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# GLOBAL EXISTENCE OF WEAK SOLUTIONS TO FOREST KINEMATIC MODEL WITH NONLINEAR DEGENERATE DIFFUSION

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**Abstract.** This article deals with the mathematical analysis of the forest kinematic model. The model is described by a system of two ordinary differential equations and one parabolic differential equation with nonlinear degenerate diffusion. Three unknown functions represent the tree densities of young and old age classes and the density of seeds. We study the initial boundary value problem for this system with nonnegative initial functions satisfying suitable conditions. We will show the existence of a weak time-global solution which is uniformly bounded.

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# 1 Introduction

This paper is concerned with the initial boundary value problem for the following system of differential equations with nonlinear degenerate diffusion:

$$\begin{cases} u_t = \beta\delta w - \gamma(v)u - fu & \text{in } \Omega \times (0,\infty), \\ v_t = fu - hv & \text{in } \Omega \times (0,\infty), \\ w_t = \Delta(w^m) - \beta w + \alpha v & \text{in } \Omega \times (0,\infty), \\ w = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$ and $u_0, v_0, w_0$ are nonnegative functions in $L^\infty(\Omega)$. This problem models the dynamics of forest: $u$ and $v$ denote the tree densities of young and old age classes, respectively, and $w$ denotes the density of seeds. In (1.1) $f, h, \alpha, \beta$ are positive constants, $0 < \delta < 1$, $m > 1$, and $\gamma \in C^1(\mathbb{R})$ is a quadratic function of the form $\gamma(v) = a(v-b)^2 + c$ with positive constants $a, b, c$. Constants $f, h, \alpha, \beta, \delta$ denote, respectively, aging rate, mortality of trees of old age class, seed production rate, seed deposition rate and seed establishment rate and, moreover, $\gamma(v)$ represents the mortality of trees of young age class.

In order to discuss the case that seeds are carried by animals, we attempt to deal with a simple nonlinear diffusion case $\Delta(w^m)$ in (1.1). As stated in Okubo–Levin [13], if we take a diffusion coefficient of the form $m\,w^{m-1}$ $(m > 1)$, then it implies that the fewer seeds are, the smaller the diffusion coefficient is (conversely, the more seeds are, the larger the diffusion coefficient is). In a forest, a lot of fruits, which ought to have a lot of seeds, attract animals, and then it is easy for seeds to spread. So we consider that the problem (1.1) seems to be more realistic for the forest kinematic model in the case that the seeds are carried and spread by animals.

The following forest kinematic model was first proposed by Kuznetsov et al. [10]:

$$\begin{cases} u_t = \beta\delta w - \gamma(v)u - fu, \\ v_t = fu - hv, \\ w_t = \Delta w - \beta w + \alpha v, \end{cases} \tag{1.2}$$

and various studies have been conducted on this model (see [5, 6], [7], [12], [14, 15, 16], [18] for (1.2) and see also [2], [17] for related models). In particular, Shirai, Chuan and Yagi [14, 15, 16] discussed (1.2) under Dirichlet boundary condition. We should also state that the stationary problem corresponding to (1.1) with nonlinear degenerate diffusion case has been studied by Yamamoto and Yamada [18].

In the work of Shirai, Chuan and Yagi [14, 15], they studied (1.2) with Dirichlet boundary condition $w = 0$ and initial conditions $u(\cdot,0) = u_0, v(\cdot,0) = v_0$ and $w(\cdot,0) = w_0$ in the case that $\Omega$ is a two-dimensional bounded domain. It was shown that, if initial functions satisfy $u_0, v_0, w_0 \geq 0$, $u_0, v_0 \in L^\infty(\Omega)$ and $w_0 \in L^2(\Omega)$, then there exists a unique global solution $(u, v, w)$ of (1.2) and that the solution is uniformly bounded in the sense that $u, v \in L^\infty(\Omega \times [0,\infty))$ and $w \in L^\infty(0,\infty; L^2(\Omega))$. They also discussed the asymptotic behavior of the solution as $t \to \infty$.

The purpose of the present paper is to study the global solvability of (1.1) under the following conditions

$$u_0, v_0, w_0 \geq 0 \ \ (\not\equiv 0), \quad u_0, v_0, w_0 \in L^\infty(\Omega) \quad \text{and} \quad w_0 \in H_0^1(\Omega). \tag{1.3}$$

In particular, this condition implies $w_0^m \in H_0^1(\Omega)$. We will look for solutions of (1.1) in the following class.

**Definition 1.1.** For $T > 0$, $(u, v, w) : \Omega \times [0, T] \to [0, \infty)^3$ is called a **weak solution** of (1.1) in $[0, T]$ if it possesses the following properties:

(i) $u, v, w \in C([0, T]; L^2(\Omega)) \cap L^\infty(\Omega \times (0, T))$ and $w^m \in L^2(0, T; H_0^1(\Omega))$,

(ii) $u$ and $v$ satisfy the first and second equations of (1.1) for $t \in [0, T]$ in the sense of $L^2(\Omega)$,

(iii) $w$ satisfies

$$\begin{aligned} &- \int_\Omega w_0(x)\varphi(x, 0)\ dx - \int_0^T \int_\Omega w\varphi_t\, dx\, dt \\ =& - \int_0^T \int_\Omega \nabla w^m \nabla \varphi\, dx\, dt + \int_0^T \int_\Omega (-\beta w + \alpha v)\varphi\, dx\, dt \end{aligned}$$

for all $\varphi \in C^1([0, T]; C_0^\infty(\Omega))$ satisfying $\varphi(\cdot, T) \equiv 0$.

Furthermore, $(u, v, w)$ is called a **global weak solution** of (1.1) if $(u, v, w)$ is a weak solution of (1.1) in $[0, T]$ for any $T > 0$.

Our first main result concerns with the global existence of weak solutions.

**Theorem 1.1.** *Suppose that $(u_0, v_0, w_0)$ satisfies (1.3). Then there exists a global weak solution $(u, v, w)$ of (1.1) such that*

(i) $u \in C^1([0, \infty); L^r(\Omega))$, $v \in C^2([0, \infty); L^r(\Omega))$ *for any* $r \geq 1$,

(ii) $w \in C([0, \infty); L^r(\Omega))$ *for any* $r \geq 1$, $w^m \in L^2_{loc}(0, \infty; H_0^1(\Omega))$,

(iii) $(u, v, w)$ *satisfies*

$$\begin{aligned} &\max\{\|u(t)\|_\infty, \|v(t)\|_\infty, \|w(t)\|_\infty\} \\ \leq& K \exp\{(\alpha\delta - (c + f)h/f)\mu t\}(\|u_0\|_\infty + \|v_0\|_\infty + \|w_0\|_\infty) \end{aligned} \tag{1.4}$$

*for $t \geq 0$, where $K$ and $\mu$ are positive constants depending only on $c, f, h, \alpha, \beta, \delta$. Here $\|\cdot\|_\infty$ denotes $L^\infty(\Omega)$-norm.*

Theorem 1.1 shows that, if $\alpha\delta - (c + f)h/f < 0$, then $(u, v, w)$ satisfies

$$\lim_{t\to\infty} u(\cdot, t) = \lim_{t\to\infty} v(\cdot, t) = \lim_{t\to\infty} w(\cdot, t) = 0 \qquad \text{in} \ \ L^\infty(\Omega).$$

Furthermore, even if $\alpha\delta - (c + f)h/f > 0$, we will show the uniform boundedness of $\|u(t)\|_\infty, \|v(t)\|_\infty$ and $\|w(t)\|_\infty$ for the weak solution given in Theorem 1.1.

**Theorem 1.2.** *Suppose that* $(u_0, v_0, w_0)$ *satisfies* (1.3)*. Let* $(u, v, w)$ *be the weak solution of* (1.1) *given in Theorem 1.1. Then it holds that*

$$\begin{aligned}&\sup_{t\geq 0}\{\|u(t)\|_\infty + \|v(t)\|_\infty + \|w(t)\|_\infty\}\\ &\leq C\max\{\,1,\ \|u_0\|_\infty + \|v_0\|_\infty + \|w_0\|_\infty\,\},\end{aligned} \tag{1.5}$$

*where* $C$ *is a positive constant depending only on* $N, \Omega, a, b, c, f, h, m, \alpha$ *and* $\beta$*.*

In order to construct a weak solution of (1.1), we will consider approximate problems with $\Delta w^m$ replaced by $\Delta(w+\varepsilon)^m$, $\varepsilon > 0$; so that the diffusion coefficient becomes non-degenerate. Our task is to estimate approximate solutions and derive their bounds independent of $\varepsilon$. Then letting $\varepsilon_n \to 0$ along some sequence $\{\varepsilon_n\}$ we will show that approximate solutions converge to a weak solution of (1.1).

The contents of the present paper are as follows. In Section 2, we prepare approximate problems and establish some basic properties of approximate solutions. In Section 3, we will prove the convergence of approximate solutions to show Theorem 1.1. Section 4 is devoted to the proof Theorem 1.2. We will take an iterative method developed by Alikakos [1] to obtain $L^\infty$ bounds of the weak solution. Finally, proofs of some auxiliary results are given in Appendix.

**Notation.** In this paper we denote $L^p(\Omega)$-norm of $u \in L^p(\Omega)$ by $\|u\|_p$ for each $p \in [1, \infty]$. For the sake of simplicity, we use the following set of parameters

$$\Lambda = \{a, b, c, h, f, \alpha, \beta, \delta\}.$$

# 2 Approximate Problem

In this section we consider the following approximate problem for any $\varepsilon > 0$,

$$\begin{cases}\partial_t u_\varepsilon = \beta\delta w_\varepsilon - \gamma(v_\varepsilon)u_\varepsilon - fu_\varepsilon & \text{in } \ \Omega\times(0,\infty),\\ \partial_t v_\varepsilon = fu_\varepsilon - hv_\varepsilon & \text{in } \ \Omega\times(0,\infty),\\ \partial_t w_\varepsilon = \Delta(w_\varepsilon+\varepsilon)^m - \beta w_\varepsilon + \alpha v_\varepsilon & \text{in } \ \Omega\times(0,\infty),\\ w_\varepsilon = 0 & \text{on } \ \partial\Omega\times(0,\infty),\\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad v_\varepsilon(x,0) = v_{0\varepsilon}(x), \quad w_\varepsilon(x,0) = w_{0\varepsilon}(x) & \text{in } \ \Omega,\end{cases} \tag{2.1}$$

where $\partial_t = \partial/\partial t$ and initial functions $u_{0\varepsilon}, v_{0\varepsilon}, w_{0\varepsilon}$ are assumed to satisfy the following conditions

$$\begin{cases}u_{0\varepsilon}, v_{0\varepsilon} \in C^\infty(\Omega), \quad w_{0\varepsilon} \in C_0^\infty(\Omega), \quad u_{0\varepsilon},\ v_{0\varepsilon},\ w_{0\varepsilon} \geq 0\ (\not\equiv 0) \ \text{ in } \ \Omega,\\ \|u_{0\varepsilon}\|_\infty \leq \|u_0\|_\infty, \quad \|v_{0\varepsilon}\|_\infty \leq \|v_0\|_\infty, \quad \|w_{0\varepsilon}\|_\infty \leq \|w_0\|_\infty,\\ u_{0\varepsilon} \to u_0, \quad v_{0\varepsilon} \to v_0, \quad w_{0\varepsilon} \to w_0 \quad \text{in } \ L^r(\Omega) \ \text{ as } \ \varepsilon \to 0 \ \text{ for any } \ r \geq 1,\\ \nabla w_{0\varepsilon} \to \nabla w_0 \quad \text{in } \ L^2(\Omega) \ \text{ as } \ \varepsilon \to 0.\end{cases} \tag{2.2}$$

We begin with the local existence of a unique classical solution to (2.1).

**Proposition 2.1.** *Assume that* $(u_{0\varepsilon}, v_{0\varepsilon}, w_{0\varepsilon})$ *satisfies* (2.2). *Then there exists a positive number* $T$ *such that* (2.1) *admits a unique nonnegative classical solution* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *in* $[0,T]$.

*Proof.* By the well-known result of Ladyženskaja, Solonnikov and Ural'ceva [11], there exists a unique classical solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ of (2.1) in $[0,T]$ with some $T > 0$. We will prove its positivity. For $\xi \in \mathbb{R}$ define

$$\xi^+ = \max\{\xi, 0\}, \quad \xi^- = \max\{-\xi, 0\};$$

then $\xi = \xi^+ - \xi^-$ and $\xi^+\xi^- = 0$. Since $w_\varepsilon(x,0) = w_{0\varepsilon}(x) \geq 0$ in $\Omega$ and $w_\varepsilon(t,x)$ is continuous with respect to $(t,x)$, there exists $T_0 \in (0,T]$ such that $w_\varepsilon(t,x) \geq -\varepsilon$ for $(x,t) \in \Omega \times [0,T_0]$. Taking $L^2(\Omega)$-inner product of the third equation of (2.1) with $-w_\varepsilon^-$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w_\varepsilon^-(t)\|_2^2 &= ((w_\varepsilon)_t, -w_\varepsilon^-)_2 \\
&= (\Delta(w_\varepsilon+\varepsilon)^m, -w_\varepsilon^-)_2 - \beta(w_\varepsilon, -w_\varepsilon^-)_2 + \alpha(v_\varepsilon, -w_\varepsilon^-)_2 \\
&= \int_\Omega m(w_\varepsilon+\varepsilon)^{m-1}\nabla w_\varepsilon \cdot \nabla w_\varepsilon^- \, dx \\
&\qquad\qquad - \beta\|w_\varepsilon^-(t)\|_2^2 - \alpha(v_\varepsilon^+ - v_\varepsilon^-, w_\varepsilon^-)_2,
\end{aligned}$$

where $(\cdot,\cdot)_2$ denotes $L^2(\Omega)$-inner product. Then it follows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w_\varepsilon^-(t)\|_2^2 &\leq -\int_\Omega m(w_\varepsilon+\varepsilon)^{m-1}|\nabla w_\varepsilon^-|^2\,dx - \beta\|w_\varepsilon^-(t)\|_2^2 \\
&\qquad\qquad + \alpha\int_\Omega v_\varepsilon^- w_\varepsilon^-\,dx \\
&\leq \alpha\|v_\varepsilon^-(t)\|_2\|w_\varepsilon^-(t)\|_2.
\end{aligned} \tag{2.3}$$

Similarly, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_\varepsilon^-(t)\|_2^2 &\leq \beta\|w_\varepsilon^-(t)\|_2\|u_\varepsilon^-(t)\|_2, \\
\frac{1}{2}\frac{d}{dt}\|v_\varepsilon^-(t)\|_2^2 &\leq f\|u_\varepsilon^-(t)\|_2\|v_\varepsilon^-(t)\|_2.
\end{aligned} \tag{2.4}$$

Thus it follows from (2.3) and (2.4) that

$$\frac{d}{dt}\Big(\|u_\varepsilon^-(t)\|_2^2 + \|v_\varepsilon^-(t)\|_2^2 + \|w_\varepsilon^-(t)\|_2^2\Big) \leq C\Big(\|u_\varepsilon^-(t)\|_2^2 + \|v_\varepsilon^-(t)\|_2^2 + \|w_\varepsilon^-(t)\|_2^2\Big)$$

with $C = 2\max\{\alpha, \beta, f\}$. Solving this inequality yields

$$\|u_\varepsilon^-(t)\|_2^2 + \|v_\varepsilon^-(t)\|_2^2 + \|w_\varepsilon^-(t)\|_2^2 \leq \Big(\|u_{0\varepsilon}^-\|_2^2 + \|v_{0\varepsilon}^-\|_2^2 + \|w_{0\varepsilon}^-\|_2^2\Big)e^{Ct} = 0,$$

which implies that $u_\varepsilon(t), v_\varepsilon(t)$ and $w_\varepsilon(t)$ are nonnegative for all $t \in [0,T_0]$ and, therefore, for all $t \in [0,T]$. $\square$

**Lemma 2.1.** *Let* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *be the solution of* (2.1) *in* $[0,T]$. *Then it holds that*

$$\begin{aligned} &\frac{d}{dt}\|u_\varepsilon(t)\|_\infty \le \beta\delta\|w_\varepsilon(t)\|_\infty - (c+f)\|u_\varepsilon(t)\|_\infty, \\ &\frac{d}{dt}\|v_\varepsilon(t)\|_\infty \le f\|u_\varepsilon(t)\|_\infty - h\|v_\varepsilon(t)\|_\infty, \\ &\frac{d}{dt}\|w_\varepsilon(t)\|_\infty \le -\beta\|w_\varepsilon(t)\|_\infty + \alpha\|v_\varepsilon(t)\|_\infty \end{aligned} \tag{2.5}$$

*for a.e.* $t \in [0,T]$.

*Proof.* By the strong maximum principle, $w_\varepsilon(x,t) > 0$ for $x \in \Omega$ and $t \in (0,T]$. Set $\Omega_{\max}(t) = \{x \in \bar{\Omega} \mid w_\varepsilon(x,t) = \|w_\varepsilon(t)\|_\infty\}$. It follows from Lemma A.1 that, for a.e. $t \in [0,T]$, $\|w_\varepsilon(t)\|_\infty$ is differentiable and

$$\frac{d}{dt}\|w_\varepsilon(t)\|_\infty = \frac{\partial w_\varepsilon}{\partial t}(x^t, t), \tag{2.6}$$

where $x^t$ is any point in $\Omega_{\max}(t)$. Since $w_\varepsilon(x,t)$ takes its maximum at $x = x^t \in \Omega$, it holds that

$$\Delta w_\varepsilon(x^t,t) \le 0, \text{ and } \nabla w_\varepsilon(x^t,t) = 0.$$

Therefore,

$$\begin{aligned} &\Delta(w_\varepsilon+\varepsilon)^m(x^t,t) \\ &= m(w_\varepsilon(x^t,t)+\varepsilon)^{m-1}\Delta w_\varepsilon(x^t,t) + m(m-1)(w_\varepsilon(x^t,t)+\varepsilon)^{m-2}|\nabla w_\varepsilon(x^t,t)|^2 \\ &\le 0. \end{aligned} \tag{2.7}$$

By using (2.1), (2.6) and (2.7) it holds that

$$\begin{aligned} \frac{d}{dt}\|w_\varepsilon(t)\|_\infty &= (\Delta(w_\varepsilon+\varepsilon)^m - \beta w_\varepsilon + \alpha u_\varepsilon)(x^t,t) \\ &\le -\beta w_\varepsilon(x^t,t) + \alpha v_\varepsilon(x^t,t) \\ &\le -\beta\|w_\varepsilon(t)\|_\infty + \alpha\|v_\varepsilon(t)\|_\infty. \end{aligned}$$

Other inequalities can be shown in a similar manner. □

We will rewrite (2.5) with use of $3 \times 3$ matrix $M$ in the following form:

$$\frac{d}{dt}\begin{pmatrix}\|u_\varepsilon(t)\|_\infty \\ \|v_\varepsilon(t)\|_\infty \\ \|w_\varepsilon(t)\|_\infty\end{pmatrix} \le M\begin{pmatrix}\|u_\varepsilon(t)\|_\infty \\ \|v_\varepsilon(t)\|_\infty \\ \|w_\varepsilon(t)\|_\infty\end{pmatrix}, \tag{2.8}$$

where

$$M = \begin{pmatrix} -(c+f) & 0 & \beta\delta \\ f & -h & 0 \\ 0 & \alpha & -\beta \end{pmatrix}. \tag{2.9}$$

Here ${}^t(u_1,v_1,w_1) \le {}^t(u_2,v_2,w_2)$ means that $u_1 \le u_2, v_1 \le v_2, w_1 \le w_2$. We will derive estimates of $\|u(t)\|_\infty$, $\|v(t)\|_\infty$, $\|w(t)\|_\infty$ in terms of the largest real eigenvalue of $M$.

**Proposition 2.2.** *Assume that* $(u_{0\varepsilon}, v_{0\varepsilon}, w_{0\varepsilon})$ *satisfies* (2.2)*. Then* (2.1) *admits a unique classical solution* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *in* $[0, \infty)$ *and it satisfies*

$$\begin{aligned} &\|u_\varepsilon(t)\|_\infty + \|v_\varepsilon(t)\|_\infty + \|w_\varepsilon(t)\|_\infty \\ &\le Ke^{\lambda_0 t}(\|u_0\|_\infty + \|v_0\|_\infty + \|w_0\|_\infty) \qquad \text{for all } \ t \ge 0, \end{aligned} \tag{2.10}$$

*where* $K$ *is a positive constant depending on* $\Lambda$ *and* $\lambda_0$ *is the largest real eigenvalue of* $M$ *which has the same sign as* $\alpha\delta - (c + f)/f$.

*Proof.* Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be a classical solution of (2.1) in $[0, T]$. We will derive a priori $L^\infty$ estimate of the solution.

Let $\lambda_0$ be the largest real eigenvalue of $M$. On account of Lemma A.3 we can take a left eigenvector of $M$ corresponding to $\lambda_0$ in the form of $(k_1, k_2, 1)$ with $k_i > 0, i = 1, 2$. Multiplying the first inequality of (2.5) by $k_1$ and the second inequality by $k_2$ one can derive

$$\begin{aligned} \begin{pmatrix} k_1 & k_2 & 1 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \|u(t)_\varepsilon\|_\infty \\ \|v(t)_\varepsilon\|_\infty \\ \|w(t)_\varepsilon\|_\infty \end{pmatrix} &\le \begin{pmatrix} k_1 & k_2 & 1 \end{pmatrix} M \begin{pmatrix} \|u(t)_\varepsilon\|_\infty \\ \|v(t)_\varepsilon\|_\infty \\ \|w(t)_\varepsilon\|_\infty \end{pmatrix} \\ &= \lambda_0 \begin{pmatrix} k_1 & k_2 & 1 \end{pmatrix} \begin{pmatrix} \|u(t)_\varepsilon\|_\infty \\ \|v(t)_\varepsilon\|_\infty \\ \|w(t)_\varepsilon\|_\infty \end{pmatrix} \end{aligned}$$

for a.e. $t \in [0, T]$. This implies that

$$\frac{d}{dt}\left\{e^{-\lambda_0 t}(k_1\|u_\varepsilon(t)\|_\infty + k_2\|v_\varepsilon(t)\|_\infty + \|w_\varepsilon(t)\|_\infty)\right\} \le 0 \qquad \text{for a.e. } t \in [0, T];$$

so that

$$\begin{aligned} k_1\|u_\varepsilon(t)\|_\infty + k_2\|v_\varepsilon(t)\|_\infty + \|w_\varepsilon(t)\|_\infty &\le e^{\lambda_0 t}(k_1\|u_{0\varepsilon}\|_\infty + k_2\|v_{0\varepsilon}\|_\infty + \|w_{0\varepsilon}\|_\infty) \\ &\le e^{\lambda_0 t}(k_1\|u_0\|_\infty + k_2\|v_0\|_\infty + \|w_0\|_\infty). \end{aligned}$$

Therefore, by putting $K = \max\{k_1, k_2, 1\}/\min\{k_1, k_2, 1\}$, we obtain (2.10).

In order to complete the proof, we combine (2.10) and regularity theory for parabolic equations with Proposition 2.1. Then it is possible to extend the local solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ in $[0, T]$ to the whole interval $[0, \infty)$ and prove (2.10) for all $t \ge 0$. $\square$

# 3 Convergence of Approximate Solutions

In this section we will show that a family of approximate solutions $\{(u_\varepsilon, v_\varepsilon, w_\varepsilon)\}_{\varepsilon > 0}$ has a subsequence which is convergent in a suitable topology.

**Lemma 3.1.** *For* $\varepsilon \in (0, 1)$ *and* $(u_{0\varepsilon}, v_{0\varepsilon}, w_{0\varepsilon})$ *satisfying* (2.2)*, the solution* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *of* (2.1) *satisfies that for every* $T > 0$

$$\left\|\frac{\partial}{\partial t}(w_\varepsilon + \varepsilon)^m\right\|^2_{L^2(\Omega\times(0,T))} + \sup_{0\le t\le T}\|\nabla(w_\varepsilon(t) + \varepsilon)^m\|_2^2 \le C(1 + (T + 1)e^{\mu T}), \tag{3.1}$$

*where* $C$ *is a positive constant depending only on* $\Lambda, m, u_0, v_0, w_0, \nabla w_0, \Omega$*, and* $\mu$ *is a nonnegative constant depending only on* $\Lambda, m$.

*Proof.* Taking $L^2(\Omega)$-inner product of the third equation of (2.1) with $\frac{\partial}{\partial t}(w_\varepsilon+\varepsilon)^m$, we get

$$\frac{4m}{(m+1)^2}\left\|\frac{\partial}{\partial t}(w_\varepsilon(t)+\varepsilon)^{\frac{m+1}{2}}\right\|_2^2 = -\frac{1}{2}\frac{d}{dt}\|\nabla(w_\varepsilon(t)+\varepsilon)^m\|_2^2 - \frac{m\beta}{m+1}\frac{d}{dt}\|w_\varepsilon(t)+\varepsilon\|_{m+1}^{m+1} + \beta\varepsilon\frac{d}{dt}\|w_\varepsilon(t)+\varepsilon\|_m^m + \alpha\int_\Omega v_\varepsilon\frac{\partial}{\partial t}(w_\varepsilon+\varepsilon)^m\,dx$$

for $t\in[0,T]$. Integrating the above identity with respect to $t$ we have

$$\begin{aligned}\frac{4m}{(m+1)^2}\int_0^t\left\|\frac{\partial}{\partial t}(w_\varepsilon(s)+\varepsilon)^{\frac{m+1}{2}}\right\|_2^2 ds + \frac{1}{2}\|\nabla(w_\varepsilon(t)+\varepsilon)^m\|_2^2 \\ \le \frac{1}{2}\|\nabla(w_{0\varepsilon}+\varepsilon)^m\|_2^2 + \frac{m\beta}{m+1}\|w_{0\varepsilon}+\varepsilon\|_{m+1}^{m+1} \\ + \beta\varepsilon\|w_\varepsilon(t)+\varepsilon\|_m^m + \alpha\int_0^t\int_\Omega v_\varepsilon\frac{\partial}{\partial t}(w_\varepsilon+\varepsilon)^m\,dx\,ds. \quad (3.2)\end{aligned}$$

Integration by parts together with the second equation of (2.1) leads us to

$$\begin{aligned}\int_0^t\int_\Omega v_\varepsilon\partial_t(w_\varepsilon+\varepsilon)^m\,dx\,ds &= -\int_0^t\int_\Omega\frac{\partial v_\varepsilon}{\partial t}(w_\varepsilon+\varepsilon)^m\,dx\,ds \\ &\quad + \int_\Omega v_\varepsilon(t)(w_\varepsilon(t)+\varepsilon)^m\,dx - \int_\Omega v_{0\varepsilon}(w_{0\varepsilon}+\varepsilon)^m\,dx \\ &= -\int_0^t\int_\Omega(fu_\varepsilon - hv_\varepsilon)(w_\varepsilon+\varepsilon)^m\,dx\,ds \\ &\quad + \int_\Omega v_\varepsilon(t)(w_\varepsilon(t)+\varepsilon)^m\,dx - \int_\Omega v_{0\varepsilon}(w_{0\varepsilon}+\varepsilon)^m\,dx \\ &\le h\int_0^t\int_\Omega v_\varepsilon(w_\varepsilon+\varepsilon)^m\,dx\,ds + \int_\Omega v_\varepsilon(t)(w_\varepsilon(t)+\varepsilon)^m\,dx\end{aligned} \tag{3.3}$$

for all $t\in[0,T]$. In (3.2)

$$\|\nabla(w_{0\varepsilon}+\varepsilon)^m\|_2 \le m\|w_{0\varepsilon}+\varepsilon\|_\infty^{m-1}\|\nabla w_{0\varepsilon}\|_2; \tag{3.4}$$

so that the right-hand side is bounded from above by a positive constant depending only on $m$, $\|w_0\|_\infty$ and $\|\nabla w_0\|_2$ because $\|w_{0\varepsilon}\|_\infty \le \|w_0\|_\infty$ and $\lim_{\varepsilon\to 0}\|\nabla w_{0\varepsilon}\|_2 = \|\nabla w_0\|_2$ (see (2.2)). Therefore, it follows from (2.2), (3.2), (3.3) and (3.4) that

$$\begin{aligned}\frac{4m}{(m+1)^2}\int_0^t\left\|\frac{\partial}{\partial t}(w_\varepsilon(s)+\varepsilon)^{\frac{m+1}{2}}\right\|_2^2 ds + \frac{1}{2}\|\nabla(w_\varepsilon(t)+\varepsilon)^m\|_2^2 \\ \le C_1\left(1+\|w_\varepsilon(t)\|_\infty^m + \|v_\varepsilon(t)\|_\infty\|w_\varepsilon(t)\|_\infty^m + t\sup_{0\le s\le t}\|v_\varepsilon(s)\|_\infty\|w_\varepsilon(s)\|_\infty^m\right),\end{aligned}$$

where $C_1$ a positive constant depending only on $\Lambda, m, u_0, v_0, w_0, \nabla w_0$ and $\Omega$. By applying

Proposition 2.2 to this inequality, we obtain

$$\begin{aligned}
&\frac{4m}{(m+1)^2}\int_0^t \left\| \frac{\partial}{\partial t}(w_\varepsilon(s)+\varepsilon)^{\frac{m+1}{2}} \right\|_2^2 ds + \frac{1}{2}\|\nabla(w_\varepsilon(t)+\varepsilon)^m\|_2^2 \\
&\le C_2\left(1+e^{m\lambda_0 t}+e^{(m+1)\lambda_0 t}+te^{(m+1)\lambda_0^+ t}\right) \\
&\le C_3\left(1+(t+1)e^{(m+1)\lambda_0^+ t}\right),
\end{aligned}$$

where $C_2$ and $C_3$ are positive constants depending only on $\Lambda, m, u_0, v_0, w_0, \nabla w_0$ and $\Omega$. In view of the following identity

$$\left|\frac{\partial}{\partial t}(w_\varepsilon+\varepsilon)^m\right| = \frac{2m}{m+1}(w_\varepsilon+\varepsilon)^{\frac{m-1}{2}}\left|\frac{\partial}{\partial t}(w_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\right|,$$

we obtain (3.1). □

**Lemma 3.2.** *Under the same assumption as Lemma 3.1, there exists a sequence* $\{\varepsilon_n\}_n$ *with* $\lim_{n\to\infty}\varepsilon_n = 0$ *such that* $\{(w_{\varepsilon_n}+\varepsilon_n)^m\}_n$ *is convergent in* $C([0,T];L^2(\Omega))$ *and* $\{w_{\varepsilon_n}\}_n$ *is convergent in* $C([0,T];L^{2m}(\Omega))$ *for any* $T>0$.

*Proof.* It follows from Lemma 3.1 and Poincaré's inequality that $\{(w_\varepsilon+\varepsilon)^m\}_\varepsilon$ is uniformly bounded in $C([0,T];H_0^1(\Omega))$ for any $T>0$. It also follows from Lemma 3.1 that, for $0\le s\le t\le T$,

$$\begin{aligned}
&\|(w_\varepsilon(t)+\varepsilon)^m-(w_\varepsilon(s)+\varepsilon)^m\|_2 \le \int_s^t \left\|\frac{\partial}{\partial t}(w_\varepsilon(\tau)+\varepsilon)^m\right\|_2 d\tau \\
&\le \left(\int_s^t \left\|\frac{\partial}{\partial t}(w_\varepsilon(\tau)+\varepsilon)^m\right\|_2^2 d\tau\right)^{1/2} |t-s|^{1/2} \le M_T|t-s|^{1/2}
\end{aligned}$$

with some $M_T>0$ independent of $\varepsilon$. This fact implies that $\{(w_\varepsilon+\varepsilon)^m\}_\varepsilon$ is equicontinuous in $L^2(\Omega)$. Since $\{(w_\varepsilon+\varepsilon)^m\}_\varepsilon$ is unifomly bounded in $C([0,T]:H_0^1(\Omega)$ for any $T>0$ by (3.1) and the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, it follows from Ascoli–Arzela's theorem that $\{(w_\varepsilon+\varepsilon)^m\}_\varepsilon$ is compact in $C([0,T];L^2(\Omega))$ for any $T>0$. Hence one can choose a sequence $\{\varepsilon_n\}_n$ with $\lim_{n\to\infty}\varepsilon_n=0$ such that $\{(w_{\varepsilon_n}+\varepsilon_n)^m\}_n$ is convergent with respect to $L^2(\Omega)$-norm in $[0,T]$ for any $T>0$. If we use the following inequality

$$|a-b|^{2m}\le|a^m-b^m|^2 \qquad \text{for all } \ a,b\ge 0,$$

we can also show that $\{w_{\varepsilon_n}+\varepsilon_n\}_n$ is convergent in $C([0,T];L^{2m}(\Omega)\}$.

Finally, note

$$\|w_\varepsilon-w_\eta\|_{2m}\le\|(w_\varepsilon+\varepsilon)-(w_\eta+\eta)\|_{2m}+|\varepsilon-\eta|\ |\Omega|^{1/2m}$$

for all $\varepsilon,\eta>0$; then it is easy to see that $\{w_{\varepsilon_n}\}_n$ is also convergent in $C([0,T];L^{2m}(\Omega)\}$ for any $T>0$. □

**Lemma 3.3.** *Under the same assumption as Lemma 3.1, let $\{u_\varepsilon, v_\varepsilon, w_\varepsilon\}$ be the solution of (2.1) in $[0,\infty)$. For any $\varepsilon$, $\eta$, $T>0$ and $r \in [1,\infty)$, it holds that*

$$\begin{aligned}&\sup_{0\le t\le T}\|u_\varepsilon(t)-u_\eta(t)\|_r+\sup_{0\le t\le T}\|v_\varepsilon(t)-v_\eta(t)\|_r\\&\le C\left(\|u_{0\varepsilon}-u_{0\eta}\|_r+\|v_{0\varepsilon}-v_{0\eta}\|_r+\sup_{0\le t\le T}\|w_\varepsilon(t)-w_\eta(t)\|_r\right),\end{aligned}\tag{3.5}$$

*where $C$ is a positive constant depending only on $\Lambda, u_0, v_0, w_0$ and $T$.*

*Proof.* For simplicity, set $U=u_\varepsilon-u_\eta$, $V=v_\varepsilon-v_\eta$ and $W=w_\varepsilon-w_\eta$. Then $U$ and $V$ satisfy the problem

$$\begin{cases}\partial_t U=\beta\delta W-\gamma(v_\varepsilon)U-\gamma'\left((v_\varepsilon+v_\eta)/2\right)u_\eta V-fU & \text{in } \Omega\times(0,\infty),\\ \partial_t V=fU-hV & \text{in } \Omega\times(0,\infty),\\ U(x,0)=u_{0\varepsilon}(x)-u_{0\eta}(x),\quad V(x,0)=v_{0\varepsilon}(x)-v_{0\eta}(x) & \text{in } \Omega.\end{cases}$$

Multiplying the first equation of the above system by $U|U|^{r-2}$ and integrating the resulting expression over $\Omega$, we see from Proposition 2.2 that

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\|U(t)\|_r^r&=\beta\delta\int_\Omega WU|U|^{r-2}\,dx-\int_\Omega\gamma(v_\varepsilon)|U|^r\,dx\\&\quad-\int_\Omega\gamma'\left((v_\varepsilon+v_\eta)/2\right)u_\eta VU|U|^{r-2}\,dx-f\|U(t)\|_r^r\\&\le\beta\delta\|W(t)\|_r\|U(t)\|_r^{r-1}+C_1\|V(t)\|_r\|U(t)\|_r^{r-1}-f\|U(t)\|_r^r\end{aligned}$$

for $t\in[0,T]$ with some $C_1>0$ depending only on $\Lambda, u_0, v_0, w_0$ and $T$. Similarly,

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\|V(t)\|_r^r&=f\int_\Omega UV|V|^{r-2}\,dx-h\|V(t)\|_r^r\\&\le f\|U(t)\|_r\|V(t)\|_r^{r-1}-h\|V(t)\|_r^r\end{aligned}$$

for $t\in[0,T]$. It follows from the above inequalities that

$$\begin{aligned}&\frac{d}{dt}\|U(t)\|_r\le\beta\delta\|W(t)\|_r+C_1\|V(t)\|_r-f\|U(t)\|_r,\\&\frac{d}{dt}\|V(t)\|_r\le f\|U(t)\|_r-h\|V(t)\|_r;\end{aligned}$$

so that, for $t\in[0,T]$,

$$\frac{d}{dt}(\|U(t)\|_r+\|V(t)\|_r)\le\beta\delta\|W(t)\|_r+C_1\|V(t)\|_r$$

for all $t\in[0,T]$. Hence

$$\begin{aligned}\|U(t)\|_r+\|V(t)\|_r&\le e^{C_1T}(\|u_{0\varepsilon}-u_{0\eta}\|_r+\|v_{0\varepsilon}-v_{0\eta}\|_r)+\beta\delta\int_0^t e^{C_1(t-s)}\|W(s)\|_r\,ds\\&\le C_2\left(\|u_{0\varepsilon}-u_{0\eta}\|_r+\|v_{0\varepsilon}-v_{0\eta}\|_r+\sup_{0\le s\le T}\|W(s)\|_r\right)\end{aligned}$$

for all $t\in[0,T]$ with positive constant $C_2=\max\{e^{C_1T},\ \beta\delta e^{C_1T}/C_1\}$. Thus we obtain (3.5). □

We can now establish the following lemma by the same argument as Ishida and Yokota [8].

**Lemma 3.4.** *Under the same assumption as Lemma 3.1, let* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *be the solution of* (2.1). *Then there esist a sequence* $\{\varepsilon_n\}_n$ *with* $\lim_{n\to\infty}\varepsilon_n = 0$ *and functions* $u, v, w$ *such that, for any* $r \geq 1$ *and* $T > 0$,

$$u_{\varepsilon_n} \to u \qquad \textit{strongly in } C([0,T]; L^r(\Omega)), \tag{3.6}$$

$$v_{\varepsilon_n} \to v \qquad \textit{strongly in } C([0,T]; L^r(\Omega)), \tag{3.7}$$

$$w_{\varepsilon_n} \to w \qquad \textit{strongly in } C([0,T]; L^r(\Omega)), \tag{3.8}$$

$$\nabla(w_{\varepsilon_n}+\varepsilon_n)^m \rightharpoonup \nabla w^m \qquad \textit{weakly in } L^2(0,T; L^2(\Omega)), \tag{3.9}$$

*as* $\varepsilon_n \to 0$.

*Proof.* It follows from Lemma 3.2 that there exist a subsequence $\{w_{\varepsilon_n}\}_n$ and a function $w \in C([0,\infty); L^{2m}(\Omega))$ such that

$$(w_{\varepsilon_n} + \varepsilon_n)^m \to w^m \qquad \text{strongly in } C([0,T]; L^2(\Omega) \ \text{ for any } \ T > 0 \tag{3.10}$$

and that

$$w_{\varepsilon_n} \to w \qquad \text{strongly in } C([0,T]; L^{2m}(\Omega)) \ \text{ for any } \ T > 0. \tag{3.11}$$

Therefore, Lemma 3.3 allows us to show that both $\{u_{\varepsilon_n}\}$ and $\{v_{\varepsilon_n}\}$ are Cauchy sequences in $C([0,T]; L^{2m}(\Omega))$ for any $T > 0$. So there exist $u, v \in C([0,\infty); L^{2m}(\Omega))$ such that

$$\begin{cases} u_{\varepsilon_n} \to u & \text{strongly in } C([0,T]; L^{2m}(\Omega)), \\ v_{\varepsilon_n} \to v & \text{strongly in } C([0,T]; L^{2m}(\Omega)) \end{cases} \tag{3.12}$$

for any $T > 0$. Here it should be noted that $u, v$ and $w$ satisfy

$$0 \leq u(x,t), v(x,t), w(x,t) \leq Ke^{\lambda_0 t}(\|u_0\|_\infty + \|v_0\|_\infty + \|w_0\|_\infty) \tag{3.13}$$

for a.e. $x \in \Omega$ and $t \geq 0$. Indeed, for each $t \geq 0$, there exist subsequences $\{u_{\varepsilon'_n}\}$, $\{v_{\varepsilon'_n}\}$, $\{w_{\varepsilon'_n}\}$ such that $u_{\varepsilon'_n}(x,t) \to u(x,t)$, $v_{\varepsilon'_n}(x,t) \to v(x,t)$, $w_{\varepsilon'_n}(x,t) \to w(x,t)$ for a.e. $x \in \Omega$ as $\varepsilon'_n \to 0$. Since $(u_{\varepsilon'_n}, v_{\varepsilon'_n}, w_{\varepsilon'_n})$ satisfy (2.10), it is easy to see that (3.13) holds ture.

Making use of Hölder's iniquality, (2.10) and (3.13) we can see that convergence properties (3.11) and (3.12) in $C([0,T]; L^{2m}(\Omega))$ yield assertions of convergence (3.6), (3.7) and (3.8) in $C([0,T]; L^r(\Omega))$ for any $r \geq 1$.

Finally it remains to prove (3.9). From Lemma 3.1 there exists a subsequence of $\{w_{\varepsilon_n}\}_n$, which is still denoted by $\{w_{\varepsilon_n}\}_n$, such that $\{\nabla(w_{\varepsilon_n} + \varepsilon_n)^m\}$ is weakly convergent in $L^2(0,T; L^2(\Omega))$ for every $T > 0$. Thus there exists a function $\xi \in L^2_{l0c}(0,\infty; L^2(\Omega))$ satisfying

$$\nabla(w_{\varepsilon_n} + \varepsilon_n)^m \rightharpoonup \xi \qquad \text{weakly in } L^2(0,T; L^2(\Omega)) \tag{3.14}$$

for every $T > 0$. Hence, by using (3.10) and (3.14), it holds that

$$\begin{aligned}\int_0^T \int_\Omega \xi\varphi \, dx \, dt &= \lim_{n\to\infty} \left( \int_0^T \int_\Omega \nabla(w_{\varepsilon_n} + \varepsilon_n)^m \varphi \, dx \, dt \right) \\ &= - \lim_{n\to\infty} \int_0^T \int_\Omega (w_{\varepsilon_n} + \varepsilon_n)^m \nabla\varphi \, dx \, dt \\ &= - \int_0^T \int_\Omega w^m \nabla\varphi \, dx \, dt\end{aligned}$$

for any $\varphi \in C_0^\infty(\Omega \times (0,T))$. The above relation implies that $w^m$ is weakly differentiable with respect to $x$, and $\xi = \nabla w^m$. Thus we complete the proof. □

*Proof of Theorem 1.1.* First we will prove that (1.1) has a weak solution. For $\varepsilon \in (0,1)$, let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the solution of (2.1) and let $\{u_{\varepsilon_n}, v_{\varepsilon_n}, w_{\varepsilon_n}\}_n$ be the subsequence given in Lemma 3.4. Let $(u,v,w)$ satisfy convergence (3.6), (3.7), (3.8) and (3.9). Then $(u_{\varepsilon_n}, v_{\varepsilon_n}, w_{\varepsilon_n})$ satisfies

$$\begin{aligned}u_{\varepsilon_n}(x,t) &= u_{0\varepsilon_n}(x) + \int_0^t \{\beta\delta w_{\varepsilon_n}(x,s) - \gamma(v_{\varepsilon_n}(x,s))u_{\varepsilon_n}(x,s) - f u_{\varepsilon_n}(x,s)\} \, ds, \\ v_{\varepsilon_n}(x,t) &= v_{0\varepsilon_n}(x) + \int_0^t \{f u_{\varepsilon_n}(x,s) - h v_{\varepsilon_n}(x,s)\} \, ds,\end{aligned}$$

and

$$\begin{aligned}&- \int_\Omega w_{0\varepsilon_n}(x)\varphi(x,0) \, dx - \int_0^T \int_\Omega w_{\varepsilon_n}\varphi_t \, dx \, dt \\ &\qquad = - \int_0^T \int_\Omega \nabla(w_{\varepsilon_n} + \varepsilon)^m \nabla\varphi \, dx \, dt - \beta \int_0^T \int_\Omega w_{\varepsilon_n}\varphi \, dx \, dt + \alpha \int_0^T \int_\Omega v_{\varepsilon_n}\varphi \, dx \, dt\end{aligned}$$

for all $\varphi \in C^1([0,T]; C_0^\infty(\Omega))$ satisfying $\varphi(\cdot, T) \equiv 0$. Letting $\varepsilon_n \to 0$ in the the above identities we find that $(u,v,w)$ satisfies the properties of Definition 1.1. Indeed, for $t \in [0,T]$,

$$\begin{aligned}&\left\| \int_0^t \gamma(v_{\varepsilon_n}(s))u_{\varepsilon_n}(s) \, ds - \int_0^t \gamma(v(s))u(s) \, ds \right\|_r \\ &\le \int_0^t \|\gamma(v_{\varepsilon_n}(s)) - \gamma(v(s))\|_r \, \|u_{\varepsilon_n}(s)\|_\infty \, ds + \int_0^t \left\|\gamma(v(s))\right\|_\infty \|u_{\varepsilon_n}(s) - u(s)\|_r \, ds \\ &\le C_T \left\{ \sup_{0\le s\le T} \|v_{\varepsilon_n}(s) - v(s)\|_r + \sup_{0\le s\le T} \|u_{\varepsilon_n}(s) - u(s)\|_r \right\}\end{aligned}$$

with some $C_T > 0$. Here we have used (2.10) and (3.13). Note that the right-hand side of the above inequalities tends to zero as $\varepsilon_n \to 0$ by Lemma 3.4.

Finally, in order to prove (1.4), we use Proposition 2.2 to derive

$$0 \le u_{\varepsilon_n}(x,t), v_{\varepsilon_n}(x,t), w_{\varepsilon_n}(x,t) \le K e^{\lambda_0 t}(\|u_0\|_\infty + \|v_0\|_\infty + \|w_0\|_\infty)$$

for all $x \in \Omega$ and $t \ge 0$. For each $t > 0$, it is possible to show with use of Lemma 3.4 that a suitable subsequence of $\{u_{\varepsilon_n}(x,t), v_{\varepsilon_n}(x,t), w_{\varepsilon_n}(x,t)\}$ converges to $(u(x,t), v(x,t), w(x,t))$ at a.e. $x \in \Omega$. Hence (1.4) can be derived from the above estimate. □

# 4 Uniformly bounded solutions

In this section we will show that the weak solution in the previous section is uniformly bounded with respect to $L^\infty(\Omega)$-norm for all $t \geq 0$. For this purpose we will derive various estimates for approximate solutions $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ of (2.1).

## 4.1 Uniform $L^2$ estimate

The first result asserts the uniform boundedness of $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ with respect to $L^2(\Omega)$-norm.

**Proposition 4.1.** *Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the solution of (2.1). Then it holds that*

$$\sup_{t\geq p}\{\|u_\varepsilon(t)\|_2, \|v_\varepsilon(t)\|_2, \|w_\varepsilon(t)\|_2\} \leq K_1(1 + \|u_0\|_\infty + \|v_0\|_\infty + \|w_0\|_\infty), \tag{4.1}$$

*where $K_1$ is a positive constant depending only on $\Omega$ and $\Lambda$.*

*Proof.* We will employ the standard energy method as in the work of Chaun and Yagi [7, Proposition 5.1] or Shirai, Chuan and Yagi [14, Proposition 5.1]. For the sake of simplicity, we write $(u, v, w)$ in place of $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ for the solution of (2.1).

There exists $\tilde{a} > 0$ such that

$$\gamma(v) \geq \tilde{a}v^2 \qquad \text{for all} \;\; v \in \mathbb{R}. \tag{4.2}$$

By taking $L^2(\Omega)$-inner product of the first equation of (2.1) with $u$, it holds that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx &= \beta\delta\int_\Omega wu\,dx - \int_\Omega \gamma(v)u^2\,dx - f\int_\Omega u^2\,dx \\ &\leq \frac{\beta^2\delta^2}{2f}\int_\Omega w^2\,dx - \tilde{a}\int_\Omega (uv)^2\,dx - \frac{f}{2}\int_\Omega u^2\,dx, \end{aligned} \tag{4.3}$$

where Young's inequality is used. Similarly,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2\,dx = f\int_\Omega uv\,dx - h\int_\Omega v^2\,dx. \tag{4.4}$$

Taking $L^2(\Omega)$-inner product of the third equation of (2.1) with $w$ leads to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2\,dx = -\int_\Omega \nabla(w+\varepsilon)^m\cdot\nabla w\,dx - \beta\int_\Omega w^2\,dx + \alpha\int_\Omega vw\,dx$$

Note that

$$\int_\Omega \nabla(w+\varepsilon)^m\cdot\nabla w\,dx = m\int_\Omega (w+\varepsilon)^{m-1}|\nabla w|^2\,dx \geq 0.$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2\,dx \leq -\beta\int_\Omega w^2\,dx + \alpha\int_\Omega vw\,dx \leq -\frac{\beta}{2}\int_\Omega w^2\,dx + \frac{\alpha^2}{2\beta}\int_\Omega v^2\,dx. \tag{4.5}$$

Let $k_1$ and $k_2$ be positive numbers to be determined later. Multiply (4.3) by $k_1$, (4.4) by $k_2$ and add the resulting expressions to (4.5):

$$\begin{aligned}&\frac{k_1}{2}\frac{d}{dt}\|u(t)\|_2^2+\frac{k_2}{2}\frac{d}{dt}\|v(t)\|_2^2+\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2\\ \le&-\frac{k_1 f}{2}\|u(t)\|_2^2-\left(k_2h-\frac{\alpha^2}{2\beta}\right)\|v(t)\|_2^2-\left(\frac{\beta}{2}-\frac{\beta^2\delta^2k_1}{2f}\right)\|w(t)\|_2^2\\ &+\int_\Omega\left\{-k_1\tilde{a}(uv)^2+k_2fuv\right\}dx.\end{aligned}$$

Here it should be noted that

$$-k_1\tilde{a}U^2+k_2fU\le C_1(k_1,k_2):=\frac{k_2^2f^2}{4\tilde{a}k_1}\qquad\text{for all}\quad U\in\mathbb{R}.$$

If we take $k_1$ and $k_2$ satisfying

$$0<k_1\le\frac{f}{2\beta\delta^2}\quad\text{and}\quad k_2\ge\frac{\alpha^2}{\beta h},$$

we can prove from the above considerations

$$\begin{aligned}&\frac{d}{dt}(k_1\|u(t)\|_2^2+k_2\|v(t)\|_2^2+\|w(t)\|_2^2)\\ &\qquad+k_1f\|u(t)\|_2^2+\frac{\alpha^2}{\beta}\|v(t)\|_2^2+\frac{\beta}{2}\|w(t)\|_2^2\le 2C_1(k_1,k_2)|\Omega|.\end{aligned}\tag{4.6}$$

Choose $\rho=\min\{f,\alpha^2/k_1\beta,\beta/2\}$ and set

$$X(t)=k_1\|u(t)\|_2^2+k_2\|v(t)\|_2^2+\|w(t)\|_2^2:$$

then it follows from (4.6) that

$$\frac{d}{dt}X(t)+\rho X(t)\le 2C_1|\Omega|\quad\text{for all}\quad t\ge 0.$$

Solving this differential inequality yields

$$X(t)\le X(0)e^{-\rho t}+2C_1|\Omega|(1-e^{-\rho t})/\rho\quad\text{for all}\ \ t\ge 0.\tag{4.7}$$

Since (2.2) implies

$$X(0)\le|\Omega|(k_1\|u_0\|_\infty^2+k_2\|v_0\|_\infty^2+\|w_0|_\infty^2),$$

(4.1) comes from (4.7). □

## 4.2 Uniform $L^\infty$ estimate

Since we have established uniform $L^2(\Omega)$-bounds for approximate solutions $(u_\varepsilon,v_\varepsilon,w_\varepsilon)$ in Proposition 4.1, we will employ an iterative method developed by Alikakos [1] in order to derive their $L^\infty(\Omega)$-estimates. The following inequality plays a key role in this subsection.

**Lemma 4.1.** *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and let $m > 1$. For each $\mu > 0$, set $\lambda = \kappa\mu + m - 1$ with*

$$\begin{cases} \kappa = 2 & \text{if} \quad N = 1, \\ 1 < \kappa < 2 & \text{if} \quad N = 2, \\ \kappa = 1 + 2/N & \text{if} \quad N \geq 3, \end{cases}$$

*where $\kappa$ can be taken as an arbitrary number in $(1, 2)$ for the case $N = 2$. For every nonegative function $w \in C^1(\Omega)$ with $w = 0$ on $\partial\Omega$, it holds that*

$$\left\|w^{\lambda}\right\|_1 \leq C_1 \left\|\nabla w^{\frac{\lambda+m-1}{2}}\right\|_2 \left\|w^{\mu}\right\|_1^{\frac{\kappa}{2}}, \tag{4.8}$$

*where $C_1$ is a positive constant depending only on $N, \Omega$ and $\kappa$.*

*Proof.* Let $w$ be any $C^1(\Omega)$ function satisfying $w|_{\partial\Omega} = 0$. We begin with the case $N \geq 3$. Since $\lambda = \frac{\lambda+m-1}{2} + \frac{N+2}{2N}\mu = \frac{\lambda+m-1}{2} + \frac{\kappa}{2}\mu$, it follows from Hölder's inequality that

$$\|w^{\lambda}\|_1 \leq \left\|w^{\frac{\lambda+m-1}{2}}\right\|_{\frac{2N}{N-2}} \left\|w^{\frac{N+2}{2N}\mu}\right\|_{\frac{2N}{N+2}} = \left\|w^{\frac{\lambda+m-1}{2}}\right\|_{\frac{2N}{N-2}} \|w^{\mu}\|_1^{\frac{\kappa}{2}}.$$

By the Gagliardo–Nirenberg–Sobolev inequality (see, e.g., [3] or [4])

$$\left\|w^{\frac{\lambda+m-1}{2}}\right\|_{\frac{2N}{N-2}} \leq C_1 \left\|\nabla w^{\frac{\lambda+m-1}{2}}\right\|_2,$$

where $C_1$ is a positive constant depending on $N$ and $\Omega$. Hence combining the above inequalities we get (4.8).

We next consider the case $N = 2$. Let $\kappa$ be an arbitrary number in the interval $(1, 2)$. In view of $\lambda = \frac{\lambda+m-1}{2} + \frac{\kappa}{2}\mu$, it is seen from Hölder's inequality that

$$\left\|w^{\lambda}\right\|_1 \leq \left\|w^{\frac{\lambda+m-1}{2}}\right\|_{\frac{2}{2-\kappa}} \left\|w^{\frac{\kappa}{2}\mu}\right\|_{\frac{2}{\kappa}} = \left\|w^{\frac{\lambda+m-1}{2}}\right\|_{\frac{2}{2-\kappa}} \|w^{\mu}\|_1^{\frac{\kappa}{2}}.$$

In order to show (4.8), it is sufficient to use the Gagliardo–Nirenberg–Sobolev inequality for the case $N = 2$:

$$\left\|w^{\frac{\lambda+m-1}{2}}\right\|_{\frac{2}{2-\kappa}} \leq C_2 \left\|\nabla w^{\frac{\lambda+m-1}{2}}\right\|_2,$$

where $C_2$ is a positive constant depending on $N$ and $\Omega$

Finally, in the case $N = 1$, note that $\kappa = 2$ and $\lambda = \frac{\lambda+m-1}{2} + \mu$. Therefore,

$$\left\|w^{\lambda}\right\|_1 \leq \left\|w^{\frac{\lambda+m-1}{2}}\right\|_{\infty} \left\|w^{\mu}\right\|_1^{\frac{\kappa}{2}}.$$

Then one can show (4.8) with use of Morrey's inequality (see Brezis [3, Theorem 9.12])

$$\left\|w^{\frac{\lambda+m-1}{2}}\right\|_{\infty} \leq C_3 \left\|\nabla w^{\frac{\lambda+m-1}{2}}\right\|_2,$$

where $C_3$ is a positive constant depending only on $\Omega$.

□

In what follows, we assume $h = 1$ without loss of generality. Indeed, if we introduce new variables $(y, s)$ by $x = y/\sqrt{h}$, $t = s/h$ and define $(\tilde{u}, \tilde{v}, \tilde{w})$ by

$$\tilde{u}(y,s) = u\left(\frac{y}{\sqrt{h}}, \frac{s}{h}\right), \quad \tilde{v}(y,s) = v\left(\frac{y}{\sqrt{h}}, \frac{s}{h}\right), \quad \tilde{w}(y,s) = w\left(\frac{y}{\sqrt{h}}, \frac{s}{h}\right),$$

then it follows from (1.1) that $(\tilde{u}, \tilde{v}, \tilde{w})$ satisfies

$$\begin{cases} \tilde{u}_t = \delta\tilde{\beta}\tilde{w} - \tilde{\gamma}(\tilde{v})\tilde{u} - \tilde{f}\tilde{u}, \\ \tilde{v}_t = \tilde{f}\tilde{u} - \tilde{v}, \\ \tilde{w}_t = \Delta_y(\tilde{w}^m) - \tilde{\beta}\tilde{w} + \tilde{\alpha}\tilde{v} \end{cases}$$

with $\tilde{f} = f/h, \tilde{\alpha} = \alpha/h, \tilde{\beta} = \beta/h$ and $\tilde{\gamma}(v) = \gamma(v)$.

We will intend to estimate general $L^p(\Omega)$-norms of approximate solutuins $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$. Hereafter, we sometimes write $(u, v, w)$ in place of $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ as in the proof of Proposition 4.1.

**Lemma 4.2.** *Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the solution of (2.1) and let $p$ be any number satisfying $p \geq 2$. Then it holds that*

$$\frac{d}{dt}\|fu_\varepsilon(t)\|_p^p \leq f\|\beta w_\varepsilon(t)\|_p^p - f\|fu_\varepsilon(t)\|_p^p, \tag{4.9}$$

$$\frac{d}{dt}\|v_\varepsilon(t)\|_p^p \leq \|fu_\varepsilon(t)\|_p^p - \|v_\varepsilon(t)\|_p^p, \tag{4.10}$$

$$\frac{d}{dt}\|\beta w_\varepsilon(t)\|_p^p \leq -c_m\beta^p \left\|\nabla w_\varepsilon^{\frac{p+m-1}{2}}(t)\right\|_2^2 + \beta(\alpha-1)p\|\beta w_\varepsilon(t)\|_p^p + \alpha\beta\|v_\varepsilon(t)\|_p^p \tag{4.11}$$

*for all $t \geq 0$, where $c_m$ is a positive number depending only on $m$.*

*Proof.* Let $p \geq 2$ be any number. We will begin with the proof of (4.11). Multiplying the third equation of (2.1) by $w^{p-1}$ and integrating the resulting expression over $\Omega$, we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega w^p\,dx = -\int_\Omega \nabla(w+\varepsilon)^m \cdot \nabla w^{p-1}\,dx + \int_\Omega \left(-\beta w^p + \alpha v w^{p-1}\right)\,dx.$$

Here it should be noted that

$$\begin{aligned} &\int_\Omega \nabla(w+\varepsilon)^m \cdot \nabla w^{p-1}\,dx = m(p-1)\int_\Omega (w+\varepsilon)^{m-1}w^{p-2}|\nabla w|^2\,dx \\ \geq &m(p-1)\int_\Omega w^{m+p-3}|\nabla w|^2\,dx = \frac{4m(p-1)}{(p+m-1)^2}\int_\Omega \left|\nabla w^{\frac{p+m-1}{2}}\right|^2\,dx \end{aligned}$$

(see [8]). By Young's inequality

$$vw^{p-1} \leq \frac{1}{p}\left(\frac{v}{\beta^{\frac{p-1}{p}}}\right)^p + \frac{p-1}{p}\left(\beta^{\frac{p-1}{p}}w^{p-1}\right)^{\frac{p}{p-1}} \leq \frac{\beta}{p\beta^p}v^p + \beta w^p. \tag{4.12}$$

Therefore,

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega w^p\,dx \le &-\frac{4mp(p-1)}{(p+m-1)^2}\int_\Omega \left|\nabla w^{\frac{p+m-1}{2}}\right|^2\,dx \\
&+\beta(\alpha-1)p\int_\Omega w^p\,dx + \frac{\alpha\beta}{\beta^p}\int_\Omega v^p\,dx.
\end{aligned}
\tag{4.13}
$$

Since $4mp(p-1)/(p+m-1)^2$ is strictly increasing with respect to $p \ge 2$, it should be noted that

$$
\frac{4mp(p-1)}{(p+m-1)^2} \ge \frac{8m}{(m+1)^2} =: c_m \qquad \text{for all } \ p \ge 2.
$$

Hence we get (4.11) by multiplying (4.13) by $\beta^p$.

Multiplying the first equation of (2.1) by $u^{p-1}$ and integrating the result over $\Omega$ one can see

$$
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,dx = \int_\Omega (\delta\beta w - \gamma(v)u - fu)u^{p-1}\,dx \le \beta\int_\Omega wu^{p-1}\,dx - f\int_\Omega u^p\,dx,
$$

where we have used $0<\delta<1$ and $\gamma(v)>0$. By Young's inequality

$$
\beta w u^{p-1} \le \frac{f}{p}\left(\frac{\beta w}{f}\right)^p + \frac{p-1}{p} f u^p
$$

in the same way as (4.12). Therefore,

$$
\frac{d}{dt}\int_\Omega u^p\,dx \le \frac{f}{f^p}\int_\Omega (\beta w)^p\,dx - f\int_\Omega u^p\,dx;
$$

which implies (4.9).

Finally, the proof of (4.10) can be carried out in the same way as above; so we omit it. □

**Lemma 4.3.** *Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the solution of (2.1). For each $q \ge 2$, set $p = \kappa q + m - 1$, where $\kappa$ is the number defined in Lemma 4.1. Then it holds that, for any $L > 0$,*

$$
\begin{aligned}
\frac{d}{dt}\|\beta w_\varepsilon(t)\|_p^p \le &-L\|\beta w_\varepsilon(t)\|_p^p + \alpha\beta\|v_\varepsilon(t)\|_p^p \\
&+K_2(L+\beta(\alpha-1)^+p)^2\beta^{m-1}\|(\beta w_\varepsilon(t))^q\|_1^\kappa,
\end{aligned}
\tag{4.14}
$$

*where $K_2$ is a positive constant depending only on $\Omega, N$ and $m$.*

*Proof.* For $q \ge 2$, set $p = \kappa q + m - 1$. Applying Lemma 4.1 and Young's inequality we see that, for any $\rho > 0$,

$$
\begin{aligned}
\rho\|w\|_p^p = \rho\|w^p\|_1 &\le C_1\rho\|\nabla w^{(p+m-1)/2}\|_2\|w^q\|_1^{\kappa/2} \\
&\le c_m\|\nabla w^{(p+m-1)/2}\|_2^2 + C_2\rho^2\|w^q\|_1^\kappa
\end{aligned}
$$

with $C_2 = C_1^2/(4c_m)$. With use of this result one can see from (4.11) that

$$
\begin{aligned}
\frac{d}{dt}\|\beta w(t)\|_p^p \le &-\{\rho - \beta(\alpha-1)p\}\|\beta w(t)\|_p^p + \alpha\beta\|v_\varepsilon(t)\|_p^p \\
&+C_2\rho^2\beta^{m-1}\|(\beta w(t))^q\|_1^\kappa.
\end{aligned}
\tag{4.15}
$$

By taking $\rho = L + \beta(\alpha-1)^+p$ with any $L > 0$ in (4.15), it is possible to obtain (4.14). □

Take any $L > \alpha\beta$ and fix it. As in Section 2, we rewrite (4.9),(4.10),(4.14) with use of $3 \times 3$ matrix $M_1$ in the following form

$$\frac{d}{dt}\begin{pmatrix} \|fu(t)\|_p^p \\ \|v(t)\|_p^p \\ \|\beta w(t)\|_p^p \end{pmatrix} \le M_1 \begin{pmatrix} \|fu(t)\|_p^p \\ \|v(t)\|_p^p \\ \|\beta w(t)\|_p^p \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ K_3 p^2 \|\beta w(t)\|_q^{q\kappa} \end{pmatrix} \tag{4.16}$$

with $p = \kappa q + m - 1$ and

$$M_1 = \begin{pmatrix} -f & 0 & f \\ 1 & -1 & 0 \\ 0 & \alpha\beta & -L \end{pmatrix}.$$

Here $K_3$ is a positive constant depending on $\Omega, N, m, \alpha$ and $\beta$.

**Remark 4.1.** On account of (4.14) , it is possible to repalce $K_3 p^2 \|\beta w(t)\|_q^{p\kappa}$ in (4.16) by $K_3 \|\beta w(t)\|_q^{p\kappa}$ in the case $\alpha \le 1$. In this case, subsequent arguments will become simpler.

By virtue of $L > \alpha\beta$, the largest real eigenvalue of $M_1$ is negative; so it is given by $-\sigma$ with $\sigma > 0$. Moreover, as in the proof of Lemma A.3, one can prove that a left eigenvector $(k_1, k_2, 1)$ of $M_1$ corresponding to $-\sigma$ satisfies $k_1 > 0$ and $k_2 > 0$. If we apply $(k_1, k_2, 1)$ to (4.16) and set

$$X_p(t) = k_1 \|fu(t)\|_p^p + k_2 \|v(t)\|_p^p + \|\beta w(t)\|_p^p, \tag{4.17}$$

then we get

$$\frac{d}{dt} X_p(t) \le -\sigma X_p(t) + K_3 p^2 \|\beta w(t)\|_q^{q\kappa}, \tag{4.18}$$

where we have used $(k_1, k_2, 1)M_1 = -\sigma(k_1, k_2, 1)$. By solving (4.18) it follows that

$$\begin{aligned} X_p(t) &\le X_p(0)e^{-\sigma t} + \frac{K_3 p^2}{\sigma}\left(\sup_{t\ge 0} \|\beta w(t)\|_q^q\right)^\kappa (1 - e^{-\sigma t}) \\ &\le \max\left\{X_p(0), \frac{K_3 p^2}{\sigma}\left(\sup_{t\ge 0} \|\beta w(t)\|_q^q\right)^\kappa\right\} \qquad \text{for all } \ t \ge 0. \end{aligned} \tag{4.19}$$

Note that, on accout of (2.2),

$$X_p(0) \le |\Omega| \max\{k_1, k_2\}(\|fu_0\|_\infty + \|v_0\|_\infty + \|\beta w_0\|_\infty)^p.$$

Therefore, if we set $A = \|fu_0\|_\infty + \|v_0\|_\infty + \|\beta w_0\|_\infty + 1$, we see from (4.19) that

$$X_p(t) \le \max\left\{K_4 A^p, \frac{K_3 p^2}{\sigma}\left(\sup_{t\ge 0} \|\beta w(t)\|_q^q\right)^\kappa\right\} \qquad \text{for all } \ t \ge 0 \tag{4.20}$$

with $K_4 = |\Omega| \max\{k_1, k_2\}$.

We will derive $L^p(\Omega)$ bounds from (4.20) combined with Proposition 4.1 by an iterative method as in the work of [1].

**Proposition 4.2.** *Let* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *be the solution of* (2.1)*. Define a sequence* $\{p_\ell\}$ *by*

$$p_1 = 2, \quad p_\ell = \kappa p_{\ell-1} + m - 1, \quad \ell = 2, 3, 4, \cdots, \tag{4.21}$$

*where* $\kappa$ *is the number defined in Lemma 4.1. If* $X_p$ *is defined by* (4.17)*, then there exists a positive constant* $K_2$*, depending only on* $\Lambda, m, N$ *and* $\Omega$*, such that*

$$\sup_{t\ge 0} X_{p_\ell}(t) \le \{K_2(\|u_0\|_\infty + \|v_0\|_\infty + \|w_0\|_\infty + 1)\}^{p_\ell} \qquad \textit{for all } \ \ell \ge 1. \tag{4.22}$$

*Proof.* By the definition (4.21)

$$p_\ell = \kappa^{\ell-1}\left(p_1 + \frac{m-1}{\kappa-1}\right) - \frac{m-1}{\kappa-1} < a^*\kappa^\ell \tag{4.23}$$

with $a^* = 2/\kappa + (m-1)/\kappa(\kappa-1)$ and

$$p_\ell = \kappa p_{\ell-1} + m - 1 > \kappa p_{\ell-1} > \kappa^2 p_{\ell-2} > \cdots > \kappa^{\ell-1}p_1 = 2\kappa^{\ell-1} \geq \kappa^\ell. \tag{4.24}$$

It follows from (4.20) and (4.23) that

$$\sup_{t\geq 0} X_{p_\ell}(t) \leq C_* \max\{A^{p_\ell}, \kappa^{2\ell}(\sup_{t\geq 0} X_{p_{\ell-1}}(t))^\kappa\} \tag{4.25}$$

with $C_* = \max\{K_4, K_3(a^*)^2/\sigma\}$. Then we can deduce from (4.24)that

$$\begin{aligned}\sup_{t\geq 0} X_{p_\ell}(t) \leq \max\Big\{ & C_* A^{p_\ell},\ C_*^{1+\kappa}\kappa^{2\ell}A^{\kappa p_{\ell-1}}, C_*^{1+\kappa+\kappa^2}\kappa^{2\ell+2(\ell-1)\kappa}A^{\kappa^2 p_{\ell-2}}, \ldots, \\ & C_*^{1+\kappa+\kappa^2+\cdots+\kappa^{\ell-2}}\kappa^{2\ell+2(\ell-1)\kappa+2(\ell-2)\kappa^2+\cdots+2\cdot 3\kappa^{\ell-3}}A^{\kappa^{\ell-2}p_2}, \\ & C_*^{1+\kappa+\cdots+\kappa^{\ell-2}}\kappa^{2S_\ell}(\sup_{t\geq 0} X_{p_1}(t))^{\kappa^{\ell-1}}\Big\},\end{aligned} \tag{4.26}$$

where

$$S_\ell = \ell + (\ell-1)\kappa + (\ell-2)\kappa^2 + \cdots + 2\kappa^{\ell-2}. \tag{4.27}$$

We use the following result whose proof can be found in Appendix:

$$S_\ell \leq \gamma_\kappa \kappa^\ell \qquad \text{for all} \quad \ell \geq 1 \tag{4.28}$$

with a positive number $\gamma_\kappa$ depending only on $\kappa$. In (4.26), recall Proposition 4.1, which gives

$$\sup_{tge0} X_{p_1}(t) = \sup_{t\geq 0} X_2(t) \leq \tilde{K}A$$

with a positive constant $\tilde{K}$ depending only on $\Lambda$ and $\Omega$. Moreover, $A > 1$ and we may assume $C_* > 1$ in (4.26). Setting $K_* = \max\{1, \tilde{K}\}$ we see from (4.26) that

$$\begin{aligned}\sup_{t\geq 0} X_{p_\ell}(t) &\leq C_*^{\kappa^{\ell-1}/(\kappa-1)}\kappa^{2S_\ell}\max\{A^{p_\ell}, A^{\kappa p_{\ell-1}}, A^{\kappa^2 p_{\ell-2}}, \ldots, A^{\kappa^{\ell-2}p_2}, (\tilde{K}A)^{\kappa^{\ell-1}}\} \\ &\leq C_*^{\kappa^{\ell-1}/(\kappa-1)}\kappa^{2S_\ell}K_*^{p_\ell}A^{p_\ell}\end{aligned} \tag{4.29}$$

Here we have used (4.24) in the last inequality. By virtue of (4.24) and (4.28), the right-hand side of (4.29) is bounded from above by $C_*^{p_\ell/\kappa(\kappa-1)}\kappa^{2\gamma_\kappa p_\ell}K_*^{p_\ell}A^{p_\ell}$. Thus we complete the proof. □

*Proof of Theorem 1.2.* Take a sequence $\{\varepsilon_n\}_n$ with $\lim_{n\to\infty}\varepsilon_n = 0$, which is given in Lemma 3.4, such that for every $r \geq 2$ and every $T > 0$

$$\lim_{\varepsilon_n\to 0} u_{\varepsilon_n} = u, \quad \lim_{\varepsilon_n\to 0} v_{\varepsilon_n} = v, \quad \lim_{\varepsilon_n\to 0} w_{\varepsilon_n} = w \qquad \text{in} \ \ C([0,T]; L^r(\Omega)),$$

where $(u, v, w)$ is the weak solution of (1.1) given in Theorem 1.1. By Proposition 4.2, $(u_{\varepsilon_n}, v_{\varepsilon_n}, w_{\varepsilon_n})$ satisfies

$$\|fu_{\varepsilon_n}(t)\|_{p_\ell}^{p_\ell} + \|v_{\varepsilon_n}(t)\|_{p_\ell}^{p_\ell} + \|\beta w_{\varepsilon_n}(t)\|_{p_\ell}^{p_\ell} < (K_2A)^{p_\ell}, \quad \ell = 1, 2, 3, \cdots ,$$

for all $t \geq 0$ with $A = \|u_0\|_\infty + \|v_0\|_\infty + \|w_0\|_\infty + 1$. Letting $\varepsilon_n \to 0$ in the above inequality leads us to

$$\|fu(t)\|_{p_\ell}^{p_\ell} + \|v(t)\|_{p_\ell}^{p_\ell} + \|\beta w(t)\|_{p_\ell}^{p_\ell} < (K_2A)^{p_\ell} \qquad \text{for} \quad t \geq 0,$$

which implies

$$\|u(t)\|_{p_\ell} \leq K_2A/f, \quad \|v(t)\|_{p_\ell} \leq K_2A, \quad \|w(t)\|_{p_\ell} \leq K_2A/\beta \quad \text{for} \quad t \geq 0.$$

Therefore, we see

$$\begin{aligned} \|u(t)\|_\infty &= \lim_{p_\ell \to \infty} \|u(t)\|_{p_\ell} \leq K_2A/f, \\ \|v(t)\|_\infty &= \lim_{p_\ell \to \infty} \|v(t)\|_{p_\ell} \leq K_2A, \\ \|w(t)\|_\infty &= \lim_{p_\ell \to \infty} \|w(t)\|_{p_\ell} \leq K_2A/\beta \end{aligned}$$

for $t \geq 0$. □

# A Appendix

**Lemma A.1.** *For $T > 0$, let $u \in C^1([0, T]; C(\bar{\Omega}))$. Then $\|u(t)\|_\infty$ is differentiable at a.e. $t \in [0, T]$ and satisfies*

$$\frac{d}{dt}\|u(t)\|_\infty = u_t(x^t, t) \qquad \textit{for a.e. } t \in [0, T],$$

*where $x^t$ is any point in $\{x \in \bar{\Omega} \,|\, u(x, t) = \|u(t)\|_\infty\}$.*

To prove this lemma, we recall the definition of the duality map and its related properties (see, e.g., Kato [9]).

**Definition A.1.** Let $X$ be a real Banach space. The **duality map** $F$ on $X$ is a multi-valued map from $X$ to its dual space $X^*$, defined by

$$v \mapsto F(v) = \left\{ g \in X^* \mid \langle g, v\rangle = \|v\|^2, \ \|g\| = \|v\| \right\},$$

where $\langle g, v\rangle$ denotes the pairing between $g \in X^*$ and $v \in X$.

**Remark A.1.** We consider the case $X = C(\bar{\Omega})$. For each $v \in C(\bar{\Omega})$ take any $x^* \in \{x \in \bar{\Omega} \,|\, v(x) = \|v\|_\infty\}$ and define a linear functional $g_v : C(\bar{\Omega}) \to \mathbb{R}$ by

$$\langle g_v, w\rangle = \|v\|_\infty w(x^*) \qquad \text{for all } w \in C(\bar{\Omega}).$$

Clearly, $g_v \in X^*$ and it satisfies

$$\langle g_v, v\rangle = \|v\|_\infty v(x^*) = \|v\|_\infty^2, \quad |\,\langle g_v, w\rangle\,| = \|v\|_\infty |w(x^*)| \leq \|v\|_\infty \|w\|_\infty;$$

so that $\|g_v\| = \|v\|_\infty$. These facts imply $g \in F(v)$.

**Lemma A.2.** *Let $X$ be a real Banach space and let $u$ be an $X$-valued function in $[0,\infty)$ such that $u(t)$ has a weak derivative at $t=t_0$ and that $\|u(t)\|$ is differentiable at $t=t_0$. Then it holds that*

$$\|u(t_0)\| \, \frac{d}{dt}\|u(t)\|\Big|_{t=t_0} = \langle\, g, u_t(t_0)\,\rangle$$

*for any $g \in F(u(t_0))$, where $F$ is the duality map on $X$.*

*Proof.* For the proof, see, for instance, [9, Lemma 1.3]. □

*Proof of Lemma A.1.* Note that

$$\big|\|u(t)\|_\infty - \|u(s)\|_\infty\big| \le \|u(t)-u(s)\|_\infty = \left\|\int_s^t u_t(\tau)\,d\tau\right\|_\infty$$
$$\le \int_s^t \|u_t(\tau)\|_\infty\,d\tau \le L\,|t-s|$$

for $t \ge s \ge 0$ with $L = \max_{0\le t\le T}\|u_t(t)\|_\infty$. Then $\|u(t)\|_\infty$ is Lipschitz continuous in $[0,T]$; so that $\|u(t)\|_\infty$ is absolutely continuous and differentiable for a.e. $t\in[0,T]$. Let $t=\tau\in[0,T]$ be a point where $\|u(t)\|_\infty$ is differentiable. Since $x \mapsto u(x,\tau)$ is continuous in $\bar\Omega$, there exists a point $x^\tau \in \bar\Omega$ such that

$$\|u(\tau)\|_\infty = u(x^\tau,\tau).$$

For such $u(\tau)\in C(\bar\Omega)$, define $g_{u(\tau)} \in C(\bar\Omega)^*$ by

$$\langle\, g_{u(\tau)}, w\,\rangle = \|u(\tau)\|_\infty\, w(x^\tau) \qquad \text{for all } w \in C(\bar\Omega).$$

By Remark A.1, $g_{u(\tau)} \in F(u(\tau))$. It follows from Lemma A.2 that

$$\|u(\tau)\|_\infty \frac{d}{dt}\|u(\tau)\|_\infty = \langle\, g_{u(\tau)}, u_t(\tau)\,\rangle$$
$$= \|u(\tau)\|_\infty\, u_t(x^\tau,\tau).$$

This fact yields $\frac{d}{dt}\|u(\tau)\|_\infty = u_t(x^\tau,\tau)$ and completes the proof. □

**Lemma A.3.** *Let $M$ be a $3\times 3$ matrix defined by (2.9) Then the largest real eigenvalue $\lambda_0$ is expressed as $\lambda_0 = (\alpha\delta - \frac{(c+f)h}{f})\,\mu$ with $\mu>0$. Moreover, any left eigenvector $(k_1,k_2,k_3)$ corresponding to $\lambda_0$ can be chosen as $k_1,k_2,k_3>0$.*

*Proof.* If we put $p(\lambda) = (\lambda+c+f)(\lambda+h)(\lambda+\beta)/f\beta$, then the eigenvalues $\lambda$ of $M$ are given by solving

$$p(\lambda) = \alpha\delta.$$

Since zeros of $p$ are all negative, the largest real eigenvalue $\lambda_0$ of $M$ has the same sign as $\alpha\delta - \frac{c+f}{f}h$ (see Figure 1). So $\lambda_0$ can be expressed as $\lambda_0 = (\alpha\delta - \frac{c+f}{f}h)\,\mu$ with $\mu>0$. Any left eigenvector $(k_1,k_2,k_3)$ corresponding to $\lambda_0$ satisfies

$$\begin{cases} (\lambda_0+(c+f))\,k_1 &= fk_2,\\ (\lambda_0+h)k_2 &= \alpha k_3,\\ (\lambda_0+\beta)k_3 &= \beta\delta k_1. \end{cases}$$

Note that $\lambda_0$ satisfies $\lambda_0 > \max\{-(c+f), -h, -\beta\}$; then we can choose $(k_1,k_2,k_3)$ satisfying $k_1,k_2,k_3>0$. □

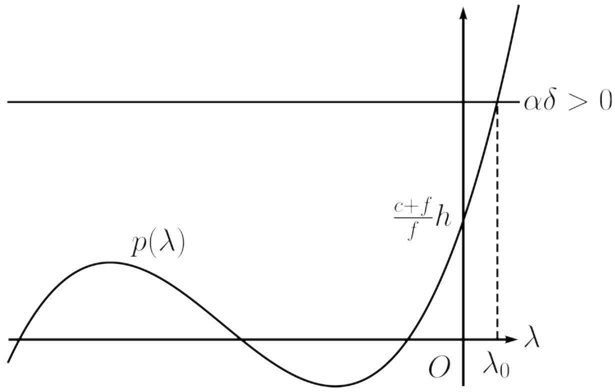


Figure 1: The largest real eigenvale $\lambda_0$

*Proof of* (4.28). In view of

$$S_\ell = \sum_{j=2}^{\ell} j\kappa^{\ell-j} = \kappa^\ell \sum_{j=2}^{\ell} j \left(\frac{1}{\kappa}\right)^j, \tag{A.1}$$

we consider

$$\begin{aligned}\sum_{j=2}^{\ell} jx^j &= x\frac{d}{dx}\left(\sum_{j=2}^{\ell} x^j\right) = x\frac{d}{dx}\left(\frac{x^2(1-x^{\ell-1})}{1-x}\right)\\ &= x\left\{\frac{2x-(\ell+1)x^\ell}{1-x} + \frac{x^2(1-x^{\ell-1})}{(1-x)^2}\right\}.\end{aligned}$$

Therefore,

$$\sum_{j=2}^{\ell} jx^j < \frac{2}{1-x} + \frac{1}{(1-x)^2} \qquad \text{for } \ 0 < x < 1;$$

so that

$$\sum_{j=2}^{\ell} \left(\frac{1}{\kappa}\right)^j < \frac{2\kappa}{\kappa-1} + \frac{\kappa^2}{(\kappa-1)^2} =: \gamma_\kappa$$

Thus (4.28) comes from (A.1). □

# References


[1] N. D. Alikakos, $L^p$ bounds of solutions of reaction-diffusion equations, Comm. Partial Differential Equations, **4** (1979), 827–868.

[2] M. Ya. Antonovsky, Impact of the factors of the environment on the dynamics of population (mathematical model), pp. 218–230 in *Proc. Soviet–American Symp. Comprehensive Analysis of the Environment*, Tbilisi 1974, Leningrad: Hydromet, (1975).

[3] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer–Verlag, New York, 2011.

[4] M. Chipot, *Elliptic Equations: An Introductory Course*, Birkhäuser, Basel, 2009.

[5] L. H. Chuan, T. Tsujikawa and A. Yagi, Asymptotic behavior of solutions for forest kinematic model, Funkcial. Ekvac., **49** (2006), 427–449.

[6] L. H. Chuan, T. Tsujikawa and A. Yagi, Stationary solutions to forest kinematic model, Glasg. Math. J., **51** (2009), 1–17.

[7] L. H. Chuan and A. Yagi, Dynamical system for forest kinematic model, Adv. Math. Sci. Appl., **16** (2006), 393–409.

[8] S. Ishida and T. Yokota, Global existence of weak solutions to quasilinear degenerate Keller-Segel systems of parabolic-parabolic type, J. Differential Equations, **252** (2012), 1421–1440.

[9] T. Kato, Nonlinear semigroups and evolution equations, J. Math. Soc. Japan, **19** (1967), 508–520.

[10] Y. A. Kuznetsov, M. Y. Antonovsky, V. N. Biktashev and E. A. Aponina, A cross-diffusion model of forest boundary dynamics, J. Math. Biol., **32** (1994), 219–232.

[11] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translation of Mathematical Monographs, Vol. **23**, Amer. Math. Soc., Rhode Island, 1968.

[12] G. Mola and A. Yagi, A forest model with memory, Funkcial. Ekvac., **52** (2009), 19–40.

[13] A. Okubo and S. A. Levin, *Diffusion and Ecological Problems: Modern Perspectives*, Interdisciplinary Appl. Math., Vol. **14**, Springer–Verlag, New York, 2nd ed., 2001.

[14] T. Shirai, L. H. Chuan and A. Yagi, Dynamical system for forest kinematic model under Dirichlet conditions, Sci. Math. Jpn., **66** (2007), 275–288.

[15] T. Shirai, L. H. Chuan and A. Yagi, Asymptotic behavior of solutions for forest kinematic model under Dirichlet conditions, Sci. Math. Jpn., **66** (2007), 289–301.

[16] T. Shirai, L. H. Chuan and A. Yagi, Stationary solutions for forest kinematic model under Dirichlet conditions, Sci. Math. Jpn., **67** (2008), 319–328.

[17] T. V. Tạ, T. H. Nguyen and A. Yagi, A sustainability condition for stochastic forest model, Commun. Pure Appl. Anal., **16** (2017), 699–718.

[18] R. Yamamoto and Y. Yamada, Analysis of forest kinematic model with degenerate diffusion, Adv. Math. Sci. Appl., **25** (2016), 307–320.