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SUM-CONNECTIVITY ENERGY OF GRAPHS

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Abstract. Energy of a graph is an important aspect related to the eigenvalues of the adjacency matrix of the graph and chemically to the intermolecular forces producing the energy of the corresponding molecule. There are different variations of the energy obtained by taking some other graph matrix instead of the adjacency matrix. Here the authors study a new type of energy called the sum-connectivity energy $SCE(G)$ of a graph G which is defined as the sum of the absolute values of the eigenvalues of the sum-connectivity matrix. In this paper we compute the sum-connectivity characteristic polynomial and the sum-connectivity energy for specific graphs, some edge deleted graphs and some specific types of complements. Some properties and bounds for $SCE(G)$ are also discussed.

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1 Introduction

Let G be a simple graph and let $\{v_1,\ v_2,\ \cdots,\ v_n\}$ be its vertices. For $i,\ j = 1,\ 2,\ \cdots,\ n$, if two vertices v_i and v_j of G are adjacent, then we use the notation $v_i \sim v_j$. For $v_i \in V(G)$, the degree of the vertex v_i, denoted by d_i, is the number of the vertices adjacent to v_i (the number of the first neighbors). In 1975, Randić defined a molecular structure descriptor

$$R = R(G) = \sum_{v_i \sim v_j} \frac{1}{\sqrt{d_i d_j}}.$$

This topological index is also known as the product-connectivity index or Randić index.

In parallel to the definition of the product-connectivity index of Randić, the sum-connectivity index of a graph G is defined as

$$S(G) = \sum_{v_i \sim v_j} \frac{1}{\sqrt{d_i + d_j}},$$

[6]. Todeschini and Consonni summarized the uses of topological indices in the structure-property-activity modelling. Topological indices play an important role in the study of complex networks on a broad spectrum of topics related to bio-informatics and proteomics. These topics cover many biomedical fields including virology, microbiology, toxicology and cancer research, to cite only some of the more intensively investigated. The main reason for the popularity of the topological indices is the high flexibility of this theory to solve in a fast way many apparently unrelated problems in all these disciplines. This determined the recent development of several interesting software and theoretical methods to handle with structure-function information and data mining in this field. Sum-connectivity index belongs to a family of Randić-like indices. Some applications of the sum-connectivity index in modelling some molecular properties is presented in [3]. The sum-connectivity index-concept suggests that it is purposeful to associate to the graph G a symmetric square matrix $SC(G)$. This sum-connectivity matrix $SC(G) = (S_{ij})_{n\times n}$ is defined as (see [5])

$$S_{ij} = \begin{cases} \frac{1}{\sqrt{d_i + d_j}} & \text{if } v_i \sim v_j, \\ 0 & otherwise. \end{cases}$$

2 The Sum-Connectivity Energy of a Graph

Let G be a simple, finite, undirected graph. The classical energy $E(G)$ is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. For more details on energy of a graph, see [1, 2].

Let G be a simple graph of order n with vertex set V and edge set E. Let $SC(G)$ be the sum-connectivity matrix of G. The characteristic polynomial of $SC(G)$ is denoted by $\phi_{SC}(G, \lambda) = det(\lambda I - SC(G))$. Since the sum-connectivity matrix is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. The sum-connectivity energy is given by

$$SCE(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

For properties of the eigenvalues of the sum-connectivity matrix, and lower and upper bounds on sum-connectivity energy, see [5]. The spectrum of a graph G is the list of distinct eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, with their multiplicities $m_1,\ m_2,\ \ldots,\ m_r$, and we write it as

$$\mathrm{Spec(G)} = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

This paper is organized as follows: In Section 3, we get some basic properties of the sum-connectivity energy of a graph. In Section 4, the sum-connectivity energy of some graphs are obtained. In Section 5, we find the sum-connectivity energy of some graphs with one edge deleted. In Section 6, we find the sum-connectivity energy of some complements of specific graphs. Finally, in Section 7, some open problems are given.

3 Some basic properties of the sum-connectivity energy of a graph

Let $d_i + d_j$ be the sum of degrees of the adjacent vertex pairs and let

$$P = \sum_{i<j} \frac{1}{d_i + d_j}.$$

Proposition 3.1. *The first three coefficients of $\phi_{SC}(G, \lambda)$ are given as follows:*
(i) *$a_0 = 1$,*
(ii) *$a_1 = 0$,*
(iii) *$a_2 = -P$.*

Proof. **(i)** From the definition, we have $\Phi_{SC}(G, \lambda) = det[\lambda I - SC(G)]$, so we get $a_0 = 1$.
(ii) The sum of determinants of all 1×1 principal submatrices of $SC(G)$ is equal to the trace of $SC(G)$. Therefore $a_1 = (-1)^1$ trace of $[SC(G)] = 0$.
(iii) Similarly proceeding, we have

$$\begin{aligned} (-1)^2 a_2 &= \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= -P. \end{aligned}$$

Proposition 3.2. *If $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ are the sum-connectivity eigenvalues of $SC(G)$, then*

$$\sum_{i=1}^{n} {\lambda_i}^2 = 2P.$$

Proof. We know that

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}a_{ji}\\ &= 2\sum_{i<j}a_{ij}^2+\sum_{i=1}^{n}a_{ii}^2\\ &= 2P.\end{aligned}$$

We now have an upper bound for the sum-connectivity energy of a graph G:

Theorem 3.3. [5] Let G be a graph with n vertices. Then

$$SCE(G) \leq \sqrt{2n\sum_{i<j}\frac{1}{d_i+d_j}}.$$

The following result gives a lower bound for the sum-connectivity energy of G:

Theorem 3.4. *Let G be a graph with n vertices. If $R=$ det $SC(G)$, then*

$$SCE(G) \geq \sqrt{2P+n(n-1)R^{\frac{2}{n}}}.$$

Proof. By definition,

$$\begin{aligned}(SCE(G))^2 &= \left(\sum_{i=1}^{n}|\lambda_i|\right)^2\\ &= \sum_{i=1}^{n}|\lambda_i|\sum_{j=1}^{n}|\lambda_j|\\ &= \sum_{i=1}^{n}|\lambda_i|^2+\sum_{i\neq j}|\lambda_i||\lambda_j|.\end{aligned}$$

Using arithmetic mean and geometric mean inequality, we have

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}.$$

Therefore,

$$\begin{aligned}[SCE(G)]^2 &\geq \sum_{i=1}^{n}|\lambda_i|^2+n(n-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}\\ &= 2P+n(n-1)R^{\frac{2}{n}}.\end{aligned}$$

Thus the result follows.

4 Sum-connectivity energy of some standard graphs

Theorem 4.1. *The sum-connectivity energy of the cycle graph C_{2n} is*

$$SCE(C_{2n}) = 2 + \sum_{m=1,\ m\neq n}^{2n-1} |\cos\frac{\pi m}{n}|.$$

Proof. The sum-connectivity matrix corresponding to the cycle graph C_{2n} is

$$SC(C_{2n}) = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 \end{bmatrix}.$$

This is a circullant matrix of order $2n$. Its eigenvalues are

$$\lambda_m = \begin{cases} 1, & \text{for } m = 0 \\ -1, & \text{for } m = n \\ \cos\frac{\pi m}{n}, & \text{for } 0 < m < n,\ n < m \leq 2n-1. \end{cases}$$

Therefore the sum-connectivity energy is

$$SCE(C_{2n}) \quad = |-1| + |1| + \sum_{m=1, m\neq n}^{2n-1} |\cos\frac{\pi m}{n}|$$

and finally we get the result.

Theorem 4.2. *The sum-connectivity energy of the complete graph K_n is*

$$SCE(K_n) = \sqrt{2n-2}.$$

The proof is similar, so we omit it.

Theorem 4.3. *The sum-connectivity energy of the star graph $K_{1,n-1}$ is*

$$SCE(K_{1,n-1}) = \frac{2\sqrt{n-1}}{\sqrt{n}}.$$

Proof. Let $K_{1,n-1}$ be the star graph with vertex set V. The sum-connectivity matrix is similarly obtained to above matrices and the characteristic equation will be

$$\lambda^{n-2}\left(\lambda^2 - \frac{n-1}{n}\right) = 0.$$

Therefore the result is obtained.

Theorem 4.4. *The sum-connectivity energy of the crown graph S_n^0 is*

$$SCE(S_n^0) = \frac{4n-4}{\sqrt{2n-2}}.$$

Proof. Let S_n^0 be a crown graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The sum-connectivity matrix is

$$SC(S_n^0) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & A & \dots & A & A \\ 0 & 0 & 0 & \dots & 0 & A & 0 & \dots & A & A \\ 0 & 0 & 0 & \dots & 0 & A & A & \dots & 0 & A \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & A & A & \dots & A & 0 \\ 0 & A & A & \dots & A & 0 & 0 & \dots & 0 & 0 \\ A & 0 & A & \dots & A & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & A & \dots & A & 0 & 0 & \dots & 0 & 0 \\ A & A & A & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

where $A = \frac{1}{\sqrt{2(n-1)}}$. Then the characteristic equation is

$$\left(\lambda + \frac{1}{\sqrt{2n-2}}\right)^{n-1} \left(\lambda - \frac{1}{\sqrt{2n-2}}\right)^{n-1} \left(\lambda + \frac{n-1}{\sqrt{2n-2}}\right) \left(\lambda - \frac{n-1}{\sqrt{2n-2}}\right) = 0$$

and therefore, the spectrum is

$$Spec_{SC}(S_n^0) = \begin{pmatrix} \frac{n-1}{\sqrt{2n-2}} & \frac{-(n-1)}{\sqrt{2n-2}} & \frac{1}{\sqrt{2n-2}} & \frac{-1}{\sqrt{2n-2}} \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Therefore,

$$SCE(S_n^0) = \frac{4n-4}{\sqrt{2n-2}}.$$

Theorem 4.5. *The sum-connectivity energy of the cocktail party graph $K_{n\times 2}$ is*

$$SCE(K_{n\times 2}) = 2\sqrt{n-1}.$$

Proof. The proof is similar and omitted.

Theorem 4.6. *The sum-connectivity energy of the complete bipartite graph $K_{n,n}$ of order $2n$ is*

$$SCE(K_{n,n}) = \sqrt{2n}.$$

Proof. Let $K_{n,n}$ be the complete bipartite graph of order $2n$ with vertex set $\{u_1,\ u_2,\ \cdots,\ u_n,\ v_1,\ v_2,\ \cdots,\ v_n\}$. The sum-connectivity matrix is

$$SC(K_{n,n}) = \begin{bmatrix} 0 & 0 & 0 & \dots & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} \\ 0 & 0 & 0 & \dots & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} \\ 0 & 0 & 0 & \dots & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \dots & 0 & 0 & 0 \\ \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \dots & 0 & 0 & 0 \\ \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Then the characteristic equation is

$$\lambda^{2n-2}\left(\lambda^2 - \frac{n}{2}\right) = 0.$$

Hence, the spectrum will be

$$Spec_{SC}(K_{n,n}) = \begin{pmatrix} \sqrt{\frac{n}{2}} & -\sqrt{\frac{n}{2}} & 0 \\ 1 & 1 & 2n-2 \end{pmatrix}.$$

Therefore, we obtain the result.

Definition 4.7. *The friendship graph, denoted by F_3^n, is the graph obtained by taking n copies of the cycle graph C_3 with a vertex in common.* It can easily be seen that $V(F_3^n) = 2n + 1$.

Theorem 4.8. *The sum-connectivity energy of the friendship graph F_3^n is*

$$SCE(F_3^n) = \frac{1}{2}\left(2n - 1 + \sqrt{\frac{17n+1}{n+1}}\right).$$

Proof. Let F_3^n be the friendship graph with $2n + 1$ vertices. The sum-connectivity matrix is

$$\begin{bmatrix} 0 & \frac{1}{\sqrt{2n+2}} & \frac{1}{\sqrt{2n+2}} & \frac{1}{\sqrt{2n+2}} & \frac{1}{\sqrt{2n+2}} & \cdots & \frac{1}{\sqrt{2n+2}} & \frac{1}{\sqrt{2n+2}} \\ \frac{1}{\sqrt{2n+2}} & 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{\sqrt{2n+2}} & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{\sqrt{2n+2}} & 0 & 0 & 0 & \frac{1}{2} & \dots & 0 & 0 \\ \frac{1}{\sqrt{2n+2}} & 0 & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{2n+2}} & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} \\ \frac{1}{\sqrt{2n+2}} & 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 \end{bmatrix}.$$

The characteristic equation is

$$\left(\lambda^2 - \frac{1}{2}\lambda - \frac{n}{n+1}\right)\left(\lambda - \frac{1}{2}\right)^{n-1}\left(\lambda + \frac{1}{2}\right)^{n} = 0$$

implying that the spectrum is

$$Spec_{SC}(F_3^n) = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{1}{4} + \frac{1}{4}\sqrt{\frac{17n+1}{n+1}} & \frac{1}{4} - \frac{1}{4}\sqrt{\frac{17n+1}{n+1}} \\ n & n-1 & 1 & 1 \end{pmatrix}.$$

Therefore we obtain the result.

Definition 4.9. *The double star graph $S_{n,m}$ is the graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0.*

It is easy to see that

$$V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1})$$

and

$$E(S_{n,m}) = \{v_0u_0; v_0v_i; u_0u_j \ : \ 1 \ \leq \ i \ \leq \ n-1, \ 1 \ \leq j \ \leq \ n-1\}.$$

Therefore, the double star graph is also a bipartite graph.

Theorem 4.10. *The sum-connectivity energy of the double star graph $S_{n,n}$ is*

$$SCE(S_{n,n}) = 2\sqrt{\frac{8n^2 - 7n + 1}{2n(n+1)}}.$$

Proof. The sum-connectivity matrix is

$$SC(S_{n,n}) = \begin{bmatrix} 0 & \frac{1}{\sqrt{n+1}} & \frac{1}{\sqrt{n+1}} & \cdots & \frac{1}{\sqrt{n+1}} & \frac{1}{\sqrt{2n}} & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{n+1}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{n+1}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{n+1}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2n}} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{\sqrt{n+1}} & \frac{1}{\sqrt{n+1}} & \cdots & \frac{1}{\sqrt{n+1}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n+1}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n+1}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n+1}} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Hence the characteristic equation is

$$\lambda^{2n-4}\left(\lambda^2 + \frac{1}{\sqrt{2n}}\lambda - \frac{n-1}{n+1}\right)\left(\lambda^2 - \frac{1}{\sqrt{2n}}\lambda - \frac{n-1}{n+1}\right) = 0$$

implying that the spectrum is

$$Spec_{SC}(S_{n,n}) = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{1}{\sqrt{2n}}+A\right) & \frac{1}{2}\left(\frac{1}{\sqrt{2n}}-A\right) & \frac{1}{2}\left(-\frac{1}{\sqrt{2n}}+A\right) & \frac{1}{2}\left(-\frac{1}{\sqrt{2n}}-A\right) \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where $A = \sqrt{\frac{8n^2-7n+1}{2n(n+1)}}$. Therefore we get the required result.

Definition 4.11. *Let n be any positive integer. The graph obtained by coalescence of n copies of the cycle graph C_4 of length 4 with a common vertex is called the Dutch windmill graph and denoted by D_4^n.*

We can easily show that the Dutch windmill graph has $3n+1$ vertices and $4n$ edges.

Theorem 4.12. *The sum-connectivity energy of the Dutch windmill graph D_4^n is*

$$SCE(D_4^n) = \sqrt{2}(n-1) + 2\sqrt{\frac{3n+1}{2n+2}}.$$

Proof. Let D_4^n be the Dutch windmill graph with $3n+1$ vertices. The sum-connectivity matrix is

$$SC(D_4^n) = \begin{bmatrix} 0 & \frac{1}{\sqrt{2n+2}} & 0 & \frac{1}{\sqrt{2n+2}} & \frac{1}{\sqrt{2n+2}} & \dots & 0 & \frac{1}{\sqrt{2n+2}} \\ \frac{1}{\sqrt{2n+2}} & 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{\sqrt{2n+2}} & 0 & 0 & 0 & \frac{1}{2} & \dots & 0 & 0 \\ \frac{1}{\sqrt{2n+2}} & 0 & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} \\ \frac{1}{\sqrt{2n+2}} & 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 \end{bmatrix}.$$

Then the characteristic equation is

$$\left(\lambda^2 - \frac{3n+1}{2n+2}\right)\lambda^{n+1}\left(\lambda - \frac{1}{\sqrt{2}}\right)^{n-1}\left(\lambda - \frac{1}{2}\right)^{n} = 0.$$

Hence, the spectrum becomes

$$Spec_{SC}(F_n^3) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \sqrt{\frac{3n+1}{2n+2}} & -\sqrt{\frac{3n+1}{2n+2}} \\ n-1 & n-1 & n+1 & 1 & 1 \end{pmatrix}$$

implying that $SCE(D_n^4) = \sqrt{2}(n-1) + 2\sqrt{\frac{3n+1}{2n+2}}$.

5 Sum-connectivity energy of specific graphs with one edge deleted

In this section we calculate the sum-connectivity energy for certain graphs with one edge deleted. Edge deletion helps to calculate several properties of large graphs in terms of smaller graphs obtained by successively deleting a number of edges:

Theorem 5.1. *Let e be an edge of the complete graph K_n. The sum-connectivity energy of $K_n - e$ is*

$$SCE(K_n - e) = \frac{n-3}{\sqrt{2n-2}} + \sqrt{\frac{2n^3+n^2-12n+5}{(2n-2)(2n-3)}}.$$

Proof. Similarly proceeding, we have the sum-connectivity matrix as

$$SC(K_n - e) = \begin{pmatrix} 0_{2\times 2} & \frac{1}{\sqrt{2n-3}} J_{2\times(n-2)} \\ \frac{1}{\sqrt{2n-3}} J_{2\times(n-2)} & \frac{1}{\sqrt{2n-2}}(J-I)_{(n-2)} \end{pmatrix}.$$

Hence the characteristic equation is

$$\lambda(\lambda - \frac{1}{\sqrt{2n-2}})^{n-3}\left(\lambda^2 - \frac{n-3}{\sqrt{2(n-1)}}\lambda - \frac{2n-4}{2n-3}\right)\left(\lambda^2 - \frac{1}{\sqrt{2n}}\lambda - \frac{n-1}{n+1}\right) = 0$$

and therefore the spectrum will be

$$Spec_{SC}(K_n - e) = \begin{pmatrix} \frac{1}{\sqrt{2n-2}} & \frac{1}{2}\left(\frac{n-3}{\sqrt{2n-2}} + B\right) & \frac{1}{2}\left(\frac{n-3}{\sqrt{2n-2}} - B\right) & 0 \\ n-3 & 1 & 1 & 1 \end{pmatrix},$$

where $B = \sqrt{\frac{2n^3+n^2-12n+5}{(2n-2)(2n-3)}}$. Therefore, the result follows.

Theorem 5.2. *Let e be an edge of the complete bipartite graph $K_{n,n}$. The sum-connectivity energy of $K_{n,n} - e$ is*

$$SCE(K_{n,n} - e) = 2\sqrt{\frac{2n^3+3n^2-4n-1}{2n(2n-1)}}.$$

Proof. Similarly

$$SC(K_{n,n} - e) = \begin{pmatrix} 0_{n\times n} & A \\ A & 0_{n\times n} \end{pmatrix},$$

where $A = \begin{pmatrix} \frac{1}{\sqrt{2n}} J_{(n-1)\times(n-1)} & \frac{1}{\sqrt{2n-1}} J_{(n-1)\times 1} \\ \frac{1}{\sqrt{2n-1}} J_{1\times(n-1)} & 0_{(1\times 1)} \end{pmatrix}$. Hence the characteristic equation is

$$\lambda^{2n-4}\left(\lambda^2 + \frac{n-1}{\sqrt{2n}}\lambda - \frac{n-1}{2n-1}\right)\left(\lambda^2 - \frac{n-1}{\sqrt{2n}}\lambda - \frac{n-1}{2n-1}\right) = 0.$$

Hence, the spectrum would be

$$Spec_{SC}(K_{n,n} - e) = \begin{pmatrix} \frac{1}{2}(-C+D) & \frac{1}{2}(-C-D) & \frac{1}{2}(C+D) & \frac{1}{2}(C-D) & 0 \\ 1 & 1 & 1 & 1 & 2n-4 \end{pmatrix}$$

where $C = \frac{n-1}{\sqrt{2\sqrt{2n}}}$ and $D = \sqrt{\frac{2n^3+3n^2-4n-1}{(2n)(2n-1)}}$ implying that

$$SCE(K_{n,n} - e) = 2\sqrt{\frac{2n^3+3n^2-4n-1}{(2n-2)2n}}.$$

6 Sum-connectivity energy of graph complements

Definition 6.1. *The complement of a graph G is a graph is denoted by $\overline{G}$ and is a graph on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.*

Definition 6.2. [4] *Let G be a graph and $P_k = \{V_1,\ V_2,\ ...,\ V_k\}$ be a partition of its vertex set V. Then the k-complement of G is obtained as follows: For all V_i and V_j in P_k, $i \neq j$, remove the edges between V_i and V_j and add the edges between the vertices of V_i and V_j which are not in G and is denoted by $\overline{G_k}$.*

Definition 6.3. [4] *Let G be a graph and $P_k = \{V_1,\ V_2,\ ...,\ V_k\}$ be a partition of its vertex set V. Then the $k(i)$-complement of G is obtained as follows: For each set V_r in P_k, remove the edges of G joining the vertices within V_r and add the edges of $\overline{G}$ (complement of G) joining the vertices of V_r, and is denoted by $\overline{G_{k(i)}}$.*

Theorem 6.4. *The sum-connectivity energy of the complement $\overline{K_n}$ of a complete graph is*

$$SCE\overline{(K_n)} = 0.$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1,\ v_2,\ ...,\ v_n\}$. The sum-connectivity matrix of the complement of K_n is a zero matrix which implies the result.

Theorem 6.5. *The sum-connectivity energy of the complement $\overline{K_{1,n-1}}$ of the star graph is*

$$SCE(\overline{K_{1,n-1}}) = \sqrt{2n-4}.$$

Proof. Let $\overline{K_{1,n-1}}$ be the complement of the star graph. The characteristic equation becomes

$$\lambda^1 \left(\lambda - \frac{n-2}{\sqrt{2n-4}}\right) \left(\lambda - \frac{1}{\sqrt{2n-4}}\right)^{n-2} = 0$$

and therefore the spectrum is

$$Spec_{SC}\overline{K_{1,n-1}} = \begin{pmatrix} \frac{n-2}{\sqrt{2n-4}} & 0 & -\frac{1}{\sqrt{2n-4}} \\ 1 & 1 & n-2 \end{pmatrix}.$$

Therefore the result is found.

Theorem 6.6. *The sum-connectivity energy of the complement $\overline{K_{n\times 2}}$ of the cocktail party graph of order $2n$ is*

$$SCE\overline{(K_{n\times 2})} = \sqrt{2}n.$$

Proof. Let $\overline{K_{n\times 2}}$ be the complement of the cocktail party graph of order $2n$ with vertex set $\{u_1,\ u_2,\ \cdots,\ u_n,\ v_1,\ v_2,\ \cdots,\ v_n\}$. Then the sum-connectivity matrix of $\overline{K_{n\times 2}}$ is

$$SC(\overline{K_{n\times 2}}) = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \dots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the characteristic equation becomes

$$(\lambda + \frac{1}{\sqrt{2}})^n(\lambda - \frac{1}{\sqrt{2}})^n = 0$$

and therefore the spectrum is

$$Spec_{SC}(K_{n,n}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ n & n \end{pmatrix}.$$

Finally we get $SCE(\overline{K_{n\times 2}}) = \sqrt{2}n$.

Theorem 6.7. *The sum-connectivity energy of the* $2(i)$*-complement of the double star graph* $S_{n,n}$ *is*

$$SCE(\overline{(S_{n,n})_{2(i)}}) = \frac{2n-4}{2\sqrt{n-1}} + \frac{1}{2}\sqrt{\frac{12n^2-20n+11}{3(n-1)}} + \frac{1}{2}\sqrt{\frac{16n-13}{3(n-1)}}.$$

Proof. Consider the sum-connectivity matrix. Then the characteristic equation becomes

$$\left(\lambda - \frac{1}{2\sqrt{n-1}}\right)^{2n-4}\left(\lambda^2 + \frac{1}{2\sqrt{n-1}}\lambda - \frac{1}{3}\right)\left(\lambda^2 - \frac{2n-3}{2\sqrt{n-1}}\lambda - \frac{1}{3}\right) = 0$$

and hence, the spectrum is

$$Spec_{SC}(\overline{(S_{n,n})_{2(i)}}) = \begin{pmatrix} \frac{1}{2\sqrt{n-1}} & A+B & A-B & C+D & C-D \\ 2n-4 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where $A = \frac{-1}{4\sqrt{n-1}}$, $B = \frac{1}{4}\sqrt{\frac{16n-13}{3(n-1)}}$, $C = \frac{2n-31}{4\sqrt{n-1}}$ and $D = \frac{1}{4}\sqrt{\frac{12n^2-20n+11}{12(n-1)}}$. Therefore,

$$SCE(\overline{(S_{n,n})_{2(i)}}) = \frac{2n-4}{2\sqrt{n-1}} + \frac{1}{2}\sqrt{\frac{12n^2-20n+11}{3(n-1)}} + \frac{1}{2}\sqrt{\frac{16n-13}{3(n-1)}}.$$

Theorem 6.8. *The sum-connectivity energy of the 2-complement of the cocktail party graph $K_{n\times 2}$ is*

$$SCE(\overline{(K_{n\times 2})_{(2)}}) = 4(n-1).$$

Proof. Consider the 2-complement $\overline{(K_{n\times 2})_{(2)}}$ of the cocktail party graph. The sum-connectivity matrix of it is

$$SC(\overline{(K_{n\times 2})_{(2)}}) = \begin{bmatrix} 0 & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \dots & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & 0 & \dots & 0 & 0 \\ \frac{1}{\sqrt{2n}} & 0 & \frac{1}{\sqrt{2n}} & \dots & \frac{1}{\sqrt{2n}} & 0 & \frac{1}{\sqrt{2n}} & \dots & 0 & 0 \\ \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & 0 & \dots & \frac{1}{\sqrt{2n}} & 0 & 0 & \dots & \frac{1}{\sqrt{2n}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \dots & 0 & 0 & 0 & \dots & 0 & \frac{1}{\sqrt{2n}} \\ \frac{1}{\sqrt{2n}} & 0 & 0 & \dots & 0 & n & 2 & \dots & -2 & -2 \\ 0 & \frac{1}{\sqrt{2n}} & 0 & \dots & 0 & \frac{1}{\sqrt{2n}} & 0 & \dots & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \frac{1}{\sqrt{2n}} & \dots & 0 & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \dots & 0 & \frac{1}{\sqrt{2n}} \\ 0 & 0 & 0 & \dots & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2n}} & \dots & \frac{1}{\sqrt{2n}} & 0 \end{bmatrix}.$$

Therefore the characteristic polynomial would be

$$\lambda^{n-1}\left(\lambda + \frac{2}{\sqrt{2n}}\right)^{n-1}\left(\lambda - \frac{n-2}{\sqrt{2n}}\right)\left(\lambda - \frac{n}{\sqrt{2n}}\right) = 0$$

implying that the sum-connectivity spectra is

$$Spec(\overline{(K_{n\times 2})_{(2)}}) = \begin{pmatrix} 0 & -\frac{2}{\sqrt{2n}} & \frac{n-2}{\sqrt{2n}} & \frac{n}{\sqrt{2n}} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}.$$

Therefore the result is obtained.

7 Some open problems

Open problem 7.1. With respect to sum-connectivity, determine the class of graphs which are co-spectral and characterize them.

Open problem 7.2. With respect to sum-connectivity, determine the class of graphs which are hyper-energetic and characterize them.

Open problem 7.3. With respect to sum-connectivity, determine the class of graphs whose sum-connectivity energy and sum-connectivity energy of their complements are equal.

Open problem 7.4. With respect to sum-connectivity, determine the class of non-co-spectral graphs which are equienergetic.

Open problem 7.5. Determine the class of graphs whose sum-connectivity energy is equal to usual energy.

8 Summary and Conclusion

Energy of a graph is a new concept in graph theory and it has a rapidly increasing importance due to both its mathematical beauty and its applications in molecular chemistry. The sum of the absolute values of the eigenvalues of the adjacency matrix of a given graph gives the energy of the graph. There are different types of graph energies with several applications and in this paper we went through a detailed search of the sum-connectivity energy. We obtain exact formulae, upper and lower bounds and some relations for the sum-connectivity energy. It is obtained for several classes of well-known graphs and also the effect of edge deletion is studied. At the end, some open problems are proposed.

References

[1] I. Gutman, The energy of a graph, Ber. Math. Stat. Sekt. Forschungsz. Graz, **103** (1978) 1-22.

[2] I. Gutman, The energy of a graph: old and new results, Combinatorics and applications, A. Betten, A. Khoner, R.Laue and A. Wassermann, eds., Springer, Berlin (2001) pp. 196-211.

[3] B. Lučić, S. Nikolić, N. Trinajstić, B. Zhou, S. Ivanić, Sum-connectivity index, I. Gutman, B. Furtula (Eds.), Novel Molecular Structure Descriptors: Theory and Applications I Univ. Kragujevac (2010) 101-136.

[4] E. Sampathkumar, L. Pushpalatha, C. V. Venkatachalam, P. Bhat, Generalized complements of a graph, Indian J. Pure Appl. Math. **29** (6) (1998) 625-639.

[5] B. Zhou, N. Trinajstić, On Sum-Connectivity Matrix and Sum-Connectivity Energy of (Molecular) Graphs, Acta Chim. Slov., **57** (2010), 518-523.

[6] B. Zhou and N. Trinajstić, On a novel connectivity index, J. Math. Chem., **46** (2009), 1252-1270.