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IMPROVED LAGUERRE MATRIX METHOD FOR SOLVING SOME NONLINEAR FUNCTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this study, a modified matrix-collocation method based on Laguerre polynomials to find the approximate solutions of the mentioned nonlinear functional differential equations under the initial or boundary conditions is proposed. These type equations are used as mathematical models in many problems in fields of engineering, mathematics, physics, chemistry, population dynamics, control theory and biology. There exists main challenges for solving the mentioned problems due to large range of variables, nonlinearity and multi-dimensionality, so on; thereby, the numerical methods have been developed by many authors. To show the effectiveness of this approach, some examples along with error estimations are illustrated by tables and figures; the consistency of the technique is analyzed.

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1 Introduction

The most mathematical models used in many problems of physics, biology, chemistry, engineering, and in other areas, are based on integral equations, nonlinear partial differential and delay partial differential equations.

The subject of this study is to apply the Laguerre matrix collocation method for solving some nonlinear functional differential equations in the general form

$$F(x,t,u(x,t),u_x(x,t),u_t(x,t),u_{xx}(x,t),u_{xt}(x,t),u_{tt}(x,t),u(x,\alpha t+\beta))=0 \tag{1}$$

with the initial and the boundary conditions

$$\begin{aligned} u(x,0) &= m(x),\ \ x\in[a,b] \\ u_t(x,0) &= n(x), \\ u(a,t) &= h(t),\ \ t\in[0,T] \\ u(b,t) &= k(t). \end{aligned} \tag{2}$$

In this study, we consider the following case of Eq.(1) which is the nonlinear delay partial differential equation of the form [1]

$$u_t(x,t)=\varepsilon u_{xx}(x,t)+u(x,t)[1-u(x,t-\tau)]+f(x,t),$$

or

$$u_t(x,t)=\varepsilon u_{xx}(x,t)+u(x,t)-u(x,t-\tau)u(x,t)+f(x,t),\ \ a\le x\le b,\ \ t>0 \tag{3}$$

where ε is the diffusion coefficient, $\tau>0$ is a constant delay, and $f(x,t)$ is the given continous function. This equation is an extension of the logic equation which is described as delayed logistic equation [2].

This model is known as the special case of the well-known Hutchinson's equation (1948) which is also called as the specific version of Wright's equation, describes delayed logistic equation with a discrete delay [3]-[7]. Lord Cherwell had the probability methods' applications to the distribution of prime numbers with related to this model. It is also utilized as a single species growth model with time delay [8]-[9].

Many real-world phenomena can be modelled by the nonlinear delay partial differential equation since processes, both natural and manmade, in biology, medicine, chemistry, physics, engineering, economics, etc., involve time delays [10]-[19].

On the other hand, most of such models can not be solved exactly. Therefore, it is necessary to design efficient numerical methods to approximate their solutions. The fundamentals and methods for such equations were developed in literature: perturbation method for the asymptotic solutions, finite element method, the complex WKB-Maslov method for the asymptotic solutions, non-oscillatory interpolation method, exponential time differencing methods, finite difference approximations, direct local boundary integral equation method, pseudo-spectral Legendre-Galerkin method, collocation method, higher order numerical methods so on [20]-[26].

In this work, we develop a numerical method, "Improved Laguerre matrix method", to get an approximate solution of Eq. (3) under the conditions (2) in the finite Laguerre series form which is given by

$$u(x,t)=\sum_{r=0}^{N}\sum_{s=0}^{N}a_{r,s}L_{r,s}(x,t);\ L_{r,s}(x,t)=L_r(x)L_s(t) \tag{4}$$

where $a_{r,s}$ $r,s=0,...,N$ are the unknown Laguerre coefficients and $L_n(x),\ n=0,1,2,...,N$ are the Laguerre polynomials defined by

$$L_n(x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n}{k}x^k, n\in\mathbb{N},\ 0\le x<\infty. \tag{5}$$

2 The Formulation of the generalized Laguerre polynomials

Generalized Laguerre polynomials $L_n(x,\alpha)$ are orthogonal in the interval $[0,+\infty)$ with respect to the weight function $\omega(x,\alpha)=x^\alpha e^{-x}$. For $\alpha=0$, these polynomials become ordinary Laguerre polynomials $L_n(x)$; $L_n(x,0)=L_n(x)$.
Polynomials $L_n(x,\alpha)$ are defined by the generating function

$$(1-t)^{-(\alpha+1)}\frac{e^{-xt}}{(1-t)}=\sum_{n=0}^{+\infty}L_n(x,\alpha)\frac{t^n}{n!}. \tag{6}$$

From the relation (5), the three term recurrence relation is obtained as

$$(n+1)L_{n+1}(x,\alpha)=(2n+\alpha+1-x)L_n(x,\alpha)-(n+\alpha)L_{n-1}(x,\alpha) \tag{7}$$

with starting values $L_0(x,\alpha)=1, L_1(x,\alpha)=\alpha+1-x$ [27].

3 Fundamental relations and Improved Laguerre matrix method

Firstly, we convert the expressions defined in (3) to matrix forms; then, by means of these matrices, we construct the Laguerre collocation method. The matrix relation of the Laguerre polynomial solution in (4) can be written as

$$[u(x,t)]=\mathbf{L}(x)\overline{\mathbf{L}}(t)\mathbf{A} \tag{8}$$

where

$$\mathbf{L}(x) = \left[\begin{array}{cccc} L_0(x) & L_1(x) & \cdots & L_N(x) \end{array}\right],$$

$$\overline{\mathbf{L}}(t) = \left[\begin{array}{cccc} \mathbf{L}(t) & 0 & \cdots & 0 \\ 0 & \mathbf{L}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{L}(t) \end{array}\right]$$

and

$$\mathbf{A} = \left[\begin{array}{ccccccccc} a_{0,0} & a_{0,1} & \cdots & a_{0,N} & \cdots & a_{N,0} & a_{N,1} & \cdots & a_{N,N} \end{array}\right]^T$$

We can write the matrix relation of Eq.(5) as

$$\mathbf{L}(x) = \mathbf{X}(x)\mathbf{H} \tag{9}$$

where

$$\mathbf{X}(x) = \left[\begin{array}{ccccc} 1 & x^1 & x^2 & \cdots & x^N \end{array}\right]$$

and

$$\mathbf{H} = \left[\begin{array}{cccccc} \frac{(-1)^0}{0!}\binom{0}{0} & \frac{(-1)^0}{0!}\binom{1}{0} & \frac{(-1)^0}{0!}\binom{2}{0} & \cdots & \frac{(-1)^0}{0!}\binom{N}{0} \\ 0 & \frac{(-1)^1}{1!}\binom{1}{1} & \frac{(-1)^1}{1!}\binom{2}{1} & \cdots & \frac{(-1)^1}{1!}\binom{N}{1} \\ 0 & 0 & \frac{(-1)^2}{2!}\binom{2}{2} & \cdots & \frac{(-1)^2}{2!}\binom{N}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(-1)^N}{N!}\binom{N}{N} \end{array}\right]$$

Also, the relations between $\mathbf{X}(x)$ and its first and second derivatives can be written as

$$\mathbf{X}'(x) = \mathbf{X}(x)\mathbf{B} \quad \text{and} \quad \mathbf{X}''(x) = \mathbf{X}(x)\mathbf{B}^2 \tag{10}$$

where

$$\mathbf{B} = \left[\begin{array}{cccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right].$$

By using the relations (9) and (10), we obtain

$$\mathbf{L}'(x) = \mathbf{X}(x)\mathbf{B}\mathbf{H} \quad \text{and} \quad \mathbf{L}''(x) = \mathbf{X}(x)\mathbf{B}^2\mathbf{H} \tag{11}$$

$$\overline{\mathbf{L}}(t) = \overline{\mathbf{X}}(t)\overline{\mathbf{H}} \quad \text{and} \quad \overline{\mathbf{L}}'(t) = \overline{\mathbf{X}}(t)\overline{\mathbf{B}\mathbf{H}} \tag{12}$$

Also, it is found from the relations (10), (11) and (12)

$$\mathbf{X}(x) = \mathbf{L}(x)\mathbf{H}^{-1} \quad ; \ \mathbf{L}'(x) = \mathbf{L}(x)\mathbf{C} \quad \text{and} \ \ \overline{\mathbf{L}}'(t) = \overline{\mathbf{L}}(t)\overline{\mathbf{C}} \tag{13}$$

where

$$\mathbf{C} = \mathbf{H}^{-1}\mathbf{B}\mathbf{H} \tag{14}$$

$$\mathbf{C} = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ 0 & 0 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \ ; \ \overline{\mathbf{C}} = diag(\mathbf{C}, \mathbf{C}, ..., \mathbf{C}).$$

Then by using (9) to (13), we have

$$\mathbf{L}''(x) = \mathbf{L}'(x)\mathbf{C} = \mathbf{L}(x)\mathbf{C}^2 \tag{15}$$

Therefore, from the relations (8), (9), (13) and (15), we reach the following:

$$[u(x,t)] = \mathbf{L}(x)\overline{\mathbf{L}}(t)\mathbf{A} \tag{16}$$

$$[u_x(x,t)] = \mathbf{L}'(x)\overline{\mathbf{L}}(t)\mathbf{A} = \mathbf{L}(x)\mathbf{C}\overline{\mathbf{L}}(t)\mathbf{A} \tag{17}$$

$$[u_{xx}(x,t)] = \mathbf{L}''(x)\overline{\mathbf{L}}(t)\mathbf{A} = \mathbf{L}(x)\mathbf{C}^2\overline{\mathbf{L}}(t)\mathbf{A} \tag{18}$$

$$[u_t(x,t)] = \mathbf{L}(x)\overline{\mathbf{L}}'(t)\mathbf{A} = \mathbf{L}(x)\overline{\mathbf{L}}(t)\overline{\mathbf{C}}\mathbf{A} \tag{19}$$

Also, we have

$$[u(x,t-\tau)u(x,t)] = \mathbf{L}(x)\overline{\mathbf{L}}(t-\tau)\overline{\overline{\mathbf{L}}}(x)\overline{\overline{\overline{\mathbf{L}}}}(t)\overline{\mathbf{A}} \tag{20}$$

where

$$\begin{aligned} \mathbf{A}_i &= \begin{bmatrix} a_{i0} & a_{i1} & \cdots & a_{iN} \end{bmatrix}^T, \ i = 0,1,...,N, \\ \mathbf{A} &= \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 & \cdots & \mathbf{A}_N \end{bmatrix}^T \\ &= \begin{bmatrix} a_{00} & \cdots & a_{0N} & a_{10} & \cdots & a_{1N} & \cdots & a_{N0} & \cdots & a_{NN} \end{bmatrix}^T, \\ \overline{\mathbf{A}}_i &= \begin{bmatrix} a_{i0}\mathbf{A} & a_{i1}\mathbf{A} & \cdots & a_{iN}\mathbf{A} \end{bmatrix}^T, \ i = 0,1,...,N, \\ \overline{\mathbf{A}} &= \begin{bmatrix} \overline{\mathbf{A}}_0 & \overline{\mathbf{A}}_1 & \cdots & \overline{\mathbf{A}}_N \end{bmatrix}^T, \end{aligned}$$

and

$$\overline{\overline{\mathbf{L}}}(x) = diag\left(\ \overline{\mathbf{L}}(x), \ \ \overline{\mathbf{L}}(x), \ \ \cdots \ \ \overline{\mathbf{L}}(x) \ \right),$$

$$\overline{\overline{\overline{\mathbf{L}}}}(t) = diag\left(\ \overline{\overline{\mathbf{L}}}(t),\ \ \overline{\overline{\mathbf{L}}}(t),\ \ \cdots\ \ \overline{\overline{\mathbf{L}}}(t)\ \right).$$

We may define $\overline{\mathbf{L}}(t-\tau)$ as

$$\overline{\mathbf{L}}(t-\tau) = \overline{\mathbf{X}}(t)\overline{\mathbf{B}}(t-\tau)\overline{\mathbf{H}}$$

where

$$\overline{\mathbf{B}}(t-\tau) = \begin{bmatrix} \binom{0}{0}(-\tau)^0 & \binom{1}{0}(-\tau)^1 & \binom{2}{0}(-\tau)^2 & \cdots & \binom{N}{0}(-\tau)^N \\ 0 & \binom{1}{1}(-\tau)^0 & \binom{2}{1}(-\tau)^1 & \cdots & \binom{N}{1}(-\tau)^{N-1} \\ 0 & 0 & \binom{2}{2}(-\tau)^0 & \cdots & \binom{N}{2}(-\tau)^{N-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}(-\tau)^0 \end{bmatrix}.$$

We use the same procedure for the initial and boundary conditions, and the matrix forms of (2) are

$$\begin{aligned} [u(x,0)] &= \mathbf{L}(x)\overline{\mathbf{L}}(0)\mathbf{A} = [m(x)] = \lambda \\ [u_t(x,0)] &= \mathbf{L}(x)\overline{\mathbf{L}}(0)\overline{\mathbf{C}}\mathbf{A} = [n(x)] = \mu \\ [u(a,t)] &= \mathbf{L}(a)\overline{\mathbf{L}}(t)\mathbf{A} = [h(t)] = \gamma \\ [u(b,t)] &= \mathbf{L}(b)\overline{\mathbf{L}}(t)\mathbf{A} = [k(t)] = \nu. \end{aligned} \tag{21}$$

Then by using (16) to (20), we obtain the matrix equation for Eq. (3) as

$$\{\mathbf{L}(x)\overline{\mathbf{L}}(t)\overline{\mathbf{C}} - \varepsilon\mathbf{L}(x)\mathbf{C}^2\overline{\mathbf{L}}(t) - \mathbf{L}(x)\overline{\mathbf{L}}(t)\}\mathbf{A} + \mathbf{L}(x)\overline{\mathbf{L}}(t-\tau)\overline{\mathbf{A}} = [f(x,t)] \tag{22}$$

or briefly

$$\mathbf{W}(x,t)\mathbf{A} + \mathbf{W}^*(x,t)\overline{\mathbf{A}} = [f(x,t)]. \tag{23}$$

The collocation points are defined by

$$x_i = \frac{l}{N}i,\ t_j = \frac{T}{N}j,\ i,j = 0,1,2,...,N \tag{24}$$

and we substitute the collocation points into Eq.(22), we have the system of matrix equations as

$$\{\mathbf{L}(x_i)\overline{\mathbf{L}}(t_j)\overline{\mathbf{C}} - \varepsilon\mathbf{L}(x_i)\mathbf{C}^2\overline{\mathbf{L}}(t_j) - \mathbf{L}(x_i)\overline{\mathbf{L}}(t_j)\}\mathbf{A} + \mathbf{L}(x_i)\overline{\mathbf{L}}(t_j-\tau)\overline{\mathbf{A}} = \mathbf{F} \tag{25}$$

Hence we can write the system of the matrix equations (25) briefly, then the fundamental matrix equation is

$$\mathbf{W}\mathbf{A} + \mathbf{W}^*\overline{\mathbf{A}} = \mathbf{F} \Longrightarrow [\mathbf{W};\mathbf{W}^*;\mathbf{F}] \tag{26}$$

Correspondingly, the matrix relations of the initial and boundary conditions are attained by the same procedure for $i, j = 0, 1, ..., N$

$$\begin{aligned} [u(x_i,0)] &= \mathbf{L}(x_i)\overline{\mathbf{L}}(0)\mathbf{A} = [m(x_i)] = \lambda_i \\ [u_t(x_i,0)] &= \mathbf{L}(x_i)\overline{\mathbf{L}}(0)\overline{\mathbf{C}}\mathbf{A} = [n(x_i)] = \mu_i \\ [u(a,t_j)] &= \mathbf{L}(a)\overline{\mathbf{L}}(t_j)\mathbf{A} = [h(t_j)] = \gamma_j \\ [u(b,t_j)] &= \mathbf{L}(b)\overline{\mathbf{L}}(t_j)\mathbf{A} = [k(t_j)] = \nu_j. \end{aligned} \tag{27}$$

Consequently, to obtain the solution of Eq. (3) under the conditions in Eq. (2), by replacing the row matrices (27) by the last rows of the augmented matrix (26), then we have the following new augmented matrix

$$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{W}}^*; \widetilde{\mathbf{F}}] \tag{28}$$

By solving the augmented matrix form, the unknown Laguerre coefficients are computed [28]-[30]. Thus, the approximate solution is found as

$$u_N(x,t) = \sum_{r=0}^{N}\sum_{s=0}^{N} a_{r,s}L_{r,s}(x,t).$$

4 Error Analysis

In this section, we can check the accuracy of the obtained solutions by resulting equation. The function $u_N(x,t)$ and its derivatives are substituted in Eq.(3), then the resulting equation must be satisfied approximately since the truncated Laguerre series (5) is approximate solution of (3). For $x = x_\alpha, t = t_\beta \in [0,l] \times [0,T] \quad \alpha, \beta = 0, 1, 2, ...$

$$E_N(x_\alpha,t_\beta) = |u_t(x_\alpha,t_\beta) - \varepsilon u_{xx}(x_\alpha,t_\beta) - u(x_\alpha,t_\beta) + u(x_\alpha,t_\beta-\tau)u(x_\alpha,t_\beta) - f(x_\alpha,t_\beta)| \cong 0$$

where $E_N(x_\alpha,t_\beta) \leq 10^{-k_{\alpha,\beta}} = 10^{-k}$ and k as a positive integer. Also, we consider the following different error norms:

- For L_2; $\quad E_N(x_\alpha,t_\beta) = (\sum_{i=1}^{n}(e_i)^2)^{1/2}$
- For L_∞; $\quad E_N(x_\alpha,t_\beta) = Max(e_i), 0 \leq i \leq n$
- For RMS; $\quad E_N(x_\alpha,t_\beta) = \sqrt{\frac{\sum_{i=1}^{n}(e_i)^2}{n+1}}$

where RMS is the Root-Mean-Square of errors and e_i is defined as $e_i = u(x_\alpha,t_\beta) - u_N(x_\alpha,t_\beta)$. We claim that $E_N(x_\alpha,t_\beta)$ is diminished and approximate to zero when the truncation limit N is accelerated [31].

5 Numerical Test

In this section, we apply our method on the nonlinear delay partial differential equation with inital and boundary conditions to show its efficiency.

Example

We deal with the nonlinear delay partial differential equation with inital and boundary conditions, namely Eq. (3) with $\varepsilon = 10^{-4}$, $\tau = 10^{-1}$, $x \in [0, 1]$, $t \in [0, 1]$ the functions $f(x, t)$, $m(x)$, $h(t)$, $k(t)$ chosen in such a way that the solution of the problem is known exactly and is equal to $u(x, t) = t\exp(x^2)$

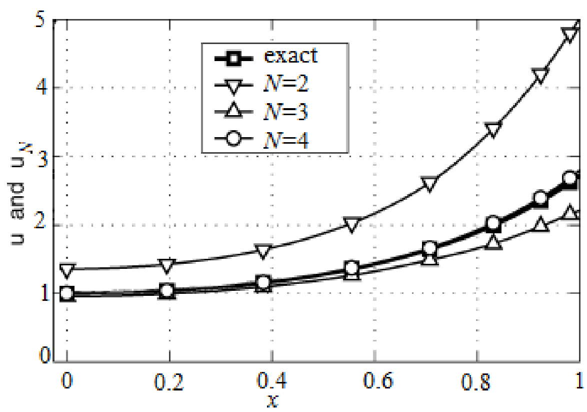

Figure 1: Exact solution and numerical solutions for $N = 2, 3, 4$.

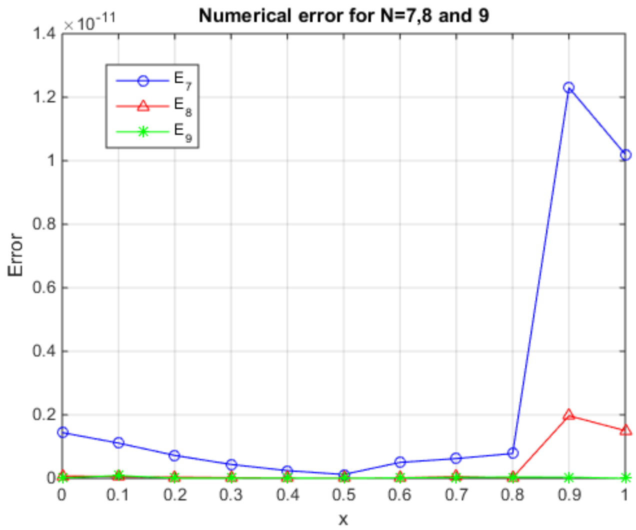

Figure 2: Numerical error for $N = 7, 8, 9$.

Table 1: L_2 and L_∞ norms of error for $x = 0.5$ and $N = 3$ and with the central processing unit (CPU).

t	L_2 Error	L_∞ Error	RMS Error
0.0	2.8171E-04	9.9782E-05	1.6039E-05
0.1	2.9221E-04	8.9279E-05	4.5636E-05
0.2	3.8885E-04	1.1055E-04	6.0728E-05
0.3	4.2964E-04	1.1711E-04	6.7098E-05
0.4	4.4344E-04	1.1751E-04	6.9253E-05
0.5	4.4216E-04	1.1473E-04	6.9054E-05
0.6	4.3204E-04	1.1021E-04	6.7473E-05
0.7	4.1670E-04	1.0478E-04	6.5078E-05
0.8	3.9825E-04	9.9165E-05	6.2197E-05
0.9	3.7824E-04	9.3371E-05	5.9071E-05
1.0	3.5750E-04	8.7583E-05	5.5832E-05

Table 2: Central processing unit (CPU) for numerical results with different N values.

N	CPU (s)
3	10.014
7	23.772
8	42.138
9	66.533
10	96.792

Table 3: Error comparison between different methods for $x = 0.5$ and $N = 10$

t	Gauss−Jacobi waveform	Present Method
0.0	2.3121E-12	9.8720E-14
0.1	2.9221E-12	8.9279E-14
0.2	3.8885E-12	1.1055E-14
0.3	4.2964E-13	1.1711E-15
0.4	4.4344E-13	1.1751E-15
0.5	4.4216E-13	1.1473E-15
0.6	4.3204E-13	1.1021E-15
0.7	4.1670E-13	1.0478E-15
0.8	3.9825E-12	9.9165E-14
0.9	3.7824E-12	9.3371E-14
1.0	3.5750E-12	8.7583E-14

6 Conclusion

We have presented and illustrated the method based on Laguerre polynomials for the solution of a class of nonlinear functional partial differential equations which are usually difficult to solve analytically. Also they play main role on biology, ecology and fluid and elastic mechanics, etc.
In many cases, it is required to obtain the approximate solutions. For this purpose, the presented novel method can be proposed. We have obtained the results efficiently when the truncation limit N is increased. The tables and figures has been shown the accuracy of the method. The method is based on computing the coefficients in the Laguerre expansion of solution of the nonlinear functional partial differential equations.
As a result, the method can also be extended to the nonlinear functional partial differential equations with the integral terms and their residual error analysis can be structured, but some modifications are required [32]-[35].

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