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MULTIPLICITY RESULTS FOR CRITICAL KIRCHHOFF PROBLEMS INVOLVING CONCAVE-CONVEX NONLINEARITIES

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Abstract. This paper is concerned with the existence and the multiplicity of solutions for elliptic Kirchhoff problems containing critical Sobolev exponent and defined on a regular bounded domain in $\mathbb{R}^3$. The results are obtained via the Nehari manifold and Ekeland's Variational Principle.

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1 Introduction

In this work, we study the existence and the multiplicity of positive solutions for the following problem:

$$(\mathcal{P}_\lambda)\begin{cases} -M\Big(\displaystyle\int_\Omega |\nabla u|^2 dx\Big)\Delta u = \lambda f(x)u^{q-1} + g(x)u^5 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $M(t) = at + b$, Ω is a regular bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, $1 < q < 2$, a, b are positive constants, λ is a positive parameter and f, g are continuous functions on $\overline{\Omega}$, $f \geq 0$, $f \neq 0$ and $g > 0$.

The problem $(\mathcal{P}_\lambda)$ is related to the stationary analogue of

$$\begin{cases} u_{tt} - \Big(a\displaystyle\int_\Omega |\nabla u|^2 dx + b\Big) \triangle u = h(x, u) & \text{in } \Omega \times (0,\ T),\\ u = 0 & \text{on } \partial\Omega \times (0,\ T),\\ u(x,\ 0) = u_0(x),\ u_t(x,\ 0) = u_1(x), & \end{cases}$$

where T is a positive constant, $u_{0,}$ u_1 are given functions. It was presented by Kirchhoff [12] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the strings produced by transverse vibrations. In such problem, the parameters have the following meanings: u denotes the displacement, $h(x,\ u)$ is the external force, b represents the initial tension and a is related to the intrinsic properties of the strings (such as Young's modulus). For more details, we refer the readers to the work [3] and the references therein.

It is pointed out that nonlocal problems model several physical and also biological systems where u describes a process depending on the average of itself as population density. After the famous article of Lions [13] where a functional analysis approach was proposed, Kirchhoff's problems began to call attention of researchers.

The following stationary elliptic version has been intensively studied:

$$(\mathcal{P}_S)\begin{cases} -\Big(a\displaystyle\int_\Omega |\nabla u|^2 dx + b\Big)\Delta u = h(x, u) & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $\Omega \subset \mathbb{R}^N$ and $h(x, u)$ is a continuous function. In the literature, there are many works and interested readers can refer to [2] where the autors obtained positive solutions for such problems by variational methods.

Let us recall a brief historic:

For $a = 0$, $b = 1$ and $f = g \equiv 1$, Ambrosetti et al. [1] established multiple results namely, they ensured the existence of a positive constant λ_0 such that the problem $(\mathcal{P}_\lambda)$ admits two positive solutions for $\lambda \in (0, \lambda_0)$, a positive solution for $\lambda = \lambda_0$ and no positive solution for $\lambda > \lambda_0$.

For $a = 0$, $b = q = 1$ and $g \equiv 1$, Tarantello [15] proved the existence of at least two solutions in a bounded domain of $\mathbb{R}^N$, $N \geq 3$, under a suitable condition on f.

In the case $a, b > 0$, $q = 1$ and $g \equiv 1$, Benmansour and Bouchekif [4] have shown the existence of two solutions under a sufficient condition on f.

Recently, Sun and liu in [14] treated the problem in the case $g \equiv f \equiv 1$; by using the Nehari decomposition, they obtained the existence of one solution and pointed out, as concluding remark, that a natural and interesting question is whether we can establish multiplicity theorems for the Kirchhoff problem with critical exponent.

Our answer is affirmative but only for a a small enough positive number. To our best knowledge, this kind of problems has not been considered before.

Throughout this paper, we use the following notation.

$$H = H_0^1(\Omega),$$

$$\int u = \int_\Omega u \, dx,$$

$$||u|| = \left(\int_\Omega |\nabla u|^2 \right)^{1/2} \quad \text{is the norm in} \quad H_0^1(\Omega),$$

C denotes generic positive constants whose exact values are not important,
B_a^r is the ball of center a and radius r and $o_n(1)$ denotes any quantity which tends to zero as n tends to infinity.
S_q is the best Sobolev constant of the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ for $1 < q < 2$ and S_6 is the best Sobolev constant of the embedding of $H_0^1(\Omega)$ into $L^6(\Omega)$.

We consider the following hypothesis:
Let

$$\begin{aligned} \lambda_1 &= \frac{2qS_q^{q/2}\sqrt{ab}}{(6-q)\,\|f\|_\infty} \left[\frac{2S_6^3\sqrt{3ab(4-q)(2-q)}}{(6-q)\,\|g\|_\infty} \right]^{(3-q)/3}, \\ \lambda_2 &= \frac{2qbS_q^{q/2}}{(6-q)\,\|f\|_\infty} \left[\frac{bS_6^3(2-q)}{(6-q)\,\|g\|_\infty} \right]^{(6-q)/4} \end{aligned}$$

and

$$\lambda_* = \max(\lambda_1, \lambda_2).$$

Our main results are:

Theorem 1.1. *Let $a, b > 0$, then there exists a positive ground state solution to the problem $(\mathcal{P}_\lambda)$ for all $\lambda \in (0, \lambda_*)$.*

Theorem 1.2. *Let $a, b > 0$ and a sufficiently small, then the problem $(\mathcal{P}_\lambda)$ admits two positive solutions for all $\lambda \in (0, \lambda_*)$.*

This work is organized as follows: In Section 2, we give some preliminary results which we will use later. Section 3 is devoted to the proof of the existence of the positive ground state solution. In section 4, we determine the level of Palais-Smale condition and give some appropriate estimations which allow us to prove the existence of a second positive solution to the problem $(\mathcal{P}_\lambda)$.

2 Preliminary results

The energy functional associated to the problem $(\mathcal{P}_\lambda)$ is given by:

$$I_\lambda(u) = \frac{1}{2}\widehat{M}\left(\|u\|^2\right) - \frac{1}{6}\int g(x)(u^+)^6 - \frac{\lambda}{q}\int f(x)(u^+)^q, \text{ for all } u \in H$$

where $\widehat{M}(t)$ is the primitive of $M(t) = at+b$ ($\widehat{M}(0) = 0$). It is clear that I_λ is well defined, of C^1on H and its critical points are weak solutions of $(\mathcal{P}_\lambda)$, that is $u \in H$ is said to be a weak solution of $(\mathcal{P}_\lambda)$ if it satisfies:

$$\left(a||u||^2 + b\right)\int \nabla u \nabla v - \int g(x)(u^+)^5 v - \lambda \int f(x)(u^+)^{q-1} v = 0\ , \text{ for all } v \in H.$$

We know that I_λ is not bounded from below on H so, we introduce the Nehari manifold given by:

$$\mathcal{N}_\lambda = \{u \in H\backslash\{0\} : \langle I'_\lambda(u),\ u\rangle = 0\}\,.$$

Let $h_u(t) = I(tu)$ for $t \in \mathbb{R}^*$ and $u \in H\backslash\{0\}$. These maps are known as fibering maps and were first introduced by Drábek and Pohozaev [10]. The set $\mathcal{N}_\lambda$ is closely linked to the behavior of $h_u(t)$, for more details see for example [8] or [9].
Let

$$h''_u(t) = 3at^2||u||^4 + b||u||^2 - \lambda(q-1)t^{q-2}\int f(x)(u^+)^q - 5t^4\int g(x)(u^+)^6.$$

It is natural to split $\mathcal{N}_\lambda$ into three subsets:

$$\begin{aligned}\mathcal{N}_\lambda^+ \ &:\ = \{u \in \mathcal{N}_\lambda\colon h''_u(1) > 0\}\,, \\ \mathcal{N}_\lambda^0 \ &:\ = \{u \in \mathcal{N}_\lambda\colon h''_u(1) = 0\}\end{aligned}$$

and

$$\mathcal{N}_\lambda^- := \{u \in \mathcal{N}_\lambda\colon h''_u(1) < 0\}\,,$$

which correspond to local minima, points of inflexion and local maxima of I_λ respectively.

We give the following useful results.

Lemma 2.1. *Suppose that u_0 is a local minimizer of I_λ in $\mathcal{N}_\lambda$ and $u_0 \notin \mathcal{N}_\lambda^0$. Then $I'_\lambda(u_0) = 0$.*

Proof If u_0 is a local minimizer for I_λ on $\mathcal{N}_\lambda$, then u_0 is a solution of the minimization problem: $\mathrm{Min}\{I_\lambda(u);\, \gamma_\lambda(u) = 0\}$, where

$$\gamma_\lambda(u) = a||u||^4 + b||u||^2 - \lambda\int f(x)(u^+)^q - \int g(x)(u^+)^6.$$

Hence, by the Theory of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that

$$I_\lambda(u_0) = \mu\gamma_\lambda(u_0).$$

Thus, $\left\langle I_\lambda'(u_0), u_0\right\rangle = \mu\left\langle \gamma_\lambda'(u_0), u_0\right\rangle$. Since $u_0 \in \mathcal{N}_\lambda$ and by the fact that $u_0 \notin \mathcal{N}_\lambda^0$, then $\left\langle \gamma_\lambda'(u_0), u_0\right\rangle \neq 0$ and we conclude that $\mu = 0$. The proof is complete.

The following lemmas play a crucial role in the sequel of this work.

Lemma 2.2. *For all $\lambda \in (0, \lambda_*)$ and for each $u \in H\backslash\{0\}$, there exists $t^+ = t^+(u) \geq t^u_{\max}$ such that $t^+u \in \mathcal{N}_\lambda^-$, and $I_\lambda(t^+u) = \max\limits_{t \geq t^u_{\max}} I_\lambda(tu)$.*

Moreover, if $\displaystyle\int f(x)(u^+)^q > 0$ then there exists an unique $t^- = t^-(u) \leq t^u_{\max}$ such that $t^-u \in \mathcal{N}_\lambda^+$ and $I_\lambda(t^-u) = \min\limits_{0 \leq t \leq t^u_{\max}} I_\lambda(tu)$.

The proof of this lemma is in [9].

Lemma 2.3. *For all $\lambda \in (0, \lambda_*)$, we have $\mathcal{N}^0 = \varnothing$.*

Proof We argue by contradiction. Suppose that there exists $u \in \mathcal{N}_\lambda^0$ i.e

$$3a\|u\|^4 + b\|u\|^2 = (q-1)\lambda\int f(x)(u^+)^q + 5\int g(x)(u^+)^6$$

and u verifies

$$a\|u\|^4 + b\|u\|^2 - \int g(x)(u^+)^6 - \lambda\int f(x)(u^+)^q = 0.$$

From this and the Sobolev inequality, we obtain:

$$\begin{aligned} 2\sqrt{(4-q)(2-q)ab}\,\|u\|^3 &\leq a(4-q)\|u\|^4 + b(2-q)\|u\|^2 \\ &\leq (6-q)S_6^{-3}\|g\|_\infty\|u\|^6, \end{aligned}$$

and

$$\begin{aligned} 2\sqrt{8ab}\,\|u\|^3 &\leq 2a\|u\|^4 + 4b\|u\|^2 \\ &\leq \lambda(6-q)\|f\|_\infty S_q^{-q/2}\|u\|^q, \end{aligned}$$

consequently we get

$$\left(\frac{2S_6^3\sqrt{(4-q)(2-q)ab}}{(6-q)\|g\|_\infty}\right)^{1/3} \leq \|u\| \leq \left(\frac{\lambda(6-q)\|f\|_\infty}{2S_q^{q/2}\sqrt{8ab}}\right)^{1/(3-q)},$$

and

$$\left(\frac{(2-q)bS_6^3}{(6-q)\|g\|_\infty}\right)^{1/4} \leq \|u\| \leq \left(\frac{\lambda(6-q)\|f\|_\infty}{4bS_q^{q/2}}\right)^{1/(2-q)}.$$

which contradicts the fact $0 < \lambda < \lambda_*$.

Lemma 2.4. *For $\lambda \in (0, \lambda_*)$ and $u \in \mathcal{N}_\lambda$, there exists $\varepsilon > 0$ and a differentiable function $\zeta : B_0^\varepsilon \subset H \longrightarrow \mathbb{R}^+$ such that $\zeta(0) = 1$, $\zeta(v)(u - v) \in \mathcal{N}_\lambda$ for $\|v\| < \epsilon$ and*

$$\langle \zeta'(0), v \rangle = \frac{\left(4a \|u\|^2 + 2b\right) \int \nabla u \nabla v - 6 \int g(x)(u^+)^5 v - \lambda q \int f(x)(u^+)^{q-1} v}{h_u''(1)}.$$

Proof Define the application $F : \mathbb{R} \times H \to \mathbb{R}$, by:

$$F(s, w) = as^3||u - w||^4 + bs||u - w||^2 - s^5 \int g(x) \left((u - w)^+\right)^6 - \lambda \int f(x) \left((u - w)^+\right)^q.$$

Since

$$F(1, 0) = 0 \text{ and } \frac{\partial F}{\partial s}(1, 0) = h_u''(1) \neq 0,$$

we obtain the desired result by applying the implicit function Theorem at $(1, 0)$.

Lemma 2.5. *The functional I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof For $u \in \mathcal{N}_\lambda$, we have

$$a \|u\|^4 + b \|u\|^2 = \int g(x)(u^+)^6 + \lambda \int f(x)(u^+)^q.$$

We obtain

$$\begin{aligned} I_\lambda(u) &= \frac{a}{4} \|u\|^4 + \frac{b}{2} \|u\|^2 - \frac{1}{6} \int g(x)(u^+)^6 - \frac{\lambda}{q} \int f(x)(u^+)^q \\ &\geq \frac{a}{12} \|u\|^4 + \frac{b}{3} \|u\|^2 - \lambda \frac{(6 - q)}{6q} \|f\|_\infty S_q^{-q/2} \|u\|^q. \end{aligned}$$

Then, I_λ is coercive and bounded from below in $\mathcal{N}_\lambda$.

Define $c_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u)$, $c_\lambda^- = \inf_{v \in \mathcal{N}_\lambda^-} I_\lambda(v)$ and $c_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u)$. Moreover if u_λ is a local minimum of I_λ, then we have

$$3a \|u_\lambda\|^4 + b \|u_\lambda\|^2 - \lambda(q - 1) \int f(x)(u_\lambda^+)^q - 5 \int g(x)(u_\lambda^+)^6 \geq 0$$

and since $\mathcal{N}_\lambda^0 = \varnothing$, we deduce that $u_\lambda \in \mathcal{N}_\lambda^+$ and so $c_\lambda^+ = c_\lambda$.

We have the following result:

Lemma 2.6. *For all $\lambda \in (0, \lambda_*)$, there exist two minimizing sequences $(u_n) \subset \mathcal{N}_\lambda^+$ and $(v_n) \subset \mathcal{N}_\lambda^-$ such that*

(i) $I_\lambda(u_n) < c_\lambda^+ + \frac{1}{n}$ *and* $I_\lambda(w_1) \geq I_\lambda(u_n) - \frac{1}{n}\|w_1 - u_n\|$ *for all* $w_1 \in \mathcal{N}_\lambda^+$.

(ii) $I_\lambda(v_n) < c_\lambda^- + \frac{1}{n}$ *and* $I_\lambda(w_2) \geq I_\lambda(v_n) - \frac{1}{n}\|w_2 - v_n\|$ *for all* $w_2 \in \mathcal{N}_\lambda^-$.

Proof It is clear that I_λ is bounded in $\mathcal{N}_\lambda$. Then by using the Ekeland variational principle, we obtain two minimizing sequences $(u_n) \subset \mathcal{N}_\lambda^+$ and $(v_n) \subset \mathcal{N}_\lambda^-$ which verify (i) and (ii) respectively.

3 Proof of Theorem 1

To establish the proof of Theorem 1, we need the following result:

Proposition 3.1. *For all* $0 < \lambda < \lambda_*$, *we have* $c_\lambda^+ < 0$.

Proof i) Let $u \in \mathcal{N}_\lambda^+$. Since

$$\begin{aligned}\lambda(6-q)\int f(x)(u^+)^q &> 2a\|u\|^4 + 4b\|u\|^2 \\ &\geq 4b\|u\|^2.\end{aligned}$$

Then,

$$\begin{aligned}I_\lambda(u) &= \frac{a}{4}\|u\|^4 + \frac{b}{2}\|u\|^2 - \frac{1}{6}\int g(x)(u^+)^6 - \frac{\lambda}{q}\int f(x)(u^+)^q \\ &\leq -\frac{b(2-q)}{3q}\|u\|^2 < 0.\end{aligned}$$

We deduce that $c_\lambda^+ < 0$.
Here, we prove by applying Ekeland's variational principle, the existence of a ground state solution. Indeed we have from i) of Lemma 2.6 the existence of a minimizing sequence $(u_n) \subset \mathcal{N}_\lambda^+$ such that

$$I(u_n) < c_\lambda^+ + \frac{1}{n} \text{ and } I(w) \geq I(u_n) - \frac{1}{n}\|w - u_n\| \text{ for all } w \in \mathcal{N}_\lambda^+.$$

Since (u_n) is bounded in H, passing to a subsequence if necessary, we have : $u_n \rightharpoonup u_0$ weakly in H.

Now we show that u_0 is not identically zero. Suppose by contradiction that $u_0 \equiv 0$.

From i) of Lemma 2.6 , we write:

$$\frac{1}{6}\int g(x)(u_n^+)^6 = \frac{a}{4}\|u_n\|^4 + \frac{b}{2}\|u_n\|^2 - c_\lambda^+.$$

Since $u_n \in \mathcal{N}_\lambda^+$, we also have:

$$0 < (4-q)a\left\|u_n\right\|^4 + (2-q)b\left\|u_n\right\|^2 - (6-q)\int g(x)(u_n^+)^6 + o_n(1).$$

Combining these last, we obtain:

$$\begin{aligned} 0 &< \frac{(q-10)}{2}a\left\|u_n\right\|^4 + (2q-16)b\left\|u_n\right\|^2 - 6(6-q)c_\lambda^+ + o_n(1) \\ &< 6(6-q)c_\lambda^+ + o_n(1) < 0. \end{aligned}$$

Which leads to a contradiction. So, we conclude that u_0 is not identically zero.

Consequently,

$$\begin{aligned} c_\lambda^+ &\leq I_\lambda(u_0) \\ &= \frac{a}{12}\left\|u_0\right\|^4 + \frac{b}{3}\left\|u_0\right\|^2 - \lambda\frac{(6-q)}{6q}\int f(x)(u_0^+)^q \\ &\leq \varliminf_{n\to\infty} I_\lambda(u_n) \\ &= c_\lambda^+, \end{aligned}$$

so

$$c_\lambda^+ = I_\lambda(u_0).$$

It follows that (u_n) converges strongly to u_0 in H and necessarily $u_0 \in \mathcal{N}_\lambda^+$. To conclude that u_0 a is local minimum of I_λ, recall that for each $u \in H$, we have

$$I_\lambda(su) \geq I_\lambda\left(t^- u\right) \text{ for all } 0 < s < t_{a,\max}^u,$$

in particular for $u = u_0 \in \mathcal{N}_\lambda^+$, we have $t^- = 1 < t_{a,\max}^{u_0}$. Choose $\varepsilon > 0$ small enough to have $1 < t_{a,\max}^{u_0-w}$ and $t(w)$ satisfying $t(w)(u_0 - w) \in \mathcal{N}_\lambda$ for all $\|w\| < \varepsilon$. Since $t(w) \to 1$ as $\|w\| \to 0$, we can suppose that

$$t(w) < t_{a,\max}^{u_0-w} \text{ for all } w \text{ such that } \|w\| < \varepsilon,$$

hence $t(w)(u_0 - w) \in \mathcal{N}_\lambda^+$ and for $0 < s < t_{a,\max}^{u_0-w}$, we have

$$I_\lambda(s(u_0 - w)) \geq I_\lambda(t(w)(u_0 - w)) \geq I_\lambda(u_0),$$

taking $s = 1$, we conclude that $I_\lambda(u_0 - w) \geq I_\lambda(u_0)$, for all $w \in H$ such that $\|w\| < \varepsilon$.

As $I_\lambda(|u|) = I_\lambda(u)$, we deduce by the maximum principle that u_0 is a positive solution.

4 Existence of a local minimizer on $\mathcal{N}_\lambda^-$

This section is devoted to the existence of a solution u_1 in $\mathcal{N}_\lambda^-$ via Mountain Pass Lemma such that $c_1 = I_\lambda(u_1)$. First we determine the good level for covering the Palais-Smale condition.

The best constant S_6 is attained in $\mathbb{R}^3$ by

$$U_{\varepsilon,x_0}(x) = \varepsilon^{1/2}\left(\varepsilon^2 + |x - x_0|^2\right)^{-1/2},$$

where $x_0 \in \Omega$ and $\varepsilon > 0$.

We have the following important result whose proof is similar to the one in [7] .

Lemma 4.1. *For all $\lambda \in (0, \lambda_*)$, the functional I_λ satisfies the (P-S)$_c$ condition for*

$$c < c^* = \frac{ab}{4} \|g\|_\infty^{-1} S_6^3 + \frac{a^3}{24} \|g\|_\infty^{-1} S_6^6 + \frac{b}{6} S_6 E_1 + \frac{a^2}{24} \|g\|_\infty^{-1} S_6^4 E_1 + c_0,$$

where $E_1 = \left(a^2 \|g\|_\infty^{-1} S_6^4 + 4b \|g\|_\infty^{-1} S_6\right)^{1/2}$.

Proof Let (u_n) be a (P-S)$_c$ sequence with $c < c^*$, then (u_n) is a bounded sequence in H. Thus it has a subsequence still denoted (u_n) such that $u_n \rightharpoonup u$ in H, $u_n \longrightarrow u$ strongly in $L^s(\Omega)$ for all $1 \leq s < 6$ and $u_n \longrightarrow u$ a.e in Ω.

Let $w_n = u_n - u$. From the Brezis-Lieb Lemma [5], one has:

$$\|u_n\|^2 = \|w_n\|^2 + \|u\|^2 + o_n(1),$$

$$\|u_n\|^4 = \|w_n\|^4 + 2\|w_n\|^2\|u\|^2 + \|u\|^4 + o_n(1),$$

and

$$\int g(x)(u_n^+)^6 = \int g(x)(w_n^+)^6 + \int g(x)(u^+)^6 + o_n(1).$$

Since $I_\lambda(u_n) = c + o_n(1)$, we get

$$\begin{aligned} \frac{a}{4} \|w_n\|^4 + \frac{b}{2} \|w_n\|^2 + \frac{a}{2} \|w_n\|^2 \|u\|^2 - \frac{1}{6} \int g(x)(w_n^+)^6 &= I_\lambda(u_n) - I_\lambda(u) \\ &= c - I_\lambda(u) + o_n(1). \end{aligned}$$

By the fact that $I_a'(u_n) = o_n(1)$ and $\langle I_a'(u), u \rangle = 0$, we obtain

$$a \|w_n\|^4 + b \|w_n\|^2 + 2a \|w_n\|^2 \|u\|^2 - \int g(x)(w_n^+)^6 = o_n(1).$$

Assume that $\|w_n\| \to l$ with $l > 0$, it follows that

$$\int g(x)(w_n)^6 = al^4 + bl^2 + 2al^2 \|u\|^2.$$

From the definition of S, we have:

$$\|w_n\|^6 \geq S_6^3 \|g\|_\infty^{-1} \int g(x)(w_n^+)^6, \text{ for all } n.$$

As $n \to +\infty$, we deduce that

$$l^2 \geq \frac{a}{2} \|g\|_\infty^{-1} S_6^3 + \frac{1}{2} S_6 \left(a^2 \|g\|_\infty^{-2} S_6^4 + 4 \|g\|_\infty^{-1} S_6 (b + 2a \|u\|^2)\right)^{1/2}.$$

Consequently we obtain

$$\begin{aligned} c &= \frac{a}{12} l^4 + \frac{b}{3} l^2 + \frac{a}{6} l^2 \|u\|^2 + I_\lambda(u) \\ &\geq \frac{a}{12} l^4 + \frac{b}{3} l^2 + c_0 \\ &\geq \frac{ab}{4} \|g\|_\infty^{-1} S_6^3 + \frac{a^3}{24} \|g\|_\infty^{-2} S_6^6 + \frac{b}{6} S_6 E_1 + \frac{a^2}{24} \|g\|_\infty^{-1} S_6^4 E_1 + c_0 \\ &= c^* \end{aligned}$$

which is a contradiction. Therefore $l=0$, then $u_n \longrightarrow u$ strongly in H. Now, we shall give some useful estimates of the extremal functions. Let $x_0 \in \Omega$ and $\phi \in C_0^{\infty}(\Omega)$ such that $\phi(x)=1$ for $x \in B_{x_0}^r$, $\phi(x)=0$ for $x \in \mathbb{R}^3 \backslash B_{x_0}^{2r}$, $0 \leq \phi \leq 1$ and $|\nabla \phi| \leq C$.

Set $u_{\varepsilon,x_0}(x)=\phi(x)U_{\varepsilon,x_0}(x)$.

Similar to the calculation of [6] and [11], we have the following estimates, as ε tends to 0 :

$$\int g(x)(u_{\varepsilon,x_0}^{+})^6=\|g\|_{\infty} A+O\left(\varepsilon^3\right), \text{ and } \|u_{\varepsilon,x_0}\|^2=B+O\left(\varepsilon\right),$$

where

$$A=\int_{\mathbb{R}^3}\left(1+|x-x_0|^2\right)^{-3}, \text{ and } B=\int_{\mathbb{R}^3}\left|\nabla U_{1,x_0}(x)\right|^2,$$

and from [15], we also have: $\displaystyle\int g(x)u_{\varepsilon,x_0}^5 u_0=O\left(\varepsilon^{1/2}\right)+o\left(\varepsilon^{1/2}\right)$.

In the search of our solution, it is natural to show that $c_1<c^*$. For this let u_0 be the ground state solution given in Section 3.

Lemma 4.2. *For $\lambda \in (0,\lambda_*)$, there exist a_0 and ε_0 small enough such that for every $0<\varepsilon<\varepsilon_0$ and $0<a<a_0$ we have $I_\lambda(u_0+tu_{\varepsilon,x_0})<c^*$ for all $t>0$.*

Proof From the above estimates and the Holder Inequality, we obtain

$$\begin{aligned}
I_\lambda(u_0+tu_{\varepsilon,x_0}) &= I_\lambda(u_0)+\frac{a}{4}t^4\|u_{\varepsilon,x_0}\|^4+\frac{b}{2}t^2\|u_{\varepsilon,x_0}\|^2-\frac{1}{6}t^6\int g(x)(u_{\varepsilon,x_0}^{+})^6 \\
&\quad -\frac{t^5}{6}\int g(x)(u_{\varepsilon,x_0}^{+})^5u_0 \\
&\quad +at^2\left[\left(\int \nabla u_0\nabla u_{\varepsilon,x_0}\right)^2+\|u_{\varepsilon,x_0}\|^2(\frac{1}{2}\|u_0\|^2+t\int \nabla u_0\nabla u_{\varepsilon,x_0})\right] \\
&\quad +o\left(\varepsilon^{1/2}\right) \\
&\leq I_\lambda(u_0)+\frac{a}{4}t^4B^2+\frac{b}{2}t^2B-\frac{1}{6}\|g\|_\infty t^6A-\frac{t^5}{6}O\left(\varepsilon^{1/2}\right)+ \\
&\quad +at^2\left[\frac{3}{2}\|u_0\|^2B+tB^{3/2}\|u_0\|\right]+o\left(\varepsilon^{1/2}\right) \\
&= c_0+Q_{\varepsilon}(t)+R(t),
\end{aligned}$$

where

$$Q_{\varepsilon}(t)=-\frac{1}{6}\|g\|_\infty At^6+\frac{a}{4}B^2t^4+\frac{b}{2}Bt^2-\frac{t^5}{6}O\left(\varepsilon^{1/2}\right)+o\left(\varepsilon^{1/2}\right),$$

and

$$R(t)=a\left[\frac{3}{2}t^2\left\|u_0\right\|^2B+t^3B^{3/2}\|u_0\|\right].$$

We know that $\lim\limits_{t\longrightarrow+\infty} Q_{\varepsilon}(t) = -\infty$, and $Q_{\varepsilon}(t) > 0$ for t near 0, so $\sup_{t\geq 0} Q_{\varepsilon}(t)$ is achieved for $t = T_\varepsilon > 0$ and T_ε verifies:

$$-A\|g\|_\infty T_\varepsilon^5 + aB^2 T_\varepsilon^3 + bBT_\varepsilon = O\left(\varepsilon^{1/2}\right).$$

Also $Q_0(t)$ attains its maximum at T_0 given by

$$T_0^2 = \frac{a\|g\|_\infty^{-1}B^2 + (a^2\|g\|_\infty^{-2}B^4 + 4b\|g\|_\infty^{-1}AB)^{1/2}}{2A}.$$

It is clear that T_ε tends to T_0 as ε goes to 0. Write $T_\varepsilon = T_0(1 \pm \delta_\varepsilon)$, hence δ_ε tends to 0 as ε goes to 0.

Moreover, since $I_\lambda(u_0 + tu_{\varepsilon,x_0}) \to -\infty$ as t goes to ∞, there exists $T_\varepsilon < T_1$ such that

$$I_\lambda(u_0 + tu_{\varepsilon,x_0}) \leq c^* + Q_{\varepsilon}(T_\varepsilon) + \sup_{t<T_1} R(t).$$

On the other hand, we have

$$\begin{aligned} Q_{\varepsilon}(T_\varepsilon) &= -\frac{1}{6}\|g\|_\infty AT_\varepsilon^6 + \frac{a}{4}B^2T_\varepsilon^4 + \frac{b}{2}BT_\varepsilon^2 - O\left(\varepsilon^{1/2}\right) + o\left(\varepsilon^{1/2}\right) \\ &= -\frac{1}{6}\|g\|_\infty AT_0^6 + \frac{a}{4}B^2T_0^4 + \frac{b}{2}BT_0^2 \\ &\quad \pm aT_0^4B^2\delta_\varepsilon \pm bT_0^2B\delta_\varepsilon \mp \|g\|_\infty T_0^6 A\delta_\varepsilon - O\left(\varepsilon^{1/2}\right) + o\left(\varepsilon^{1/2}\right) \\ &= -\frac{1}{6}\|g\|_\infty AT_0^6 + \frac{a}{4}B^2T_0^4 + \frac{b}{2}BT_0^2 - O\left(\varepsilon^{1/2}\right) + o\left(\varepsilon^{1/2}\right) \end{aligned}$$

Now substituting the expression of T_0, we obtain

$$\begin{aligned} Q_{\varepsilon}(T_\varepsilon) &= \frac{ab\|g\|_\infty^{-1}B^3}{4A} + \frac{b(a^2\|g\|_\infty^{-2}B^6 + 4b\|g\|_\infty^{-1}B^3A)^{1/2}}{6A} + \frac{a^3\|g\|_\infty^{-2}B^6}{24A^2} + \\ &\quad \frac{a^2\|g\|_\infty^{-1}(a^2\|g\|_\infty^{-2}B^{12} + 4b\|g\|_\infty^{-1}B^9A)^{1/2}}{24A^2} - O\left(\varepsilon^{1/2}\right) + o\left(\varepsilon^{1/2}\right) \\ &= \frac{ab}{4}\|g\|_\infty^{-1}S_6^3 + \frac{a^3}{24}\|g\|_\infty^{-2}S_6^6 + \left(\frac{b}{6}S_6 + \frac{a^2}{24}\|g\|_\infty^{-1}S_6^4\right)(a^2\|g\|_\infty^{-2}S_6^4 + 4b\|g\|_\infty^{-1}S_6)^{1/2} \\ &\quad - O\left(\varepsilon^{1/2}\right) + o\left(\varepsilon^{1/2}\right) \\ &\leq c^* - c_0 - O\left(\varepsilon^{1/2}\right) + o\left(\varepsilon^{1/2}\right). \end{aligned}$$

Thus we have

$$\begin{aligned} I_\lambda(u_0 + tu_{\varepsilon,x_0}) &\leq c^* - O\left(\varepsilon^{1/2}\right) + o\left(\varepsilon^{1/2}\right) + \sup_{t<T_1} R(t) \\ &\leq c^* - O\left(\varepsilon^{1/2}\right) + o\left(\varepsilon^{1/2}\right) + aK, \end{aligned}$$

where $K := \frac{3}{2}T_1^2\|u_0\|^2 B + T_1^3 B^{3/2}\|u_0\|$.

Consequently, there exist a_0 and ε_0 small enough such that

$$I_\lambda(u_0 + tu_{\varepsilon,x_0}) < c^*, \text{ for every } 0 < \varepsilon < \varepsilon_0 \text{ and } 0 < a < a_0.$$

By Lemma Lemma 2.2., there exists unique $t^{+}(u) > 0$ such that $t^{+}(u)\,u \in \mathcal{N}_{\lambda}^{-}$ and $I_{\lambda}(t^{+}u) \geq I_{\lambda}(tu)$, for all $|t| \geq t_{a,\max}^{u}$ and every $u \in H$ such that $\|u\| = 1$.

The extremal property of $t^{+}(u)$ and its uniqueness give that it is a continuous function of u.

Set
$$V_1 = \{0\} \cup \left\{ u \; / \; \|u\| < t^{+}\left(\frac{u}{\|u\|}\right) \right\}$$
and
$$V_2 = \left\{ u \; / \; \|u\| > t^{+}\left(\frac{u}{\|u\|}\right) \right\}.$$
As in [15], we remark that for $\lambda \in (0, \lambda_*)$, we have $H \backslash \mathcal{N}_{\lambda}^{-} = V_1 \cup V_2$ and $\mathcal{N}_{\lambda}^{+} \subset V_1$, $u_0 \in V_1$ and $u_0 + t_0 u_{\varepsilon,x_0} \in V_2$ for a $t_0 > 0$, carefully choosen.

Let $\Gamma = \{h : [0,1] \to H$ continuous, $h(0) = u_0,\; h(1) = u_0 + t_0 u_{\varepsilon,x_0}\}$ .It is obvious that $h : [0,1] \to H$ given by $h(t) = u_0 + t t_0 u_{\varepsilon,x_0}$ belongs to Γ. We conclude that:
$$c = \inf_{h\in\Gamma} \max_{t\in[0,1]} I_{\lambda}(h(t)) < c^{*}.$$
As the range of any $h \in \Gamma$ intersects $\mathcal{N}_{\lambda}^{-}$, one has $c \geq c_1$.

Applying again the Ekeland Variational Principle, we obtain a minimizing sequence $(u_n) \subset \mathcal{N}_{\lambda}^{-}$ such that
$$I_{\lambda}(u_n) \to c_1$$
and
$$\|I_{\lambda}'(u_n)\| \to 0.$$

We also deduce that $c_1 < c^{*}$.

Consequently, we get a subsequence (u_{n_k}) of (u_n) and $u_1 \in H$ such that
$$u_{n_k} \to u_1 \text{ strongly in } H.$$
This implies that u_1 is a critical point for I_{λ}, $u_1 \in \mathcal{N}_{\lambda}^{-}$ and $I_{\lambda}(u_1) = c_1$.

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