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DOUBLE LAPLACE TRANSFORM IN BICOMPLEX SPACE WITH APPLICATIONS

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Abstract. Motivated by the work of Eltayeb and Killicman in this paper we generalize complex double Laplace transform to bicomplex double Laplace transform. Also, we derive some of its basic properties and inversion theorem in bicomplex space. Applications of bicomplex Double Laplace transform have been discussed in finding the solution of two-dimensional time-dependent bicomplex Schrödinger equation for free particle by using two different approaches.

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1 Introduction

In this paper, we extend the complex double Laplace transform to bicomplex double Laplace transform in two bicomplex variables. In 1892, Corrado [6] defined bicomplex numbers as

$$\mathbb{C}_2 = \{\xi : \xi = x_0 + i_1x_1 + i_2x_2 + jx_3 |\ x_0, x_1, x_2, x_3 \in \mathbb{R}\},$$

or

$$\mathbb{C}_2 = \{\xi : \xi = z_1 + i_2z_2 |\ z_1, z_2 \in \mathbb{C}_1\}.$$

where i_1 and i_2 are imaginary units such that $i_1^2 = i_2^2 = -1,\ i_1i_2 = i_2i_1 = j,\ j^2 = 1$ and $\mathbb{R}$, $\mathbb{C}_1$ and $\mathbb{C}_2$ are sets of real numbers, complex numbers and bicomplex numbers, respectively. The set of bicomplex numbers is a commutative ring with unit and zero divisors. Hence, contrary to quaternions, bicomplex numbers are commutative with some non-invertible elements situated on the null cone.

In 1928 and 1932, Futagawa originated the concept of holomorphic functions of a bicomplex variable in a series of papers [15], [16]. In 1934, Dragoni [9] gave some basic results in the theory of bicomplex holomorphic functions while Price [8] and Rönn [23] have developed the bicomplex algebra and function theory.

In recent developments, authors have done efforts to extend Polygamma function [22], inverse Laplace transform, its convolution theorem [20], Stieltjes transform [18], Tauberian Theorem of Laplace-Stieltjes transform [21] and Bochner Theorem of Fourier-Stieltjes transform in the bicomplex variable from their complex counterpart. In their procedure, the idempotent representation of bicomplex plays a vital role.

In 1936, Van der Pol [25] introduced about the double Laplace transform. This has been used by Humbert [17] in the study of hypergeometric functions; by Jaeger [12] to solve boundary value problems in heat conduction. In 1951, Estrin et al. [24] extend the complex double Laplace transform to multiple Laplace transform in n independent complex variables. In 2008, Elatayeb and Kilicman [11] used double Laplace transform for solving a second-order partial differential equations. In 2010, Kilicman and Gaddin [4] discussed relationship between double Laplace transform and double Sumudu transform. In 2013, Kashuri et al. [3] used double Laplace transform and double new integral transform in solving partial differential equation.

For solving the large class of bicomplex partial differential equations, we need integral transforms defined for large class. In this process we derive bicomplex double Laplace transform with convergence conditions that can be capable the transferring signals from real-valued (x, t) domain to bicomplexified frequency (ξ, η) domain.

Idempotent Representation: Every bicomplex number can be uniquely expressed as a complex combination of e_1 and e_2, viz.

$$\xi = (z_1 + i_2z_2) = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2,$$

(where $e_1 = \frac{1+j}{2}, e_2 = \frac{1-j}{2}; e_1 + e_2 = 1$ and $e_1e_2 = e_2e_1 = 0$).

This representation of a bicomplex number is known as Idempotent Representation of ξ. The coefficients $(z_1 - i_1z_2)$ and $(z_1 + i_1z_2)$ are called the Idempotent Components of the bicomplex number $\xi = z_1 + i_2z_2$ and $\{e_1, e_2\}$ is called idempotent basis.

Cartesian Set: The Auxiliary complex spaces A_1 and A_2 are defined as follows:

$$A_1 = \{z_1 - i_1 z_2,\ \forall\ z_1, z_2 \in \mathbb{C}_1\},\ A_2 = \{z_1 + i_1 z_2,\ \forall\ z_1, z_2 \in \mathbb{C}_1\}.$$

A cartesian set determined by X_1 and X_2 in A_1 and A_2 respectively is denoted as $X_1 \times_e X_2$ and is defined as:

$$X_1 \times_e X_2 = \{z_1 + i_2 z_2 \in \mathbb{C}_2 : z_1 + i_2 z_2 = w_1 e_1 + w_2 e_2,\ w_1 \in X_1, w_2 \in X_2\}.$$

With the help of idempotent representation, we define functions $P_1 : \mathbb{C}_2 \to A_1 \subseteq \mathbb{C}_1$, $P_2 : \mathbb{C}_2 \to A_2 \subseteq \mathbb{C}_1$ as follows:

$$P_1(z_1 + i_2 z_2) = P_1[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 - i_1 z_2) \in A_1,\ \forall\ z_1 + i_2 z_2 \in \mathbb{C}_2,$$

$$P_2(z_1 + i_2 z_2) = P_2[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 + i_1 z_2) \in A_2,\ \forall\ z_1 + i_2 z_2 \in \mathbb{C}_2.$$

In the following theorem, Price discuss the convergence of bicomplex function with respect to its idempotent complex component functions. This theorem is useful in proving our results.

Theorem 1.1 (Price [8]). *$F(\xi) = F_{e_1}(\xi_1)e_1 + F_{e_2}(\xi_2)e_2$ is convergent in domain $D \subseteq \mathbb{C}_2$ iff $F_{e_1}(\xi_1)$ and $F_{e_2}(\xi_2)$ under functions $P_1 : D \to D_1 \subseteq \mathbb{C}_1$ and $P_2 : D \to D_2 \subseteq \mathbb{C}_1$ are convergent in domains D_1 and D_2, respectively.*

The organization of this paper is as follows:
In Section 2, we establish bicomplex double Laplace transform with convergence conditions. In Section 3, we present some useful properties of bicomlex double Laplace transform. In Section 4, we establish the inversion theorem for bicomplex double Laplace transform. In section 5, we discuss applications of bicomplex double Laplace transform in finding the solution of two-dimensional time-dependent bicomplex Schrödinger equation and last Section 6 contains the conclusion.

2 Bicomplex Double Laplace Transform

Let $f(x,t)$ be a bicomplex-valued function of two variables $x, t > 0$, which is piecewise continuous and has exponential order K_1 and K_2 w.r.t. x and t respectively. The bicomplex Laplace transform (see, Kumar and Kumar [5]) w.r.t. x is

$$L_x[f(x,t)] = \int_0^\infty e^{-\xi x} f(x,t)dx = \bar{f}(\xi, t), \qquad \xi \in \Omega_1 \subset \mathbb{C}_2 \tag{1}$$

where

$$\Omega_1 = \{\xi = s_1 e_1 + s_2 e_2 \in \mathbb{C}_2 :\ \mathrm{Re}(P_1 : \xi) > K_1 \text{ and } \mathrm{Re}(P_2 : \xi) > K_1\} \tag{2}$$

or

$$\Omega_1 = \{\xi \in \mathbb{C}_2\ :\ \mathrm{Re}(\xi) > K_1 + |\mathrm{Im}_j(\xi)|\} \tag{3}$$

where $\mathrm{Im}_j(\xi)$ denotes the imaginary part of ξ w.r.t. j and (1) is convergent and analytic in Ω_1. Similarly, bicomplex Laplace transform of $f(x,t)$ w.r.t. t is

$$L_t[f(x,t)] = \int_0^\infty e^{-\eta t} f(x,t)dt = \bar{f}(x,\eta), \qquad \eta \in \Omega_2 \subset \mathbb{C}_2 \tag{4}$$

where

$$\Omega_2 = \{\eta = p_1e_1 + p_2e_2 \in \mathbb{C}_2 :\ \mathrm{Re}(P_1 : \eta) > K_2 \text{ and } \mathrm{Re}(P_2 : \eta) > K_2\} \tag{5}$$

or

$$\Omega_2 = \{\eta \in \mathbb{C}_2 \ :\ \mathrm{Re}(\eta) > K_2 + |\mathrm{Im}_j(\eta)|\} \tag{6}$$

where (4) is convergent and analytic in Ω_2. Now, taking the bicomplex Laplace transform of (1) w.r.t. t and using (4), we have

$$\begin{aligned} L_{xt}[f(x,t)] = L_t[\bar{f}(\xi,t)] &= \int_0^\infty e^{-\eta t}\bar{f}(\xi,t)dt \\ &= \int_0^\infty e^{-\eta t}\int_0^\infty e^{-\xi x} f(x,t)dxdt = \bar{\bar{f}}(\xi,\eta), \quad (\xi,\eta)\in\Omega \end{aligned} \tag{7}$$

the integral on right hand side is convergent and analytic in

$$\Omega = \left\{(\xi,\eta) \in \mathbb{C}_2^2 \ :\ \xi \in \Omega_1 \text{ and } \eta \in \Omega_2\right\}. \tag{8}$$

Now, we define the bicomplex double Laplace transform as follows:

Definition 2.1. *Let $f(x,t)$ be a bicomplex-valued function of two variables $x,t > 0$, which is piecewise continuous and has exponential order K_1 and K_2 w.r.t. x and t respectively. Then bicomplex double Laplace transform is defined as*

$$L_{xt}[f(x,t)](\xi,\eta) = \int_0^\infty\int_0^\infty e^{-\xi x-\eta t} f(x,t)dxdt = \bar{\bar{f}}(\xi,\eta), \qquad (\xi,\eta)\in\Omega$$

which exists and is convergent for all $(\xi,\eta) \in \Omega$ as defined in (8).

3 Properties of Bicomplex Double Laplace Transform

In this section, we discuss some properties of bicomplex double Laplace transform viz. linearity property, change of scale property, shifting property etc.

Theorem 3.1 (Linearity Property)**.** *Let $f(x,t)$ and $g(x,t)$ be two bicomplex-valued function of $x,t > 0$ such that*

$L_{xt}[f(x,t)] = \bar{\bar{f}}(\xi,\eta), \qquad (\xi,\eta)\in\Omega$
where $\Omega = \left\{(\xi,\eta) \in \mathbb{C}_2^2 \ :\ Re(\xi) > K_1 + |Im_j(\xi)| \text{ and } Re(\eta) > K_2 + |Im_j(\eta)|\right\}$
and $L_{xt}[g(x,t)] = \bar{\bar{g}}(\xi,\eta), \qquad (\xi,\eta)\in\Omega$
where $\Omega = \left\{(\xi,\eta) \in \mathbb{C}_2^2 \ :\ Re(\xi) > K_3 + |Im_j(\xi)| \text{ and } Re(\eta) > K_4 + |Im_j(\eta)|\right\}.$

Then,

$$L_{xt}[c_1 f(x,t) + c_2 g(x,t)] = c_1 L_{xt}[f(x,t)] + c_2 L_{xt}[g(x,t)], \qquad (\xi,\eta) \in \Omega$$

where $\Omega = \{(\xi,\eta) \in \mathbb{C}_2^2 \ : \ Re(\xi) > \max(K_1, K_3) + |Im_j(\xi)|$ *and* $Re(\eta) > \max(K_2, K_4) + |Im_j(\eta)|\}$ *and* c_1, c_2 *are constants.*

Proof. Applying the definition of bicomplex double Laplace transform,

$$\begin{aligned}
L_{xt}[c_1 f(x,t) + c_2 g(x,t)] &= \int_0^\infty \int_0^\infty e^{-\xi x - \eta t}[c_1 f(x,t) + c_2 g(x,t)]dxdt \\
&= c_1 \int_0^\infty \int_0^\infty e^{-\xi x - \eta t} f(x,t)dxdt + c_2 \int_0^\infty \int_0^\infty e^{-\xi x - \eta t} g(x,t)dxdt \\
&= c_1 \bar{\bar{f}}(\xi,\eta) + c_2 \bar{\bar{g}}(\xi,\eta).
\end{aligned}$$

Thus,

$$L_{xt}[c_1 f(x,t) + c_2 g(x,t)] = c_1 L_{xt}[f(x,t)] + c_2 L_{xt}[g(x,t)].$$

□

Theorem 3.2 (Change of Scale Property)**.** *Let* $\bar{\bar{f}}(\xi,\eta)$ *be the bicomplex double Laplace transform of bicomplex-valued function* $f(x,t)$*. Then*

$$L_{xt}[f(\alpha x, \beta t)](\xi,\eta) = \frac{1}{\alpha\beta} \bar{\bar{f}}\left(\frac{\xi}{\alpha}, \frac{\eta}{\beta}\right), \qquad (\xi,\eta) \in \Omega \text{ and } \alpha, \beta > 0$$

where Ω *defined in (8).*

Proof. From the definition of bicomplex double Laplace transform,

$$\begin{aligned}
L_{xt}[f(\alpha x, \beta t)](\xi,\eta) &= \int_0^\infty \int_0^\infty e^{-\xi x - \eta t} f(\alpha x, \beta t)dxdt \\
&= \int_0^\infty e^{-\eta t} \left(\int_0^\infty e^{-\xi x} f(\alpha x, \beta t)dx \right) dt \\
&= \frac{1}{\alpha} \int_0^\infty e^{-\eta t} \left(\int_0^\infty e^{-\frac{\xi}{\alpha} r} f(r, \beta t)dr \right) dt \quad [\text{Taking } \alpha x = r] \\
&= \frac{1}{\alpha} \int_0^\infty e^{-\eta t} \bar{f}\left(\frac{\xi}{\alpha}, \beta t\right) dt \\
&= \frac{1}{\alpha\beta} \int_0^\infty e^{-\frac{\eta}{\beta} s} \bar{f}\left(\frac{\xi}{\alpha}, s\right) ds \quad [\text{Taking } \beta t = s] \\
&= \frac{1}{\alpha\beta} \bar{\bar{f}}\left(\frac{\xi}{\alpha}, \frac{\eta}{\beta}\right).
\end{aligned}$$

Thus,

$$L_{xt}[f(\alpha x, \beta t)](\xi,\eta) = \frac{1}{\alpha\beta} \bar{\bar{f}}\left(\frac{\xi}{\alpha}, \frac{\eta}{\beta}\right).$$

□

Theorem 3.3 (First Shifting Property). *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$L_{xt}\left[e^{ax+bt}f(x,t)\right](\xi,\eta)=\bar{\bar{f}}(\xi-a,\eta-b),\qquad (\xi-a,\eta-b)\in\Omega$$

where Ω defined in (8).

Proof. Applying the definition of bicomplex double Laplace transform,

$$\begin{aligned}L_{xt}\left[e^{ax+bt}f(x,t)\right](\xi,\eta)&=\int_0^\infty\int_0^\infty e^{-\xi x-\eta t}e^{ax+bt}f(x,t)dxdt\\&=\int_0^\infty e^{-(\eta-b)t}\left(\int_0^\infty e^{-(\xi-a)x}f(x,t)dx\right)dt\\&=\int_0^\infty e^{-(\eta-b)t}\bar{f}(\xi-a,t)dt\\&=\bar{\bar{f}}(\xi-a,\eta-b).\end{aligned}$$

Thus,

$$L_{xt}\left[e^{ax+bt}f(x,t)\right](\xi,\eta)=\bar{\bar{f}}(\xi-a,\eta-b).$$

□

Theorem 3.4 (Double Laplace Transform of Derivatives). *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$L_{xt}\left[f_{xt}(x,t)\right](\xi,\eta)=\xi\eta\bar{\bar{f}}(\xi,\eta)-\xi\bar{f}(\xi,0)-\eta\bar{f}(0,\eta)+f(0,0),\quad (\xi,\eta)\in\Omega$$

where Ω defined in (8) and $f_{xt}(x,t)=\frac{\partial^2}{\partial x\partial t}f(x,t)$.

Proof. Applying the definition of bicomplex double Laplace transform,

$$\begin{aligned}L_{xt}\left[f_{xt}(x,t)\right]&=\int_0^\infty e^{-\eta t}\left(\int_0^\infty e^{-\xi x}f_{xt}(x,t)dx\right)dt\\&=\int_0^\infty e^{-\eta t}\left[\left(e^{-\xi x}f_t(x,t)\right)_{x=0}^{\infty}+\xi\int_0^\infty e^{-\xi x}f_t(x,t)dx\right]dt\\&=-\int_0^\infty e^{-\eta t}f_t(0,t)dt+\xi\int_0^\infty e^{-\eta t}\int_0^\infty f_t(x,t)dxdt\\&=f(0,0)-\eta\int_0^\infty e^{-\eta t}f(0,t)dt+\xi\int_0^\infty e^{-\eta t}\left(\int_0^\infty f_t(x,t)dt\right)dx\\&=f(0,0)-\eta\bar{f}(0,\eta)+\xi\int_0^\infty e^{-\xi x}\left[\left(e^{-\eta t}f(x,t)\right)_{t=0}^{\infty}+\eta\int_0^\infty e^{-\eta t}f(x,t)dt\right]dx\\&=f(0,0)-\eta\bar{f}(0,\eta)-\xi\int_0^\infty e^{-\xi x}f(x,0)dx+\xi\eta\int_0^\infty\int_0^\infty e^{-\xi x-\eta t}f(x,t)dxdt\\&=f(0,0)-\eta\bar{f}(0,\eta)-\xi\bar{f}(\xi,0)+\xi\eta\bar{\bar{f}}(\xi,\eta).\end{aligned}$$

Thus,

$$L_{xt}\left[f_{xt}(x,t)\right](\xi,\eta)=\xi\eta\bar{\bar{f}}(\xi,\eta)-\xi\bar{f}(\xi,0)-\eta\bar{f}(0,\eta)+f(0,0).$$

□

Theorem 3.5 (Multiplication by xt)**.** *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$L_{xt}[xtf(x,t)](\xi,\eta)=\frac{\partial^2}{\partial\xi\partial\eta}\bar{\bar{f}}(\xi,\eta),\qquad (\xi,\eta)\in\Omega \text{ (as } \Omega \text{ defined in (8))}$$

Proof. Applying the definition of bicomplex double Laplace transform,

$$\begin{aligned}\frac{\partial^2}{\partial\xi\partial\eta}\bar{\bar{f}}(\xi,\eta)&=\left(\frac{\partial^2}{\partial\xi_1\partial\eta_1}\bar{\bar{f}}_{e_1}(\xi_1,\eta_1)\right)e_1+\left(\frac{\partial^2}{\partial\xi_2\partial\eta_2}\bar{\bar{f}}_{e_2}(\xi_2,\eta_2)\right)e_2\\&=\left(\frac{\partial^2}{\partial\xi_1\partial\eta_1}\int_0^\infty\int_0^\infty e^{-\xi_1x-\eta_1t}f_{e_1}(x,t)dxdt\right)e_1\\&\quad+\left(\frac{\partial^2}{\partial\xi_2\partial\eta_2}\int_0^\infty\int_0^\infty e^{-\xi_2x-\eta_2t}f_{e_2}(x,t)dxdt\right)e_2\end{aligned}$$

$\Big($where $\bar{\bar{f}}(\xi,\eta)=\bar{\bar{f}}_{e_1}(\xi_1,\eta_1)e_1+\bar{\bar{f}}_{e_2}(\xi_2,\eta_2)e_2,\ \xi=\xi_1e_1+\xi_2e_2$ and $\eta=\eta_1e_1+\eta_2e_2\Big)$.

Applying Leibniz's rule for complex functions [13, p. 243], we have

$$\begin{aligned}\frac{\partial^2}{\partial\xi\partial\eta}\bar{\bar{f}}(\xi,\eta)&=(-1)^2\left\{\left(\int_0^\infty\int_0^\infty e^{-\xi_1x-\eta_1t}xtf_{e_1}(x,t)dxdt\right)e_1+\right.\\&\quad\left.\left(\int_0^\infty\int_0^\infty e^{-\xi_2x-\eta_2t}xtf_{e_2}(x,t)dxdt\right)e_2\right\}\\&=\int_0^\infty\int_0^\infty e^{-(\xi_1e_1+\xi_2e_2)x-(\eta_1e_1+\eta_2e_2)t}\left(f_{e_1}(x,t)e_1+f_{e_2}(x,t)e_2\right)dxdt\\&=\int_0^\infty\int_0^\infty e^{-\xi x-\eta t}xtf(x,t)dxdt.\end{aligned}$$

Thus,

$$L_{xt}[xtf(x,t)](\xi,\eta)=\frac{\partial^2}{\partial\xi\partial\eta}\bar{\bar{f}}(\xi,\eta).$$

In general,

$$L_{xt}\left[x^mt^nf(x,t)\right](\xi,\eta)=(-1)^{m+n}\frac{\partial^{m+n}}{\partial\xi^m\partial\eta^n}\bar{\bar{f}}(\xi,\eta).$$

□

Theorem 3.6 (Division by xt)**.** *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$L_{xt}\left[\frac{f(x,t)}{xt}\right](\xi,\eta)=\int_\xi^\infty\int_\eta^\infty\bar{\bar{f}}(\xi,\eta)d\xi d\eta,\qquad(\xi,\eta)\in\Omega$$

provided the integral on right hand exists.

Proof. Applying the definition of bicomplex double Laplace transform,

$$\bar{\bar{f}}(\xi,\eta) = \int_0^\infty \int_0^\infty e^{-\xi x-\eta t} f(x,t)dxdt \tag{9}$$

Integrating (9) w.r.t. ξ from ξ to ∞ and η from η to ∞, we have

$$\begin{aligned}
\int_\xi^\infty \int_\eta^\infty \bar{\bar{f}}(\xi,\eta)d\xi d\eta &= \int_\xi^\infty \int_\eta^\infty \int_0^\infty \int_0^\infty e^{-\xi x} e^{-\eta t} f(x,t)dxdtd\xi d\eta \\
&= \int_\eta^\infty \int_0^\infty \int_0^\infty \left(\frac{e^{-\xi x}}{-x}\right)_{\xi=\xi}^\infty e^{-\eta t} f(x,t)dxdtd\eta \\
&= \int_0^\infty \int_0^\infty \left(0+\frac{e^{-\xi x}}{x}\right)\left(\frac{e^{-\eta t}}{-t}\right)_{\eta=\eta}^\infty f(x,t)dxdt \\
&= \int_0^\infty \int_0^\infty e^{-\xi x} e^{-\eta t} \frac{f(x,t)}{xt} dxdt \\
&= L_{xt}\left[\frac{f(x,t)}{xt}\right](\xi,\eta).
\end{aligned}$$

Thus,

$$L_{xt}\left[\frac{f(x,t)}{xt}\right](\xi,\eta) = \int_\xi^\infty \int_\eta^\infty \bar{\bar{f}}(\xi,\eta)d\xi d\eta.$$

□

Theorem 3.7 (Double Laplace Transform of Integrals)**.** *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$L_{xt}\left[\int_0^x \int_0^t f(u,v)dudv\right] = \frac{\bar{\bar{f}}(\xi,\eta)}{\xi\eta}, \quad Re(\xi) > |Im_j(\xi)|\,,\ Re(\eta) > |Im_j(\eta)|\,.$$

Proof. Let

$$g(x,t) = \int_0^x \int_0^t f(u,v)dudv.$$

Hence, we have

$$\begin{aligned}
&g_{xt}(x,t) = f(x,t) \text{ and } g(0,0) = 0 \\
&\therefore \quad L_{xt}[g_{xt}(x,t)] = L[f(x,t)] = \bar{\bar{f}}(\xi,\eta).
\end{aligned}$$

Now from the Theorem 3.4 we have

$$\begin{aligned}
&L_{xt}\left[g_{xt}(x,t)\right] = \xi\eta\bar{\bar{g}}(\xi,\eta) - \xi\bar{g}(\xi,0) - \eta\bar{g}(0,\eta) + g(0,0) \\
\Rightarrow &\bar{f}_2(\xi,\eta) = \xi\eta\bar{g}_2(\xi,\eta) - \xi\bar{g}_1(\xi,0) - \eta\bar{g}_1(0,\eta) \\
\therefore \quad &\bar{g}(\xi,\eta) = \frac{\bar{\bar{f}}(\xi,\eta)}{\xi\eta} + \frac{\bar{g}(\xi,0)}{\eta} + \frac{\bar{g}(0,\eta)}{\xi}.
\end{aligned}$$

But $\bar{g}(\xi,0)=0$ and $\bar{g}(0,\eta)=0$, therefore

$$\bar{\bar{g}}(\xi,\eta)=\frac{\bar{\bar{f}}(\xi,\eta)}{\xi\eta}$$
$$\therefore\quad L_{xt}[g(x,t)]=\frac{\bar{\bar{f}}(\xi,\eta)}{\xi\eta}.$$

Hence,

$$L_{xt}\left[\int_0^x\int_0^t f(u,v)dudv\right]=\frac{\bar{\bar{f}}(\xi,\eta)}{\xi\eta}.$$

□

Theorem 3.8. *Let $f(x,t)$ be a periodic function of period K and T w.r.t. x and t respectively. Then the bicomplex double Laplace transform is given by*

$$L_{xt}[f(x,t)]=\frac{\int_0^K\int_0^T e^{-\xi x-\eta t}f(x,t)dxdt}{\left(1-e^{-K\xi}\right)\left(1-e^{-T\eta}\right)},\quad Re(\xi)>|Im_j(\xi)|\ \ and\ Re(\eta)>|Im_j(\eta)|\,.$$

Proof. Let $f(x,t)$ be a periodic function with period K w.r.t. x. Then for $\xi\in\mathbb{C}_2$ and $\mathrm{Re}(\xi)>|\mathrm{Im}_j(\xi)|$, see Agarwal et al. [20]

$$L_x[f(x,t)]=\frac{\int_0^K e^{-\xi x}f(x,t)dx}{1-e^{-K\xi}}=\bar{f}(\xi,t). \tag{10}$$

Similarly, for $\eta\in\mathbb{C}_2$ and $\mathrm{Re}(\eta)>|\mathrm{Im}_j(\eta)|$ taking the bicomplex Laplace transform of (10) w.r.t. t, we have

$$\begin{aligned}L_t[\bar{f}(\xi,t)]=\bar{\bar{f}}(\xi,\eta)&=\frac{\int_0^T e^{-\eta t}\bar{f}_1(\xi,t)dt}{1-e^{-T\eta}}\\&=\frac{1}{1-e^{-T\eta}}\int_0^T e^{-\eta t}\frac{\int_0^K e^{-\xi x}f(x,t)dx}{1-e^{-K\xi}}dt\\&=\frac{\int_0^K\int_0^T e^{-\xi x-\eta t}f(x,t)dxdt}{\left(1-e^{-K\xi}\right)\left(1-e^{-T\eta}\right)}.\end{aligned}$$

Thus,

$$L_{xt}[f(x,t)]=\frac{\int_0^K\int_0^T e^{-\xi x-\eta t}f(x,t)dxdt}{\left(1-e^{-K\xi}\right)\left(1-e^{-T\eta}\right)}.$$

□

4 Inversion

In this section, we derive the inversion theorem for bicomplex double Laplace transform.

Theorem 4.1. *Let $\bar{\bar{f}}(\xi,\eta)$ be the bicomplex double Laplace transform of bicomplex-valued function $f(x,t)$. Then*

$$f(x,t) = -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} e^{\xi x+\eta t}\bar{\bar{f}}(\xi,\eta)d\xi d\eta, \quad (\xi,\eta)\in\Omega$$

where Ω defined in (8) and Γ_1 and Γ_2 are Bromwich closed contours in bicomplex space.

Proof. Taking the inverse bicomplex Laplace transform [2] of $\bar{\bar{f}}(\xi,\eta)$ w.r.t. ξ, we have

$$L_\xi^{-1}[\bar{\bar{f}}(\xi,\eta)] = \bar{f}(x,\eta) = \frac{1}{2\pi i_1}\int_{\Gamma_2} e^{\xi x}\bar{\bar{f}}(\xi,\eta)d\xi. \tag{11}$$

Similarly, taking inverse bicomplex Laplace transform of (11) w.r.t. η, we have

$$\begin{aligned} L_\eta^{-1}[\bar{f}(x,\eta)] = f(x,t) &= \frac{1}{2\pi i_1}\int_{\Gamma_1} e^{\eta t}\bar{f}(x,\eta)d\eta \\ &= \frac{1}{(2\pi i_1)^2}\int_{\Gamma_1} e^{\eta t}\int_{\Gamma_2} e^{\xi x}\bar{\bar{f}}(\xi,\eta)d\xi d\eta. \end{aligned}$$

Hence,

$$f(x,t) = -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} e^{\xi x+\eta t}\bar{\bar{f}}(\xi,\eta)d\xi d\eta.$$

□

5 Applications

In this section, we discuss applications of bicomplex double Laplace transform in finding the solution of two-dimensional time-dependent bicomplex Schrödinger equation for free particle by two different approaches. In first approach, we find the solution of above equation by taking the bicomplex double Laplace transform under suitable initial and boundary conditions w.r.t. spaces variables x and y and in second approach, w.r.t. space variable x and time variable t.

Rochon and Tremblay [7] discussed the extension of time dependent time-dependent complex Schrödinger equation in bicomplex form. In [20], Agarwal et al. discussed the solution of one-dimensional time-dependent bicomplex Schrödinger equation for free particle using by bicomplex Laplace transform. In [1], Arnold discussed the solution of two-dimensional time-dependent complex Schrödinger equation using by Fourier-Laplace transform. In [14], Dehghan et al. discussed the numerical solution of two-dimensional time-dependent Schrödinger equation.

Here, we discuss the solution of two-dimensional time-dependent bicomplex Schrödinger equation for free particle. The one-dimensional time-dependent bicomplex Schrödinger equation [7] is defined as

$$i_1\hbar\partial_t\psi(x,t) + \frac{\hbar^2}{2m}\partial_x^2\psi(x,t) - V(x,t)\psi(x,t) = 0, \tag{12}$$

where

$$\psi : \mathbb{R}^2 \to \mathbb{C}_2 \text{ and } V : \mathbb{R}^2 \to \mathbb{R}.$$

We extend the above one-dimensional equation in two dimensions as

$$i_1\hbar\partial_t\psi(x,y,t) + \frac{\hbar^2}{2m}\left(\partial_x^2\psi(x,y,t) + \partial_y^2\psi(x,y,t)\right) - V(x,y,t)\psi(x,y,t) = 0, \tag{13}$$

where

$$\psi : \mathbb{R}^3 \to \mathbb{C}_2 \text{ and } V : \mathbb{R}^3 \to \mathbb{R}.$$

with initial and boundary conditions

$$\begin{aligned} &\psi(x,y,0) = h(x,y),\ \psi(0,y,t) = f_1(y,t),\ \psi(x,0,t) = g_1(x,t), \\ &\psi_x(0,y,t) = f_2(y,t),\ \psi_y(x,0,t) = g_2(x,t),\ x > 0,\ y > 0,\ t > 0. \end{aligned} \tag{14}$$

For free particle $V(x,y,t) = 0$, (13) becomes

$$i_1\hbar\partial_t\psi(x,y,t) + \frac{\hbar^2}{2m}\left(\partial_x^2\psi(x,y,t) + \partial_y^2\psi(x,y,t)\right) = 0. \tag{15}$$

(a) Taking bicomplex double Laplace transform of (15) w.r.t. x and y, we have

$$\begin{aligned} &\int_0^\infty\int_0^\infty e^{-\xi x-\eta y} i_1\hbar\partial_t\psi(x,y,t)dxdy + \int_0^\infty\int_0^\infty e^{-\xi x-\eta y}\frac{\hbar^2}{2m}\left(\partial_x^2\psi(x,y,t)\right. \\ &\left.+\ \partial_y^2\psi(x,y,t)\right) dxdy = 0 \\ \Rightarrow\ & i_1\hbar\frac{d}{dt}\bar{\bar{\psi}}(\xi,\eta,t) + \frac{\hbar^2}{2m}\Big(\left(\xi^2+\eta^2\right)\bar{\bar{\psi}}(\xi,\eta,t) - \xi\bar{\psi}(0,\eta,t) - \eta\bar{\psi}(\xi,0,t) \\ &-\bar{\psi}_x(0,\eta,t) - \bar{\psi}_y(\xi,0,t)\Big) = 0, \\ &\Big(\text{where } \bar{\bar{\psi}}(\xi,\eta,t) = L_{xy}\left[\psi(x,y,t)\right] \text{ is bicomplex double Laplace transform of } \psi(x,y,t)\Big). \end{aligned}$$

Applying the boundary conditions (14), we get

$$\begin{aligned} & i_1\hbar\frac{d\bar{\bar{\psi}}}{dt} + \frac{\hbar^2}{2m}\Big(\left(\xi^2+\eta^2\right)\bar{\bar{\psi}} - \xi\bar{f}_1(\eta,t) - \eta\bar{g}_1(\xi,t) - \bar{f}_2(\eta,t) - \bar{g}_2(\xi,t)\Big) = 0 \\ \Rightarrow\ & \frac{d\bar{\bar{\psi}}}{dt} - i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right) = -i_1\frac{\hbar}{2m}\left(\xi\bar{f}_1(\eta,t) + \eta\bar{g}_1(\xi,t) + \bar{f}_2(\eta,t) + \bar{g}_2(\xi,t)\right). \end{aligned}$$

Rearranging the terms and simplifying, we get

$$\begin{aligned} \bar{\bar{\psi}}(\xi,\eta,t) = &-i_1\frac{\hbar}{2m}\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\int\exp\left(-i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\left(\xi\bar{f}_1(\eta,t)\right. \\ &\left.+\eta\bar{g}_1(\xi,t) + \bar{f}_2(\eta,t) + \bar{g}_2(\xi,t)\right)dt + c\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right). \end{aligned} \tag{16}$$

Letting $\bar{\bar{\psi}}(\xi,\eta,0) = \bar{\bar{h}}(\xi,\eta)$ in (16) we have

$$\begin{aligned} c &= \bar{\bar{h}}(\xi,\eta) + i_1\frac{\hbar}{2m}\int\exp\left(-i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\left(\xi\bar{f}_1(\eta,t) + \eta\bar{g}_1(\xi,t) + \bar{f}_2(\eta,t) + \bar{g}_2(\xi,t)\right)dt\Big|_{t=0} \\ &= \bar{\bar{p}}(\xi,\eta) \qquad \text{(say)}. \end{aligned} \tag{17}$$

Therefore, (16) becomes

$$\bar{\bar{\psi}}(\xi,\eta,t) = -i_1\frac{\hbar}{2m}\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\int\exp\left(-i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\left(\xi\bar{f}_1(\eta,t)+\right.$$
$$\left.\eta\bar{g}_1(\xi,t)+\bar{f}_2(\eta,t)+\bar{g}_2(\xi,t)\right)dt+\bar{\bar{p}}(\xi,\eta)\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right). \tag{18}$$

Taking the inverse bicomplex Laplace transform of (18), we have

$$\psi(x,y,t) = -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2}e^{\xi x+\eta y}\left[\bar{\bar{p}}(\xi,\eta)\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)+A(\xi,\eta,t)\right]d\xi d\eta \tag{19}$$

where,

$$A(\xi,\eta,t) = -i_1\frac{\hbar}{2m}\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\int\exp\left(-i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)\left(\xi\bar{f}_1(\eta,t)\right.$$
$$\left.+\eta\bar{g}_1(\xi,t)+\bar{f}_2(\eta,t)+\bar{g}_2(\xi,t)\right)dt$$

and Γ_1 and Γ_2 are closed contours in bicomplex space w.r.t. ξ and η respectively. (19) is the solution of two-dimensional time-dependent bicomplex Schrödinger equation for free particle.

For illustration, let us consider the initial and boundary conditions for equation (15) as

$$\psi(x,y,0)=\sin\left(\frac{2\pi}{\lambda}x\right)\cos\left(\frac{2\pi}{\lambda}y\right),\ \psi_x(0,y,t)=\frac{2\pi}{\lambda}\exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}y\right),$$
$$\psi(0,y,t)=0,\ \psi(x,0,t)=0,\ \psi_y(x,0,t)=\frac{2\pi}{\lambda}\exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}x\right). \tag{20}$$

Then (19) becomes

$$\psi(x,y,t) = -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2}e^{\xi x+\eta y}\frac{2\pi}{\lambda}\frac{\eta\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)}{\left(\xi^2+\left(\frac{2\pi}{\lambda}\right)^2\right)\left(\eta^2+\left(\frac{2\pi}{\lambda}\right)^2\right)}d\xi d\eta$$
$$= -\frac{1}{2\pi\lambda}\int_{\Gamma_1}\frac{e^{\xi x}}{\left(\xi^2+\left(\frac{2\pi}{\lambda}\right)^2\right)}2\pi i_1\left(\lim_{\eta\to i_1\frac{2\pi}{\lambda}}e^{\eta y}\frac{\eta\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)}{\left(\eta+i_1\frac{2\pi}{\lambda}\right)}\right.$$
$$\left.+\lim_{\eta\to -i_1\frac{2\pi}{\lambda}}e^{\eta y}\frac{\eta\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2+\eta^2\right)t\right)}{\left(\eta-i_1\frac{2\pi}{\lambda}\right)}\right)d\xi$$
$$= -\frac{i_1}{\lambda}\int_{\Gamma_1}\frac{e^{\xi x}}{\left(\xi^2+\left(\frac{2\pi}{\lambda}\right)^2\right)}\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2-\frac{4\pi^2}{\lambda^2}\right)t\right)\cos\left(\frac{2\pi}{\lambda}y\right)d\xi$$
$$= -\frac{i_1}{\lambda}\cos\left(\frac{2\pi}{\lambda}y\right)2\pi i_1\left(\lim_{\xi\to i_1\frac{2\pi}{\lambda}}\frac{e^{\xi x}}{\left(\xi+i_1\frac{2\pi}{\lambda}\right)}\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2-\frac{4\pi^2}{\lambda^2}\right)t\right)\right.$$
$$\left.+\lim_{\xi\to -i_1\frac{2\pi}{\lambda}}\frac{e^{\xi x}}{\left(\xi-i_1\frac{2\pi}{\lambda}\right)}\exp\left(i_1\frac{\hbar}{2m}\left(\xi^2-\frac{4\pi^2}{\lambda^2}\right)t\right)\right)$$
$$= \exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}x\right)\cos\left(\frac{2\pi}{\lambda}y\right).$$

Therefore,

$$\psi(x,y,t)=\exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}x\right)\cos\left(\frac{2\pi}{\lambda}y\right). \tag{21}$$

Expression in (21) is the solution of two-dimensional time-dependent bicomplex Schrödinger equation (15) for initial and boundary conditions (20).

(b) Consider the two-dimensional time-dependent bicomplex Schrödinger equation (15) for free particle with the condition

$$\psi(x,y,t) \text{ is bounded as } |y|\to\infty \text{ and } x>0,\ t>0. \tag{22}$$

Taking the bicomplex double Laplace transform of (15) w.r.t. x and t, we get

$$\begin{aligned}
&\int_0^\infty\int_0^\infty e^{-\xi x-\eta t} i_1\hbar\partial_t\psi(x,y,t)dxdy+\int_0^\infty\int_0^\infty e^{-\xi x-\eta t}\frac{\hbar^2}{2m}\left(\partial_x^2\psi(x,y,t)\right.\\
&\quad\left.+\,\partial_y^2\psi(x,y,t)\right)dxdy=0\\
&\Rightarrow\ i_1\hbar\left(\eta\bar{\bar{\psi}}(\xi,y,\eta)-\bar{\psi}(\xi,y,0)\right)+\frac{\hbar^2}{2m}\left(\xi^2\bar{\bar{\psi}}(\xi,y,\eta)-\xi\bar{\psi}(0,y,\eta)-\bar{\psi}_x(0,y,\eta)\right)\\
&\quad+\frac{\hbar^2}{2m}\frac{d^2}{dy^2}\bar{\bar{\psi}}(\xi,y,\eta)=0\\
&\Rightarrow\ \frac{d^2\bar{\bar{\psi}}}{dy^2}+\left(\xi^2+i_1\frac{2m\eta}{\hbar}\right)\bar{\bar{\psi}}=i_1\frac{2m}{\hbar}\bar{\psi}(\xi,y,0)+\xi\bar{\psi}(0,y,\eta)+\bar{\psi}_x(0,y,\eta).
\end{aligned}$$

Rearranging the terms and solving, we get

$$\begin{aligned}
\bar{\bar{\psi}}(\xi,y,\eta)&=c_1\exp\left(y\sqrt{-\xi^2-i_1\frac{2m\eta}{\hbar}}\right)+c_2\exp\left(-y\sqrt{-\xi^2-i_1\frac{2m\eta}{\hbar}}\right)\\
&\quad+\frac{1}{\frac{d^2}{dy^2}+\left(\xi^2+i_1\frac{2m\eta}{\hbar}\right)}\left(i_1\frac{2m}{\hbar}\bar{\psi}(\xi,y,0)+\xi\bar{\psi}(0,y,\eta)+\bar{\psi}_x(0,y,\eta)\right),
\end{aligned} \tag{23}$$

$$\left[\text{where Re}\left(P_1:\sqrt{-\xi^2-i_1\frac{2m\eta}{\hbar}}\right)>0 \text{ and Re}\left(P_2:\sqrt{-\xi^2-i_1\frac{2m\eta}{\hbar}}\right)>0\right].$$

$\because\ \bar{\bar{\psi}}(\xi,y,\eta)$ is bounded as $|y|\to\infty\Rightarrow c_1=c_2=0$. Then (23) becomes

$$\bar{\bar{\psi}}(\xi,y,\eta)=\frac{1}{\frac{d^2}{dy^2}+\left(\xi^2+i_1\frac{2m\eta}{\hbar}\right)}\left(i_1\frac{2m}{\hbar}\bar{\psi}(\xi,y,0)+\xi\bar{\psi}(0,y,\eta)+\bar{\psi}_x(0,y,\eta)\right). \tag{24}$$

Taking the inverse bicomplex Laplace transform of (24), we get

$$\psi(x,y,t)=-\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2}e^{\xi x+\eta t}\bar{\bar{\psi}}(\xi,y,\eta)d\xi d\eta \tag{25}$$

where Γ_1 and Γ_2 are closed contours in bicomplex space w.r.t. ξ and η respectively. (25) is the solution of two-dimensional time-dependent bicomplex Schrödinger equation (15).

For illustration, let us consider the initial and boundary conditions for equation (15) as

$$\psi(x,y,0)=\sin\left(\frac{2\pi}{\lambda}x\right)\sin\left(\frac{2\pi}{\lambda}y\right),\ \psi(0,y,t)=0$$
$$\psi_x(0,y,t)=\frac{2\pi}{\lambda}\exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}y\right). \tag{26}$$

Then (25) becomes

$$\begin{aligned}\psi(x,y,t)&=-\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2}\frac{2\pi}{\lambda}\sin\left(\frac{2\pi}{\lambda}y\right)\frac{e^{\xi x+\eta t}}{\xi^2+i_1\frac{2m\eta}{\hbar}-\frac{4\pi^2}{\lambda^2}}\left(i_1\frac{2m}{\hbar}\frac{1}{\xi^2+\frac{4\pi^2}{\lambda^2}}+\frac{1}{\eta-i_1\frac{4\hbar\pi^2}{m\lambda^2}}\right)d\xi d\eta\\
&=-i_1\frac{m}{\pi\hbar\lambda}\sin\left(\frac{2\pi}{\lambda}y\right)\int_{\Gamma_1}\int_{\Gamma_2}\frac{e^{\xi x+\eta t}}{\left(\xi^2+i_1\frac{2m\eta}{\hbar}-\frac{4\pi^2}{\lambda^2}\right)\left(\xi^2+\frac{4\pi^2}{\lambda^2}\right)}d\xi d\eta\\
&\quad-\frac{1}{2\pi\lambda}\sin\left(\frac{2\pi}{\lambda}y\right)\int_{\Gamma_1}\int_{\Gamma_2}\frac{e^{\xi x+\eta t}}{\left(\xi^2+i_1\frac{2m\eta}{\hbar}-\frac{4\pi^2}{\lambda^2}\right)\left(\eta-i_1\frac{4\hbar\pi^2}{m\lambda^2}\right)}d\xi d\eta\\
&=I_1+I_2.\quad\text{(say)}\end{aligned} \tag{27}$$

Now,

$$\begin{aligned}I_1&=-i_1\frac{m}{\pi\hbar\lambda}\sin\left(\frac{2\pi}{\lambda}y\right)\int_{\Gamma_1}\int_{\Gamma_2}\frac{e^{\xi x+\eta t}}{\left(\xi^2+i_1\frac{2m\eta}{\hbar}-\frac{4\pi^2}{\lambda^2}\right)\left(\xi^2+\frac{4\pi^2}{\lambda^2}\right)}d\xi d\eta\\
&=-i_1\frac{m}{\pi\hbar\lambda}\sin\left(\frac{2\pi}{\lambda}y\right)\int_{\Gamma_1}2\pi i_1\frac{\hbar}{2mi_1}e^{\xi x}\lim_{\eta\to -i_1\left(\frac{4\pi^2}{\lambda^2}-\xi^2\right)\frac{\hbar}{2m}}\left(e^{\eta t}\frac{1}{\xi^2+\frac{4\pi^2}{\lambda^2}}\right)d\xi\\
&=-\frac{i_1}{\lambda}\sin\left(\frac{2\pi}{\lambda}y\right)\int_{\Gamma_1}\frac{e^{\xi x}}{\xi^2+\frac{4\pi^2}{\lambda^2}}\exp\left(-i_1\frac{\hbar}{2m}\left(\frac{4\pi^2}{\lambda^2}-\xi^2\right)t\right)d\xi\\
&=-\frac{i_1}{\lambda}\sin\left(\frac{2\pi}{\lambda}y\right)2\pi i_1\left[\lim_{\xi\to i_1\frac{2\pi}{\lambda}}\frac{e^{\xi x}}{\xi+i_1\frac{2\pi}{\lambda}}\exp\left(-i_1\frac{\hbar}{2m}\left(\frac{4\pi^2}{\lambda^2}-\xi^2\right)t\right)\right.\\
&\quad\left.+\lim_{\xi\to -i_1\frac{2\pi}{\lambda}}\frac{e^{\xi x}}{\xi-i_1\frac{2\pi}{\lambda}}\exp\left(-i_1\frac{\hbar}{2m}\left(\frac{4\pi^2}{\lambda^2}-\xi^2\right)t\right)\right]\\
&=\exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}x\right)\sin\left(\frac{2\pi}{\lambda}y\right).\end{aligned}$$

Therefore,

$$I_1=\exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}x\right)\sin\left(\frac{2\pi}{\lambda}y\right). \tag{28}$$

Similarly,

$$\begin{aligned} I_2 &= -\frac{1}{2\pi\lambda}\sin\left(\frac{2\pi}{\lambda}y\right)\int_{\Gamma_1}\int_{\Gamma_2}\frac{e^{\xi x+\eta t}}{\left(\xi^2+i_1\frac{2m\eta}{\hbar}-\frac{4\pi^2}{\lambda^2}\right)\left(\eta-i_1\frac{4\hbar\pi^2}{m\lambda^2}\right)}d\xi d\eta \\ &= -i_1\lambda\sin\left(\frac{2\pi}{\lambda}y\right)\int_{\Gamma_1}\frac{\left(\exp\left(i_1\frac{4\pi^2\hbar}{\lambda^2 m}t\right)-\exp\left(i_1\frac{\lambda^2\xi^2\hbar-4\pi^2\hbar}{2\lambda^2 m}t\right)\right)}{\lambda^2\xi^2-12\pi^2}d\xi \\ &= 0. \end{aligned} \tag{29}$$

Using (28) and (29) in (27), we get

$$\psi(x,y,t)=\exp\left(-i_1\frac{4\hbar\pi^2}{m\lambda^2}t\right)\sin\left(\frac{2\pi}{\lambda}x\right)\sin\left(\frac{2\pi}{\lambda}y\right). \tag{30}$$

Expression in (30) is the solution of two-dimensional time-dependent bicomplex Schrödinger equation (15) under the conditions given in (22) and (26).

6 Conclusion

In this paper, we derived the bicomplex double Laplace transform and its inverse. The applications have been illustrated to find the solution of two-dimensional time-dependent bicomplex Schrödinger equation by using two different approaches. Moreover, similar to work of Rochon and Tremblay [7], under some discrete symmetries two-dimensional time-dependent bicomplex Schrödinger equation can be decomposed into two standard two-dimensional Schrödinger equation. Therefore, solution of two standard two-dimensional Schrödinger equations can be obtained by separating the solution of two-dimensional time-dependent bicomplex Schrödinger equation.

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