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ISOLATED LARGE DIFFUSION

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Abstract. In this paper we formulate a prototype problem of isolated large diffusion, in the sense that, we consider large diffusion in finite disjoint parts of a domain where there is no interaction between the diffusion on each part. We show that the solutions present spatial homogenization and we analyze all asymptotic behavior, as well as convergence of solutions, eigenvalues and spectral projections.

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1 Introduction

Reaction-diffusion equations is an important type of partial differential equations because they have a lot of applications, as we can see in [1] and [4]. Among them, the equations with large diffusion model situations where the heat distribution blows up in the physical domain of the equation, see [3] and [5]. Thus it is interesting to consider situations where it is possible to isolate the large diffusion.

In this paper we present a prototype problem when the diffusion coefficient is large and isolated in finite bounded parts of $\mathbb{R}^N$. We will study the well posedness and obtain the convergence of the solutions in an appropriate energy space.

Let $\Omega = \cup_{i=1}^m \Omega_i \subset \mathbb{R}^N$ be a smooth open set with m connected smooth components such that $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$ if $i \neq j$. We denote $\Gamma_i = \partial\Omega_i$, for $i = 1, ..., m$, and $\Gamma = \cup_{i=1}^m \Gamma_i$. For $\varepsilon > 0$, we define $\Omega_i^\varepsilon = \{x \in \Omega_i : \text{dist}(x, \Gamma_i) > \varepsilon\}$, $i = 1, ..., m$, $\Omega^\varepsilon = \cup_{i=1}^m \Omega_i^\varepsilon$ and, for each $i = 1, ..., n$, we take the family of molifiers $\eta_\varepsilon^i \in C^\infty(\mathbb{R}^N)$, with $\int_{\mathbb{R}^N} \eta_\varepsilon^i \, dx = 1$ and $\text{supp}(\eta_\varepsilon^i) \subset B(0, \varepsilon) \subset \mathbb{R}^N$. If $\chi_{\Omega_i^\varepsilon}$ is the characteristic function of Ω_i^ε, then

$$\frac{1}{\varepsilon}\eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}(x) = \frac{1}{\varepsilon}\int_{\Omega_i^\varepsilon} \eta_\varepsilon^i(x-y)\chi_{\Omega_i^\varepsilon}(y)\, dy \to \infty, \quad \text{as} \quad \varepsilon \to 0, \tag{1.1}$$

$$\text{supp}(\eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}) \subset \overline{\text{supp}(\eta_\varepsilon^i) + \text{supp}(\chi_{\Omega_i^\varepsilon})} = \overline{B(0,\varepsilon) + \Omega_i^\varepsilon} \subset \bar{\Omega}_i. \tag{1.2}$$

We consider the family of second order equations

$$\begin{cases} u_t^\varepsilon - Div\Big(\dfrac{1}{\varepsilon}\displaystyle\sum_{i=1}^m \eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}(x)\nabla u^\varepsilon\Big) + \lambda u^\varepsilon = f(u^\varepsilon) \quad \text{in} \quad \Omega, \\ \frac{\partial u^\varepsilon}{\partial \nu_i^\varepsilon} = 0 \quad \text{in} \quad \Gamma_i, \quad i = 1, ..., m, \\ u^\varepsilon(0) = u_0^\varepsilon, \end{cases} \tag{1.3}$$

where $\lambda > 0$ and f is continuously differentiable and $\frac{\partial u^\varepsilon}{\partial \nu_i^\varepsilon} = \big\langle \frac{1}{\varepsilon}\sum_{i=1}^m \eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}(x)\nabla u^\varepsilon, \nu_i^\varepsilon \big\rangle$ is the co-normal derivative in Γ_i and ν_i^ε denotes the unit outward normal vector to Γ_i.

Now we informally try to guess what happens when $\varepsilon \to 0$. For this we fix j and integrate (1.3) in Ω_j. By Divergence Theorem,

$$\int_{\Omega_j} u_t^\varepsilon \, dx - \int_{\Gamma_j} \frac{1}{\varepsilon}\eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}(x)\frac{\partial u^\varepsilon}{\partial \nu_j^\varepsilon}\, d\sigma_j + \int_{\Omega_j} \lambda u^\varepsilon \, dx = \int_{\Omega_j} f(u^\varepsilon)\, dx,$$

where $d\sigma_j$ indicates the usual surface measure in Γ_j. And by Neumann boundary condition on Γ_j,

$$\int_{\Omega_j} u_t^\varepsilon \, dx + \int_{\Omega_j} \lambda u^\varepsilon \, dx = \int_{\Omega_j} f(u^\varepsilon)\, dx.$$

Since large diffusivity implies fast homogenization in the spatial variable (see [5]), we expect that, when $\varepsilon \to 0$, the limiting solution u_j is spatially constant in each Ω_j, thus

$$\int_{\Omega_j} \dot{u}_j \, dx + \int_{\Omega_j} \lambda u_j \, dx = \int_{\Omega_j} f(u_j)\, dx.,$$

and we obtain that the limiting problem of (1.3) as $\varepsilon \to 0$ is given by the ordinary differential systems

$$\begin{cases} \dot{u}_i + \lambda u_i = f(u_i), \quad i = 1, ..., m, \\ u(0) = u_0, \end{cases} \tag{1.4}$$

where $u_0 = \lim_{\varepsilon \to 0} \sum_{i=1}^{m} \frac{1}{|\Omega_i|} \int_{\Omega_i} u^\varepsilon(0)\, dx$.

Our goal is to show that the solutions of (1.3) converges to solutions of (1.4) in an appropriate functional space. To accomplish this task we divide this paper as follows. In Section 2 we establish the functional space to treat the problems (1.3) and (1.4). In Section 3 we prove the convergence of the spectral properties of the related elliptic operators. Finally, in Section 4 we consider the well posedness of the parabolic problems and prove that the solutions of (1.3) converges to the solutions of (1.4) as ε goes to zero.

2 Functional Setting

In this section we define the variational formulation of the problem (1.3). We will consider the fractional power space of the linear operators related with (1.3) and we will see that this energy space with an appropriate inner product generates a norm, which we will state in what sense the solution of (1.3) converges to the solution of (1.4).

We define the operator $A_\varepsilon : \mathcal{D}(A_\varepsilon) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$\mathcal{D}(A_\varepsilon) = \{u \in H^2(\Omega)\,;\, \frac{\partial u^\varepsilon}{\partial \nu_i^\varepsilon} = 0,\ i = 1, ..., m\}, \quad A_\varepsilon u = -Div\Big(\frac{1}{\varepsilon} \sum_{i=1}^{m} \eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}(x) \nabla u\Big) + \lambda u.$$

We denote $L^2_\Omega = \{u \in H^1(\Omega)\,;\, \nabla u = 0 \text{in } \Omega\}$, note that if $u \in L^2_\Omega$ then u is constant a.e in each Ω_i, thus we define the operator $A_0 : L^2_\Omega \subset L^2(\Omega) \to L^2(\Omega)$ by $A_0 u = \sum_{i=1}^{m} \lambda u_i$, where u_i denotes the constant value of u in Ω_i.

Since the diffusion coefficient $\frac{1}{\varepsilon} \sum_{i=1}^{m} \eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}(x)$ is smooth, we known that A_ε is a positive invertible operator with compact resolvent for each $\varepsilon \in (0, \varepsilon_0]$, hence we define in the usual way, see [6], the fractional power space $X_\varepsilon^{\frac{1}{2}} = H^1(\Omega)$ and we denote $X_0^{\frac{1}{2}} = L^2_\Omega$. In this spaces we consider the following inner products

$$\langle u, v \rangle_{X_\varepsilon^{\frac{1}{2}}} = \int_\Omega \frac{1}{\varepsilon} \sum_{i=1}^{m} \eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}(x) \nabla u \nabla v\, dx + \int_\Omega \lambda u v\, dx, \quad u, v \in X_\varepsilon^{\frac{1}{2}},\ \varepsilon \in (0, \varepsilon_0]; \tag{2.1}$$

$$\langle u, v \rangle_{X_0^{\frac{1}{2}}} = |\Omega|^{-1} \lambda \sum_{i=1}^{m} u_i v_i, \quad u, v \in X_0^{\frac{1}{2}}. \tag{2.2}$$

The space $X_0^{\frac{1}{2}}$ is a finite dimensional closed subspace of $X_\varepsilon^{\frac{1}{2}}$, $\varepsilon \in (0, \varepsilon_0]$ and $H^1(\Omega) \hookrightarrow X_\varepsilon^{\frac{1}{2}}$ but the injection is not uniform, in fact is valid, as we can see of (2.1) that for ε sufficiently small,

$$\|u\|^2_{X_\varepsilon^{\frac{1}{2}}(\Omega^\varepsilon)} \leq \frac{1}{\varepsilon} \|u\|^2_{H^1(\Omega^\varepsilon)}. \tag{2.3}$$

Thus estimates in the H^1-norm does not produce well estimates in $X_\varepsilon^{\frac{1}{2}}$, hence we consider the fractional power space $X_\varepsilon^{\frac{1}{2}}$ as the functional space to deal with the problems (1.3) and (1.4).

3 The Elliptic Problem

In this section we study the convergence of the elliptic problems. We prove the convergence of eigenvalue, eigenfunctions and spectral projections. This approach is essentials to obtain the convergence of the solutions of the parabolic problem (1.3) and the limiting one.

We consider the following elliptic problem

$$\begin{cases} -Div\Big(\dfrac{1}{\varepsilon}\displaystyle\sum_{i=1}^{m}\eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}(x)\nabla u^\varepsilon\Big) + \lambda u^\varepsilon = f(u^\varepsilon) \quad \text{in} \quad \Omega, \\ \frac{\partial u^\varepsilon}{\partial \nu_i^\varepsilon} = 0 \quad \text{in} \quad \Gamma_i, \quad i = 1, ..., m, \end{cases} \tag{3.1}$$

where $f \in L^2(\Omega)$.

To compare the problems we need to project functions of $X_\varepsilon^{\frac{1}{2}}$ onto $X_0^{\frac{1}{2}}$, thus we consider the projection

$$Pu = \sum_{i=1}^{m}\Big(\frac{1}{|\Omega_i|}\int_{\Omega_i} u\,dx\Big)\chi_{\Omega_i}, \quad u \in L^2(\Omega) \quad \text{or} \quad u \in X_\varepsilon^{\frac{1}{2}}.$$

We have P is an orthogonal projection acting on L^2 onto L^2_Ω or $X_\varepsilon^{\frac{1}{2}}$ onto $X_0^{\frac{1}{2}}$.

Theorem 3.1. *Let u^ε solution of* (3.1)*. Then*

$$\lim_{\varepsilon\to 0}\|u^\varepsilon - u\|_{X_\varepsilon^{\frac{1}{2}}} = 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon}\frac{1}{\varepsilon}\sum_{i=1}^{m}\eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}(x)|\nabla u^\varepsilon|^2\,dx = 0, \tag{3.2}$$

where $u = \lambda^{-1}Pf$ and $f \in L^2(\Omega)$. Furthermore

$$\lim_{\varepsilon\to 0}\|A_\varepsilon^{-1} - A_0^{-1}P\|_{\mathcal{L}(L^2(\Omega), X_\varepsilon^{\frac{1}{2}})} = 0. \tag{3.3}$$

Proof. The solutions u^ε and u satisfies

$$\int_\Omega \frac{1}{\varepsilon}\sum_{i=1}^{m}\eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}(x)\nabla u^\varepsilon\nabla\varphi\,dx + \int_\Omega \lambda u^\varepsilon\varphi\,dx = \int_\Omega f\varphi\,dx, \quad \forall\,\varphi \in X_\varepsilon^{\frac{1}{2}}, \tag{3.4}$$

$$\int_\Omega \lambda u\varphi\,dx = \int_\Omega Pf\varphi\,dx, \quad \forall\,\varphi \in X_0^{\frac{1}{2}}. \tag{3.5}$$

Taking $\varphi = u^\varepsilon - u$ in (3.4) and $\varphi = Pu^\varepsilon - u$ in (3.5), we obtain by inner product (2.1) that

$$\|u^\varepsilon - u\|^2_{X_\varepsilon^{\frac{1}{2}}} \leq \int_\Omega |f(I-P)u^\varepsilon|\,dx.$$

By Poincar 迣's inequality for average, we have

$$\int_\Omega |f(I-P)u^\varepsilon|\,dx \le \sum_{i=1}^m \|f\|_{L^2(\Omega_i)}\Big(\int_{\Omega_i}|\nabla u^\varepsilon|^2\,dx\Big)^{\frac12},$$

but

$$\int_{\Omega_i}|\nabla u^\varepsilon|^2\,dx = \int_{\Omega_i^\varepsilon}|\nabla u^\varepsilon|^2\,dx + \int_{\Omega_i\setminus\Omega_i^\varepsilon}|\nabla u^\varepsilon|^2\,dx.$$

We estimate each above integrals in the following way,

$$\int_{\Omega_i\setminus\Omega_i^\varepsilon}|\nabla u^\varepsilon|^2\,dx \le \Big(\int_{\Omega_i\setminus\Omega_i^\varepsilon}|\nabla u^\varepsilon|^2\,dx\Big)^{\frac12}\Big(\int_{\Omega_i\setminus\Omega_i^\varepsilon}dx\Big)^{\frac12} \le C|\Omega_i\setminus\Omega_i^\varepsilon|^{\frac12}\to 0 \quad \text{as} \quad \varepsilon\to 0,$$

$$\frac{1}{\varepsilon}\int_{\Omega_i^\varepsilon}|\nabla u^\varepsilon|^2\,dx \le \int_{\Omega_i^\varepsilon}\frac{1}{\varepsilon}\sum_{i=1}^m \eta_\varepsilon^i * \chi_{\Omega_i^\varepsilon}(x)|\nabla u^\varepsilon|^2\,dx \le \|u^\varepsilon\|^2_{X_\varepsilon^{\frac12}} \le C.$$

Put this estimates together we obtain $\lim_{\varepsilon\to 0}\|u^\varepsilon - u\|_{X_\varepsilon^{\frac12}} = 0$. The last inequality proves the second limiting in (3.2).

To prove (3.3) we note that, for each $g \in L^2(\Omega)$ with $\|g\|_{L^2(\Omega)} \le 1$, the equations $A_\varepsilon u^\varepsilon = g$ and $A_0 u = Pg$ implies that $u^\varepsilon = A_\varepsilon^{-1}g$ and $u = A_0^{-1}Pg$. The result follows from the same argument as above. □

Next we analyse the spectral properties of operator A_ε.

Theorem 3.2. *If we denote, for each $\varepsilon \in (0,\varepsilon_0]$ the spectra of A_ε by $\sigma(A_\varepsilon) = \{\lambda_1^\varepsilon, \lambda_2^\varepsilon, ...\}$ and the $\{\varphi_1^\varepsilon, \varphi_2^\varepsilon, ...\}$ the related set of eigenfunctions. Then*

(i) $\lambda_1^\varepsilon = \lambda$ and $\varphi_1^\varepsilon = |\Omega|^{-\frac12}$.

(ii) $\lambda_i^\varepsilon \to \infty$ as $\varepsilon \to 0$ for all $i > 2$.

(iii) given $\delta > 0$ sufficiently small, the operator

$$Q_\varepsilon = Q_\varepsilon(\lambda) = \frac{1}{2\pi i}\int_{|\mu+\lambda|=\delta}(\mu + A_\varepsilon)^{-1}\,d\mu,\ \ \varepsilon\in(0,\varepsilon_0],$$

are compact projections of $L^2(\Omega)$ onto $X_\varepsilon^{\frac12}$ and

$$\lim_{\varepsilon\to 0}\|Q_\varepsilon - I\|_{\mathcal{L}(L^2(\Omega),X_\varepsilon^{\frac12})} = 0. \tag{3.6}$$

Proof. The statement (i) is immediate from definition of A_ε. If (ii) fails, we can take $R > 0$ and sequences $\varepsilon_k \to 0$ and $\{\lambda_j^{\varepsilon_k}\}$, $j > 1$ such that $|\lambda_j^{\varepsilon_k}| \le R$. We can assume $\lambda_j^{\varepsilon_k} \to \mu$. Let $\varphi_j^{\varepsilon_k}$ be the corresponding eigenfunction to $\lambda_j^{\varepsilon_k}$ with $\|\varphi_j^{\varepsilon_k}\|_{X_{\varepsilon_k}^{\frac12}} = 1$. Then $\varphi_j^{\varepsilon_k} = \lambda_j^{\varepsilon_k}A_{\varepsilon_k}^{-1}\varphi_j^{\varepsilon_k}$. Since $A_{\varepsilon_k}^{-1}$ converges to A_0^{-1}, we can assume $\varphi_j^{\varepsilon_k} \to u^0$ as $\varepsilon_k \to 0$ for some $u^0 \in X_0^{\frac12}$. Thus

$$\varphi_j^{\varepsilon_k} = \lambda_j^{\varepsilon_k}A_{\varepsilon_k}^{-1}\varphi_j^{\varepsilon_k} \to \mu A_0^{-1}u^0,$$

as $\varepsilon_k \to 0$. Since $\varphi_j^{\varepsilon_k} \to u^0$, we get $u^0 = \mu A_0^{-1} u^0$, which implies $\mu \in \sigma(A_0)$, thus $\mu = \lambda$ and then $\lambda_j^{\varepsilon_k} \to \lambda$ as $\varepsilon_k \to 0$, $j > 1$, which is an absurd.

To prove (iii) note that, since $\sigma(A_\varepsilon)$ is a sequence that goes to infinity as ε goes to zero, for δ sufficiently small Q_ε is well defined and

$$\begin{aligned}\|Q_\varepsilon - I\|_{\mathcal{L}(L^2(\Omega),X_\varepsilon^{\frac{1}{2}})} &= \left\| \frac{1}{2\pi i} \int_{|\mu-\lambda|=\delta} (\mu + A_\varepsilon)^{-1} - (\mu + A_0)^{-1} P \, d\mu \right\|_{\mathcal{L}(L^2(\Omega),X_\varepsilon^{\frac{1}{2}})} \\ &\leq \frac{1}{2\pi} \int_{|\mu-\lambda|=\delta} \|(\mu + A_\varepsilon)^{-1} - (\mu + A_0)^{-1} P\|_{\mathcal{L}(L^2(\Omega),X_\varepsilon^{\frac{1}{2}})} \, |d\mu|,\end{aligned}$$

where we have used that $\frac{1}{2\pi i} \int_{|\mu-\lambda|=\delta} (\mu + A_0)^{-1} P \, d\mu = I_{\mathcal{L}(L^2(\Omega))}$. But, we claim that it is valid the following identity

$$A_\varepsilon^{\frac{1}{2}} \Big((\mu + A_\varepsilon)^{-1} - (\mu + A_0)^{-1} P \Big) = A_\varepsilon (\mu + A_\varepsilon)^{-1} A_\varepsilon^{\frac{1}{2}} (A_\varepsilon^{-1} - A_0^{-1} P)(\mu + A_0)^{-1} \quad (3.7)$$

and then the result follows from convergence (3.3) by noting that $A_\varepsilon(\mu + A_\varepsilon)^{-1} = I - \mu(\mu + A_\varepsilon)^{-1}$ and $A_0(\mu + A_0)^{-1} = I - \mu(\mu + A_0)^{-1}$ are uniformly bounded.

To verify (3.7) note that

$$\begin{aligned}(\mu + A_\varepsilon)^{-1} - (\mu + A_0)^{-1} P &= (\mu + A_\varepsilon)^{-1} [(\mu + A_0) - (\mu + A_\varepsilon)](\mu + A_0)^{-1} P \\ &= (\mu + A_\varepsilon)^{-1} A_\varepsilon (A_\varepsilon^{-1} - A_0^{-1} P) A_0 (\mu + A_0)^{-1} P,\end{aligned}$$

where we have used that $A_0^{-1} = A_0^{-1} P$. □

4 The Parabolic Problem

In this section we study the well-posedness of (1.3) in the fractional power space $X_\varepsilon^{\frac{1}{2}}$. We prove that the solutions of (1.3) converges to the solution of (1.4) in the $X_\varepsilon^{\frac{1}{2}}$-norm.

The results in existence and uniqueness of this section follow from the standard theory of existence and uniqueness for parabolic equations developed in [2] and [6]. However, due to the diffusion coefficient in (1.3) and the domain type considered, we will revisit the theorems addressing the main steps in the statements. In fact we need to ensure that the constants are independent of ε, since we have the blow up condition (1.1).

Definition 4.1. We say that (1.3) is locally well posed if, for any $u_0^\varepsilon \in L^2(\Omega)$, $\|u_0^\varepsilon\|_{L^2(\Omega)} < r$, for some $r > 0$, there is a map $u^\varepsilon(\cdot, u_0^\varepsilon) : [0, \tau] \subset \mathbb{R} \to L^2(\Omega)$ such that

(i) $u^\varepsilon(\cdot, u_0^\varepsilon) \in C([0, \tau], L^2(\Omega))$,

(ii) $u^\varepsilon(\cdot, u_0^\varepsilon)$ satisfies the variation of constants formula

$$u^\varepsilon(t, u_0^\varepsilon) = e^{-A_\varepsilon t} u_0^\varepsilon + \int_0^t e^{-A_\varepsilon (t-s)} f(u^\varepsilon(s, u_0^\varepsilon)) \, ds, \quad t \in [0, \tau],$$

where $e^{-A_\varepsilon t}$ is the strongly continuous linear semigroup generated by $-A_\varepsilon$.

In this case we say that $u^\varepsilon(\cdot, u_0^\varepsilon)$ is a mild solution of (1.3). If we can take $\tau = \infty$ we say that (1.3) is globally well posed.

The local well posed is obtained in the following way.

Theorem 4.2. *Assume that for each $R > 0$, there is a constant $C = C(R) > 0$ such that, the Neminski functional $f : X_\varepsilon^{\frac{1}{2}} \to L^2(\Omega)$ satisfies*

$$\|f(u) - f(v)\|_{X_\varepsilon^{\frac{1}{2}}} \leq C\|u - v\|_{L^2(\Omega)}, \quad \text{and} \quad \|f(u^\varepsilon)\|_{X_\varepsilon^{\frac{1}{2}}} \leq C,$$

as long as $\|u\|_{L^2}, \|v\|_{L^2L^2(\Omega)} \leq R$. Then (1.3) is locally well posed and the solutions depend continuously on the initial data, that is, if $\|u_0^\varepsilon\|_{L^2}, \|v_0^\varepsilon\|_{L^2} \leq R$, then

$$\|u^\varepsilon(t, u_0^\varepsilon) - u^\varepsilon(t, v_0^\varepsilon)\|_{L^2(\Omega)} \leq C\|u_0^\varepsilon - v_0^\varepsilon\|_{L^2(\Omega)}.$$

Proof. Since A_ε has compact resolvent, we define, for each $u_0^\varepsilon \in L^2(\Omega)$, the linear semigroup

$$e^{-A_\varepsilon t} u_0^\varepsilon = \sum_{j=1}^{\infty} e^{-\lambda_j^\varepsilon t} Q_j^\varepsilon u_0^\varepsilon = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} (\mu + A_\varepsilon)^{-1} u_0^\varepsilon \, d\mu, \tag{4.1}$$

where γ is the boundary of the set $\{\mu \in \mathbb{C} : |\arg(\mu)| \leq \phi\} \setminus \{\mu \in \mathbb{C} : |\mu| \leq r\}$, for some $\phi \in (\pi/2, \pi)$ and $r > 0$, oriented in such a way that the imaginary part is increasing. it is valid the following inequality (see [2])

$$\|e^{-A_\varepsilon t} u_0^\varepsilon\|_{\mathcal{L}(L^2(\Omega))} \leq M t^{-\frac{1}{2}} \|u_0^\varepsilon\|_{X_\varepsilon^{\frac{1}{2}}}. \tag{4.2}$$

Thus, we consider the map Φ_ε

$$(\Phi_\varepsilon u)(t) = e^{-A_\varepsilon t} u_0^\varepsilon + \int_0^t e^{-A_\varepsilon (t-s)} f(u(s)) \, ds, \ t > 0, \tag{4.3}$$

in the space

$$\mathcal{K}_\tau = \{u(\cdot) \in C([0, \tau], L^2(\Omega)) : u(0) = u_0^\varepsilon, \ \|u(t)\|_{L^\infty([0,0+\tau], L^2(\Omega))} \leq M\|u_0^\varepsilon\|_{L^2(\Omega)} + 1\}.$$

We can choose $\tau > 0$ uniformly for all $\|u_0^\varepsilon\|_{L^2(\Omega)} < r$, such that

$$\begin{aligned} \|(\Phi_\varepsilon u)(t)\|_{L^2(\Omega)} &\leq M\|u_0^\varepsilon\|_{L^2(\Omega)} + C(Mr + 1) \int_0^t (t - s)^{-\frac{1}{2}} \, ds \\ &\leq M\|u_0^\varepsilon\|_{L^2(\Omega)} + 1, \quad t \in [0, \tau], \end{aligned}$$

which prove that $\Phi_\varepsilon(\mathcal{K}_\tau) \subset \mathcal{K}_\tau$.

To prove that Φ_ε is a contraction, take $u, v \in \mathcal{K}_\tau$ to

$$\begin{aligned} \|(\Phi_\varepsilon u)(t) - (\Phi_\varepsilon v)(t)\|_{L^2(\Omega)} &\leq C(Mr + 1)M \int_0^t (t - s)^{-\frac{1}{2}} \, ds \sup_{0 \leq s \leq t} \{\|u(s) - v(s)\|_{L^2(\Omega)}\} \\ &\leq \frac{1}{2} \sup_{0 \leq s \leq t} \{\|u(s) - v(s)\|_{L^2(\Omega)}\}. \end{aligned}$$

Thus Φ_ε is a contraction on $\mathcal{K}_\tau$, which implies that Φ_ε has a unique fixed point in $\mathcal{K}_\tau$.

Now, let $u(\cdot, u_0^\varepsilon)$ be the unique mild solution of (1.3) For $\|u_0^\varepsilon\|_{L^2(\Omega)}, \|v_0^\varepsilon\|_{L^2(\Omega)} < r$, we can prove that

$$\|u(t, u_0^\varepsilon) - v(t, v_0^\varepsilon)\|_{L^2(\Omega)} \leq r\|u_0^\varepsilon - v_0^\varepsilon\|_{L^2(\Omega)} + \frac{1}{2} \sup_{0\leq s\leq t} \{\|u(s, u_0^\varepsilon) - v(s, v_0^\varepsilon)\|_{L^2(\Omega)}\},$$

which implies

$$\|u(t, u_0^\varepsilon) - v(t, v_0^\varepsilon)\|_{L^2(\Omega)} \leq C\|u_0^\varepsilon - v_0^\varepsilon\|_{L^2(\Omega)},$$

for $C > 0$ independent of ε. □

Corollary 4.3. *Suppose that f as in Theorem 4.2. Then, for each $u_0 \in L^2(\Omega)$ there is a maximal time of existence $\tau_M > 0$ such that*

(i) the solution $u^\varepsilon(\cdot, u_0^\varepsilon)$ is defined in $[0, \tau_M])$,

(ii) either $\tau_M = \infty$ or $\liminf_{t\to\tau_M} \|u^\varepsilon(t, u_0^\varepsilon)\|_{L^2} = \infty$.

Proof. For each $u_0^\varepsilon \in L^2(\Omega)$, let

$$\tau_M = \sup\{t_1 : \text{there is a solition of (1.3) defined on } [0, t_1]\}$$

We have that if the solutions remains bounded in an interval of the form $[0, \tau_0]$, then there is $\sigma > 0$ such that the solution $u(\cdot, \tau, u(\tau, u_0^\varepsilon))$ is defined in $[\tau, \tau + \sigma]$. Let $\tau \in [0, \tau_0)$ be such that $\tau + \sigma > \tau_0$. Since the function $z : [0, \tau_0] \to L^2(\Omega)$ is defined by $z(t) = u(t, u_0^\varepsilon)$ for $t \in [0, \tau]$ and $z(t) = u(t, \tau, u(\tau, u_0^\varepsilon))$ for $t \in [\tau, \tau + \sigma]$ is a solution, we get a contradiction. □

Theorem 4.4. *Suppose that f as in Theorem 4.2 and for $u_0^\varepsilon \in L^2(\Omega)$ we have*

$$\sup_{t\in[0,\tau]} \|u^\varepsilon(t, u_0^\varepsilon)\|_{L^2(\Omega)} < \infty, \quad \text{for all } \tau > 0. \tag{4.4}$$

Then $\tau_M = \infty$. That is, (1.3) is globally well posed. Moreover if $u_0^\varepsilon \in X_\varepsilon^{\frac{1}{2}}$, then

$$u(\cdot, u_0^\varepsilon) \in C([0, \infty), X_\varepsilon^{\frac{1}{2}}). \tag{4.5}$$

Proof. Since we have the energy estimate, it follows from Gronwall Lemma that the solutions are defined for all positive time. This solutions satisfy the variation constants formula. Hence (1.3) is globally well posed if we assume some appropriate conditions in f. For example, if f satisfies

$$uf(u) \leq u^2 + C|u|, \quad t, u \in \mathbb{R},$$

for some constant $C > 0$, then (4.4) is true. See [2] Chapter 12 for more general conditions.

For the regularity (4.5) we need to consider the relation between the dimension N and the exponent $p = 2$ and injection theorem. All details can be found in [2] or [6]. □

Now we are in position to state our main result in this work.

Theorem 4.5. *For each* $u_0^\varepsilon \in X_\varepsilon^{\frac{1}{2}}$ *and* f *continuously differentiable, the parabolic equation* (1.3) *and the ODE* (1.4) *are globally well posed in* $X_\varepsilon^{\frac{1}{2}}$ *and* $X_0^{\frac{1}{2}}$ *respectively. Moreover if* u^ε *is the solution of* (1.3) *and* u *the solution of* (1.4), *then*

$$\lim_{\varepsilon\to 0} \|u^\varepsilon - u\|_{X_\varepsilon^{\frac{1}{2}}} = 0.$$

Proof. The solution of (1.4) satisfies

$$u(t) = e^{-\lambda t}u_0 + \int_0^t e^{-\lambda(t-s)} f(u(s))\, ds,$$

and the solutions of (1.3) satisfy

$$u^\varepsilon(t, u_0^\varepsilon) = e^{-A_\varepsilon t}u_0^\varepsilon + \int_0^t e^{-A_\varepsilon(t-s)} f(u^\varepsilon(s, u_0^\varepsilon))\, ds.$$

It follows form expression (4.1) and the convergence (3.3) that

$$\lim_{\varepsilon\to 0} \|e^{-A_\varepsilon t} - e^{-A_0 t}P\|_{\mathcal{L}(L^2(\Omega), X_\varepsilon^{\frac{1}{2}})} = 0.$$

Thus we assume, for simplicity, $u_0^\varepsilon = 0$. We decompose $X_\varepsilon^{\frac{1}{2}} = Y_\varepsilon + Z_\varepsilon$, where $Y_\varepsilon = Q_\varepsilon X_\varepsilon^{\frac{1}{2}}$ and $Z_\varepsilon = (I - Q_\varepsilon)X_\varepsilon^{\frac{1}{2}}$. We write the above solutions as $u^\varepsilon(t) = v^\varepsilon(t) + w^\varepsilon(t)$, where

$$v^\varepsilon(t) = \int_0^t e^{-\lambda(t-s)} f(u(s)) \quad \text{and} \quad w^\varepsilon(t) = \int_0^t e^{-A_\varepsilon(t-s)} f(u^\varepsilon(s, u_0^\varepsilon))\, ds.$$

The result is proved if we show that $\|w^\varepsilon(t)\|_{X_\varepsilon^{\frac{1}{2}}} \to 0$ as $\varepsilon \to 0$. Indeed, we have from (4.2) that

$$\|w^\varepsilon(t)\|_{X_\varepsilon^{\frac{1}{2}}} \leq \int_0^t M(t-s)^{-\frac{1}{2}} \|(I - Q_\varepsilon) f(w^\varepsilon(s))\|_{X_\varepsilon^{\frac{1}{2}}}\, ds.$$

It follows from (3.6) that $\|w^\varepsilon(t)\|_{X_\varepsilon^{\frac{1}{2}}} \to 0$ as $\varepsilon \to 0$. □

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