_{\alpha(y)}^{x} e^{\frac{\lambda z}{\mathrm{h}(y)}} |g(z,y)| \, \mathrm{d}z \right]^p \mathrm{d}x, \qquad y \in (\mathrm{a}, \mathrm{b}).$$

Integrating by parts we get, for all $y \in (\mathrm{a}, \mathrm{b})$, that

$$\begin{aligned} \mathrm{g}_\lambda(y) &= \left[-\frac{\mathrm{h}(y)}{p\lambda} e^{-\frac{p\lambda\beta(y)}{\mathrm{h}(y)}} \right] \left[\frac{1}{\mathrm{h}(y)} \int_{\alpha(y)}^{\beta(y)} e^{\frac{\lambda z}{\mathrm{h}(y)}} |g(z,y)| \, \mathrm{d}z \right]^p \\ &\quad + \frac{1}{\lambda} \int_{\alpha(y)}^{\beta(y)} e^{-\frac{(p-1)\lambda x}{\mathrm{h}(y)}} |g(x,y)| \left[\frac{1}{\mathrm{h}(y)} \int_{\alpha(y)}^{x} e^{\lambda \frac{z}{\mathrm{h}(y)}} |g(z,y)| \, \mathrm{d}z \right]^{p-1} \mathrm{d}x \\ &\leqslant \frac{1}{\lambda} \int_{\alpha(y)}^{\beta(y)} |g(x,y)| \left[\frac{1}{\mathrm{h}(y)} \int_{\alpha(y)}^{x} e^{-\lambda \frac{(x-z)}{\mathrm{h}(y)}} |g(z,y)| \, \mathrm{d}z \right]^{p-1} \mathrm{d}x \end{aligned}$$

which can be written as

$$\mathrm{g}_\lambda(y) \leqslant \frac{1}{\lambda} \int_{\alpha(y)}^{\beta(y)} |g(x,y)| \left[\mathrm{L}_\lambda \, |g| \, \right]^{p-1} \mathrm{d}x$$

and therefore (15) becomes

$$\|\mathrm{L}_\lambda \, |g|\|_p^p \leqslant \frac{1}{\lambda} \int_\Omega |g(x,y)| \left[\mathrm{L}_\lambda \, |g| \, \right]^{p-1} \mathrm{d}x \mathrm{d}y. \tag{16}$$

If $p = 1$ then (16) yields that

$$\|\mathrm{L}_\lambda |g|\|_1 \leqslant \frac{1}{\lambda}\int_\Omega |g(x,y)|\,\mathrm{d}x\mathrm{d}y = \frac{1}{\lambda}\|g\|_1. \tag{17}$$

However, if $p > 1$ then Hölder inequality (with $\frac{1}{p}+\frac{1}{q}=1$) applied to (16) yields that

$$\begin{aligned}\|\mathrm{L}_\lambda |g|\|_p^p &\leqslant \frac{1}{\lambda}\left[\int_\Omega |g(x,y)|^p\,\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{p}}\left[\int_\Omega \left[\mathrm{L}_\lambda |g|\,\right]^{q(p-1)}\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{q}} \\ &= \frac{1}{\lambda}\left[\int_\Omega |g(x,y)|^p\,\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{p}}\left[\int_\Omega \left[\mathrm{L}_\lambda |g|\,\right]^{p}\mathrm{d}x\mathrm{d}y\right]^{\frac{p-1}{p}} \\ &= \frac{1}{\lambda}\|g\|_p\|\mathrm{L}_\lambda |g|\|_p^{p-1}\end{aligned}$$

and therefore

$$\|\mathrm{L}_\lambda |g|\|_p \leqslant \frac{1}{\lambda}\|g\|_p. \tag{18}$$

Finally, combining (14) together with (17)–(for $p = 1$) and (18)–(for $p > 1$) we get, for all $g \in \mathrm{X}_p$ ($p \geqslant 1$), that

$$\|\mathrm{L}_\lambda g\|_p \leqslant \frac{1}{\lambda}\|g\|_p \tag{19}$$

and therefore L_λ is a bounded linear operator from X_p ($p \geqslant 1$) into itself.

Step II (Second Auxiliary Operator).

Let $\lambda \geqslant 0$ and let K_λ be the following linear operator

$$\mathrm{K}_\lambda\psi := \mathrm{K}\left(e^{-\lambda\frac{(\beta-\alpha)}{\mathrm{h}}}\psi\right). \tag{20}$$

For all $\psi \in \mathrm{Y}_p$ ($p \geqslant 1$) we have

$$\|\mathrm{K}_\lambda\psi\|_{\mathrm{Y}_p} = \left\|\mathrm{K}\left(e^{-\lambda\frac{(\beta-\alpha)}{\mathrm{h}}}\psi\right)\right\|_{\mathrm{Y}_p} \leqslant \|\mathrm{K}\|\,e^{-\lambda\frac{\mathrm{m}}{\mathrm{M}}}\,\|\psi\|_{\mathrm{Y}_p}$$

which proves that K_λ is a bounded linear operator from Y_p ($p \geqslant 1$) into itself and by the way

$$\|\mathrm{K}_\lambda\| \leqslant \|\mathrm{K}\|\,e^{-\lambda\frac{\mathrm{m}}{\mathrm{M}}}.$$

So, if $\|\mathrm{K}\| \leqslant 1$ then $\|\mathrm{K}_\lambda\| < 1$ for all $\lambda > 0$. However, if $\|\mathrm{K}\| \geqslant 1$ then $\|\mathrm{K}_\lambda\| < 1$ for all $\lambda > \frac{\mathrm{M}}{\mathrm{m}}\ln\|\mathrm{K}\|$. Accordingly, in both cases we can write that

$$\|\mathrm{K}_\lambda\| < 1 \qquad \text{for all} \qquad \lambda > \frac{\mathrm{M}}{\mathrm{m}}\ln\mathrm{k}. \tag{21}$$

Step III (Proof of Point 1).

Let $\lambda > \frac{\mathrm{M}}{\mathrm{m}}\ln\mathrm{k}$ and let $g \in \mathrm{X}_p$ ($p \geqslant 1$). Solving $(\lambda - \mathrm{A}_\mathrm{K})\varphi = g$ means to find φ fulfilling

$$\lambda\varphi = -\mathrm{h}\frac{\partial\varphi}{\partial x} + g \tag{22}$$

and

$$\varphi \in \mathrm{W}_p \tag{23}$$

and

$$\Phi_\alpha \varphi = \mathrm{K}\Phi_\beta \varphi. \tag{24}$$

Firstly, easy computations show that (22) admits the following solution

$$\varphi(x,y) = e^{-\frac{\lambda x}{\mathrm{h}(y)}} \psi(y) + \mathrm{L}_\lambda g(x,y) \qquad (x,y) \in \Omega \tag{25}$$

where ψ is an arbitrary function and L_λ is defined by (13).

Next, let $\psi \in \mathrm{Y}_p$ $(p \geqslant 1)$. Integrating (25) and then using (18) we obtain

$$\begin{aligned}
\|\varphi\|_p &\leqslant \left[\int_{\mathrm{a}}^{\mathrm{b}} \left(\int_{\alpha(y)}^{\beta(y)} e^{-p\frac{\lambda x}{\mathrm{h}(y)}} \mathrm{d}x\right) |\psi(y)|^p \,\mathrm{d}y\right]^{\frac{1}{p}} + \|\mathrm{L}_\lambda g\|_p \\
&= \left[\int_{\mathrm{a}}^{\mathrm{b}} \frac{\mathrm{h}(y)}{p\lambda} \left(e^{-p\lambda\frac{\alpha(y)}{\mathrm{h}(y)}} - e^{-p\lambda\frac{\beta(y)}{\mathrm{h}(y)}}\right) |\psi(y)|^p \,\mathrm{d}y\right]^{\frac{1}{p}} + \|\mathrm{L}_\lambda g\|_p \\
&\leqslant \left[\frac{1}{p\lambda} \int_{\mathrm{a}}^{\mathrm{b}} |\psi(y)|^p \,\mathrm{h}(y)\mathrm{d}y\right]^{\frac{1}{p}} + \frac{1}{\lambda}\|g\|_p \\
&= \frac{1}{(p\lambda)^{\frac{1}{p}}} \|\psi\|_{\mathrm{Y}_p} + \frac{1}{\lambda}\|g\|_p
\end{aligned}$$

which implies that $\varphi \in \mathrm{X}_p$ and by (22),

$$\left\|\mathrm{h}\frac{\partial \varphi}{\partial x}\right\|_p = \|-\lambda\varphi + g\|_p \leqslant \lambda\|\varphi\|_p + \|g\|_p < \infty$$

and theretofore $\varphi \in \mathrm{W}_p$.

Finally, φ fulfills (24) if ψ is solution of

$$\psi = \mathrm{K}_\lambda \psi + \mathrm{K}\Phi_\beta \left(\mathrm{L}_\lambda g\right)$$

where K_λ is defined by (20). Using (21) we get that

$$\psi = \left(\mathrm{I}_{\mathrm{Y_p}} - \mathrm{K}_\lambda\right)^{-1} \mathrm{K}\Phi_\beta \mathrm{L}_\lambda g \in \mathrm{Y}_p$$

which we put in (25) to obtain

$$\varphi(x,y) = e^{-\frac{\lambda x}{\mathrm{h}(y)}} \left(\mathrm{I}_{\mathrm{Y_p}} - \mathrm{K}_\lambda\right)^{-1} \mathrm{K}\Phi_\beta \mathrm{L}_\lambda g(y) + \mathrm{L}_\lambda g(x,y), \qquad (x,y) \in \Omega. \tag{26}$$

Now we can say that (26) is the unique solution of $(\lambda - \mathrm{A}_\mathrm{K})\varphi = g$ which leads to $\lambda \in \rho\left(\mathrm{A}_\mathrm{K}\right)$ and proves (11).

Step IV (Proof of Point 2).

Let $\lambda > \frac{\mathrm{M}}{\mathrm{m}} \ln \mathrm{k}$. Then (11) yields that $(\lambda - \mathrm{A_K})^{-1}$ is a bounded linear operator from X_p $(p \geqslant 1)$ into itself which implies that $(\lambda - \mathrm{A_K})$ is a closed operator and therefore $\mathrm{A_K} = \lambda - (\lambda - \mathrm{A_K})$ is closed too.

$\mathrm{A_K}$ is densely defined because $\mathcal{C}_c(\Omega) \subset \mathrm{D_K} \subset \mathrm{X}_p$ $(p \geqslant 1)$ ($\mathcal{C}_c(\Omega)$ is the subspace of all continuous functions with compact support in Ω).

Step IV (Proof of Point 3).

Let $\lambda > \frac{\mathrm{M}}{\mathrm{m}} \ln \mathrm{k}$ and let $g \in \mathrm{X}_p$ $(p \geqslant 1)$ be such that $g \geqslant 0$. According to (11) and (26) we get that

$$(\lambda - \mathrm{A_K})^{-1} g(x,y) = e^{-\frac{\lambda x}{\mathrm{h}(y)}} \left(\mathrm{I}_{\mathrm{Y_p}} - \mathrm{K}_\lambda\right)^{-1} \mathrm{K}\Phi_\beta \mathrm{L}_\lambda g(y) + \mathrm{L}_\lambda g(x,y), \quad (x,y) \in \Omega$$

which leads, by virtue of (21), to

$$(\lambda - \mathrm{A_K})^{-1} g(x,y) = e^{-\frac{\lambda x}{\mathrm{h}(y)}} \sum_{n \geqslant 1} \mathrm{K}_\lambda^n \mathrm{K}\Phi_\beta \mathrm{L}_\lambda g(y) + \mathrm{L}_\lambda g(x,y) \quad (x,y) \in \Omega.$$

All the terms in the right hand side are positive and therefore $(\lambda - \mathrm{A_K})^{-1} g$ is a positive function. □

The second proposition can be announced as follows

Proposition 2. *Suppose that assumptions* $\left(\mathcal{A}^1_{\alpha,\beta}\right)$ *and* $\left(\mathcal{A}^2_{\alpha,\beta}\right)$, *and* $(\mathcal{A}_\mathrm{h})$ *hold true. Suppose also that* K *is a bounded linear operator from* Y_p $(p \geqslant 1)$ *into itself. Then*

$$\left\|(\lambda - \mathrm{A_K})^{-n} g\right\|_p \leqslant \frac{\mathrm{k}}{\left(\lambda - \frac{\mathrm{M}}{\mathrm{m}} \ln \mathrm{k}\right)^n} \|g\|_p, \qquad \lambda > \frac{\mathrm{M}}{\mathrm{m}} \ln \mathrm{k}, \qquad n = 1, 2, 3, \cdots$$

for all $g \in \mathrm{X}_p$ $(p \geqslant 1)$ *where* k *is defined by* (12).

Proof. For convenience, we divide the proof in several steps.

Step I. Let us define on X_p $(p \geqslant 1)$ the following norm

$$|||\varphi|||_p := \left[\int_\Omega |\varphi(x,y)|^p \, \mathrm{f}^p(x,y) \mathrm{d}x\mathrm{d}y\right]^{\frac{1}{p}}$$

where

$$\mathrm{f}(x,y) := \mathrm{k}^{\frac{x - \alpha(y)}{\beta(y) - \alpha(y)}} \qquad (x,y) \in \Omega.$$

Both norms $|||\varphi|||_p$ and $\|\cdot\|_p$ are equivalent because of

$$\|\varphi\|_p \leqslant |||\varphi|||_p \leqslant \mathrm{k}\|\varphi\|_p \tag{27}$$

for all $\varphi \in \mathrm{X}_p$ $(p \geqslant 1)$.

Step II. Let $\lambda > \frac{\mathrm{M}}{\mathrm{m}} \ln \mathrm{k}$ and let $g \in \mathrm{X}_p$ $(p \geqslant 1)$. According to (11) we can say that

$$\varphi := (\lambda - \mathrm{A_K})^{-1} g \tag{28}$$

is the unique solution of

$$\lambda\varphi = -\mathrm{h}\frac{\partial\varphi}{\partial x} + g \tag{29}$$

satisfying

$$\Phi_{\alpha}\varphi = \mathrm{K}\Phi_{\beta}\varphi. \tag{30}$$

So multiplying (29) by $((\operatorname{sgn}\varphi)\,|\varphi|^{p-1}\,\mathrm{f}^p)$ and then integrating it over Ω we get that

$$\begin{aligned}\lambda\|\!|\varphi\|\!|_p^p &= -\frac{1}{p}\int_\Omega \mathrm{h}(y)\frac{\partial\,|\varphi|^p}{\partial x}(x,y)\mathrm{f}^p(x,y)\mathrm{d}x\mathrm{d}y \\ &\qquad + \int_\Omega (\operatorname{sgn}\varphi)(x,y)\,|\varphi(x,y)|^{p-1}\,\mathrm{f}^p(x,y)g(x,y)\mathrm{d}x\mathrm{d}y \\ &:= \qquad \mathrm{I_p} \qquad + \qquad \mathrm{J_p}.\end{aligned} \tag{31}$$

Firstly, integrating $\mathrm{I_p}$ by parts,

$$\begin{aligned}\mathrm{I_p} &= -\frac{1}{p}\int_{\mathrm{a}}^{\mathrm{b}}\left[\int_{\alpha(y)}^{\beta(y)}\frac{\partial\,|\varphi|^p}{\partial x}(x,y)\mathrm{f}^p(x,y)\mathrm{d}x\right]\mathrm{h}(y)\mathrm{d}y \\ &= \frac{1}{p}\|\Phi_\alpha\varphi\|_{\mathrm{Y}_p}^p - \frac{1}{p}\mathrm{k}^p\left\|\Phi_\beta\varphi\right\|_{\mathrm{Y}_p}^p \\ &\qquad + \frac{1}{p}\,(p\ln\mathrm{k})\int_{\mathrm{a}}^{\mathrm{b}}\left[\int_{\alpha(y)}^{\beta(y)}|\varphi(x,y)|^p\,\mathrm{f}^p(x,y)\mathrm{d}x\right]\frac{\mathrm{h}(y)}{\beta(y)-\alpha(y)}\mathrm{d}y \\ &\leqslant \frac{1}{p}\|\Phi_\alpha\varphi\|_{\mathrm{Y}_p}^p - \frac{1}{p}\mathrm{k}^p\left\|\Phi_\beta\varphi\right\|_{\mathrm{Y}_p}^p + \left(\frac{\mathrm{M}}{\mathrm{m}}\ln\mathrm{k}\right)\|\!|\varphi\|\!|_p^p\end{aligned}$$

which leads, by virtue of (30), to

$$\begin{aligned}\mathrm{I_p} &\leqslant \frac{1}{p}\left\|\mathrm{K}\Phi_\beta\varphi\right\|_{\mathrm{Y}_p}^p - \frac{1}{p}\mathrm{k}^p\left\|\Phi_\beta\varphi\right\|_{\mathrm{Y}_p}^p + \left(\frac{\mathrm{M}}{\mathrm{m}}\ln\mathrm{k}\right)\|\!|\varphi\|\!|_p^p \\ &\leqslant \frac{1}{p}\left(\|\mathrm{K}\|^p - \mathrm{k}^p\right)\left\|\Phi_\beta\varphi\right\|_{\mathrm{Y}_p}^p + \left(\frac{\mathrm{M}}{\mathrm{m}}\ln\mathrm{k}\right)\|\!|\varphi\|\!|_p^p\end{aligned}$$

and therefore

$$\mathrm{I_p} \leqslant \left(\frac{\mathrm{M}}{\mathrm{m}}\ln\mathrm{k}\right)\|\!|\varphi\|\!|_p^p. \tag{32}$$

Next, for the term $\mathrm{J_p}$, Hölder inequality (with $\frac{1}{p}+\frac{1}{q}=1$) yields that

$$\begin{aligned}\mathrm{J_p} &\leqslant \int_\Omega |\varphi(x,y)|^{p-1}\,\mathrm{f}^p(x,y)\,|g(x,y)|\,\mathrm{d}x\mathrm{d}y \\ &= \int_\Omega \left(|\varphi(x,y)|\,\mathrm{f}(x,y)\right)^{p-1}\left(|g(x,y)|\,\mathrm{f}(x,y)\right)\mathrm{d}x\mathrm{d}y \\ &\leqslant \left[\int_\Omega \left(|\varphi(x,y)|\,\mathrm{f}(x,y)\right)^{q(p-1)}\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{q}}\left[\int_\Omega \left(|g(x,y)|\,\mathrm{f}(x,y)\right)^p\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{p}} \\ &= \left[\int_\Omega |\varphi(x,y)|^p\,\mathrm{f}^p(x,y)\mathrm{d}x\mathrm{d}y\right]^{\frac{p-1}{p}}\left[\int_\Omega |g(x,y)|^p\,\mathrm{f}^p(x,y)\mathrm{d}x\mathrm{d}y\right]^{\frac{1}{p}}\end{aligned}$$

and therefore

$$\mathrm{J_p} \leqslant \|\varphi\|_p^{p-1}\|g\|_p. \tag{33}$$

Combining now (31) together with (32) and (33) we obtain

$$\lambda\|\varphi\|_p^p \leqslant \left(\frac{\mathrm{M}}{\mathrm{m}}\ln \mathrm{k}\right)\|\varphi\|_p^p + \|\varphi\|_p^{p-1}\|g\|_p$$

which leads to

$$\left(\lambda - \frac{\mathrm{M}}{\mathrm{m}}\ln \mathrm{k}\right)\|\varphi\|_p \leqslant \|g\|_p$$

and by (28),

$$\left\|(\lambda - \mathrm{A_K})^{-1} g\right\|_p \leqslant \frac{1}{\left(\lambda - \frac{\mathrm{M}}{\mathrm{m}}\ln \mathrm{k}\right)}\|g\|_p.$$

By induction on the integer $n \geqslant 1$ we get that

$$\left\|(\lambda - \mathrm{A_K})^{-n} g\right\|_p \leqslant \frac{1}{\left(\lambda - \frac{\mathrm{M}}{\mathrm{m}}\ln \mathrm{k}\right)^n}\|g\|_p \qquad n = 1, 2, 3, \cdots$$

Finally (27) yields that

$$\left\|(\lambda - \mathrm{A_K})^{-n} g\right\|_p \leqslant \frac{\mathrm{k}}{\left(\lambda - \frac{\mathrm{M}}{\mathrm{m}}\ln \mathrm{k}\right)^n}\|g\|_p \qquad n = 1, 2, 3, \cdots$$

which ends the proof. □

Now the main result of this section can be formulated as follows

Theorem 1. *Suppose that assumptions* $\left(\mathcal{A}^1_{\alpha,\beta}\right)$ *and* $\left(\mathcal{A}^2_{\alpha,\beta}\right)$, *and* $(\mathcal{A}_\mathrm{h})$ *hold true. Suppose also that* K *is a bounded linear operator from* Y_p $(p \geqslant 1)$ *into itself. Then the unbounded linear operator* $(\mathrm{A_K}, \mathrm{D_K})$ *generates, on* X_p $(p \geqslant 1)$, *a strongly continuous semigroup* $\mathbb{A}_\mathrm{K} = (\mathbb{A}_\mathrm{K}(t))_{t\geqslant 0}$ *satisfying,*

$$\|\mathbb{A}_\mathrm{K}(t)\varphi\|_p \leqslant \mathrm{k}^{\left(1+\frac{\mathrm{M}}{\mathrm{m}}t\right)}\|\varphi\|_p \qquad t \geqslant 0 \tag{34}$$

for all $\varphi \in \mathrm{X}_p$ $(p \geqslant 1)$ *where* k *is defined by* (12). *Furthermore, if* K *is positive then so is the generated semigroup* $\mathbb{A}_\mathrm{K} = (\mathbb{A}_\mathrm{K}(t))_{t\geqslant 0}$.

Proof. Proposition 1 and Proposition 2 yield that all the required conditions of Hille–Yosida Theorem (see [7, Thm. II.3.8]) are fulfilled. Accordingly, $(\mathrm{A_K}, \mathrm{D_K})$ generates, on X_p $(p \geqslant 1)$, a strongly continuous semigroup $\mathbb{A}_\mathrm{K} = (\mathbb{A}_\mathrm{K}(t))_{t\geqslant 0}$ satisfying

$$\|\mathbb{A}_\mathrm{K}(t)\varphi\|_p \leqslant \mathrm{k}e^{t\left(\frac{\mathrm{M}}{\mathrm{m}}\ln \mathrm{k}\right)} = \mathrm{k}^{\left(1+\frac{\mathrm{M}}{\mathrm{m}}t\right)}\|\varphi\|_p \qquad t \geqslant 0$$

for all $\varphi \in \mathrm{X}_p$ $(p \geqslant 1)$. Finally, the positivity of $\mathbb{A}_\mathrm{K} = (\mathbb{A}_\mathrm{K}(t))_{t\geqslant 0}$ follows from that of $(\lambda - \mathrm{A_K})^{-1}$ (See Proposition 1 (3)) because of [5, Prop. 7.1]. □

About the full model (1)–(2), an immediate consequence of Theorem 1 can be formulated as follows

Corollary 1. *Suppose that assumptions* $\left(\mathcal{A}^1_{\alpha,\beta}\right)$ *and* $\left(\mathcal{A}^2_{\alpha,\beta}\right)$, *and* $(\mathcal{A}_\mathrm{h})$ *hold true. Suppose also that* K *is a bounded linear operator from* Y_p $(p \geqslant 1)$ *into itself. If* P *is a bounded linear operator from* X_p $(p \geqslant 1)$ *into itself then the full model* (1)–(2) *is well-posed in* X_p $(p \geqslant 1)$.

Proof. The full model (1)–(2) is governed by the unbounded linear operator $(\mathbb{A}_\mathrm{K} + \mathrm{P}, \mathrm{D}_\mathrm{K})$. The boundeness of P yields that $(\mathbb{A}_\mathrm{K} + \mathrm{P}, \mathrm{D}_\mathrm{K})$ is a linear bounded perturbation of the infinitesimal generator $(\mathbb{A}_\mathrm{K}, \mathrm{D}_\mathrm{K})$ (Theorem 1). Now [7, Thm. III.1.3] ends the proof. □

Remark 1. *Assumptions* $\left(\mathcal{A}^1_{\alpha,\beta}\right)$ *and* $\left(\mathcal{A}^2_{\alpha,\beta}\right)$, *and* $(\mathcal{A}_\mathrm{h})$ *were crucial to our computations. It is then legitimate to ask what happens if* $\mathrm{m} = 0$ *or* $\mathrm{M} = \infty$*?*

4 Application 1

In this application we are concerned with a mathematical model describing a structured cell population. Each cell is distinguished by its cell cycle length l and by its age a. If $f = f(t, a, l)$ denotes, at time t, the cell density with respect to age a and cell cycle length l, the cell population is then governed by the following equation

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial a} - \mu f + \int_0^l \int_0^a \eta(a, l, a', l') f(t, a', l') \mathrm{d}a' \mathrm{d}l' \tag{35}$$

where $\mu = \mu(a, l)$ stands for the *cell mortality rate* and $\eta(a, l, a', l')$ denotes the *transition rate* at which cells change their cell cycle length from l' to l and its age from a' to a.

The cell cycle length l is the time between cell birth and cell division. As no cell can be rejuvenated, then the cell cycle length l must be positive and therefore $0 < l_1 < l < l_2 \leqslant \infty$ where l_1 and l_2 denote the minimum and the maximum cell cycle lengths. The age a of each cell is null $(a = 0)$ at birth and equals to its cell cycle length l $(a = l)$ at division. Between birth and division, we have then $0 \leqslant a \leqslant l$.

During each mitosis, there may be a positive correlation $\tau = \tau(l, l')$ between the cell cycle length of a mother cell l' and that of a daughter cell l. This correlation, called *Transition Biological Rule*, is mathematically described by

$$f(t, 0, l) = \delta \int_{l_1}^{l_2} \tau(l, l') f(t, l', l') \mathrm{d}l' \tag{36}$$

where $\delta \geqslant 0$ is the average number of daughter cells viable per mitosis. To ensure the continuity of the cell flux for $\delta = 1$, the kernel of correlation τ must fulfill the following *normalization condition*,

$$\int_{l_1}^{l_2} \mathrm{k}(l, l') \mathrm{d}l = 1 \qquad \text{for all } l' \in (l_1, l_2). \tag{37}$$

There also may be a total inheritance of the cell cycle length l between a mother cell and its daughters, which leads to *Perfect Memory Rule*

$$f(t, 0, l) = \gamma f(t, l, l). \tag{38}$$

where $\gamma \geqslant 0$ is the average number of daughter cells viable per mitosis.

Actually, at each mitosis, the cell population is divided into two distinct subpopulation in most observed cases. The first one obeys to the Transition Biological Rule described by (36) while the second one obeys to the Perfect Memory Rules described by (38). In other words, both Biological Rules cohabit, at each mitosis, and lead to a third one, mathematically described by the following boundary condition

$$f(t,0,l) = \delta \int_{l_1}^{l_2} \tau(l,l') f(t,l',l') \mathrm{d}l' + \gamma f(t,l,l) \tag{39}$$

where $\delta \geqslant 0$ and $\gamma \geqslant 0$ denote the average number of daughter cells viable per mitosis into the corresponding cell subpopulation.

The model (35)–(36)–(with $\eta = 0$) has been introduced for the first time in [8] and mathematically studied in [10, 11] and then in [3]. We have then proved that this model is governed by a strongly continuous semigroup.

In order to give a general study, we have recently considered the full model described by both equations (35) and (39). We have then proved that this model is governed by a strongly continuous semigroup (see [1]).

Now we can say that

Lemma 2. *Let l_1 and l_2 and, δ and γ be such that*

$$0 < l_1 < l_2 \leqslant \infty \qquad \text{and} \qquad 0 \leqslant \delta\ ,\ \gamma < \infty. \tag{40}$$

Suppose that the kernel $\tau(v,v')$ is positive and fulfills (37).

The unperturbed model (35),(39)*–(with $\eta = 0$) is governed by a strongly continuous semigroup $\mathbb{A}_{\delta,\gamma} = (\mathbb{A}_{\delta,\gamma}(t))_{t\geqslant 0}$ satisfying,*

$$\left\|\mathbb{A}_{\delta,\gamma}(t)\varphi\right\|_1 \leqslant \Big(\max\left\{1, \delta+\gamma\right\}\Big)^{\left(1+\frac{1}{l_1}t\right)} \|\varphi\|_1 \qquad\qquad t \geqslant 0$$

for all $\varphi \in \mathrm{X}_1$. Furthermore, $\mathbb{A}_{\delta,\gamma} = (\mathbb{A}_{\delta,\gamma}(t))_{t\geqslant 0}$ is a positive semigroup.

Proof. In order to apply Theorem 1 we firstly put

$$\mathrm{a} := l_1 \qquad \text{and} \qquad \mathrm{b} := l_2.$$

Next, we define the functions

$$\alpha(l) := 0 \qquad \text{and} \qquad \beta(l) := l \qquad l \in (\mathrm{a}, \mathrm{b})$$

and then

$$\mathrm{h}(l) := 1 \qquad l \in (\mathrm{a}, \mathrm{b}).$$

Finally, let $\mathrm{K}_{\delta,\gamma}$ be such that

$$\mathrm{K}_{\delta,\gamma}\psi(l) := \delta \int_{l_1}^{l_2} \tau(l,l')\psi(l',l')\mathrm{d}l' + \gamma\psi(l) \qquad l \in (\mathrm{a}, \mathrm{b}).$$

The operator $\mathrm{K}_{\delta,\gamma}$ is obviously positive. Furthermore (37) leads to

$$\begin{aligned}\left\|\mathrm{K}_{\delta,\gamma}\psi\right\|_{\mathrm{Y}_1} &= \delta\int_{l_1}^{l_2}\left[\int_{l_1}^{l_2}\tau(l,l')\mathrm{d}l\right]\psi(l')\mathrm{d}l' + \gamma\int_{l_1}^{l_2}\psi(l)\mathrm{d}l \\ &= (\delta+\gamma)\int_{l_1}^{l_2}\psi(l)\mathrm{d}l \\ &= (\delta+\gamma)\left\|\psi\right\|_{\mathrm{Y}_1}\end{aligned}$$

for all nonnegative $\psi \in \mathrm{Y}_1$ and therefore $\left\|\mathrm{K}_{\delta,\gamma}\right\| = \delta+\gamma$.

Due to (40), assumptions $\left(\mathcal{A}^1_{\alpha,\beta}\right)$ and $\left(\mathcal{A}^2_{\alpha,\beta}\right)$–(with $\mathrm{m} = l_1$), and $(\mathcal{A}_\mathrm{h})$–(with $\mathrm{M} = 1$) hold true. The linear operator $\mathrm{K} := \mathrm{K}_{\delta,\gamma}$ is bounded from Y_1 into itself whose norm is $\|\mathrm{K}\| = \delta+\gamma$.

Now, all the required conditions of Theorem 1 are fulfilled. Accordingly, $\left(\mathrm{A}_{\delta,\gamma}, \mathrm{D}_{\delta,\gamma}\right) := (\mathrm{A}_\mathrm{K}, \mathrm{D}_\mathrm{K})$–(with $\mathrm{K} := \mathrm{K}_{\delta,\gamma}$) generates a strongly continuous semigroup $\mathbb{A}_{\delta,\gamma} = (\mathbb{A}_{\delta,\gamma}(t))_{t\geqslant 0}$ satisfying (34)–(with $\|\mathrm{K}\| = \delta+\gamma$). The positivity of $\mathbb{A}_{\delta,\gamma} = (\mathbb{A}_{\delta,\gamma}(t))_{t\geqslant 0}$ follows from that of $\mathrm{K}_{\delta,\gamma}$. □

Suppose now that the cell mortality rate $\mu = \mu(a,l)$ and the transition rate $\eta(a,l,a',l')$ are subject to following assumptions

$$\left(\mathcal{A}_\mu\right) : \qquad \overline{\mu} := \operatorname*{ess\,sup}_{(a,l)\in\Omega} |\mu(a,l)| < \infty$$

$$\left(\mathcal{A}_\eta\right) : \qquad \overline{\eta} := \operatorname*{ess\,sup}_{(a',l')\in\Omega} \int_\Omega |\eta(a,l,a',l')|\, \mathrm{d}a\mathrm{d}l < \infty.$$

Corollary 2. *Let l_1 and l_2 and, δ and γ be such that* (40) *holds true and let $\tau(v,v')$ be a positive kernel fulfilling* (37). *If both assumptions $\left(\mathcal{A}_\mu\right)$ and $\left(\mathcal{A}_\eta\right)$ hold true then the full model* (35), (39) *is well-posed on* X_1.

Proof. According to Lemma 2 and [7, Thm. III.1.3], it suffices to prove that

$$\mathrm{P}\varphi(a,l) := -\mu(a,l)\varphi(a,l) + \int_0^l\int_0^a \eta(a,l,a',l')\varphi(a',l')\mathrm{d}a'\mathrm{d}l'$$

is a bounded linear operator from X_1 into itself. This is true because of $\|\mathrm{P}\varphi\|_1 \leqslant (\overline{\mu}+\overline{\eta})\,\|\varphi\|_1$ for all $\varphi \in \mathrm{X}_1$. □

5 Application 2

We consider a structured bacterial population in which each individual is distinguished by its degree of maturity μ and its maturation velocity v. If $f = f(t,\mu,v)$ denotes the bacterial density with respect to the degree of maturity μ and the maturation velocity v, at time t, then

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial \mu} - \left[\int_{\mathrm{v_{min}}}^{\mathrm{v_{max}}} r(\mu,v',v)\mathrm{d}v'\right] f + \int_{\mathrm{v_{min}}}^{\mathrm{v_{max}}} r(\mu,v,v')f(t,\mu,v')\mathrm{d}v' \tag{41}$$

where $r(\mu, v, v')$ stands for the *transition rate* at which bacteria change their velocities from v' to v.

The degree of maturity of a daughter bacteria is $\mu = 0$ while that of a mother bacteria is $\mu = 1$. Between birth and division, the degree of maturity of each bacteria is $0 < \mu < 1$. As each bacteria may not become less mature, its maturation velocity v must be positive and thus $0 \leqslant \mathrm{v}_{\min} < v < \mathrm{v}_{\max} < \infty$ where $\mathrm{v}_{\min}$ is the minimum velocity while $\mathrm{v}_{\max}$ is the maximum velocity.

During mitosis, there may be a correlation $\mathrm{k} := \mathrm{k}(v, v')$ between the maturation velocity v' of a mother bacteria and that of its daughter v. If $\overline{p} \geqslant 0$ denotes the average number of bacteria daughter viable per mitotic, then this correlation (called *Transition Rule*) is described by

$$v f(t, 0, v) = \overline{p} \int_{\mathrm{v}_{\min}}^{\mathrm{v}_{\max}} \mathrm{k}(v, v') f(t, 1, v') v' \mathrm{d}v'. \tag{42}$$

To ensure the continuity of the bacterial flux for $\overline{p} = 1$, the kernel of correlation k must fulfils the following *normalization condition*

$$\int_{\mathrm{v}_{\min}}^{\mathrm{v}_{\max}} \mathrm{k}(v, v') \mathrm{d}v = 1 \qquad \text{for all } v' \in (\mathrm{v}_{\min}, \mathrm{v}_{\max}). \tag{43}$$

There also may be a total inheritance of the maturation velocity v between a mother bacteria and its daughters. If $\overline{q} \geqslant 0$ denotes the average number of bacteria daughter viable per mitotic, then this inheritance (called *Perfect Memory Rule*) is mathematically described by

$$f(t, 0, v) = \overline{q} f(t, 1, v). \tag{44}$$

In most observed mitosis, the bacterial population is divided into two distinct subpopulation. The first one obeys to the Transition Rule described by (42) while the second one obeys to the Perfect Memory Rules described by (44). At each mitosis, both Biological Rules cohabit and lead obviously to a third one, mathematically described by

$$f(t, 0, v) = \frac{\overline{p}}{v} \int_{\mathrm{v}_{\min}}^{\mathrm{v}_{\max}} \mathrm{k}(v, v') f(t, 1, v') v' \mathrm{d}v' + \overline{q} f(t, 1, v) \tag{45}$$

where $\overline{p} \geqslant 0$ and $\overline{q} \geqslant 0$ denote the average number of daughter bacteria viable per mitosis into the corresponding bacterial subpopulation.

The model (41)–(42) has been introduced and numerically studied for the first time in [9]. The first theoretical studies of the model (41)-(42) were given in [4]. We have then proved that this model is governed by a strongly continuous semigroup.

Recently, we have proved in [2] that the full model (41), (45) is governed by a strongly continuous semigroup (see also [6] for a different method).

Now we can say that

Lemma 3. *Let* $\mathrm{v}_{\min}$ *and* $\mathrm{v}_{\max}$ *and,* $\overline{p}$ *and* $\overline{q}$ *be such that*

$$0 \leqslant \mathrm{v}_{\min} < \mathrm{v}_{\max} < \infty \qquad \textit{and} \qquad 0 \leqslant \overline{p}\ ,\ \overline{q} < \infty. \tag{46}$$

Suppose that the kernel $\mathrm{k}(v,v')$ *is positive and fulfills* (43).

The unperturbed model (41)–(45)–*(with* $r=0$*) is governed by a strongly continuous semi-group* $\mathbb{A}_{\overline{\mathrm{p}},\overline{\mathrm{q}}} = (\mathbb{A}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}(t))_{t\geqslant 0}$ *satisfying,*

$$\left\|\mathbb{A}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}(t)\varphi\right\|_{1} \leqslant \Big(\max\left\{1,\overline{p}+\overline{q}\right\}\Big)^{(1+\mathrm{v}_{\max}t)}\left\|\varphi\right\|_{1} \qquad t\geqslant 0 \tag{47}$$

for all $\varphi\in \mathrm{X}_1$. *Furthermore,* $\mathbb{A}_{\overline{\mathrm{p}},\overline{\mathrm{q}}} = (\mathbb{A}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}(t))_{t\geqslant 0}$ *is a positive semigroup.*

Proof. In order to apply Theorem 1 we firstly put

$$\mathrm{a} := \mathrm{v}_{\min} \qquad \text{and} \qquad \mathrm{b} := \mathrm{v}_{\max}.$$

Next, we define the functions

$$\alpha(v) := 0 \qquad \text{and} \qquad \beta(v) := 1 \qquad v\in(\mathrm{a},\mathrm{b})$$

and then

$$\mathrm{h}(v) := v \qquad v\in(\mathrm{a},\mathrm{b}).$$

Finally, let $\mathrm{K}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}$ be such that

$$\mathrm{K}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}\psi(v) := \frac{\overline{p}}{v}\int_{\mathrm{v}_{\min}}^{\mathrm{v}_{\max}} \mathrm{k}(v,v')\psi(v')v'\mathrm{d}v' + \overline{q}\psi(v) \qquad y\in(\mathrm{a},\mathrm{b}).$$

The operator $\mathrm{K}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}$ is obviously positive. Using (43) we get that

$$\begin{aligned}
\left\|\mathrm{K}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}\psi\right\|_{\mathrm{Y}_1} &= \overline{p}\int_{\mathrm{v}_{\min}}^{\mathrm{v}_{\max}}\left[\int_{\mathrm{v}_{\min}}^{\mathrm{v}_{\max}} \mathrm{k}(v,v')\mathrm{d}v\right]\psi(v')v'\mathrm{d}v' + \overline{q}\int_{\mathrm{v}_{\min}}^{\mathrm{v}_{\max}}\psi(v)v\mathrm{d}v \\
&= (\overline{p}+\overline{q})\int_{\mathrm{v}_{\min}}^{\mathrm{v}_{\max}}\psi(v)v\mathrm{d}v \\
&= (\overline{p}+\overline{q})\left\|\psi\right\|_{\mathrm{Y}_1}
\end{aligned}$$

for all nonnegative $\psi\in\mathrm{Y}_1$ and therefore $\left\|\mathrm{K}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}\right\| = \overline{p}+\overline{q}$.

Due to (46), assumptions $\left(\mathcal{A}^1_{\alpha,\beta}\right)$ and $\left(\mathcal{A}^2_{\alpha,\beta}\right)$–(with $\mathrm{m}=1$), and $(\mathcal{A}_{\mathrm{h}})$–(with $\mathrm{M}=\mathrm{v}_{\max}$) hold true. The linear operator $\mathrm{K} := \mathrm{K}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}$ is bounded from Y_1 int itself whose norm is $\|\mathrm{K}\| = \overline{p}+\overline{q}$.

Now, all the required conditions of Theorem 1 are fulfilled and therefore $\left(\mathrm{A}_{\overline{\mathrm{p}},\overline{\mathrm{q}}},\mathrm{D}_{\overline{p},\overline{q}}\right) := (\mathrm{A}_{\mathrm{K}},\mathrm{D}_{\mathrm{K}})$–(with $\mathrm{K} := \mathrm{K}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}$) generates a strongly continuous semigroup $\mathbb{A}_{\overline{\mathrm{p}},\overline{\mathrm{q}}} = (\mathbb{A}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}(t))_{t\geqslant 0}$ satisfying (34)–(with $\|\mathrm{K}\| = \overline{p}+\overline{q}$). The positivity of $\mathbb{A}_{\overline{\mathrm{p}},\overline{\mathrm{q}}} = (\mathbb{A}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}(t))_{t\geqslant 0}$ follows from that of $\mathrm{K}_{\overline{\mathrm{p}},\overline{\mathrm{q}}}$. □

Suppose now that the transition rate $r(\mu,v,v')$ is subject to the following assumption

$$(\mathcal{A}_{\mathbf{r}})\ : \qquad \overline{r} := \operatorname*{ess\,sup}_{(\mu,v)\in\Omega}\int_{\mathrm{v}_{\min}}^{\mathrm{v}_{\max}} \left|r(\mu,v',v)\right|\mathrm{d}v' < \infty.$$

Corollary 3. *Let* $\mathrm{v}_{\min}$ *and* $\mathrm{v}_{\max}$ *and,* $\overline{p}$ *and* $\overline{q}$ *be such that* (46) *holds true and let* $\mathrm{k}(v, v')$ *be a positive kernel fulfilling* (43)*. If the assumption* $(\mathcal{A}_{\mathbf{r}})$ *holds true then the full model* (41)*,* (45) *is well-posed on* X_1*.*

Proof. According to Lemma 3 and [7, Thm. III.1.3], it suffices to prove that

$$\mathrm{P}\varphi(\mu, v) := -\left[\int_{\mathrm{v}_{\min}}^{\mathrm{v}_{\max}} r(\mu, v', v)\mathrm{d}v'\right]\varphi(\mu, v) + \int_{\mathrm{v}_{\min}}^{\mathrm{v}_{\max}} r(\mu, v, v')\varphi(\mu, v')\mathrm{d}v'$$

is a bounded linear operator from X_1 into itself. This is true because of $\|\mathrm{P}\varphi\|_1 \leqslant 2\overline{r}\|\varphi\|_1$ for all $\varphi \in \mathrm{X}_1$. □

6 Application 3

This application deals with the following one-dimensional transport equation

$$\frac{\partial f}{\partial t} = -y\frac{\partial f}{\partial x} \tag{48}$$

where $f = f(t, x, y)$ stands for the particles density having at time t the position $x \in \big(-\mathrm{x}_{\max}, \mathrm{x}_{\max}\big)$ $(0 < \mathrm{x}_{\max} < \infty)$ and the velocity $y \in \big(0, \mathrm{y}_{\max}\big)$ $(0 < \mathrm{y}_{\max} < \infty)$. This equation describes the transport of particles (e.g. neutrons, photons, molecules of gas) in a plane parallel domain with a width of $2\mathrm{x}_{\max}$ mean free paths. We endow (48) with the following boundary condition

$$f\big(t, -\mathrm{x}_{\max}, y\big) = \xi f\big(t, \mathrm{x}_{\max}, y\big) \qquad y \in \big(0, \mathrm{y}_{\max}\big) \tag{49}$$

where $\xi \geqslant 0$ is a given real. Boundary condition (49) is periodic (or vacuum) whenever $\xi = 1$ (or $\xi = 0$).

Now we can say that

Lemma 4. *Let* $\mathrm{x}_{\max}$ *and* $\mathrm{y}_{\max}$ *and* ξ *be such that*

$$0 < \mathrm{x}_{\max} < \infty \quad \text{and} \quad 0 < \mathrm{y}_{\max} < \infty \quad \text{and} \quad 0 \leqslant \xi < \infty. \tag{50}$$

The model (48)*–*(49) *is governed by a positive strongly continuous semigroup* $\mathbb{A}_\xi = (\mathbb{A}_\xi(t))_{t\geqslant 0}$ *satisfying,*

$$\big\|\mathbb{A}_\xi(t)\varphi\big\|_p \leqslant \Big(\max\big\{1, \xi\big\}\Big)^{\left(1+\frac{\mathrm{y}_{\max}}{2\mathrm{x}_{\max}}t\right)}\|\varphi\|_p \qquad\qquad t \geqslant 0$$

for all $\varphi \in \mathrm{X}_p$ $(p \geqslant 1)$*.*

Proof. In order to apply Theorem 1 we firstly put

$$\mathrm{a} := 0 \qquad \text{and} \qquad \mathrm{b} := \mathrm{y}_{\max}.$$

Next, we define the functions

$$\alpha(y) := -\mathrm{x}_{\max} \qquad \text{and} \qquad \beta(y) := \mathrm{x}_{\max} \qquad y \in (\mathrm{a}, \mathrm{b})$$

and then

$$\mathrm{h}(y) := y \qquad y \in (\mathrm{a}, \mathrm{b}).$$

Finally, let $\mathrm{K}_\xi := \xi \mathrm{I}_{\mathrm{Y_p}}$ where $\mathrm{I}_{\mathrm{Y_p}}$ is the identity operator into Y_p ($p \geqslant 1$). Obviously, K_ξ is a bounded linear operator from Y_p ($p \geqslant 1$) into itself whose norm is $\|\mathrm{K}\| = \xi$.

Due to (50), assumptions $\left(\mathcal{A}^1_{\alpha,\beta}\right)$ and $\left(\mathcal{A}^2_{\alpha,\beta}\right)$–(with $\mathrm{m} = 2\mathrm{x}_{\max}$), and $(\mathcal{A}_{\mathrm{h}})$–(with $\mathrm{M} = \mathrm{y}_{\max}$) hold true. The linear operator $\mathrm{K} := \mathrm{K}_\xi$ is bounded from Y_p ($p \geqslant 1$) into itself whose norm is $\|\mathrm{K}\| = \xi$.

Now, all the required conditions of Theorem 1 are fulfilled. Accordingly, $\left(\mathrm{A}_\xi, \mathrm{D}_\xi\right) := (\mathrm{A}_{\mathrm{K}}, \mathrm{D}_{\mathrm{K}})$–(with $\mathrm{K} := \xi \mathrm{I}_{\mathrm{Y_p}}$) generates a strongly continuous semigroup $\mathbb{A}_\xi = (\mathbb{A}_\xi(t))_{t \geqslant 0}$ satisfying (34)–(with $\|\mathrm{K}\| = \xi$). The positivity of $\mathbb{A}_\xi = (\mathbb{A}_\xi(t))_{t \geqslant 0}$ follows from that of $\mathrm{K} := \xi \mathrm{I}_{\mathrm{Y_p}}$ because of $\xi \geqslant 0$. □

Acknowledgement

The author is grateful and indebted to the professors G. Chaigne, M. Chateau and S. Loubignac of L. Bourliaguet Institute (Thiviers. France). This research was supported by LMCM-RSA.

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