Advances in Mathematical
Sciences and Applications
Vol. 28, No. 1 (2019), pp. 29–57

GAKKOTOSHO
TOKYO JAPAN

EXISTENCE OF WEAK SOLUTIONS TO STATIONARY AND EVOLUTIONARY MAXWELL-STOKES TYPE PROBLEMS AND THE ASYMPTOTIC BEHAVIOR OF THE SOLUTION

Junichi Aramaki *
Division of Science, Tokyo Denki University
Hatoyama-machi, Saitama, 350-0394, Japan
(E-mail: aramaki@hctv.ne.jp)

Abstract. We consider the existence of a weak solution to a class of evolutionary Maxwell-Stokes type problems containing a p-curlcurl system in a multiply-connected domain. Moreover, we show that the weak solution converges to a weak solution of a stationary Maxwell-Stokes type problem as time tending to infinity.

Communicated by Editors; Received October 10, 2018
AMS Subject Classification: 35K55, 35K60, 35K65, 35K90.
Keywords: Maxwell-Stokes system, weak solution, p-curlcurl operator, asymptotic behavior, multiply-connected domain.

1 Introduction

Generalized Maxwell equations in an electromagnetic field are written by

$$\begin{cases} \varepsilon\partial_t \boldsymbol{E} + \sigma\boldsymbol{j} = \operatorname{curl} \boldsymbol{H}, \\ \mu\partial_t \boldsymbol{H} + \operatorname{curl} \boldsymbol{E} = \boldsymbol{F}, \\ \varepsilon \operatorname{div} \boldsymbol{E} = q, \\ \operatorname{div} \boldsymbol{H} = 0 \end{cases} \tag{1.1}$$

in $\Omega_T := \Omega \times (0,T)$, where Ω is a bounded domain in $\mathbb{R}^3$ with a boundary Γ, $\boldsymbol{E}$ and $\boldsymbol{H}$ denote the electric and magnetic fields, respectively, ε is the permittivity of the electric field, μ is the permeability of the magnetic field, σ is the electric conductivity of the material, $\boldsymbol{j}$ is the total current density and q is the density of the electric charge. Since the displace current $\varepsilon\partial_t\boldsymbol{E}$ is small in comparison with the eddy currents, we neglect the term. We use the nonlinear extension of Ohm's law $|\boldsymbol{j}|^{p-2}\boldsymbol{j} = \sigma\boldsymbol{E}$ $(1 < p < \infty)$. Then $\boldsymbol{H}$ satisfies the following equations containing the p-curlcurl equation

$$\begin{cases} \mu\partial_t \boldsymbol{H} + \operatorname{curl} [\frac{1}{\sigma}|\operatorname{curl} \boldsymbol{H}|^{p-2}\operatorname{curl} \boldsymbol{H}] = \boldsymbol{F}, \\ \operatorname{div} \boldsymbol{H} = 0 \end{cases} \tag{1.2}$$

in Ω_T. The solvability of such a system depends on the nature of the boundary conditions and the shape of the domain. In the article by Yin et al. [16], they impose the natural boundary condition

$$\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma_T := \Gamma \times (0,T), \tag{1.3}$$

where $\boldsymbol{n}$ denotes the outward normal unit vector field to Γ and they also impose the initial condition

$$\boldsymbol{H}(0) = \boldsymbol{H}_0 \text{ on } \Omega. \tag{1.4}$$

Putting $\nu = 1/\sigma$, we consider the following system.

$$\begin{cases} \mu\partial_t \boldsymbol{H} + \operatorname{curl} [\nu|\operatorname{curl} \boldsymbol{H}|^{p-2}\operatorname{curl} \boldsymbol{H}] = \boldsymbol{F} & \text{in } \Omega_T, \\ \operatorname{div} \boldsymbol{H} = 0 & \text{in } \Omega_T, \\ \boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0}, & \text{on } \Gamma_T, \\ \boldsymbol{H}(0) = \boldsymbol{H}_0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

As a necessary condition for the existence of a solution to this problem, the external field $\boldsymbol{F}$ must satisfy $\operatorname{div} \boldsymbol{F} = 0$ in Ω_T.

The authors of [16] obtained the existence theorem of a weak solution of (1.5) in the case where $p \geq 2$, and Ω is a bounded simply connected domain without holes. The model system (1.5) provides a good approximation of Bean's model. Moreover, we can analyze the phase-change process between normal and superconductor regions. For precision, see [16] and references therein. Miranda et al. [12] considered the problem (1.5) under a more general setup in the case where Ω is simply connected domain with the boundary condition $\boldsymbol{H} \cdot \boldsymbol{n} = \boldsymbol{0}$ on Γ_T instead of (1.3). They obtained a "weak" solution. However, their "weak" solution is not the solution of (1.5) in the distribution sense, so the "weak" solution is not a weak solution strictly speaking.

In this paper, we consider a more general Maxwell system than (1.5) in the case where Ω is a multiply-connected domain.

$$\begin{cases} \partial_t \boldsymbol{u} + \operatorname{curl}[S_s(x,t,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] = \boldsymbol{F} & \text{in } \Omega_T, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega_T, \\ \boldsymbol{u}\times\boldsymbol{n} = 0 & \text{on } \Gamma_T, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & i = 1,\dots,I, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega, \end{cases} \tag{1.6}$$

where the function $S(x,t,s)$ satisfies some structural conditions (2.4a)-(2.4c) below, and Γ_i $(i = 0,1,\dots,I)$ are connected components of the boundary Γ with Γ_0 denoting the boundary of the unbounded connected component of $\mathbb{R}^3\setminus\overline{\Omega}$, and $\langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i}$ denotes some duality bracket. More precisely, these are defined in section 2. Under the hypothesis $\operatorname{div}\boldsymbol{F} = 0$ in Ω_T, we show the existence of a weak solution of (1.6). Our weak solution satisfies (1.6) in the distribution sense. To show this, we must extend the space of test functions in the weak formulation of (1.6) to a wider class of the space of test functions than that considered in [12].

In the case where the condition $\operatorname{div}\boldsymbol{F} = 0$ in Ω_T is not satisfied, it is natural to consider the following Maxwell-Stokes type problem: to find $(\boldsymbol{u},\pi)$ in an appropriate space such that

$$\begin{cases} \partial_t \boldsymbol{u} + \operatorname{curl}[S_s(x,t,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] + \nabla\pi = \boldsymbol{f} & \text{in } \Omega_T, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega_T, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma_T, \\ \pi = 0 & \text{on } \Gamma_T, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & i = 1,\dots,I, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega, \end{cases} \tag{1.7}$$

for any $\boldsymbol{f}$ in an appropriate space. In this paper, we derive the existence of a unique weak solution $(\boldsymbol{u},\pi)$ of (1.7) by using the existence of a weak solution of (1.6).

In our previous paper Aramaki [4], we considered the following stationary Maxwell type problem.

$$\begin{cases} \operatorname{curl}[S_s(x,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] = \boldsymbol{F} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = 0 & \text{on } \Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & i = 1,\dots,I. \end{cases}$$

Under the hypothesis $\operatorname{div}\boldsymbol{F} = 0$ in Ω, we showed the existence of a weak solution and its regularity under stronger conditions than (2.4a)-(2.4c). The existence of a weak solution is reconsidered in section 3 in which we treat the stationary Maxwell-Stokes type problem: to find $(\boldsymbol{u},\pi)$ such that

$$\begin{cases} \operatorname{curl}[S_s(x,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi = 0 & \text{on } \Gamma, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & i = 1,\dots,I. \end{cases} \tag{1.8}$$

For the stationary case (1.8) or the evolutionary case (1.7), the existence theory of a weak solution is interesting and it seems to be new in our understanding.

We should mention a well-known Stokes problem.

$$\begin{cases} \partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \Omega_T, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega_T, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma_T, \\ \pi = 0 & \text{on } \Gamma_T, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \quad . \tag{1.9}$$

For the system (1.9), Baba et al. [5] showed the existence of a weak solution, by using the semi-group theory.

It is also interesting to consider the asymptotic behavior of a weak solution $(\boldsymbol{u}, \pi) = (\boldsymbol{u}(t), \pi(t))$ of (1.7) as $t \to \infty$. We show that $(\boldsymbol{u}(t), \pi(t))$ converges to a weak solution of the stationary version (1.8).

The paper is organized as follows. In section 2, we give some preliminaries on the shape of the domain Ω, some spaces of functions and the structural conditions of a Carathéodory function $S(x, t, s)$. In section 3, we consider the stationary Maxwell-Stokes type problem. Section 4 consists of two subsections. In subsection 4.1, we derive the existence of a weak solution to an evolutionary Maxwell type problem (1.6), and in subsection 4.2, we consider an evolutionary Maxwell-Stokes type problem (1.7). Finally, in section 5, we consider the asymptotic behavior of the solution obtained in section 4 as time t tends to infinity. In Appendix A, we give a proof of Lemma 4.4, and in Appendix B we give a different example satisfying the structural conditions (2.4a)-(2.4c) from the function $S(x, t, s)$ corresponding to p-curlcurl operator.

2 Preliminaries

In this section, we state some preliminaries. Let Ω be a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$ boundary Γ, and $1 < p < \infty$. Throughout this paper, we denote the conjugate exponent of p by p', i.e., $(1/p) + (1/p') = 1$. From now on we use $L^p(\Omega)$, $W^{m,p}(\Omega)$, $H^m(\Omega)$, $W_0^{m,p}(\Omega)$, $H_0^m(\Omega)$ ($m \geq 0$ integer), $W^{s,p}(\Gamma)$ and $H^s(\Gamma)$ ($s \in \mathbb{R}$) for the standard L^p and Sobolev spaces of functions defined in Ω and Γ, respectively, and $C_0^m(\Omega)$ is the set of C^m functions with compact support in Ω ($m \geq 0$ integer or $m = \infty$). For any Banach space B, we denote $B \times B \times B$ by boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard Euclidean inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$ by $\boldsymbol{a} \cdot \boldsymbol{b}$. For the dual space $\boldsymbol{B}'$ of $\boldsymbol{B}$, we denote $\langle \cdot, \cdot \rangle_{\boldsymbol{B}', \boldsymbol{B}}$ for the duality bracket.

Since we allow Ω to be a multiply-connected domain with holes in $\mathbb{R}^3$, we assume that Ω satisfies the following conditions as in Amrouche and Seloula [1] (cf. Amrouche and Seloula [2], Dautray and Lions [6] and Girault and Raviart [8]). Ω is locally situated on one side of Γ and satisfies the following (O1) and (O2).

(O1) Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_I$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

(O2) There exist J connected open surfaces Σ_j, ($j = 1, \dots, J$), called cuts, contained in Ω such that

(a) each surface Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$,

(b) $\partial\Sigma_j \subset \Gamma$ $(j = 1, \ldots, J)$, where $\partial\Sigma_j$ denotes the boundary of Σ_j, and Σ_j is non-tangential to Γ,

(c) $\overline{\Sigma_j} \cap \overline{\Sigma_k} = \emptyset \, (j \neq k)$,

(d) the open set $\Omega^\circ = \Omega \setminus (\cup_{j=1}^J \Sigma_j)$ is pseudo $C^{1,1}$ simply connected (cf. [1, Definition 1.1]).

The number J is called the first Betti number and I the second Betti number. We say that Ω is simply connected if $J = 0$ and Ω has no holes if $I = 0$. If we define

$$\mathbb{K}_T^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{curl} \boldsymbol{v} = \boldsymbol{0}, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$$

and

$$\mathbb{K}_N^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega); \operatorname{curl} \boldsymbol{v} = \boldsymbol{0}, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\},$$

then it is well known that $\dim \mathbb{K}_T^p(\Omega) = J$ and $\dim \mathbb{K}_N^p(\Omega) = I$. In the latter, we need a basis of $\mathbb{K}_N^p(\Omega)$ defined as follows. Let $q_i^N \in W^{2,p}(\Omega)$ $(1 \leq i \leq I)$ be a unique solution of the problem

$$\begin{cases} -\Delta q_i^N = 0 & \text{in } \Omega, \\ q_i^N \big|_{\Gamma_0} = 0 \text{ and } q_i^N \big|_{\Gamma_k} = \text{const.}, & \text{for } 1 \leq k \leq I, \\ \langle \partial_{\boldsymbol{n}} q_i^N, 1 \rangle_{\Gamma_k} = \delta_{ik} \text{ for } 1 \leq k \leq I \text{ and } \langle \partial_{\boldsymbol{n}} q_i^N, 1 \rangle_{\Gamma_0} = -1, & \end{cases}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_k} = \langle \cdot, \cdot \rangle_{W^{-1/p,p}(\Gamma_k), W^{1-1/p',p'}(\Gamma_k)}$. Then we can see that $\{\nabla q_i^N\}_{i=1}^I$ is a basis of $\mathbb{K}_N^p(\Omega)$ (cf. [1, Corollary 4.2]).

We introduce some spaces of vector functions. Define a space

$$\mathbb{X}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{curl} \boldsymbol{v} \in \boldsymbol{L}^p(\Omega), \operatorname{div} \boldsymbol{v} \in L^p(\Omega)\}$$

with the norm

$$\|\boldsymbol{v}\|_{\mathbb{X}^p(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{curl} \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)}.$$

We note that $\mathbb{X}^p(\Omega)$ is a Banach space with respect to the given norm as above. Moreover, we note that if $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ and $\operatorname{curl} \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$, then the tangent trace $\boldsymbol{n} \times \boldsymbol{v} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$ is well defined and

$$\langle \boldsymbol{n} \times \boldsymbol{v}, \boldsymbol{\varphi} \rangle_\Gamma := \langle \boldsymbol{n} \times \boldsymbol{v}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma), \boldsymbol{W}^{1-1/p',p'}(\Gamma)} = \int_\Omega \operatorname{curl} \boldsymbol{v} \cdot \boldsymbol{\varphi} dx - \int_\Omega \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{\varphi} dx$$

for any $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$, and that if $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ and $\operatorname{div} \boldsymbol{v} \in L^p(\Omega)$, then the normal trace $\boldsymbol{v} \cdot \boldsymbol{n} \in W^{-1/p,p}(\Gamma)$ is well defined and

$$\langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle_\Gamma := \langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle_{W^{-1/p,p}(\Gamma), W^{1-1/p',p'}(\Gamma)} = \int_\Omega (\operatorname{div} \boldsymbol{v}) \varphi dx + \int_\Omega \boldsymbol{v} \cdot \nabla \varphi dx$$

for any $\varphi \in W^{1,p'}(\Omega)$ (cf. [1, p. 45]).

Furthermore, define a closed subspace of $\mathbb{X}^p(\Omega)$ by

$$\mathbb{X}_N^p(\Omega) = \{\boldsymbol{v} \in \mathbb{X}^p(\Omega); \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}.$$

From [1, (1.5)], we see that $\boldsymbol{W}_0^{1,p}(\Omega) \hookrightarrow \mathbb{X}_N^p(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$, where the symbol $\hookrightarrow$ means that the embedding mapping is continuous, and there exists a constant $C > 0$ depending only on p and Ω such that

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{v}\|_{\mathbb{X}^p(\Omega)} \text{ for all } \boldsymbol{v} \in \mathbb{X}_N^p(\Omega).$$

For $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$, $\operatorname{div} \boldsymbol{v} \in W^{-1,p}(\Omega)$ and $\operatorname{curl} \boldsymbol{v} \in \mathbb{X}_N^{p'}(\Omega)' (\subset \boldsymbol{W}^{-1,p}(\Omega))$ are defined by

$$\langle \operatorname{div} \boldsymbol{v}, \varphi \rangle = -\int_\Omega \boldsymbol{v} \cdot \nabla\varphi dx \text{ for all } \varphi \in W_0^{1,p'}(\Omega),$$

and

$$\langle \operatorname{curl} \boldsymbol{v}, \boldsymbol{\varphi} \rangle = \int_\Omega \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{\varphi} dx \text{ for all } \boldsymbol{\varphi} \in \mathbb{X}_N^{p'}(\Omega), \tag{2.1}$$

respectively. In fact, it follows from the Hölder inequality that

$$|\langle \operatorname{div} \boldsymbol{v}, \varphi \rangle| = \left| \int_\Omega \boldsymbol{v} \cdot \nabla\varphi dx \right| \leq \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \|\nabla\varphi\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \|\varphi\|_{W_0^{1,p'}(\Omega)}$$

for all $\varphi \in W_0^{1,p'}(\Omega)$. Thus $\operatorname{div} \boldsymbol{v} \in W^{-1,p}(\Omega)$ is well defined and the inequality

$$\|\operatorname{div} \boldsymbol{v}\|_{W^{-1,p}(\Omega)} \leq \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$$

holds. Similarly, we see that

$$|\langle \operatorname{curl} \boldsymbol{v}, \boldsymbol{\varphi} \rangle| = \left| \int_\Omega \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{\varphi} dx \right| \leq \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \|\operatorname{curl} \boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{\varphi}\|_{\mathbb{X}_N^{p'}(\Omega)}$$

for all $\boldsymbol{\varphi} \in \mathbb{X}_N^{p'}(\Omega)$. Thus $\operatorname{curl} \boldsymbol{v} \in \mathbb{X}_N^{p'}(\Omega)'$ is well defined and the inequality

$$\|\operatorname{curl} \boldsymbol{v}\|_{\mathbb{X}_N^{p'}(\Omega)'} \leq \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$$

holds. Thus the linear operators $\operatorname{div} : \boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \to \operatorname{div} \boldsymbol{v} \in W^{-1,p}(\Omega)$ and $\operatorname{curl} : \boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \to \operatorname{curl} \boldsymbol{v} \in \mathbb{X}_N^{p'}(\Omega)' (\hookrightarrow \boldsymbol{W}^{-1,p}(\Omega))$ are continuous.

Moreover, we define a space

$$\mathbb{V}_N^p(\Omega) = \{\boldsymbol{v} \in \mathbb{X}_N^p(\Omega); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 \text{ for } i = 1, \ldots, I\}.$$

The following inequalities are used frequently (cf. [1]). If we define

$$\mathbb{X}^{1,p}(\Omega) = \{\boldsymbol{v} \in \mathbb{X}^p(\Omega); \boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)\},$$

then we can see that $\mathbb{X}^{1,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and there exists a constant $C > 0$ depending only on p and Ω such that

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\|\boldsymbol{v}\|_{\mathbb{X}^p(\Omega)} + \|\boldsymbol{v} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}). \tag{2.2}$$

Moreover, we can deduce the following (cf. [1, p. 40]): for any $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ, we have

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C\big(\|\operatorname{curl} \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i}|\big). \tag{2.3}$$

Thus we have the following.

Lemma 2.1. $\mathbb{V}^p_N(\Omega)$ *is a reflexive, separable Banach space with the norm*

$$\|\boldsymbol{v}\|_{\mathbb{V}^p_N(\Omega)} := \|\mathrm{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$$

which is equivalent to the $\boldsymbol{W}^{1,p}(\Omega)$*-norm.*

We note that it follows from the Sobolev embedding theorem that there exists a constant $C > 0$ depending only on p and Ω such that for all $\boldsymbol{v} \in \mathbb{V}^p_N(\Omega)$,

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Gamma)} \le C\|\boldsymbol{v}\|_{\mathbb{V}^p_N(\Omega)},$$

where the second term on the left-hand side denotes the norm of the trace of $\boldsymbol{v}$ on Γ. If $6/5 \le p < \infty$, then we have $\mathbb{X}^p_N(\Omega) \subset \boldsymbol{L}^2(\Omega)$ and define a Hilbert space

$$\boldsymbol{L}^2_\sigma(\Omega) = \text{ the closure of } \mathbb{V}^p_N(\Omega) \text{ in } \boldsymbol{L}^2(\Omega).$$

We have $\mathbb{V}^p_N(\Omega) \hookrightarrow \mathbb{X}^p_N(\Omega)$ and so $\mathbb{X}^p_N(\Omega)' \hookrightarrow \mathbb{V}^p_N(\Omega)'$. If $6/5 \le p < \infty$, then we see that $\mathbb{V}^p_N(\Omega)$ is dense in $\boldsymbol{L}^2_\sigma(\Omega)$, and that $\mathbb{V}^p_N(\Omega) \hookrightarrow \boldsymbol{L}^2_\sigma(\Omega) = \boldsymbol{L}^2_\sigma(\Omega)' \hookrightarrow \mathbb{V}^p_N(\Omega)'$ by F. Riesz identification of $\boldsymbol{L}_\sigma(\Omega)$ and $\boldsymbol{L}_\sigma(\Omega)'$.

Let $0 < T < \infty$, and let $\Omega_T = \Omega \times (0, T)$ and $\Gamma_T = \Gamma \times (0, T)$. Assume that $S(x, t, s)$ is a Carathéodoty function in $\Omega_T \times [0, \infty)$ satisfying the following structural conditions.

For a.e. $(x, t) \in \Omega_T$, $S(x, t, s) \in C^2((0, \infty)) \cap C([0, \infty))$ as a function of s, and there exist $1 < p < \infty$ and constants $0 < \lambda \le \Lambda < \infty$ such that for a.e. $(x, t) \in \Omega_T$ and $s > 0$,

$$S(x, t, 0) = 0 \text{ and } \lambda s^{(p-2)/2} \le S_s(x, t, s) \le \Lambda s^{(p-2)/2}. \tag{2.4a}$$

$$\lambda s^{(p-2)/2} \le S_s(x, t, s) + 2sS_{ss}(x, t, s) \le \Lambda s^{(p-2)/2}. \tag{2.4b}$$

$$S_{ss}(x, t, s) < 0 \text{ if } 1 < p < 2 \text{ and } S_{ss}(x, t, s) \ge 0 \text{ if } p \ge 2, \tag{2.4c}$$

where $S_s = \partial S/\partial s, S_{ss} = \partial^2 S/\partial s^2$. We note that (2.4a) implies that

$$\frac{2}{p}\lambda s^{p/2} \le S(x, t, s) \le \frac{2}{p}\Lambda s^{p/2} \text{ for a.e. } (x, t) \in \Omega_T \text{ and } s \in [0, \infty). \tag{2.5}$$

Example 2.2. *If* $S(x, t, s) = \nu(x, t)g(s)s^{p/2}$*, where* ν *is a measurable function in* Ω_T *and satisfies*

$$0 < \nu_* \le \nu(x, t) \le \nu^* < \infty \text{ for a.e. } (x, t) \in \Omega_T \tag{2.6}$$

for some constants ν_* *and* ν^**, and a function* $g(s) \in C^\infty([0, \infty))$*.*

When $g(s) \equiv 1$*, it follows from elementary calculations that* (2.4a)-(2.4c) *hold. This case corresponds to the p-curlcurl operator.*

As an another example, we can take

$$g(s) = \begin{cases} e^{-1/s} + 1 & \text{for } s > 0, \\ 1 & \text{for } s = 0. \end{cases} \tag{2.7}$$

Then we can see that $S(x, t, s) = \nu(x, t)g(s)s^{p/2}$ *satisfies* (2.4a)-(2.4c) *if* $p \ge 2$*, for which we give a proof in Appendix B.*

Lemma 2.3. *If $S(x,t,s)$ satisfies* (2.4a) *and* (2.4b)*, then for a.e.* $(x,t) \in \Omega_T$*,* $J[\boldsymbol{a}] = S(x,t,|\boldsymbol{a}|^2)$ *is strictly convex.*

Proof. First, it follows from (2.4a) and (2.4b) that for a.e. $(x,t) \in \Omega_T$, a function $G(s) = S(x,t,s^2)$ is a strictly monotone increasing and strictly convex function with respect to $s \in [0,\infty)$. Indeed, $G'(s) = 2sS_s(x,t,s^2) \geq 2\lambda s^{p-1} > 0$ for $s > 0$ from (2.4a), and

$$G''(s) = 2(S_s(x,t,s^2) + 2s^2 S_{ss}(x,t,s^2)) > 0$$

for $s > 0$ from (2.4b).

Thus, for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ and $0 \leq \lambda \leq 1$, we have

$$J[\lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b}] \leq S(x,t,(\lambda|\boldsymbol{a}| + (1-\lambda)|\boldsymbol{b}|)^2) \leq \lambda J[\boldsymbol{a}] + (1-\lambda)J[\boldsymbol{b}].$$

Thus J is convex. Moreover, let $\boldsymbol{a} \neq \boldsymbol{b}$ and $0 < \lambda < 1$. When $|\lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b}| < \lambda|\boldsymbol{a}| + (1-\lambda)|\boldsymbol{b}|$, since $S(x,t,s)$ is strictly monotone increasing with respect to s, we have

$$\begin{aligned} J[\lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b}] &= S(x,t,|\lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b}|^2) \\ &< S(x,t,(\lambda|\boldsymbol{a}| + (1-\lambda)|\boldsymbol{b}|)^2) \leq \lambda J[\boldsymbol{a}] + (1-\lambda)J[\boldsymbol{b}]. \end{aligned}$$

When $|\lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b}| = \lambda|\boldsymbol{a}| + (1-\lambda)|\boldsymbol{b}|$, we see that $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent since $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$. We may assume that $\boldsymbol{b} \neq \boldsymbol{0}$, so we can write $\boldsymbol{a} = c\boldsymbol{b}$. Since $c|\boldsymbol{b}|^2 = |\boldsymbol{a}||\boldsymbol{b}|$, we see that $c \geq 0$ and $c \neq 1$. Thus we have $|\boldsymbol{a}| \neq |\boldsymbol{b}|$. Since $S(x,t,s^2)$ is strictly convex with respect to s, we have

$$\begin{aligned} S(x,t,|\lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b}|^2) &= S(x,t,(\lambda|\boldsymbol{a}| + (1-\lambda)|\boldsymbol{b}|)^2) \\ &< \lambda S(x,t,|\boldsymbol{a}|^2) + (1-\lambda)S(x,t,|\boldsymbol{b}|^2). \end{aligned}$$

Therefore, we have $J[\lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b}] < \lambda J[\boldsymbol{a}] + (1-\lambda)J[\boldsymbol{b}]$. □

We give the following lemma with respect to the monotonicity of S_s.

Lemma 2.4. *There exists a constant $c > 0$ such that for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,*

$$\big(S_s(x,t,|\boldsymbol{a}|^2)\boldsymbol{a} - S_s(x,t,|\boldsymbol{b}|^2)\boldsymbol{b}\big) \cdot (\boldsymbol{a} - \boldsymbol{b}) \geq \begin{cases} c|\boldsymbol{a} - \boldsymbol{b}|^p & \text{if } p \geq 2, \\ c(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p-2}|\boldsymbol{a} - \boldsymbol{b}|^2 & \text{if } 1 < p < 2. \end{cases}$$

In particular, S_s is strictly monotone, that is,

$$\big(S_s(x,t,|\boldsymbol{a}|^2)\boldsymbol{a} - S_s(x,t,|\boldsymbol{b}|^2)\boldsymbol{b}\big) \cdot (\boldsymbol{a} - \boldsymbol{b}) > 0 \text{ if } \boldsymbol{a} \neq \boldsymbol{b}.$$

For the proof, see [4, Lemma 3.6].

3 Stationary Maxwell-Stokes type problem

In this section, we consider a stationary Maxwell-Stokes type problem. To do so, let $S(x, s)$ be a Carathéodory function in $\Omega \times [0, \infty)$ satisfying (2.4a)-(2.4c) without t-variable. We consider the following problem: to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p'}(\Omega)$ such that

$$\begin{cases} \operatorname{curl}[S_s(x, |\operatorname{curl} \boldsymbol{u}|^2)\operatorname{curl} \boldsymbol{u}] + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \pi = 0 & \text{on } \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 & i = 1, \dots I. \end{cases} \tag{3.1}$$

We give the notion of weak solutions for (3.1).

Definition 3.1. *We say that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p'}(\Omega)$ is a weak solution of (3.1), if $(\boldsymbol{u}, \pi) \in \mathbb{V}_N^p(\Omega) \times W_0^{1,p'}(\Omega)$ and $(\boldsymbol{u}, \pi)$ satisfies*

$$\int_\Omega S_s(x, |\operatorname{curl} \boldsymbol{u}|^2)\operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} dx + \int_\Omega \nabla\pi \cdot \boldsymbol{v} dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx \tag{3.2}$$

for all $\boldsymbol{v} \in \mathbb{X}_N^p(\Omega)$.

We have the following.

Theorem 3.2. *Assume that $1 < p < \infty$ and $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$. Then the problem (3.1) has a weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p'}(\Omega)$, if and only if*

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{z} dx = 0 \text{ for all } \boldsymbol{z} \in \mathbb{K}_N^p(\Omega). \tag{3.3}$$

In this situation, the weak solution $(\boldsymbol{u}, \pi)$ is unique, and there exists a constant $C > 0$ depending only on p, λ and Ω such that

$$\|\boldsymbol{u}\|_{\mathbb{V}_N^p(\Omega)}^p + \|\pi\|_{W_0^{1,p'}(\Omega)}^{p'} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}^{p'}.$$

The proof consists of the following lemma and proposition. First, we consider the following Dirichlet problem for the Poisson equation.

$$\begin{cases} \Delta\pi = \operatorname{div} \boldsymbol{f} & \text{in } \Omega, \\ \pi = 0 & \text{on } \Gamma. \end{cases} \tag{3.4}$$

Lemma 3.3. *If $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$, then the problem (3.4) has a unique weak solution $\pi \in W_0^{1,p'}(\Omega)$, and there exists a constant $C > 0$ depending only on p' and Ω such that*

$$\|\pi\|_{W_0^{1,p'}(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Proof. The proof of Lemma 3.3 follows from the following proposition (for example, see [1, Theorem 4.2]).

Proposition 3.4. *Let X and M be reflexive Banach spaces with the dual spaces X' and M', respectively, and let $a(v,w)$ be a continuous bilinear form on $X \times M$. Define an operator $A \in \mathcal{L}(X, M')$ by*

$$a(v,w) = \langle Av, w\rangle_{M',M} \text{ for any } v \in X \text{ and } w \in M,$$

and put $V = \mathrm{Ker}A$. Then the following (i) and (ii) are equivalent.

(i) The inf-sup condition holds, that is, there exists $\beta > 0$ such that

$$\inf_{0\neq w\in M} \sup_{0\neq v\in X} \frac{|a(v,w)|}{\|v\|_X \|w\|_M} \geq \beta.$$

(ii) $A : X/V \to M'$ is an isomorphism, and

$$\|A^{-1}f\|_{X/V} \leq \frac{1}{\beta}\|f\|_{M'} \text{ for all } f \in M'.$$

We continue the proof of Lemma 3.3. We apply this Proposition with $X = W_0^{1,p'}(\Omega)$, $M = W_0^{1,p}(\Omega)$ and

$$a(\pi, w) = \int_\Omega \nabla\pi \cdot \nabla w dx \text{ for any } \pi \in X \text{ and } w \in M.$$

The operator $A \in \mathcal{L}(X, M')$ is defined by

$$\langle A\pi, w\rangle_{M',M} = a(\pi, w) = \int_\Omega \nabla\pi \cdot \nabla w dx \text{ for any } \pi \in X \text{ and } w \in M.$$

From Kozono and Yanagisawa [10, (2.18)], there exists a constant $c > 0$ depending only on p and Ω such that

$$\|\nabla\pi\|_{\boldsymbol{L}^{p'}(\Omega)} \leq c \sup_{0\neq w\in W_0^{1,p}(\Omega)} \frac{\left|\int_\Omega \nabla\pi \cdot \nabla w dx\right|}{\|\nabla w\|_{\boldsymbol{L}^{p}(\Omega)}}. \tag{3.5}$$

From the Poincaré inequality, for $f \in W_0^{1,p'}(\Omega)$, $\|f\|_{W_0^{1,p'}(\Omega)}$ and $\|\nabla f\|_{\boldsymbol{L}^{p'}(\Omega)}$ are equivalent. Thus we can easily see that $\mathrm{Ker}A = \{0\}$, and that the inf-sup condition holds. We know that if $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$, then $\mathrm{div}\, \boldsymbol{f} \in W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)'$, and $\|\mathrm{div}\, \boldsymbol{f}\|_{W^{-1,p'}(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}$. Therefore, it follows from Proposition 3.4 (ii) that there exists a unique $\pi \in W_0^{1,p'}(\Omega)$ such that

$$\int_\Omega \nabla\pi \cdot \nabla w dx = -\langle \mathrm{div}\, \boldsymbol{f}, w\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \text{ for all } w \in W_0^{1,p}(\Omega). \tag{3.6}$$

If we take $w \in C_0^\infty(\Omega)$ as a test function, then it follows that $\Delta\pi = \mathrm{div}\, \boldsymbol{f}$ in $W^{-1,p'}(\Omega)$. It follows from (3.5) and (3.6) that there exist positive constants C_1, C_2 and C_3 dependent only on p' and Ω such that

$$\begin{aligned}\|\pi\|_{W_0^{1,p'}(\Omega)} &\leq C_1 \sup_{0\neq w\in W_0^{1,p}(\Omega)} \frac{|\langle \mathrm{div}\, \boldsymbol{f}, w\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}|}{\|\nabla w\|_{\boldsymbol{L}^{p}(\Omega)}} \\ &\leq C_2 \|\mathrm{div}\, \boldsymbol{f}\|_{W^{-1,p'}(\Omega)} \leq C_3 \|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}. \end{aligned} \tag{3.7}$$

This completes the proof of Lemma 3.3. □

Remark 3.5. *In our previous paper Aramaki [3], we derived the existence and regularity of a weak solution to the Poisson equation in* (3.4) *with the Neumann boundary condition.*

Proof of Theorem 3.2

Step 1 (Necessity). When $(\boldsymbol{u},\pi)\in\mathbb{V}^p_N(\Omega)\times W^{1,p'}_0(\Omega)$ satisfies (3.2), for every $\boldsymbol{z}\in\mathbb{K}^p_N(\Omega)$, since $\pi=0$ on Γ and $\operatorname{div}\boldsymbol{z}=0$ in Ω, we have

$$\int_\Omega \nabla\pi\cdot\boldsymbol{z}dx=\int_\Gamma \pi(\boldsymbol{z}\cdot\boldsymbol{n})dS-\int_\Omega \pi\operatorname{div}\boldsymbol{z}dx=0 \tag{3.8}$$

where dS denotes the surface measure, and since $\operatorname{curl}\boldsymbol{z}=\boldsymbol{0}$ in Ω, equality (3.3) easily follows from (3.2) with $\boldsymbol{v}=\boldsymbol{z}\in\mathbb{K}^p_N(\Omega)$.

Step 2 (Sufficiency), For given $\boldsymbol{f}\in\boldsymbol{L}^{p'}(\Omega)$, choose $\pi\in W^{1,p'}_0(\Omega)$ as a weak solution of (3.4), and put $\boldsymbol{F}=\boldsymbol{f}-\nabla\pi\in\boldsymbol{L}^{p'}(\Omega)$. Then $\boldsymbol{F}$ satisfies that $\operatorname{div}\boldsymbol{F}=0$ in $W^{-1,p'}(\Omega)$ and from the hypothesis (3.3) and (3.8),

$$\int_\Omega \boldsymbol{F}\cdot\boldsymbol{z}dx=0 \text{ for all } \boldsymbol{z}\in\mathbb{K}^p_N(\Omega). \tag{3.9}$$

Thus we can reduce problem (3.1) to

$$\begin{cases}\operatorname{curl}[S_s(x,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}]=\boldsymbol{F} & \text{in }\Omega,\\ \operatorname{div}\boldsymbol{u}=0 & \text{in }\Omega,\\ \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0} & \text{on }\Gamma,\\ \langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0 & i=1,\dots,I,\end{cases} \tag{3.10}$$

where $\boldsymbol{F}\in\boldsymbol{L}^{p'}(\Omega)$ satisfies $\operatorname{div}\boldsymbol{F}=0$ in $W^{-1,p'}(\Omega)$ and (3.9).

We consider the following minimization problem: to find $\boldsymbol{u}\in\mathbb{V}^p_N(\Omega)$ such that $\boldsymbol{u}$ is a minimizer of

$$R_*:=\inf_{\boldsymbol{v}\in\mathbb{V}^p_N(\Omega)}R[\boldsymbol{v}] \tag{3.11}$$

where

$$R[\boldsymbol{v}]=\frac{1}{2}\int_\Omega S(x,|\operatorname{curl}\boldsymbol{v}|^2)dx-\int_\Omega \boldsymbol{F}\cdot\boldsymbol{v}dx.$$

Here we say that $\boldsymbol{u}\in\mathbb{V}^p_N(\Omega)$ is a minimizer of (3.11), if $R[\boldsymbol{u}]=R_*$. We claim that (3.11) has a unique minimizer $\boldsymbol{u}\in\mathbb{V}^p_N(\Omega)$, and that there exists a constant $C>0$ dependent only on p,λ and Ω such that

$$\|\boldsymbol{u}\|^p_{\mathbb{V}^p_N(\Omega)}\le C\|\boldsymbol{F}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)}.$$

Indeed, if $\boldsymbol{u},\boldsymbol{v}\in\mathbb{V}^p_N(\Omega)$ and $\boldsymbol{u}\neq\boldsymbol{v}$, then we have $\operatorname{curl}\boldsymbol{u}\neq\operatorname{curl}\boldsymbol{v}$, otherwise, since $\operatorname{curl}(\boldsymbol{u}-\boldsymbol{v})=\boldsymbol{0},\operatorname{div}(\boldsymbol{u}-\boldsymbol{v})=0$ in Ω and $\langle(\boldsymbol{u}-\boldsymbol{v})\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0$ for $i=1,\dots,I$, we have $\boldsymbol{u}=\boldsymbol{v}$ from (2.3). Thus, it follows from Lemma 2.3 that R is a strictly convex and proper functional. From Aramaki [4], we can see that R is lower semi-continuous on $\mathbb{V}^p_N(\Omega)$. By (2.5) and the Hölder and Young inequalities, we have

$$R[\boldsymbol{v}]\ge\frac{\lambda}{p}\|\boldsymbol{v}\|^p_{\mathbb{V}^p_N(\Omega)}-C(\varepsilon)\|\boldsymbol{F}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)}-\varepsilon\|\boldsymbol{v}\|^p_{\mathbb{V}^p_N(\Omega)}$$

for any $\varepsilon > 0$. If we choose $\varepsilon > 0$ so that $\varepsilon = \lambda/2p$, we can see that R is coercive on $\mathbb{V}_N^p(\Omega)$. Hence it follows from Ekeland and Temam [7, Proposition 1.2] that the minimization problem has a unique minimizer $\boldsymbol{u} \in \mathbb{V}_N^p(\Omega)$.

By the Euler-Lagrange equation

$$\frac{d}{d\tau} R(\boldsymbol{u} + \tau \boldsymbol{v}) \bigg|_{\tau=0} = 0 \text{ for any } \boldsymbol{v} \in \mathbb{V}_N^p(\Omega),$$

we can see that $\boldsymbol{u}$ satisfies

$$\int_\Omega S_s(x, |\operatorname{curl} \boldsymbol{u}|^2) \operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} dx = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{v} dx \text{ for all } \boldsymbol{v} \in \mathbb{V}_N^p(\Omega). \tag{3.12}$$

In order to show that $\boldsymbol{u}$ is a weak solution of (3.10), we have to extend the space $\mathbb{V}_N^p(\Omega)$ of test functions to $\mathbb{X}_N^p(\Omega)$ in (3.12). For any $\boldsymbol{w} \in \mathbb{X}_N^p(\Omega) (\subset \boldsymbol{W}^{1,p}(\Omega))$, the Dirichlet problem

$$\begin{cases} \Delta \chi = \operatorname{div} \boldsymbol{w} & \text{in } \Omega, \\ \chi = 0 & \text{on } \Gamma \end{cases}$$

has a unique solution $\chi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Define

$$\boldsymbol{v} = \boldsymbol{w} - \nabla \chi - \sum_{i=1}^{I} \langle (\boldsymbol{w} - \nabla \chi) \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} \nabla q_i^N,$$

where $\{\nabla q_i^N\}_{i=1}^I$ is a basis of $\mathbb{K}_N^p(\Omega)$ defined in the Section 2, then we have $\operatorname{curl} \boldsymbol{v} = \operatorname{curl} \boldsymbol{w}, \operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{w} - \Delta \chi = 0$ in Ω, $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{w} \times \boldsymbol{n} + \boldsymbol{n} \times \nabla \chi = \boldsymbol{0}$ on Γ, because $\boldsymbol{n} \times \nabla$ has only tangential derivatives (cf. Mitreau et al. [13, p. 138]) and $\chi = 0$ on Γ, and

$$\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = \langle (\boldsymbol{w} - \nabla \chi) \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} - \sum_{k=1}^{I} \langle (\boldsymbol{w} - \nabla \chi) \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_k} \delta_{ik} = 0$$

for $i = 1, \dots, I$ by the definition of $\boldsymbol{v}$. Thus $\boldsymbol{v} \in \mathbb{V}_N^p(\Omega)$. From the fact that $\operatorname{div} \boldsymbol{F} = 0$ in Ω and (3.9), we have

$$\int_\Omega \boldsymbol{F} \cdot \boldsymbol{v} dx = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{w} dx - \int_\Omega \boldsymbol{F} \cdot \nabla \chi dx = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{w} dx.$$

Here we used

$$\int_\Omega \boldsymbol{F} \cdot \nabla \chi dx = -\int_\Omega (\operatorname{div} \boldsymbol{F}) \chi dx + \int_\Gamma (\boldsymbol{F} \cdot \boldsymbol{n}) \chi dS = 0.$$

Thus $\boldsymbol{u}$ is a weak solution of (3.9), that is,

$$\int_\Omega S_s(x, |\operatorname{curl} \boldsymbol{u}|^2) \operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{w} dx = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{w} dx \text{ for all } \boldsymbol{w} \in \mathbb{X}_N^p(\Omega). \tag{3.13}$$

If we take $\boldsymbol{w} = \boldsymbol{u}$ as a test function of (3.13), it follows from (2.4a), the Hölder and Young inequalities that for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$\frac{\lambda}{p} \|\operatorname{curl} \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}^p \le C(\varepsilon) \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}^{p'} + \varepsilon \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}^p.$$

If we choose small enough $\varepsilon > 0$, then there exist positive constants C and C_1 depending only on p, λ and Ω such that

$$\|\boldsymbol{u}\|^p_{\mathbb{V}^p_N(\Omega)} \le C\|\boldsymbol{F}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} \le C_1\|\boldsymbol{f}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)}. \tag{3.14}$$

From the above arguments, we can see that $(\boldsymbol{u}, \pi) \in \mathbb{V}^p_N(\Omega) \times W^{1,p'}_0(\Omega)$ is a weak solution of (3.1).

We show the uniqueness of a weak solution for (3.1). Let $(\boldsymbol{u}_1, \pi_1)$ and $(\boldsymbol{u}_2, \pi_2)$ be two weak solutions of (3.1). Taking $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ as a test function of (3.2), we have

$$\int_\Omega S_s(x, |\mathrm{curl}\,\boldsymbol{u}_i|^2)\mathrm{curl}\,\boldsymbol{u}_i \cdot \mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx + \int_\Omega \nabla\pi_i \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)dx = \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)dx \text{ for } i = 1, 2..$$

Therefore, we can see that

$$\int_\Omega \big(S_s(x, |\mathrm{curl}\,\boldsymbol{u}_1|^2)\mathrm{curl}\,\boldsymbol{u}_1 - S_s(x, |\mathrm{curl}\,\boldsymbol{u}_2|^2)\mathrm{curl}\,\boldsymbol{u}_2\big) \cdot \mathrm{curl}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2)dx + \int_\Omega \nabla(\pi_1 - \pi_2) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)dx = 0.$$

Since $\mathrm{div}\,(\boldsymbol{u}_1 - \boldsymbol{u}_2) = 0$ in Ω and $\pi_1 - \pi_2 = 0$ on Γ, if we apply the divergence theorem, then we can see that the last integral on the left-hand side vanishes. Hence, by the strict monotonicity of S_s (Lemma 2.4), we have $\boldsymbol{u}_1 = \boldsymbol{u}_2$. Furthermore, if we choose $\boldsymbol{v} \in \boldsymbol{C}^\infty_0(\Omega)$ as a test function of (3.2), we have

$$\int_\Omega \nabla(\pi_1 - \pi_2) \cdot \boldsymbol{v}dx = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{C}^\infty_0(\Omega).$$

Since $\nabla(\pi_1 - \pi_2) = \boldsymbol{0}$ in $\boldsymbol{L}^{p'}(\Omega)$, we can see that $\pi_1 - \pi_2$ is equal to a constant. Since $\pi_i \in W^{1,p'}_0(\Omega)$, we see that the constant is equal to zero, so we have $\pi_1 = \pi_2$ in Ω.

Finally, we show the estimate. From (3.14) and (3.7), there exists a positive constant C dependent only on p, λ and Ω such that

$$\|\boldsymbol{u}\|^p_{\mathbb{V}^p_N(\Omega)} + \|\pi\|^{p'}_{W^{1,p'}_0(\Omega)} \le C\|\boldsymbol{f}\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)}.$$

Remark 3.6. *In our previous paper [4], we showed that the equation* (3.10) *has a weak solution under the hypothesis* $\mathrm{div}\,\boldsymbol{F} = 0$ *in* Ω *and proved the regularity of the weak solution under the additional hypothesis* $\boldsymbol{F} \in C^\alpha(\overline{\Omega})$ *for some* $0 < \alpha < 1$*. However, the existence theory of a weak solution for* (3.1) *seems to be new.*

4 Evolutionary Maxwell and Maxwell-Stokes type problems

In and after this section, we assume that $6/5 \le p < \infty$. In this case, we note that $\mathbb{V}^p_N(\Omega) \subset \boldsymbol{L}^2(\Omega)$. We consider an evolutionary Maxwell type problem in subsection 4.1 and an evolutionary Maxwell-Stokes type problem in subsection 4.2..

4.1 Existence of a weak solution to an evolutionary Maxwell type problem

Assume that $\boldsymbol{F} \in \boldsymbol{L}^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))$ satisfies $\operatorname{div}\boldsymbol{F} = 0$ in Ω_T and $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\Omega)$. We rewrite the Maxwell type problem (1.6).

$$\begin{cases} \partial_t \boldsymbol{u} + \operatorname{curl}[S_s(x,t,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}] = \boldsymbol{F} & \text{in } \Omega_T, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega_T, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma_T, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & \text{for } i = 1,\dots,I, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \tag{4.1}$$

We give a notion of weak solutions to (4.1).

Definition 4.1. *We say that $\boldsymbol{u} = \boldsymbol{u}(t)$ is a weak solution of (4.1), if $\boldsymbol{u} \in L^p(0,T;\mathbb{V}^p_N(\Omega)) \cap C([0,T);\boldsymbol{L}^2_\sigma(\Omega))$ with $\partial_t\boldsymbol{u} \in L^{p'}(0,T;\mathbb{X}^p_N(\Omega)')$, and $\boldsymbol{u}$ satisfies that for a.e. $t \in (0,T)$,*

$$\langle \partial_t \boldsymbol{u}(t), \boldsymbol{w}\rangle_{\mathbb{X}^p_N(\Omega)',\mathbb{X}^p_N(\Omega)} + \int_\Omega S_s(x,t,|\operatorname{curl}\boldsymbol{u}(t)|^2)\operatorname{curl}\boldsymbol{u}(t)\cdot\operatorname{curl}\boldsymbol{w}dx$$
$$= \int_\Omega \boldsymbol{F}(t)\cdot\boldsymbol{w}dx \text{ for all } \boldsymbol{w}\in\mathbb{X}^p_N(\Omega), \tag{4.2}$$

and $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

To solve (4.1), the next proposition takes an important role. In order to do so, we define a nonlinear operator $A(t) : \mathbb{V}^p_N(\Omega) \to \boldsymbol{W}^{1,p}(\Omega)'(\hookrightarrow \mathbb{X}^p_N(\Omega)')$ by

$$\langle A(t)\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega S_s(x,t,|\operatorname{curl}\boldsymbol{u}|^2)\operatorname{curl}\boldsymbol{u}\cdot\operatorname{curl}\boldsymbol{v}dx \text{ for } \boldsymbol{u}\in\mathbb{V}^p_N(\Omega) \text{ and } \boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$$

and define a functional $\boldsymbol{L}(t) \in \boldsymbol{W}^{1,p}(\Omega)'(\hookrightarrow \mathbb{X}^p_N(\Omega)')$ by

$$\langle \boldsymbol{L}(t), \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{F}(t)\cdot\boldsymbol{v}dx \text{ for all } \boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega).$$

For $\boldsymbol{u} \in \mathbb{V}^p_N(\Omega)$ and $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$, from (2.4a) and the Hölder inequality, we have

$$\begin{aligned} |\langle A(t)\boldsymbol{u},\boldsymbol{v}\rangle| &\le \Lambda\int_\Omega |\operatorname{curl}\boldsymbol{u}|^{p-1}|\operatorname{curl}\boldsymbol{v}|dx \\ &\le \Lambda\|\operatorname{curl}\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}\|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \\ &\le C\Lambda\|\boldsymbol{u}\|^{p-1}_{\mathbb{V}^p_N(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \text{ for all } \boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega). \end{aligned} \tag{4.3}$$

Thus $A(t)\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)' \hookrightarrow \mathbb{X}^p_N(\Omega)'$ is well defined and

$$\|A(t)\boldsymbol{u}\|_{\mathbb{X}^p_N(\Omega)'} \le C_1\|A(t)\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)'} \le C_2\|\boldsymbol{u}\|^{p-1}_{\mathbb{V}^p_N(\Omega)}.$$

In particular, if we take $\boldsymbol{v}\in\mathbb{V}^p_N(\Omega)$ in (4.3), then we can see that

$$\|A(t)\boldsymbol{u}\|_{\mathbb{V}^p_N(\Omega)'} \le C_3\|\boldsymbol{u}\|^{p-1}_{\mathbb{V}^p_N(\Omega)}. \tag{4.4}$$

Here we note that the above constants C_1, C_2 and C_3 are independent of t.

On the other hand, for $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$, since

$$|\langle \boldsymbol{L}(t), \boldsymbol{v}\rangle| \le \|\boldsymbol{F}(t)\|_{\boldsymbol{L}^{p'}(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)},$$

we see that

$$\boldsymbol{L}(t) \in \boldsymbol{W}^{1,p}(\Omega)' \hookrightarrow \mathbb{X}^p_N(\Omega)' \hookrightarrow \mathbb{V}^p_N(\Omega)'.$$

We consider the problem.

$$\begin{cases} \partial_t \boldsymbol{u} + A(t)\boldsymbol{u} = \boldsymbol{L}(t) & \text{in } \Omega_T, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \tag{4.5}$$

Then we have the following.

Proposition 4.2. *Assume that* $\boldsymbol{F} \in L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))$ *and* $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\Omega)$. *Then the problem* (4.5) *has a unique solution* $\boldsymbol{u} \in L^p(0,T;\mathbb{V}^p_N(\Omega)) \cap C([0,T];\boldsymbol{L}^2_\sigma(\Omega))$ *with* $\partial_t\boldsymbol{u} \in L^{p'}(0,T;\mathbb{V}^p_N(\Omega)')$. *The first equation of* (4.5) *holds in the sense of* $L^{p'}(0,T;\mathbb{V}^p_N(\Omega)')$.

Proof. We can see that $A(t)$ satisfies (4.4) and is hemi-continuous on $\mathbb{V}^p_N(\Omega)$, that is, for any $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\phi} \in \mathbb{V}^p_N(\Omega)$ and $\tau \in \mathbb{R}$, $\langle A(t)(\boldsymbol{v}+\tau\boldsymbol{w}), \boldsymbol{\phi}\rangle$ is continuous in τ. Furthermore, from (2.4a), we have

$$\langle A(t)\boldsymbol{u}, \boldsymbol{u}\rangle = \int_\Omega S_t(x,t,|\operatorname{curl}\boldsymbol{u}|^2)|\operatorname{curl}\boldsymbol{u}|^2 dx \ge \lambda\|\operatorname{curl}\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} = \lambda\|\boldsymbol{u}\|^p_{\mathbb{V}^p_N(\Omega)}$$

for $\boldsymbol{u} \in \mathbb{V}^p_N(\Omega)$. We emphasize that the constant λ is independent of t, that is, $A(t)$ is coercive uniformly in t. Therefore we can apply the celebrated result of Lions [11, Chapter 2, Theorem 1.2 and Remark 1.8] (cf. Zheng [17, Theorem 3.2.1]). So, system (4.5) has a unique solution $\boldsymbol{u} \in L^p(0,T;\mathbb{V}^p_N(\Omega)) \cap C([0,T];\boldsymbol{L}^2_\sigma(\Omega))$ with $\partial_t\boldsymbol{u} \in L^{p'}(0,T;\mathbb{V}^p_N(\Omega)')$. □

We obtain the following theorem.

Theorem 4.3. *Assume that* $\boldsymbol{F} \in L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))$ *satisfies that* $\operatorname{div}\boldsymbol{F} = 0$ *in* Ω_T *and that for a.e.* $t \in (0,T)$,

$$\int_\Omega \boldsymbol{F}(t)\cdot\boldsymbol{z}dx = 0 \textit{ for all } \boldsymbol{z} \in \mathbb{K}^p_N(\Omega),$$

and assume that $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\Omega)$. *Then the problem* (4.1) *has a unique weak solution* $\boldsymbol{u} \in L^p(0,T;\mathbb{V}^p_N(\Omega)) \cap C([0,T);\boldsymbol{L}^2_\sigma(\Omega))$ *with* $\partial_t\boldsymbol{u} \in L^{p'}(0,T;\mathbb{X}^p_N(\Omega)')$, *and there exists a constant* $C > 0$ *depending only on* p, λ *and* Ω *such that*

$$\|\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{u}\|^p_{L^p(0,T;\mathbb{V}^p_N(\Omega))} \le C(\|\boldsymbol{F}\|^{p'}_{L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))} + \|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)}). \tag{4.6}$$

Proof. From Proposition 4.2, problem (4.1) has a unique solution $\boldsymbol{u} \in L^p(0,T;\mathbb{V}^p_N(\Omega)) \cap C([0,T);\boldsymbol{L}^2_\sigma(\Omega))$ with $\partial_t\boldsymbol{u} \in L^{p'}(0,T;\mathbb{V}^p_N(\Omega)')$ in the sense that for a.e. $t \in (0,T)$,

$$\langle \partial_t\boldsymbol{u}(t), \boldsymbol{v}\rangle_{\mathbb{V}^p_N(\Omega)',\mathbb{V}^p_N(\Omega)} + \int_\Omega S_s(x,t,|\operatorname{curl}\boldsymbol{u}(t)|^2)\operatorname{curl}\boldsymbol{u}(t)\cdot\operatorname{curl}\boldsymbol{v}dx$$
$$= \int_\Omega \boldsymbol{F}(t)\cdot\boldsymbol{v}dx \text{ for all } \boldsymbol{v} \in \mathbb{V}^p_N(\Omega)$$

and $\boldsymbol{u}(0) = \boldsymbol{u}_0$. However, since we know that $A(t)\boldsymbol{u} \in \mathbb{X}_N^p(\Omega)'$ and $\boldsymbol{L}(t) \in \mathbb{X}_N^p(\Omega)'$, we have $\partial_t \boldsymbol{u}(t) \in \mathbb{X}_N^p(\Omega)'$ for a.e. $t \in (0, T)$, and we can see that the first equation of (4.1) holds in the sense of $L^{p'}(0, T; \mathbb{X}_N^p(\Omega)')$. That is

$$\langle \partial_t \boldsymbol{u}(t), \boldsymbol{v} \rangle_{\mathbb{X}_N^p(\Omega)', \mathbb{X}_N^p(\Omega)} + \int_\Omega S_s(x, t, |\mathrm{curl}\, \boldsymbol{u}(t)|^2) \mathrm{curl}\, \boldsymbol{u}(t) \cdot \mathrm{curl}\, \boldsymbol{v} dx$$
$$= \int_\Omega \boldsymbol{F}(t) \cdot \boldsymbol{v} dx \text{ for all } \boldsymbol{v} \in \mathbb{V}_N^p(\Omega). \quad (4.7)$$

Since $\boldsymbol{C}_0^\infty(\Omega)$ is not contained in $\mathbb{V}_N^p(\Omega)$, we can not show that (4.7) implies (4.2), or (4.1) in the distribution sense. To overcome this, we show that we can replace the space $\mathbb{V}_N^p(\Omega)$ of test functions in (4.7) with $\mathbb{X}_N^p(\Omega)$. For any $\boldsymbol{w} \in \mathbb{X}_N^p(\Omega)$, we consider the following Dirichlet problem for the Poisson equation.

$$\begin{cases} \Delta\theta = \mathrm{div}\, \boldsymbol{w} & \text{in } \Omega, \\ \theta = 0 & \text{on } \Gamma. \end{cases} \quad (4.8)$$

Since $\mathrm{div}\, \boldsymbol{w} \in L^p(\Omega)$, (4.8) has a unique solution $\theta \in W^{2,p}(\Omega)$. Hence, $\nabla\theta \in \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{n} \times \nabla\theta = \boldsymbol{0}$ on Γ, because the operator $\boldsymbol{n} \times \nabla$ contains only tangent derivatives (cf. [13, p. 138]) and $\theta = 0$ on Γ. Thus $\nabla\theta \in \mathbb{X}_N^p(\Omega)$. If we define

$$\boldsymbol{v} = \boldsymbol{w} - \nabla\theta - \sum_{i=1}^I \langle (\boldsymbol{w} - \nabla\theta) \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} \nabla q_i^N,$$

then we see that $\mathrm{div}\, \boldsymbol{v} = \mathrm{div}\, \boldsymbol{w} - \Delta\theta = 0$ in Ω, $\boldsymbol{v} \times \boldsymbol{n} = 0$ on Γ and $\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0$ for $i = 1, \ldots, I$, so $\boldsymbol{v} \in \mathbb{V}_N^p(\Omega)$ and $\mathrm{curl}\, \boldsymbol{v} = \mathrm{curl}\, \boldsymbol{w}$ in Ω.

Here we use the following lemma whose proof is given in the Appendix A.

Lemma 4.4. *Assume that* $\boldsymbol{z} \in \boldsymbol{L}^{p'}(\Omega)$. *Then* $\mathrm{curl}\, \boldsymbol{z} \in \mathbb{X}_N^p(\Omega)'$ *and* $\boldsymbol{n} \times \boldsymbol{z} \in \boldsymbol{W}^{-1/p', p'}(\Gamma)$ *are well defined, and the following Green formula holds.*

$$\langle \boldsymbol{n} \times \boldsymbol{z}, \boldsymbol{\phi} \rangle_{W^{-1/p', p'}(\Gamma), W^{1-1/p, p}(\Gamma)} = \langle \mathrm{curl}\, \boldsymbol{z}, \boldsymbol{\phi} \rangle_{\mathbb{X}_N^p(\Omega)', \mathbb{X}_N^p(\Omega)} - \int_\Omega \boldsymbol{z} \cdot \mathrm{curl}\, \boldsymbol{\phi} dx$$

for all $\boldsymbol{\phi} \in \mathbb{X}_N^p(\Omega)$.

We continue the proof of Theorem 4.3. For a.e. $t \in (0, T)$, we apply this lemma with

$$\boldsymbol{v} = S_s(x, t, |\mathrm{curl}\, \boldsymbol{u}(t)|^2) \mathrm{curl}\, \boldsymbol{u}(t) \in \boldsymbol{L}^{p'}(\Omega).$$

Since $\mathrm{curl}\, \boldsymbol{v} = \boldsymbol{F}(t) - \partial_t \boldsymbol{u}(t) \in \mathbb{X}_N^p(\Omega)'$, it follows from Lemma 4.4 that we have

$$S_s(x, t, |\mathrm{curl}\, \boldsymbol{u}(t)|^2) \mathrm{curl}\, \boldsymbol{u}(t) \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p', p'}(\Gamma),$$

so

$$\boldsymbol{n} \cdot \mathrm{curl}\, [S_s(x, t, |\mathrm{curl}\, \boldsymbol{u}(t)|^2) \mathrm{curl}\, \boldsymbol{u}(t)]$$
$$= \mathrm{div}_\Gamma (S_s(x, t, |\mathrm{curl}\, \boldsymbol{u}(t)|^2) \mathrm{curl}\, \boldsymbol{u}(t) \times \boldsymbol{n}) \in W^{-1-1/p', p'}(\Gamma),$$

where div_Γ denotes the surface divergence (cf. [13]). Thus, since $\theta = 0$ on Γ and $\mathrm{div}\,\boldsymbol{F} = 0$ in Ω, we have

$$\begin{aligned}
&\langle \partial_t \boldsymbol{u}(t), \nabla\theta\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} \\
&= \int_\Omega \boldsymbol{F}(t)\cdot\nabla\theta dx - \langle \mathrm{curl}\,[S_s(x,t,|\mathrm{curl}\,\boldsymbol{u}(t)|^2)\mathrm{curl}\,\boldsymbol{u}(t)], \nabla\theta\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} \\
&= \langle \boldsymbol{F}(t)\cdot\boldsymbol{n}, \theta\rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} \\
&\quad - \langle \mathrm{div}_\Gamma (S_s(x,t,|\mathrm{curl}\,\boldsymbol{u}(t)|^2)\mathrm{curl}\,\boldsymbol{u}(t)\times\boldsymbol{n}), \theta\rangle_{W^{-1-1/p',p'}(\Gamma), W^{2-1/p,p}(\Gamma)} = 0.
\end{aligned}$$

Similarly, from the hypothesis, we have

$$\begin{aligned}
\langle \partial_t \boldsymbol{u}(t), \nabla q_i^N\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} &= -\int_\Omega S_s(x,t,|\mathrm{curl}\,\boldsymbol{u}(t)|^2)\mathrm{curl}\,\boldsymbol{u}(t)\cdot\mathrm{curl}\,\nabla q_i^N dx \\
&\quad + \int_\Omega \boldsymbol{F}(t)\cdot\nabla q_i^N dx \\
&= \int_\Omega \boldsymbol{F}(t)\cdot\nabla q_i^N dx = 0.
\end{aligned}$$

Thus we have

$$\langle \partial_t \boldsymbol{u}(t), \boldsymbol{v}\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} = \langle \partial_t \boldsymbol{u}(t), \boldsymbol{w}\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)}.$$

Also we have

$$\int_\Omega \boldsymbol{F}(t)\cdot\nabla\theta dx = \langle \boldsymbol{F}(t)\cdot\boldsymbol{n}, \theta\rangle_\Gamma = 0$$

and the hypothesis implies

$$\int_\Omega \boldsymbol{F}(t)\cdot\nabla q_j^N dx = 0.$$

Therefore, we can replace $\mathbb{V}^p_N(\Omega)$ of test functions in (4.7) with $\mathbb{X}^p_N(\Omega)$, so we get (4.2).

We show the uniqueness of a weak solution. Let $\boldsymbol{u}_i = \boldsymbol{u}_i(t)$ $(i = 1, 2)$ be two weak solutions of (4.1). Taking $\boldsymbol{u}_1 - \boldsymbol{u}_2$ as a test function of (4.2) and integrating over $(0, t)$, we have

$$\begin{aligned}
\frac{1}{2}\int_\Omega |\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2 dx + \int_0^t \int_\Omega \big(S_s(x,\tau,|\mathrm{curl}\,\boldsymbol{u}_1(\tau)|^2)\mathrm{curl}\,\boldsymbol{u}_1(\tau) \\
- S_s(x,\tau,|\mathrm{curl}\,\boldsymbol{u}_2(\tau)|^2)\mathrm{curl}\,\boldsymbol{u}_2(\tau)\big)\cdot\mathrm{curl}\,(\boldsymbol{u}_1(\tau) - \boldsymbol{u}_2(\tau)) dx d\tau = 0.
\end{aligned}$$

By the strict monotonicity (Lemma 2.3) of S_s, we have $\boldsymbol{u}_1(t) = \boldsymbol{u}_2(t)$ in $\mathbb{V}^p_T(\Omega)$ for a.e. $t \in (0, T)$.

Finally we show the estimate (4.6). Taking $\boldsymbol{u} = \boldsymbol{u}(t)$ as a test function of (4.2), integrating over $(0, t)$, and using (2.4a) and the Hölder and Young inequalities, we have

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} + \lambda \int_0^t \|\mathrm{curl}\,\boldsymbol{u}(\tau)\|^p_{\boldsymbol{L}^p(\Omega)} d\tau \\
&\quad \le \int_0^t \int_\Omega \boldsymbol{F}(\tau)\cdot\boldsymbol{u}(\tau) dx d\tau + \frac{1}{2}\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\quad \le C(\lambda)\int_0^t \|\boldsymbol{F}(\tau)\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} d\tau + \frac{\lambda}{2}\int_0^t \|\mathrm{curl}\,\boldsymbol{u}(\tau)\|^p_{\boldsymbol{L}^p(\Omega)} d\tau + \frac{1}{2}\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

Therefore, there exists a constant $C>0$ depending only on p,λ and Ω such that

$$\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)}+\int_0^t\|\operatorname{curl}\boldsymbol{u}(\tau)\|^p_{\boldsymbol{L}^p(\Omega)}d\tau\le C(\|\boldsymbol{F}\|^{p'}_{\boldsymbol{L}^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))}+\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)}).$$

Taking the sup over $(0,T)$, we get the estimate. This completes the proof of Theorem 4.3. □

Remark 4.5. *In the case where Ω has no holes, the forth conditions of* (4.1) *do not exist. Furthermore, since we showed that we can take $\mathbb{X}^p_N(\Omega)$ as the space of test functions, our weak solution is a solution of* (4.1) *in the distribution sense. Hence we must assume that the compatibility condition* $\operatorname{div}\boldsymbol{F}=0$ *in* Ω_T.

4.2 Existence of a weak solution to an evolutionary Maxwell-Stokes type problem

In this subsection, we consider the Maxwell-Stokes type problem (1.7). We give the notion of a weak solution for the system (1.7).

Definition 4.6. *We say that $(\boldsymbol{u},\pi)=(\boldsymbol{u}(t),\pi(t))$ is a weak solution of* (1.7)*, if*

$$(\boldsymbol{u},\pi)\in\left(L^p(0,T;\mathbb{V}^p_N(\Omega))\cap C([0,T);\boldsymbol{L}^2_\sigma(\Omega))\right)\times L^{p'}(0,T;W^{1,p'}_0(\Omega))$$

with $\partial_t\boldsymbol{u}\in L^{p'}(0,T;\mathbb{X}^p_N(\Omega)')$, and $(\boldsymbol{u},\pi)$ satisfies that, for a.e. $t\in(0,T)$,

$$\langle\partial_t\boldsymbol{u}(t),\boldsymbol{v}\rangle_{\mathbb{X}^p_N(\Omega)',\mathbb{X}^p_N(\Omega)}+\int_\Omega S_s(x,t,|\operatorname{curl}\boldsymbol{u}(t)|^2)\operatorname{curl}\boldsymbol{u}(t)\cdot\operatorname{curl}\boldsymbol{v}dx$$
$$+\int_\Omega\nabla\pi(t)\cdot\boldsymbol{v}dx=\int_\Omega\boldsymbol{f}(t)\cdot\boldsymbol{v}dx\ \text{for all}\ \boldsymbol{v}\in\mathbb{X}^p_N(\Omega)\quad(4.9)$$

and $\boldsymbol{u}(0)=\boldsymbol{u}_0$.

We have the following theorem.

Theorem 4.7. *Assume that $S(x,t,s)$ satisfies the structural conditions* (2.4a)-(2.4c)*, and that $\boldsymbol{f}\in L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))$ and $\boldsymbol{u}_0\in\boldsymbol{L}^2_\sigma(\Omega)$. Furthermore, we assume that for a.e. $t\in(0,T)$,*

$$\int_\Omega\boldsymbol{f}(t)\cdot\boldsymbol{z}dx=0\ \text{for all}\ \boldsymbol{z}\in\mathbb{K}^p_N(\Omega).\quad(4.10)$$

Then (1.7) *has a unique weak solution $(\boldsymbol{u},\pi)$, and there exists a constant $C>0$ depending only on p,λ,Λ and Ω such that*

$$\|\boldsymbol{u}\|^2_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\|\operatorname{curl}\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega_T)}+\|\pi\|^{p'}_{L^{p'}(0,T;W^{1,p'}_0(\Omega))}$$
$$\le C(\|\boldsymbol{f}\|^{p'}_{\boldsymbol{L}^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))}+\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)}).\quad(4.11)$$

Proof. For a.e. $t \in (0,T)$, we first consider the following Dirichlet problem for the Poisson equation.

$$\begin{cases} \Delta\pi = \operatorname{div} \boldsymbol{f}(t) & \text{in } \Omega, \\ \pi = 0 & \text{on } \Gamma. \end{cases} \tag{4.12}$$

By Lemma 3.3, (4.12) has a unique weak solution $\pi = \pi(t) \in W_0^{1,p'}(\Omega)$ which is measurable with respect to t, and there exists a constant $C > 0$ depending only on p' and Ω such that

$$\|\pi(t)\|_{W_0^{1,p'}(\Omega)} \leq C(\|\boldsymbol{f}(t)\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Hence $\pi \in L^{p'}(0,T; W_0^{1,p'}(\Omega))$, and

$$\|\pi\|_{L^{p'}(0,T;W_0^{1,p'}(\Omega))} \leq C\|\boldsymbol{f}\|_{L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))}. \tag{4.13}$$

If we define $\boldsymbol{F} = \boldsymbol{f} - \nabla\pi \in L^{p'}(0,T; \boldsymbol{L}^{p'}(\Omega))$, then we have $\operatorname{div} \boldsymbol{F} = \operatorname{div} \boldsymbol{f} - \Delta\pi = 0$ in Ω_T. Using the hypothesis (4.10) and the fact that $\pi = 0$ on Γ, for a.e. $t \in (0,T)$,

$$\int_\Omega \boldsymbol{F}(t) \cdot \boldsymbol{z} dx = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{z} dx - \int_\Omega \nabla\pi \cdot \boldsymbol{z} dx = \int_\Gamma \pi(\boldsymbol{z} \cdot \boldsymbol{n}) dS = 0$$

for all $\boldsymbol{z} \in \mathbb{K}_N^p(\Omega)$. Therefore, it follows from Theorem 4.3 that system (1.7) has a unique weak solution $(\boldsymbol{u}, \pi) \in L^p(0,T; \mathbb{V}_N^p(\Omega)) \cap C([0,T); \boldsymbol{L}_\sigma^2(\Omega))$ with $\partial_t \boldsymbol{u} \in \boldsymbol{L}^{p'}(0,T; \mathbb{X}_N^p(\Omega)')$, and there exists a constant $C > 0$ depending only on p, λ and Ω such that (4.6) holds. Since π satisfies (4.13), we have

$$\|\boldsymbol{F}\|_{L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))} \leq \|\boldsymbol{f}\|_{L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))} + \|\nabla\pi\|_{L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))} \leq C(\|\boldsymbol{f}\|_{L^{p'}(0,T;\boldsymbol{L}^{p'}(\Omega))}.$$

Hence $(\boldsymbol{u}, \pi)$ is a weak solution of (1.7) and satisfies (4.11).

Finally we show the uniqueness of the weak solution. Let $(\boldsymbol{u}_1, \pi_1) = (\boldsymbol{u}_1(t), \pi_1(t))$ and $(\boldsymbol{u}_2, \pi_2) = (\boldsymbol{u}_2(t), \pi_2(t))$ be two weak solutions of (1.7). For a.e. $t \in (0,T)$, taking $\boldsymbol{v} = \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)$ as a test function of (4.9), we have

$$\begin{aligned} &\langle \partial_t(\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)), \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) \rangle_{\mathbb{X}_N^p(\Omega)', \mathbb{X}_N^p(\Omega)} \\ &+ \int_\Omega \big(S_s(x,t,|\operatorname{curl} \boldsymbol{u}_1(t)|^2) \operatorname{curl} \boldsymbol{u}_1(t) - S_s(x,t,|\operatorname{curl} \boldsymbol{u}_2(t)|^2) \operatorname{curl} \boldsymbol{u}_2(t) \big) \cdot \operatorname{curl} (\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)) dx \\ &\qquad + \int_\Omega \nabla(\pi_1(t) - \pi_2(t)) \cdot (\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)) dx = 0. \end{aligned}$$

Since $\pi_i(t) = 0$ on Γ and $\operatorname{div} \boldsymbol{u}_i(t) = 0$ in Ω, the last integral on the left-hand side vanishes. Integrating this equality over $(0,t)$ and using $\boldsymbol{u}_1(0) - \boldsymbol{u}_2(0) = \boldsymbol{0}$, we have

$$\begin{aligned} \frac{1}{2} \int_\Omega |\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2 dx + \int_0^t \int_\Omega \big(& S_s(x,\tau,|\operatorname{curl} \boldsymbol{u}_1(\tau)|^2) \operatorname{curl} \boldsymbol{u}_1(\tau) \\ & - S_s(x,\tau,|\operatorname{curl} \boldsymbol{u}_2(\tau)|^2) \operatorname{curl} \boldsymbol{u}_2(\tau) \big) \cdot \operatorname{curl} (\boldsymbol{u}_1(\tau) - \boldsymbol{u}_2(\tau)) dx d\tau = 0. \end{aligned}$$

By the strict monotonicity of S_t (Lemma 2.4), we have $\boldsymbol{u}_1 = \boldsymbol{u}_2$. Moreover, taking $\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega)$ as a test function of (4.9), we have

$$\int_\Omega \nabla(\pi_1 - \pi_2) \cdot \boldsymbol{v} dx = 0,$$

so $\nabla(\pi_1 - \pi_2) = 0$ in the distribution sense. Therefore, we have $\pi_1 - \pi_2$ is equal to a constant with respect to x. Since $\pi_1(t) = \pi_2(t) = 0$ on Γ_T, we have $\pi_1 = \pi_2$. $\square$

When $S(x,t,s) = S(x,s)$ is independent of t, we can improve the previous Theorem 4.7, provided that the given data $\boldsymbol{f}$ and $\boldsymbol{u}_0$ are more regular.

Proposition 4.8. *Let $S(x,s)$ satisfy the structural conditions* (2.4a)-(2.4c) *with the same constants λ and Λ. Assume that $\boldsymbol{f} \in L^\infty(0,T;\boldsymbol{L}^{p'}(\Omega)) \cap \boldsymbol{L}^2(\Omega_T)$ satisfies that for a.e. $t \in (0,T)$,*

$$\int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{z} dx = 0 \text{ for all } \boldsymbol{z} \in \mathbb{K}_N^p(\Omega),$$

and assume that $\boldsymbol{u}_0 \in \mathbb{V}_N^p(\Omega)$. Then the weak solution $(\boldsymbol{u},\pi)$ of (1.7) *satisfies that $\partial_t \boldsymbol{u} \in \boldsymbol{L}^2(\Omega_T)$, $\operatorname{curl}\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{L}^p(\Omega))$ and*

$$\pi \in L^{p'}(0,T;W_0^{1,p'}(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),$$

and that there exists a constant $C > 0$ depending only on p, λ, Λ and Ω such that

$$\|\partial_t \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega_T)} + \|\boldsymbol{u}\|^p_{L^\infty(0,T;\mathbb{V}_N^p(\Omega))} + \|\pi\|_{L^2(0,T;H_o^1(\Omega)} \leq C(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega_T)} + \|\boldsymbol{u}_0\|^p_{\mathbb{V}_N^p(\Omega)}), \tag{4.14}$$

in addition to (4.11).

Proof. Using the Galerkin approximation (cf. [17, Chapter 3]), we may choose formally $\partial_t \boldsymbol{u}(t)$ as a test function in (4.9). Integrating (4.9) with $\boldsymbol{v} = \partial_t \boldsymbol{u}(t)$ over $(0,t)$ leads to

$$\int_0^t \int_\Omega |\partial_\tau \boldsymbol{u}(\tau)|^2 dx d\tau + \int_0^t \int_\Omega S_s(x, |\operatorname{curl}\boldsymbol{u}(\tau)|^2) \operatorname{curl}\boldsymbol{u}(\tau) \cdot \operatorname{curl}\partial_\tau \boldsymbol{u}(\tau) dx d\tau$$
$$+ \int_0^t \int_\Omega \nabla\pi(\tau) \cdot \partial_\tau \boldsymbol{u}(\tau) dx d\tau = \int_0^t \int_\Omega \boldsymbol{f}(\tau) \cdot \partial_\tau \boldsymbol{u}(\tau) dx d\tau.$$

For any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that we have

$$\left| \int_0^t \int_\Omega \boldsymbol{f}(\tau) \cdot \partial_\tau \boldsymbol{u}(\tau) dx d\tau \right| \leq C(\varepsilon) \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega_T)} + \varepsilon \int_0^t \int_\Omega |\partial_\tau \boldsymbol{u}(\tau)|^2 dx d\tau.$$

Since $\boldsymbol{f}(t) \in \boldsymbol{L}^2(\Omega)$ for a.e. $t \in (0,T)$, the solution π of (4.12) satisfies

$$\|\pi\|_{L^2(0,T;H_0^1(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega_T)}. \tag{4.15}$$

Hence, we have

$$\left| \int_0^t \int_\Omega \nabla\pi(\tau) \cdot \partial_\tau \boldsymbol{u}(\tau) dx d\tau \right| \leq C(\varepsilon) \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega_T)} + \varepsilon \int_0^t \int_\Omega |\partial_\tau \boldsymbol{u}(\tau)|^2 dx d\tau.$$

On the other hand, it follows from the structural condition (2.4a) that we have

$$\begin{aligned}
&\int_0^t \int_\Omega S_s(x, |\operatorname{curl}\boldsymbol{u}(\tau)|^2) \operatorname{curl}\boldsymbol{u}(\tau) \cdot \partial_\tau \operatorname{curl}\boldsymbol{u}(\tau) dx d\tau \\
&= \frac{1}{2} \int_0^t \int_\Omega \frac{d}{d\tau} S(x, |\operatorname{curl}\boldsymbol{u}(\tau)|^2) dx d\tau \\
&= \frac{1}{2} \int_\Omega S(x, |\operatorname{curl}\boldsymbol{u}(t)|^2) dx - \frac{1}{2} \int_\Omega S(x, |\operatorname{curl}\boldsymbol{u}_0|^2) dx \\
&\geq \frac{\lambda}{2} \|\boldsymbol{u}(t)\|^p_{\mathbb{V}_N^p(\Omega)} - \frac{\Lambda}{2} \|\boldsymbol{u}_0\|^p_{\mathbb{V}_N^p(\Omega)}.
\end{aligned}$$

Here we used the fact that $S(x,s)$ is independent of t. Therefore, if we choose small enough $\varepsilon > 0$, then there exists a constant $C > 0$ depending only on λ, Λ, p and Ω such that

$$\int_0^t \int_\Omega |\partial_\tau \boldsymbol{u}(\tau)|^2 dx d\tau + \|\boldsymbol{u}(t)\|^p_{\mathbb{V}^p_N(\Omega)} \le C(\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega_T)} + \|\boldsymbol{u}_0\|^p_{\mathbb{V}^p_N(\Omega)}).$$

This and (4.15) imply (4.14). □

5 Asymptotic behavior of the weak solution as time tends to infinity

In this section, we show that the weak solution $(\boldsymbol{u}, \pi) = (\boldsymbol{u}(t), \pi(t))$ of the evolutionary problem (1.7) converges to a weak solution of a stationary problem as $t \to \infty$ in $\boldsymbol{L}^2(\Omega) \times W^{1,p'}_0(\Omega)$ under some conditions. In order to proceed, let a Carathéodory function $S^{(\infty)}(x,s)$ satisfy the same structural conditions (2.4a)-(2.4c) with the same constants. Assume that $\boldsymbol{f}_\infty \in \boldsymbol{L}^{p'}(\Omega)$ satisfies

$$\int_\Omega \boldsymbol{f}_\infty \cdot \boldsymbol{z} dx = 0 \text{ for all } \boldsymbol{z} \in \mathbb{K}^p_N(\Omega).$$

We write the unique weak solution of the following stationary system by $(\boldsymbol{u}_\infty, \pi_\infty) \in \mathbb{V}^p_N(\Omega) \times W^{1,p'}_0(\Omega)$ whose existence is guaranteed in Theorem 3.2, that is,

$$\begin{cases} \operatorname{curl}[S^{(\infty)}_s(x, |\operatorname{curl}\boldsymbol{u}_\infty|^2)\operatorname{curl}\boldsymbol{u}_\infty] + \nabla\pi_\infty = \boldsymbol{f}_\infty & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u}_\infty = 0 & \text{in } \Omega, \\ \boldsymbol{u}_\infty \times \boldsymbol{n} = 0 & \text{on } \Gamma, \\ \pi = 0 & \text{on } \Gamma, \\ \langle \boldsymbol{u}_\infty \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 & i = 1, \dots, I. \end{cases} \tag{5.1}$$

Furthermore, we assume that $\boldsymbol{f} = \boldsymbol{f}(t) \in L^\infty(0,\infty; \boldsymbol{L}^{p'}(\Omega))$ satisfies that for a.e. $t \in (0,\infty)$,

$$\int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{z} dx = 0 \text{ for all } \boldsymbol{z} \in \mathbb{K}^p_N(\Omega),$$

and assume that $(\boldsymbol{u}, \pi) = (\boldsymbol{u}(t), \pi(t)) \in L^p(0,\infty; \mathbb{V}^p_N(\Omega)) \cap C([0,\infty); \boldsymbol{L}^2_\sigma(\Omega))$ with $\partial_t \boldsymbol{u} \in L^{p'}(0,\infty; \mathbb{X}^p_N(\Omega)')$ is the unique weak solution of (1.7) with $T = \infty$. Then for a.e. $t \in (0,\infty)$, $\pi(t) - \pi_\infty$ is a weak solution of

$$\begin{cases} \Delta(\pi(t) - \pi_\infty) = \operatorname{div}(\boldsymbol{f}(t) - \boldsymbol{f}_\infty) & \text{in } \Omega, \\ \pi(t) - \pi_\infty = 0 & \text{on } \Gamma. \end{cases}$$

Therefore, there exists a constant $C > 0$ depending only on p' and Ω such that

$$\|\pi(t) - \pi_\infty\|_{W^{1,p'}_0(\Omega)} \le C\|\boldsymbol{f}(t) - \boldsymbol{f}_\infty\|_{\boldsymbol{L}^{p'}(\Omega)}. \tag{5.2}$$

For a.e. $t \in (0,\infty)$, taking $\boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{u}_\infty$ as a test function of (3.2) and (4.9), we have

$$\int_\Omega S^{(\infty)}_s(x, |\operatorname{curl}\boldsymbol{u}_\infty|^2)\operatorname{curl}\boldsymbol{u}_\infty \cdot \operatorname{curl}\boldsymbol{w}(t) dx + \int_\Omega \nabla\pi_\infty \cdot \boldsymbol{w}(t) dx = \int_\Omega \boldsymbol{f}_\infty \cdot \boldsymbol{w}(t) dx$$

and

$$\langle \partial_t \boldsymbol{u}(t), \boldsymbol{w}(t)\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} + \int_\Omega S_s(x,t,|\mathrm{curl}\, \boldsymbol{u}(t)|^2)\mathrm{curl}\, \boldsymbol{u}(t)\cdot \mathrm{curl}\, \boldsymbol{w}(t)dx \\ + \int_\Omega \nabla \pi(t)\cdot \boldsymbol{w}(t)dx = \int_\Omega \boldsymbol{f}(t)\cdot \boldsymbol{w}(t)dx.$$

Hence, we can write

$$\langle \partial_t \boldsymbol{w}(t), \boldsymbol{w}(t)\rangle_{\mathbb{X}^p_N(\Omega)', \mathbb{X}^p_N(\Omega)} \\ + \int_\Omega \big(S_s(x,t,|\mathrm{curl}\, \boldsymbol{u}(t)|^2)\mathrm{curl}\, \boldsymbol{u}(t) - S_s(x,t,|\mathrm{curl}\, \boldsymbol{u}_\infty|^2)\mathrm{curl}\, \boldsymbol{u}_\infty\big)\cdot \mathrm{curl}\, \boldsymbol{w}(t)dx = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= -\int_\Omega \nabla(\pi(t)-\pi_\infty)\cdot \boldsymbol{w}(t)dx, \\ I_2 &= \int_\Omega (\boldsymbol{f}(t)-\boldsymbol{f}_\infty)\cdot \boldsymbol{w}(t)dx, \\ I_3 &= \int_\Omega \big(S_s^{(\infty)}(x,|\mathrm{curl}\, \boldsymbol{u}_\infty|^2)\mathrm{curl}\, \boldsymbol{u}_\infty - S_s(x,t,|\mathrm{curl}\, \boldsymbol{u}_\infty|^2)\mathrm{curl}\, \boldsymbol{u}_\infty\big)\cdot \mathrm{curl}\, \boldsymbol{w}(t)dx. \end{aligned}$$

We estimate I_1, I_2 and I_3. For a.e $t \in (0,\infty)$, if we use the Hölder and Young inequalities and (5.2), then we can see that for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$\begin{aligned} |I_1| &\le C(\varepsilon)\|\nabla(\pi(t)-\pi_\infty)\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} + \varepsilon \|\boldsymbol{w}(t)\|^p_{\boldsymbol{L}^p(\Omega)} \\ &\le C(\varepsilon, p', \Omega)(\|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\|^{p'}_{L^{p'}(\Omega)} + \varepsilon \|\boldsymbol{w}(t)\|^p_{\boldsymbol{L}^p(\Omega)}, \\ |I_2| &\le C(\varepsilon)(\|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} + \varepsilon \|\boldsymbol{w}(t)\|^p_{\boldsymbol{L}^p(\Omega)}, \end{aligned}$$

and

$$\begin{aligned} |I_3| \le\ & C(\varepsilon)\|S_s^{(\infty)}(x,|\mathrm{curl}\, \boldsymbol{u}_\infty|^2)\mathrm{curl}\, \boldsymbol{u}_\infty - S_s(x,t,|\mathrm{curl}\, \boldsymbol{u}_\infty|^2)\mathrm{curl}\, \boldsymbol{u}_\infty\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} \\ & + \varepsilon \|\mathrm{curl}\, \boldsymbol{w}(t)\|^p_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

Put

$$\xi(t) = \|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\|^{p'\wedge 2}_{L^{p'}(\Omega)},$$

where $p' \wedge 2 = \min\{p', 2\}$, and

$$\zeta(t) = \|S_s^{(\infty)}(x,|\mathrm{curl}\, \boldsymbol{u}_\infty|^2)\mathrm{curl}\, \boldsymbol{u}_\infty - S_s(x,t,|\mathrm{curl}\, \boldsymbol{u}_\infty|^2)\mathrm{curl}\, \boldsymbol{u}_\infty\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)}.$$

When $p \ge 2$, from Lemma 2.4, we have

$$\int_\Omega \big(S_s(x,t,|\mathrm{curl}\, \boldsymbol{u}(t)|^2)\mathrm{curl}\, \boldsymbol{u}(t) - S_s(x,t,|\mathrm{curl}\, \boldsymbol{u}_\infty|^2)\mathrm{curl}\, \boldsymbol{u}_\infty\big)\cdot \mathrm{curl}\, \boldsymbol{w}(t)dx \\ \ge \lambda \int_\Omega |\mathrm{curl}\, \boldsymbol{w}(t)|^p dx = \lambda \|\boldsymbol{w}(t)\|^p_{\mathbb{V}^p_T(\Omega)}.$$

It follows from the Hölder inequality that

$$\int_\Omega |\boldsymbol{w}(t)|^2 dx \le C(p,\Omega)\left(\int_\Omega |\boldsymbol{w}(t)|^p dx\right)^{2/p},$$

so we have

$$\left(\int_\Omega |\boldsymbol{w}(t)|^2 dx\right)^{p/2} \le C\int_\Omega |\boldsymbol{w}(t)|^p dx \le C_2\|\boldsymbol{w}(t)\|^p_{\mathbb{V}^p_T(\Omega)}.$$

Hence, if we choose small enough $\varepsilon > 0$, then there exists positive constants c, C and D such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{w}(t)|^2 dx + c\left(\int_\Omega |\boldsymbol{w}(t)|^2 dx\right)^{p/2} \le C\xi(t) + D\zeta(t). \tag{5.3}$$

If we put

$$\phi(t) = \int_\Omega |\boldsymbol{w}(t)|^2 dx \quad \text{and} \quad l(t) = 2C\xi(t) + 2D\zeta(t),$$

then inequality (5.3) means that

$$\phi'(t) + 2c\phi(t)^{p/2} \le l(t). \tag{5.4}$$

5.1 The degenerate case $p > 2$

When $p > 2$, we use the following lemma from Simon [14, p. 600].

Lemma 5.1. *Assume that $\phi(t)$ is a continuous positive function in an interval $I \subset \mathbb{R}$, and differentiable for a.e. $t \in I$, and satisfies that*

$$\phi'(t) + c(t)\phi(t)^{p/2} \le l(t) \ \text{a.e.}\ t \in I,$$

where $p > 2$, $c(t) \ge 0$ and $l \in L^1_{\mathrm{loc}}(I)$. Then for any $t_0, t \in I$ with $t_0 \le t$,

$$\phi(t) \le \left(\frac{p-2}{2}\int_{t_0}^t c(\sigma)d\sigma\right)^{-2/(p-2)} + \int_{t_0}^t l(\sigma)d\sigma.$$

Applying this lemma with $I = (0,\infty), t_0 = t/2$, $c(t) = 2c$ and

$$\phi(t) = \int_\Omega |\boldsymbol{w}(t)|^2 dx = \int_\Omega |\boldsymbol{u}(t) - \boldsymbol{u}_\infty|^2 dx,$$

we have the following.

Theorem 5.2. *When $p > 2$, if we assume that*

$$\int_{t/2}^t (\xi(\tau) + \zeta(\tau))d\tau \to 0 \ \text{as}\ t \to \infty,$$

then we have

$$\|\boldsymbol{u}(t) - \boldsymbol{u}_\infty\|_{\boldsymbol{L}^2(\Omega)} \to 0 \ \text{as}\ t \to \infty.$$

Furthermore, if $\xi(t) \to 0$ as $t \to \infty$, then we have

$$\|\pi(t) - \pi_\infty\|_{W_0^{1,p'}(\Omega)} \to 0 \ \text{as}\ t \to \infty.$$

5.2 The case $p = 2$

From (5.3), we have

$$\frac{d}{dt}\int_\Omega |\boldsymbol{w}(t)|^2 dx + 2c\int_\Omega |\boldsymbol{w}(t)|^2 dx \le l(t). \tag{5.5}$$

Since $\boldsymbol{f} \in L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$ and

$$|S_s^\infty(x, |\mathrm{curl}\,\boldsymbol{u}_\infty|^2)\mathrm{curl}\,\boldsymbol{u}_\infty - S_s(x,t,|\mathrm{curl}\,\boldsymbol{u}_\infty|^2)\mathrm{curl}\,\boldsymbol{u}_\infty| \le 2\Lambda|\mathrm{curl}\,\boldsymbol{u}_\infty|,$$

there exists a constant l_0 independent of t such that $l(t) \le l_0$. We show that $\boldsymbol{w} \in L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$. Indeed, multiplying e^{2ct} to the above inequality (5.5), and then integrating over (σ,τ), we have

$$\int_\sigma^\tau\int_\Omega e^{2ct}\partial_t|\boldsymbol{w}(t)|^2 dxdt + 2c\int_\sigma^\tau\int_\Omega e^{2ct}|\boldsymbol{w}(t)|^2 dxdt \le l_0\int_\sigma^\tau e^{2ct}dt = \frac{l_0}{2c}(e^{2c\tau} - e^{2c\sigma}).$$

Since

$$\int_\sigma^\tau\int_\Omega e^{2ct}\partial_t|\boldsymbol{w}(t)|^2 dxdt = e^{2c\tau}\int_\Omega |\boldsymbol{w}(\tau)|^2 dx - e^{2c\sigma}\int_\Omega |\boldsymbol{w}(\sigma)|^2 dx - 2c\int_\sigma^\tau\int_\Omega e^{2ct}|\boldsymbol{w}(t)|^2 dxdt,$$

we have

$$e^{2c\tau}\int_\Omega |\boldsymbol{w}(\tau)|^2 dx \le \frac{l_0}{2c}(e^{2c\tau} - e^{2c\sigma}) + e^{2c\sigma}\int_\Omega |\boldsymbol{w}(\sigma)|^2 dx.$$

If we put $\tau = t$ and $\sigma = 0$, we have

$$\int_\Omega |\boldsymbol{w}(t)|^2 dx \le \frac{l_0}{2c} + \int_\Omega |\boldsymbol{u}_0 - \boldsymbol{u}_\infty|^2 dx =: l_0.$$

Here we use the following lemma (cf. Heraux [9, p. 286]).

Lemma 5.3. *Let $\phi(t)$ be a non-negative function, and absolutely continuous in any compact interval of $(0,\infty)$ and let $c > 0$ be a constant, and assume that $l(t)$ is a non-negative function that belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^+)$. If*

$$\phi'(t) + c\phi(t) \le l(t) \text{ for all } t \ge 0,$$

then for any $t_0, t \in \mathbb{R}^+$ with $t_0 \le t$,

$$\phi(t) \le e^{c(t_0-t)}\phi(t_0) + \frac{1}{1-e^{-c}}\sup_{\tau\ge t_0}\int_\tau^{\tau+1} l(\sigma)d\sigma.$$

Applying this lemma to (5.5) with

$$\phi(t) = \int_\Omega |\boldsymbol{w}(t)|^2 dx,$$

and setting t_0 as fixed, then for any $t > t_0$, we have

$$\phi(t) \le e^{2c(t_0-t)}\phi(t_0) + \frac{1}{1-e^{-2c}}\sup_{\tau\ge t_0}\int_\tau^{\tau+1} l(\sigma)d\sigma.$$

Thus we have the following.

Theorem 5.4. *When $p = 2$, if*

$$\int_t^{t+1} (\xi(\tau) + \zeta(\tau))d\tau \to 0 \text{ as } t \to \infty,$$

then we have

$$\|\boldsymbol{u}(t) - \boldsymbol{u}_\infty\|_{\boldsymbol{L}^2(\Omega)} \to 0 \text{ as } t \to \infty.$$

Furthermore, if $\xi(t) \to 0$ as $t \to \infty$,

$$\|\pi(t) - \pi_\infty\|_{H_0^1(\Omega)} \to 0 \text{ as } t \to \infty.$$

5.3 The singular case $6/5 \le p < 2$

In this case, since we use Proposition 4.8, we assume that $S(x,t,s) = S^{(\infty)}(x,s)$ is independent of t. Moreover, we assume that $\boldsymbol{f} \in L^\infty(0,\infty;\boldsymbol{L}^{p'}(\Omega)) \cap \boldsymbol{L}^2(\Omega_\infty)$ satisfies that for a.e $t \in (0,T)$,

$$\int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{z} dx = 0 \text{ for all } \boldsymbol{z} \in \mathbb{K}_N^p(\Omega)$$

and $\boldsymbol{u}_0 \in \mathbb{V}_N^p(\Omega)$. From Proposition 4.8 and Lemma 2.4, we have

$$\frac{d}{dt}\int_\Omega |\boldsymbol{w}(t)|^2 dx + \lambda \int_\Omega (|\mathrm{curl}\,\boldsymbol{u}(t)| + |\mathrm{curl}\,\boldsymbol{u}_\infty|)^{p-2} |\mathrm{curl}\,\boldsymbol{w}(t)|^2 dx \le I_1 + I_2.$$

We use the reverse Hölder inequality (cf. Sobolev [15, p. 8]) with $0 < s = p/2 < 1$ and $s' = p/(p-2) < 0$. We have

$$\int_{\widehat{\Omega}} (|\mathrm{curl}\,\boldsymbol{u}(t)| + |\mathrm{curl}\,\boldsymbol{u}_\infty|)^{p-2} |\mathrm{curl}\,\boldsymbol{w}(t)|^2 dx$$
$$\ge \left(\int_{\widehat{\Omega}} |\mathrm{curl}\,\boldsymbol{w}(t)|^p dx\right)^{2/p} \left(\int_{\widehat{\Omega}} (|\mathrm{curl}\,\boldsymbol{u}(t)| + |\mathrm{curl}\,\boldsymbol{u}_\infty|)^p dx\right)^{(p-2)/2},$$

where

$$\widehat{\Omega} = \{x \in \Omega; |\mathrm{curl}\,\boldsymbol{u}(x,t)| + |\mathrm{curl}\,\boldsymbol{u}_\infty(x)| \ne 0\}.$$

From Theorem 4.7, there exists a constant $C > 0$ independent of t such that

$$\int_\Omega (|\mathrm{curl}\,\boldsymbol{u}(t)| + |\mathrm{curl}\,\boldsymbol{u}_\infty|)^p dx \le C.$$

Therefore there exists a constant $c > 0$ independent of t such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{w}(t)|^2 dx + c\left(\int_\Omega |\mathrm{curl}\,\boldsymbol{w}(t)|^p dx\right)^{2/p} \le I_1 + I_2.$$

Since $\mathbb{V}_N^p(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$, and I_1 and I_2 are estimated by similar method as above, if we choose small enough $\varepsilon > 0$, then there exist constants $c_1 > 0$ and $C > 0$ independent of t such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{w}(t)|^2 dx + c_1 \int_\Omega |\boldsymbol{w}(t)|^2 dx \le Cl(t).$$

By method similar to that for the case $p = 2$, we get the following.

Theorem 5.5. *Assume that* $S(x,t,s) = S^{(\infty)}(x,s)$ *and* $6/5 \le p < 2$. *Furthermore, assume that* $\boldsymbol{f} \in L^\infty(0,\infty;\boldsymbol{L}^{p'}(\Omega)) \cap \boldsymbol{L}^2(\Omega_\infty)$ *and* $\boldsymbol{u}_0 \in \mathbb{V}^p_N(\Omega)$. *If*

$$\int_t^{t+1} l(\tau)d\tau \to 0 \ \text{as}\ t \to \infty,$$

then we have

$$\|\boldsymbol{u}(t) - \boldsymbol{u}_\infty\|_{\boldsymbol{L}^2(\Omega)} \to 0 \ \text{as}\ t \to \infty.$$

Furthermore, if $\xi(t) \to 0$ *as* $t \to \infty$, *then we have*

$$\|\pi(t) - \pi_\infty\|_{W_0^{1,p'}(\Omega)} \to 0 \ \text{as}\ t \to \infty.$$

A Proof of Lemma 4.4

In this appendix, we prove Lemma 4.4. In the Section 2, we showed that if $\boldsymbol{z} \in \boldsymbol{L}^{p'}(\Omega)$, then $\operatorname{curl}\boldsymbol{z} \in \mathbb{X}^p_N(\Omega)'$ is well defined by (2.1). We see that $X^p_N(\Omega)$ is a subspace of $\boldsymbol{W}^{1,p}(\Omega)$ equipped with the norm $\|\cdot\|_{\boldsymbol{W}^{1,p}(\Omega)}$. Since $\operatorname{curl}\boldsymbol{z} \in \mathbb{X}^p_N(\Omega)'$, it follows from the Hahn-Banach theorem that there exists a $\widetilde{\operatorname{curl}}\,\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)'$ such that

$$\langle \widetilde{\operatorname{curl}}\,\boldsymbol{z}, \boldsymbol{\phi}\rangle_{\boldsymbol{W}^{1,p}(\Omega)',\boldsymbol{W}^{1,p}(\Omega)} = \langle \operatorname{curl}\boldsymbol{z}, \boldsymbol{\phi}\rangle_{\mathbb{X}^p_N(\Omega)',\mathbb{X}^p_N(\Omega)} \ \text{for}\ \boldsymbol{\phi} \in \mathbb{X}^p_N(\Omega).$$

By the classical Gagliardo lemma, for any $\boldsymbol{\phi} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, there exists $\widetilde{\boldsymbol{\phi}} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\widetilde{\boldsymbol{\phi}} = \boldsymbol{\phi}$ on Γ and satisfies

$$\|\widetilde{\boldsymbol{\phi}}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\|\boldsymbol{\phi}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}.$$

Define

$$\langle \boldsymbol{n}\times\boldsymbol{z}, \boldsymbol{\phi}\rangle = \langle \widetilde{\operatorname{curl}}\,\boldsymbol{z}, \widetilde{\boldsymbol{\phi}}\rangle_{\boldsymbol{W}^{1,p}(\Omega)',\boldsymbol{W}^{1,p}(\Omega)} - \int_\Omega \boldsymbol{z}\cdot\operatorname{curl}\widetilde{\boldsymbol{\phi}}dx. \tag{A.1}$$

We have to show that the right-hand side of (A.1) is independent of the choice of $\widetilde{\boldsymbol{\phi}} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\widetilde{\boldsymbol{\phi}} = \boldsymbol{\phi}$ on Γ. To do so, let $\widetilde{\boldsymbol{\phi}} \in \boldsymbol{W}^{1,p}_0(\Omega)(\subset \mathbb{X}^p_N(\Omega))$. Since $\boldsymbol{C}^1_0(\Omega)$ is dense in $\boldsymbol{L}^{p'}(\Omega)$, we can choose $\boldsymbol{z}_j \in \boldsymbol{C}^1_0(\Omega)$ such that $\boldsymbol{z}_j \to \boldsymbol{z}$ in $\boldsymbol{L}^{p'}(\Omega)$. Since the operator $\operatorname{curl} : \boldsymbol{L}^{p'}(\Omega) \to \mathbb{X}^p_N(\Omega)'$ is continuous, we have $\operatorname{curl}\boldsymbol{z}_j \to \operatorname{curl}\boldsymbol{z}$ in $\mathbb{X}^p_N(\Omega)'$.

Thus, we have

$$\begin{aligned}\langle \widetilde{\operatorname{curl}}\,\boldsymbol{z}_j, \widetilde{\boldsymbol{\phi}}\rangle_{\boldsymbol{W}^{1,p}(\Omega)',\boldsymbol{W}^{1,p}(\Omega)} &= \langle \operatorname{curl}\boldsymbol{z}_j, \widetilde{\boldsymbol{\phi}}\rangle_{\mathbb{X}^p_N(\Omega)',\mathbb{X}^p_N(\Omega)} \\ &\to \langle \operatorname{curl}\boldsymbol{z}, \widetilde{\boldsymbol{\phi}}\rangle_{\mathbb{X}^p_N(\Omega)',\mathbb{X}^p_N(\Omega)} = \langle \widetilde{\operatorname{curl}}\,\boldsymbol{z}, \widetilde{\boldsymbol{\phi}}\rangle_{\boldsymbol{W}^{1,p}(\Omega)',\boldsymbol{W}^{1,p}(\Omega)}\end{aligned}$$

as $j \to \infty$. On the other hand, it follows from the Green formula that

$$\begin{aligned}\langle \widetilde{\operatorname{curl}}\,\boldsymbol{z}_j, \widetilde{\boldsymbol{\phi}}\rangle_{\boldsymbol{W}^{1,p}(\Omega)',\boldsymbol{W}^{1,p}(\Omega)} &= \langle \operatorname{curl}\boldsymbol{z}_j, \widetilde{\boldsymbol{\phi}}\rangle_{\mathbb{X}^p_N(\Omega)',\mathbb{X}^p_N(\Omega)} \\ &= \int_\Omega \boldsymbol{z}_j\cdot\operatorname{curl}\widetilde{\boldsymbol{\phi}}dx + \int_\Gamma \boldsymbol{z}_j\cdot(\widetilde{\boldsymbol{\phi}}\times\boldsymbol{n})dS = \int_\Omega \boldsymbol{z}_j\cdot\operatorname{curl}\widetilde{\boldsymbol{\phi}}dx \to \int_\Omega \boldsymbol{z}\cdot\operatorname{curl}\widetilde{\boldsymbol{\phi}}dx\end{aligned}$$

as $j \to \infty$. So we see that

$$\langle \widetilde{\operatorname{curl}} \boldsymbol{z}, \widetilde{\boldsymbol{\phi}} \rangle_{\boldsymbol{W}^{1,p}(\Omega)', \boldsymbol{W}^{1,p}(\Omega)} = \int_\Omega \boldsymbol{z} \cdot \operatorname{curl} \widetilde{\boldsymbol{\phi}} dx \text{ for all } \widetilde{\boldsymbol{\phi}} \in \boldsymbol{W}_0^{1,p}(\Omega).$$

This implies that the right-hand side of (A.1) is independent of the choice of $\widetilde{\boldsymbol{\phi}}$ such that $\widetilde{\boldsymbol{\phi}} = \boldsymbol{\phi}$ on Γ. Since

$$\begin{aligned} |\langle \boldsymbol{n} \times \boldsymbol{z}, \boldsymbol{\phi} \rangle| &\leq C(\|\widetilde{\operatorname{curl}} \boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)'} + \|\boldsymbol{z}\|_{\boldsymbol{L}^{p'}(\Omega)}) \|\widetilde{\boldsymbol{\phi}}\|_{\boldsymbol{W}^{1,p}(\Omega)} \\ &\leq C_1(\|\widetilde{\operatorname{curl}} \boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)'} + \|\boldsymbol{z}\|_{\boldsymbol{L}^{p'}(\Omega)}) \|\boldsymbol{\phi}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} \text{ for all } \boldsymbol{\phi} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \end{aligned}$$

we can see that $\boldsymbol{z} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p',p'}(\Gamma)$ is well defined and the Green formula (A.1) holds. In particular, we have

$$\langle \boldsymbol{n} \times \boldsymbol{z}, \boldsymbol{\phi} \rangle_{W^{-1/p',p'}(\Gamma), W^{1-1/p,p}(\Gamma)} = \langle \operatorname{curl} \boldsymbol{z}, \boldsymbol{\phi} \rangle_{\mathbb{X}_N^p(\Omega)', \mathbb{X}_N^p(\Omega)} - \int_\Omega \boldsymbol{z} \cdot \operatorname{curl} \boldsymbol{\phi} dx$$

for all $\boldsymbol{\phi} \in \mathbb{X}_N^p(\Omega)$.

B Proof of the function in (2.7) to satisfy the structural conditions

In this appendix, we show that if a function $\nu(x,t)$ is measurable in Ω_T and satisfies (2.6) and a function $g(s)$ is defined by (2.7), then the function $S(x,t,s) = \nu(x,t) g(t) s^{p/2}$ satisfies (2.4a)-(2.4c) if $p \geq 2$. To do so, by putting $S(s) = g(s) s^{p/2}$, it suffices to prove that $S(s)$ satisfies (2.4a)-(2.4c) for some constants $0 < \lambda \leq \Lambda < \infty$. We note that the function $g(s)$ defined by (2.7) is infinitely differentiable in $[0,\infty)$ and satisfies $1 \leq g(s) \leq 2$ in $[0,\infty)$. We have, for $s > 0$,

$$S_s = g'(s) s^{p/2} + \frac{p}{2} g(s) s^{(p-2)/2} = \big(s g'(s) + \frac{p}{2} g(s)\big) s^{(p-2)/2}, \tag{B.1}$$

and

$$S_{ss} = \left(g''(s) s^2 + p g'(s) s + \frac{p(p-2)}{4} g(s) \right) s^{(p-4)/2}. \tag{B.2}$$

On the other hand, for $s > 0$,

$$g'(s) = \frac{1}{s^2} e^{-1/s} \tag{B.3}$$

and

$$g''(s) = -2 \frac{1}{s^3} e^{-1/s} + \frac{1}{s^4} e^{-1/s}. \tag{B.4}$$

Here we note that

$$\lim_{s \to +0} \frac{1}{s^k} e^{-1/s} = 0 \text{ for every integer } k \geq 0,$$

and so $0 \leq \frac{1}{s^k} e^{-1/s} \leq C_k$ for $s \geq 0$ with some constant $C_k > 0$. If we substitute (2.7) and (B.3) for (B.1), then we have

$$S_s = \left(\frac{1}{s} e^{-1/s} + \frac{p}{2}(1 + e^{-1/s}) \right) s^{(p-2)/2}.$$

Since

$$\frac{p}{2} \leq \frac{1}{s} e^{-1/s} + \frac{p}{2}(1 + e^{-1/s}) = sg'(s) + \frac{p}{2} g(s) \leq C_1 + \frac{p}{2}(1 + C_0),$$

we see that (2.4a) holds.

If we substitute (2.7), (B.3) and (B.4) for (B.2), then we have

$$S_s + 2sS_{ss} = \left\{ (2\frac{1}{s^2} + (2p-3)\frac{1}{s} + \frac{p(p-1)}{2})e^{-1/s} + \frac{p(p-1)}{2} \right\} s^{(p-2)/2}.$$

If $p \geq 2$, then

$$\begin{aligned} \frac{p(p-1)}{2} &\leq 2\frac{1}{s^2} e^{-1/s} + (2p-3)\frac{1}{s} e^{-1/s} + \frac{p(p-1)}{2} e^{-1/s} + \frac{p(p-1)}{2} \\ &\leq 2C_2 + (2p-3)C_1 + \frac{p(p-1)}{2} C_0 + \frac{p(p-1)}{2}. \end{aligned}$$

This implies that (2.4b) holds.

Similarly, we have

$$S_{ss} = \left\{ (\frac{1}{s^2} + (p-2)\frac{1}{s} + \frac{p(p-2)}{4})e^{-1/s} + \frac{p(p-2)}{4} \right\} e^{(p-4)/2}.$$

Here the discriminant D of the quadratic $(\frac{1}{s})^2 + (p-2)\frac{1}{s} + \frac{p(p-2)}{4}$ of $1/s$ satisfies

$$D = (p-2)^2 - 4 \cdot \frac{p(p-2)}{4} = -2(p-2) \leq 0 \text{ if } p \geq 2.$$

Thus $\frac{1}{s^2} + (p-2)\frac{1}{s} + \frac{p(p-2)}{4} \geq 0$ for $s > 0$. Therefore, we have (2.4c) if $p \geq 2$.

Acknowledgments

We would like to thank the anonymous referee for his or her very kind advice about a previous version of this article.

References

[1] C. Amrouche and N. H. Seloula, L^p-theory for vector potentials and Sobolev's inequality for vector fields. Application to the Stokes equations with pressure boundary conditions, Math. Models and Methods in Appl. Sci., 23, (2013), 37-92.

[2] C. Amrouche and N. H. Seloula, L^p-theory for vector potentials and Sobolev's inequality for vector fields., C. R. Acad. Sci. Paris, Ser. 1., 349, (2011), 529-534.

[3] J. Aramaki, Existence and regularity for the Neumann problem to the Poisson equation and an application to the Maxwell-Stokes type equation, Comm. Math. Anal., 21(1), (2018), 54-66.

[4] J. Aramaki, Existence and regularity of a weak solution to a class of systems in a multi-connected domain, J. Part. Diff. Eq., 32(1), (2019), 1-19.

[5] H. A. Baba, C. Amrouche and N. Seloula, Instationary Stokes problem with pressure boundary condition in L^p-spaces, J. Evol. Equ., 17, (2017), 641-667.

[6] R. Dautray and J. L. Lions, Mathematical Analysis and Numerical Method for Science and Technology Vol. 3, Springer Verlag, New York, 1990.

[7] I. Ekeland and R. Temam, Convex Analysis and Variational Problems, SIAM, Philadelphia, 1999.

[8] V. Girault and P. A. Raviart, Finite Element Methods for Navier-Stokes equations, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1979.

[9] A. Haraux, Nonlinear Evolution Equations-Global Behavior of Solutions, Lecture Notes in Math., 841, Springer-Verlag, Berlin-New York, 1981.

[10] H. Kozono and T. Yanagisawa, Global div-curl lemma on bounded domains in $\mathbb{R}^3$, J. Funct. Anal., 256, (2009), 3847-3859.

[11] J. L. Lions, Quelque Méthodes de Résolution des Problémes aux Limites Non Linéaires, Dunod Gauthier-Villars, 1969.

[12] F. Miranda, J. F. Rodrigues and L. Santos, On a p-curl system arising in electromagnetism, Discrete and Cont. Dyn. Systems Ser. S, 6(3), (2012), 605-629.

[13] D. Mitrea, M. Mitrea and J. Pipher, Vector potential theory on nonsmooth domains in $\mathbb{R}^3$ and applications to electromagnetic scattering, J. Fourier Anal. Appl., 3(2), (1997), 131-192.

[14] J. Simon, Quelque propriétés de solutions d'équations et d'évolution paraboliques non linéires, Ann. Scuola Norm. Pisa Cl. Sci., 2(4), (1975), 585-609.

[15] S. Sobolev, Applications of Functional Analysis in Mathematical Physics, Translation of Mathematical Monographs, Vol. 7, AMS, Providence, R. I., 1963.

[16] H. Yin, B. Q. Li and J. Zou, A degenerate evolution system modeling Bean's critical state type-II superconductors, Discrete and Continuous Dynamical Systems, 8, (2002), 781-794.

[17] S. Zheng, Nonlinear Evolution Equations, Chapman and Hall/CRC, 2004.