Advances in Mathematical
Sciences and Applications
Vol. 28, No. 2 (2019), pp. 235–254

GAKKOTOSHO
TOKYO JAPAN

ON LOCAL STRONG SOLUTIONS TO THE CAUCHY PROBLEM OF TWO-DIMENSIONAL NONHOMOGENEOUS NAVIER-STOKES-KORTEWEG EQUATIONS WITH VACUUM

Huanyuan Li
Graduate School of Mathematical Sciences, University of Tokyo,
Komaba 3-8-1, Meguro-Ku, Tokyo, 153-8914, Japan
(E-mail: lihuanyuan1111@163.com)

Abstract. In this paper, we consider the Cauchy problem of nonhomogeneous incompressible Navier-Stokes-Korteweg equations on the two-dimensional space with vacuum as the far field density. We establish the local existence and uniqueness of strong solutions to the 2D Cauchy problem of nonhomogeneous incompressible Navier-Stokes-Korteweg equations provided the initial density decays not too slow at infinity.

Communicated by Yoshikazu Giga ; Received January 7, 2019
AMS Subject Classification :35D35, 35Q30, 76D45
Keywords: Navier-Stokes-Korteweg equations, 2D Cauchy problem, Vacuum, Strong solutions, Capillary fluids

1 Introduction and main result

The motion of a viscous incompressible capillary fluid in a two-dimensional space is governed by the nonhomogeneous incompressible Navier-Stokes-Korteweg equations which read as follows:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \triangle u + \nabla P + \kappa \mathrm{div}(\nabla \rho \otimes \nabla \rho) = 0, \\ \mathrm{div} u = 0, \end{cases} \tag{1.1}$$

where $t \geq 0$ is time, $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ is spatial coordinate. The unknown functions $\rho(x,t), u(x,t) = (u_1(x,t), u_2(x,t))$ and $P(x,t)$ represent the density, velocity field and pressure of the fluid, respectively. The constants $\kappa > 0$ and $\mu > 0$ stand for the capillary and viscosity coefficients of the fluid respectively.

Let $\Omega = \mathbb{R}^2$ and we consider the Cauchy problem for (1.1) with the far field behavior condition(in the weak sense):

$$(\rho, u) \to (0,0), \quad \text{as} \quad |x| \to \infty, \tag{1.2}$$

and initial data:

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \quad \text{in } \mathbb{R}^2. \tag{1.3}$$

The Navier-Stokes-Korteweg equations are widely studied by many mathematicians since of its physical importance and mathematical complexity, especially a great of efforts have been devoted to the mathematical theory for compressible capillary fluids, see the references [4, 7, 8] and therein. In particular, if there is no capillary effect, that is $\kappa \equiv 0$, the Navier-Stokes-Korteweg system reduces to the well-konown Navier-Stokes equations, which have been studied extensively, see J. Simon [14], Cho, Choe-Kim [2, 3] and Huang-Wang [6] for more details on the Navier-Stokes model. When the capillary coefficient $\kappa > 0$, the study of the Navier-Stokes-Korteweg becomes rather difficult than the Navier-Stokes model since of the appearance of capillary effect. For the nonhomogeneous incompressible Navier-Stokes-Korteweg equations (1.1) over a bounded smooth domain $\Omega \subset \mathbb{R}^3$, under the following compatibility condition on the initial data:

$$-\mathrm{div}(\mu(\rho_0)(\nabla u_0 + (\nabla u_0)^T)) + \nabla P_0 + \mathrm{div}(\kappa(\rho_0)\nabla \rho_0 \otimes \nabla \rho_0) = \rho_0^{\frac{1}{2}} g, \quad \mathrm{div} u_0 = 0, \quad \text{in } \Omega, \tag{1.4}$$

for some $(P_0, g) \leq H^1(\Omega) \times L^2(\Omega)$, Tan-Wang [15] and Wang [16] established the local strong solutions to the initial and boundary value problem when the capillary $\kappa(\rho)$ and viscosity $\mu(\rho)$ are positive constants and variable functions of the density, respectively. And very recently, the author [9] established a blow-up critrion for the strong solutions to the initial boundary value probelm of two-dimensional nonhomogeneous Navier-Stokes-Korteweg equations.

To my best knowledge, there is no any further results of establishing solutions to the Cauchy problem of the Navier-Stokes-Korteweg equations (1.1). Using the ideas of Chen-Tan-Wang [1] for 3D Cauchy problem of the nonhomogeneous MHD system, the local strong solutions of the 3D Cauchy problem of Navier-Stokes-Korteweg (1.1) can be established in a similar way. However, some difficulties will bring out when we apply these ideas to the 2D case, since the Sobolev inequality is critical. Recently, Li-Liang [10] established the local strong solutions to the 2D Cauchy problem of the compressible Navier-Stokes equations with vacuum as far field density by deriving some spatial weighted energy estimates. Motivated by their work, Liang [11] proved the local existence of strong solutions to the 2D Cauchy problem of the nonhomogeneous incompressible Navier-Stokes equations, that is (1.1) with $\kappa \equiv 0$. The purpose of this paper is to establish local strong solutions to the Cauchy problem (1.1)-(1.3) as an extension of Liang's work [11] to Navier-Stokes-Korteweg model. First we give the definition of strong solutions to the Cauchy problem (1.1)-(1.3) as follows.

Definition 1.1 (Strong solutions)**.** If all derivatives involved in (1.1) are regular distributions, and equations (1.1) hold almost everywhere in $\mathbb{R}^2 \times (0,T)$, then (ρ, u, P) is called a strong solution to (1.1).

Now we are ready to state the main result of this paper, and we would like to point out that, in this section, for $1 \le r \le \infty$ and $k \in \mathbb{N}$, we denote the standard Lesbegue and Sobolev spaces as follows:

$$L^r = L^r(\mathbb{R}^2), \quad W^{k,r} = W^{k,r}(\mathbb{R}^2), \quad H^k = W^{k,2}.$$

Theorem 1.2. *Let η_0 be a positive constant and*

$$\bar{x} := (e + |x|^2)^{\frac{1}{2}} \log^{1+\eta_0}(e + |x|^2). \tag{1.5}$$

For constants $q > 2$ and $a > 1$, assume that the initial data (ρ_0, u_0) satisfies

$$0 \le \bar{x}^a \rho_0 \in L^1 \cap H^2 \cap W^{2,q}, \sqrt{\rho} u_0 \in L^2, \nabla u_0 \in L^2 \text{ and } \operatorname{div} u_0 = 0. \tag{1.6}$$

Then there exist a small time T_0 and a unique strong solution (ρ, u, P) to the Cauchy problem (1.1)-(1.3) on $\mathbb{R}^2 \times (0, T_0]$ satisfying

$$\begin{cases} 0 \le \rho \in C([0,T_0]; L^1 \cap H^2 \cap W^{2,q}), \\ \bar{x}^a \rho \in L^\infty(0,T_0; L^1 \cap H^2 \cap W^{2,q}), \\ \sqrt{\rho} u, \nabla u, \bar{x}^{-1} u, \sqrt{t}\sqrt{\rho} u_t, \sqrt{t}\nabla P, \sqrt{t}\nabla^2 u \in L^\infty(0,T_0;L^2), \\ \nabla u \in L^2(0,T_0;H^1) \cap L^{\frac{q+1}{q}}(0,T_0;W^{1,q}), \\ \nabla P \in L^2(0,T_0;L^2) \cap L^{\frac{q+1}{q}}(0,T_0;L^q), \\ \sqrt{t}\nabla u \in L^2(0,T_0;W^{1,q}), \\ \sqrt{\rho} u_t, \sqrt{t}\nabla u_t, \sqrt{t}\bar{x}^{-1} u_t \in L^2(\mathbb{R}^2 \times (0,T_0)), \end{cases} \tag{1.7}$$

and

$$\inf_{0 \le t \le T_0} \int_{B_N} \rho(x,t) dx \ge \frac{1}{4} \int_{\mathbb{R}^2} \rho_0(x) dx, \tag{1.8}$$

for some constant $N > 0$ and $B_N = \{x \in \mathbb{R}^2 | |x| < N\}$.

We now make some comments on the key ingredients of the analysis of this paper. It should be pointed out that, for the whole two-dimensional space, it seems difficult to bound the $L^p(\mathbb{R}^2)$-norm of u just in terms of $\|\sqrt{\rho}u\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla u\|_{L^2(\mathbb{R}^2)}$. Furthermore, the appearance of capillary term will bring out some new difficulties. In order to overcome these difficulties, we will make goof use of some key ideas due to [10, 11] where they deal with the compressible and nonhomogeneous Navier-Stokes equations, repectively. On the other hand, motivated by [10], it is enough to bound the $L^p(\mathbb{R}^2)$-norm of the momentum ρu instead just of u. More precisely, using a Hardy-type inequality which is originally due to Lions [12], together with some careful analysis on the spatial weighted estimate of the density, we can obtain the desired estimates on the $L^p(\mathbb{R}^2)$-norm of the momentum ρu. Next, we then construct approximate solutions to (1.1) with density strictly positive, consider an initial and boundary value problem in any bounded ball B_R with radius R. Finally, combining all key points mentioned before with the similar arguments as in [3, 10, 11], we derive the desired bounds on the gradient of velocity and spatial weighted density, which are independent of both the radius of the balls B_R and the lower bound of the initial density.

Remark 1. After this work was completed, we found a recent work of Y. Liu, W. Wang and S. N. Zheng [13] closely related to ours. They also prove the local well-posedness of strong solution with vacuum to the Cauchy problem of two-dimensional nonhomogeneous incompressible Navier-Stokes-Korteweg equations. However, as is discussed in detail, see Remark 2, they need a stronger assumption on the initial data than ours, that is, except for the same regularity condition (1.6), the following compatibility condition on (ρ_0, u_0) is also necessary.

$$-\mu\triangle u_0 + \nabla P_0 + \kappa \mathrm{div}(\nabla\rho_0 \otimes \nabla\rho_0) = \rho_0^{\frac{1}{2}} g, \tag{1.9}$$

for some $(P_0, g) \in H^1 \times L^2(\mathbb{R}^2)$.

The rest of the paper is arranged as follows. In Section 2, we collect some elementary facts and inequalities which will be needed in the later analysis. In Section 3, we will derive some a priori estimates which are used to obtain the local existence and uniqueness of strong solutions. The proof of main result Theorem 1.2 will be given in Section 4.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be frequently used later. First of all, if the initial density is strictly away from vacuum, the following local existence theorem on bounded balls can be shown by similar arguments as in [3, 15].

Lemma 2.1. *For $R > 0$ and $B_R = \{x \in \mathbb{R}^2 \,|\, |x| < R\}$, assume that (ρ_0, u_0) satisfies*

$$\rho_0 \in H^3(B_R), u_0 \in H^2(B_R), \quad \inf_{x\in B_R} \rho_0(x) > 0, \quad \mathrm{div} u_0 = 0. \tag{2.1}$$

Then there exists a small time $T_R > 0$ such that the equations (1.1) *with the following initial and boundary conditions*

$$\begin{aligned} &(\rho, u)(x, t = 0) = (\rho_0, u_0), && x \in B_R, \\ &u(x, t) = 0, && x \in \partial B_R, t > 0, \end{aligned} \tag{2.2}$$

has a unique strong solution (ρ, u, P) *on* $B_R \times (0, T_R]$ *satisfying*

$$\rho \in C([0, T_R]; H^3), \quad (\nabla u, P) \in C([0, T_R]; H^2) \cap L^2(0, T_R; H^3), \tag{2.3}$$

where we denote $H^k = H^k(B_R)$ *for positive integer* k.

Next, for $\Omega \subset \mathbb{R}^2$, the following weighted L^m-bounds for elements of the Hilbert space $\tilde{D}^{1,2}(\Omega) := \{v \in H^1_{loc}(\Omega) | \nabla v \in L^2(\Omega)\}$ can be found in [12].

Lemma 2.2. *For* $m \in [2, \infty)$ *and* $\theta \in (1 + \frac{m}{2}, \infty)$, *there exists a positive constant* C *independent of* Ω *such that for either* $\Omega = \mathbb{R}^2$ *or* $\Omega = B_R$ *with* $R \geq 1$ *and for any* $v \in \tilde{D}^{1,2}(\Omega)$,

$$\left(\int_\Omega \frac{|v|^m}{e + |x|^2} (\log(e + |x|^2))^{-\theta} dx \right)^{\frac{1}{m}} \leq C\|v\|_{L^2(B_1)} + C\|\nabla v\|_{L^2(\Omega)}. \tag{2.4}$$

A useful consequence of Lemma 2.2 is the following crucial weighted bounds for elements of $\tilde{D}^{1,2}(\Omega)$, which has been proved in [10].

Lemma 2.3. *Let* Ω *be as in Lemma 2.2, and* $\bar{x}$ *and* η_0 *be as in* (1.5). *Assume that* $\rho \in L^1 \cap L^\infty(\Omega)$ *is a non-negative function such that*

$$\int_{B_{N_1}} \rho dx \geq M_1, \quad \|\rho\|_{L^1 \cap L^\infty(\Omega)} \leq M_2, \tag{2.5}$$

for positive constants M_1, M_2 *and* $N_1 \geq 1$ *with* $B_{N_1} \subset \Omega$. *Then for* $\epsilon > 0$ *and* $\eta > 0$, *there is a positive constant* C *depending only on* $\epsilon, \eta, M_1, M_2, N_1$, *and* η_0 *such that every* $v \in \tilde{D}^{1,2}(\Omega)$ *satisfies*

$$\|v\bar{x}^{-\eta}\|_{L^{(2+\epsilon)/\tilde{\eta}}(\Omega)} \leq C\|\rho^{1/2} v\|_{L^2(\Omega)} + C\|\nabla v\|_{L^2(\Omega)} \tag{2.6}$$

with $\tilde{\eta} = \min\{1, \eta\}$.

3 A priori estimates

Throughout this section, we omit the integration domain B_R with $R > 0$ below for notations simplicity. For $1 \leq r \leq \infty$ and $k \in \mathbb{N}$, the Lesbegue and Sobolev spaces on some ball B_R are defined in a standard way:

$$L^r = L^r(B_R), \quad W^{k,r} = W^{k,r}(B_R), \quad H^k = W^{k,2}.$$

Moreover, for $R > 4N_0 \geq 4$, assume that (ρ_0, u_0) satisifes, in addition to (2.1), that

$$\frac{1}{2} \leq \int_{B_{N_0}} \rho_0(x) dx \leq \int_{B_R} \rho_0(x) dx \leq \frac{3}{2}. \tag{3.1}$$

Lemma 2.1 thus yield that there exists some $T_R > 0$ such that the initial and boundary value problem (1.1) and (2.2) has a unique strong solution (ρ, u, P) on $B_R \times [0, T_R]$ satisfying (2.3).

Let $\bar{x}, \eta_0, a$ and q be as in Theorem 1.2, the main goal of this section is to derive the following key a priori estimate on $\psi(t)$ defined by

$$\psi(t) := 1 + \|\rho^{1/2} u\|_{L^2} + \|\nabla u\|_{L^2} + \|\bar{x}^a \rho\|_{L^1 \cap H^2 \cap W^{2,q}}. \tag{3.2}$$

Proposition 1. *Assume that* (ρ_0, u_0) *satisfies* (2.1) *and* (3.1). *Let* (ρ, u, P) *be the solution to the initial and boundary value problem* (1.1) *and* (2.2) *on* $B_R \times (0, T_R]$ *obtained by Lemma 2.1. Then there exist positive constants* T_0 *and* M *both depending only on* $\mu, \kappa, q, a, \eta_0, N_0$ *and* E_0 *such that*

$$\begin{aligned}
&\sup_{0\le t\le T_0}\Big(\psi(t)+\sqrt{t}\|\sqrt{\rho}u_t\|_{L^2}+\sqrt{t}\|\nabla^2 u\|_{L^2}+\sqrt{t}\|\nabla P\|_{L^2}\Big)\\
&+\int_0^{T_0}\Big(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}}+\|\nabla P\|_{L^q}^{\frac{q+1}{q}}\Big)dt\\
&+\int_0^{T_0}\big(t\|\nabla^2 u\|_{L^q}^2+t\|\nabla P\|_{L^q}^2+t\|\nabla u_t\|_{L^2}^2\big)dt\le M,
\end{aligned}\tag{3.3}$$

where

$$E_0 := \|\sqrt{\rho_0}u_0\|_{L^2}+\|\nabla u_0\|_{L^2}+\|\bar{x}^a\rho_0\|_{L^1\cap H^2\cap W^{2,q}}.$$

The proof of Proposition 1 is composed of some lemmas. First, we give the following standard energy estimate for (ρ, u, P) and the estimate on the L^p-norm of the density.

Lemma 3.1. *Under the conditions of Proposition 1, let* (ρ, u, P) *be a solution to the initial and boundary problem* (1.1) *and* (2.2). *Then for any* $t > 0$,

$$\sup_{0\le s\le t}(\|\rho\|_{L^1\cap L^\infty}+\|\sqrt{\rho}u\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2)+\int_0^t\int|\nabla u|^2dxds\le C. \tag{3.4}$$

Proof. First, it is easy to deduce from $(1.1)_1$ and $\mathrm{div} u = 0$ that

$$\sup_{0\le s\le t}\|\rho\|_{L^1\cap L^\infty}\le C. \tag{3.5}$$

Then applying the standard energy estimate to (1.1) gives

$$\sup_{0\le s\le t}(\|\sqrt{\rho}u\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2)+\int_0^t\int|\nabla u|^2dxds\le C. \tag{3.6}$$

This together with (3.5) yields (3.4) and completes the proof of Lemma 3.1. □

Next, we will derive the key estimate on the $\|\nabla u\|_{L^2(0,t;L^2)}$.

Lemma 3.2. *Under the conditions of Proposition 1, let* (ρ, u, P) *be a solution to the initial and boundary problem* (1.1) *and* (2.2). *Then there exists a* $T_1 = T_1(N_0, E_0) > 0$ *such that for all* $t \in (0, T_1]$,

$$\sup_{0\le s\le t}(\|\bar{x}^a\rho\|_{L^1}+\|\nabla u\|_{L^2}^2)+\int_0^t\|\sqrt{\rho}u_s\|_{L^2}^2ds\le C+C\int_0^t\psi^\alpha(s)ds. \tag{3.7}$$

Proof. First, for $N > 1$, define a family of functions $\phi_N \in C_0^\infty(B_N)$ satisfying

$$0\le\phi_N\le 1,\quad \phi_N(x)=1,\ \text{ if } |x|\le N/2,\quad |\nabla^k\phi_N|\le CN^{-k},\quad k\in\mathbb{N}, \tag{3.8}$$

it follows from $(1.1)_1$ and (3.4) that

$$\begin{aligned} \frac{d}{dt}\int \rho\phi_{2N_0}dx =& \int \rho u\cdot\nabla\phi_{2N_0}dx \\ &\geq -CN_0^{-1}\left(\int \rho dx\right)^{1/2}\left(\int \rho|u|^2dx\right)^{1/2} \\ &\geq -\tilde{C}(E_0,N_0), \end{aligned} \tag{3.9}$$

where we used the fact $\int \rho dx = \int \rho_0 dx$ in the last inequality.

Integrating (3.9) over the time interval $(0,t)$ and using (3.1) gives

$$\inf_{0\leq t\leq T_1}\int_{B_{2N_0}}\rho dx \geq \inf_{0\leq t\leq T_1}\int \rho\phi_{2N_0}dx \geq \int \rho_0\phi_{2N_0}dx - \tilde{C}T_1 \geq \frac{1}{4}. \tag{3.10}$$

where we take $T_1 := \min\{1, (4\tilde{C})^{-1}\}$. From now on, we will always assume that $t \leq T_1$. The combination of (3.10), (3.4) and (2.6) yields that for $\epsilon > 0$ and $\eta > 0$, every $v \in \tilde{D}^{1,2}(B_R)$ satisfies

$$\|v\bar{x}^{-\eta}\|_{L^{(2+\epsilon)/\tilde{\eta}}} \leq C\|\rho^{1/2}v\|_{L^2} + C\|\nabla v\|_{L^2} \tag{3.11}$$

with $\tilde{\eta} = \min\{1,\eta\}$.

Next, multiplying $(1.1)_1$ by $\bar{x}^a$ and integrating by parts imply that

$$\begin{aligned} \frac{d}{dt}\int \bar{x}^a\rho dx &\leq C\int \rho|u|\bar{x}^{a-1}\log^{1+\eta_0}(e+|x|^2)dx \\ &\leq C\|\rho\bar{x}^{a-1+\frac{8}{8+a}}\|_{L^{\frac{8+a}{7+a}}}\|u\bar{x}^{-\frac{4}{8+a}}\|_{L^{8+a}} \\ &\leq C\|\rho\|_{L^\infty}^{\frac{1}{8+a}}\|\rho\bar{x}^a\|_{L^1}^{\frac{7+a}{8+a}}(\|\rho^{1/2}u\|_{L^2} + \|\nabla u\|_{L^2}) \\ &\leq C(1+\|\rho\bar{x}^a\|_{L^1})(1+\|\nabla u\|_{L^2}) \end{aligned} \tag{3.12}$$

due to (3.4) and (3.11). This combined with Gronwall inequality and (3.4) lead to

$$\sup_{0\leq s\leq t}\|\rho\bar{x}^a\|_{L^1} \leq C\exp\left\{C\int_0^t(1+\|\nabla u\|_{L^2}^2)ds\right\} \leq C. \tag{3.13}$$

Now we are prepared to estimate the first order derivatives of the velocity. Multiplying $(1.1)_2$ by u_t and integrating by parts, one has

$$\begin{aligned} &\int \rho|u_t|^2dx + \mu\frac{d}{dt}\int |\nabla u|^2dx \\ =& -\int (\rho u\cdot\nabla u)\cdot u_t dx + \kappa\int \nabla\rho\otimes\nabla\rho : \nabla u_t dx \\ =&\kappa\frac{d}{dt}\int \nabla\rho\otimes\nabla\rho : \nabla u dx + 2\int \kappa\nabla(u\cdot\nabla\rho)\otimes\nabla\rho : \nabla u dx \\ &-\int (\rho u\cdot\nabla u)\cdot u_t dx. \end{aligned} \tag{3.14}$$

First, it follows from (3.4), (3.11), and (3.13) that for any $\epsilon > 0$ and $\eta > 0$,

$$\begin{aligned}
\|\rho^\eta v\|_{L^{(2+\epsilon)/\tilde\eta}} &\leq C\|\rho^\eta \bar{x}^{\frac{3\tilde\eta a}{4(2+\epsilon)}}\|_{L^{4(2+\epsilon)/3\tilde\eta}}\|v\bar{x}^{-\frac{3\tilde\eta a}{4(2+\epsilon)}}\|_{L^{4(2+\epsilon)/\tilde\eta}} \\
&\leq C\left(\int \rho^{\frac{4(2+\epsilon)\eta}{3\tilde\eta}-1}\rho\bar{x}^a dx\right)^{\frac{3\tilde\eta}{4(2+\epsilon)}}\|v\bar{x}^{-\frac{3\tilde\eta a}{4(2+\epsilon)}}\|_{L^{4(2+\epsilon)/\tilde\eta}} \\
&\leq C\|\rho\|_{L^\infty}^{\frac{4(2+\epsilon)\eta-3\tilde\eta}{4(2+\epsilon)}}\|\rho\bar{x}^a\|_{L^1}^{\frac{3\tilde\eta}{4(2+\epsilon)}}(\|\rho^{1/2}v\|_{L^2}+\|\nabla v\|_{L^2}) \\
&\leq C\|\rho^{\frac12}v\|_{L^2}+C\|\nabla v\|_{L^2},
\end{aligned} \tag{3.15}$$

where $\tilde\eta = \min\{1, \eta\}$ and $v \in \tilde{D}^{1,2}(B_R)$. In particular, this together with (3.4) and (3.11) derives

$$\|\rho^\eta u\|_{L^{(2+\epsilon)/\tilde\eta}} + \|u\bar{x}^{-\eta}\|_{L^{(2+\epsilon)/\tilde\eta}} \leq C(1+\|\nabla u\|_{L^2}). \tag{3.16}$$

Then we estimate the terms in the right hand side of (3.14). First, the combination of the Cauchy-Schwarz and Gagliardo-Nirenberg inequalities yields

$$\begin{aligned}
\int(\rho u\cdot\nabla u)\cdot u_t dx &\leq \frac14\int\rho|u_t|^2dx + C\int\rho|u|^2|\nabla u|^2dx \\
&\leq \frac14\int\rho|u_t|^2dx + C\|\rho^{\frac12}u\|_{L^8}^2\|\nabla u\|_{L^{\frac83}}^2 \\
&\leq \frac14\int\rho|u_t|^2dx + C\|\rho^{\frac12}u\|_{L^8}^2\|\nabla u\|_{L^2}^{\frac32}\|\nabla u\|_{H^1}^{\frac12} \\
&\leq \frac14\int\rho|u_t|^2dx + C\psi^\alpha + \epsilon\|\nabla^2 u\|_{L^2}^2,
\end{aligned} \tag{3.17}$$

where (and in what follows) we use $\alpha > 1$ to denote a generic constant, which may different from line to line.

For the second term on the right hand side of (3.14), integration by parts together with Gagliardo-Nirenberg inequality deduces that

$$\begin{aligned}
&\int\kappa\nabla(u\cdot\nabla\rho)\otimes\nabla\rho:\nabla u dx \\
\leq& C\int|\nabla\rho|^2|\nabla u|^2dx + C\int|\nabla^2\rho||\nabla\rho||u||\nabla u|dx \\
\leq& C\|\nabla\rho\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + C\|\nabla\rho\|_{L^\infty}\|\bar{x}^a\nabla^2\rho\|_{L^q}\|\bar{x}^{-a}u\|_{L^{q^*}}\|\nabla u\|_{L^2} \\
\leq& C\psi^\alpha.
\end{aligned} \tag{3.18}$$

Here $\frac1q + \frac1{q^*} = 1.$ and $q^* > 2$. Inserting (3.17) and (3.18) into (3.14) gives

$$\begin{aligned}
&\frac12\int\rho|u_t|^2dx + \frac{d}{dt}\int(\mu|\nabla u|^2 - \kappa\nabla\rho\otimes\nabla\rho:\nabla u)dx \\
\leq&\epsilon\|\nabla^2u\|_{L^2}^2 + C\psi^\alpha.
\end{aligned} \tag{3.19}$$

Differentiating the continuity equation $(1.1)_1$ with respect to $x_i, i = 1, 2$, we get

$$(\partial_{x_i}\rho)_t + u\cdot\nabla(\partial_{x_i}\rho) + \partial_{x_i}u\cdot\nabla\rho = 0, \tag{3.20}$$

multiplying (3.20) by $4|\partial_{x_i}\rho|^2\partial_{x_i}\rho$, integration by parts over the domain B_R yields

$$\begin{aligned}\frac{d}{dt}\|\partial_{x_i}\rho\|_{L^4}^4 &\le C\int |\nabla u||\nabla\rho||\partial_{x_i}\rho|^3dx\\ &\le C\|\nabla u\|_{L^2}\|\nabla\rho\|_{L^2}\|\partial_{x_i}\rho\|_{L^\infty}^3\\ &\le C\psi^\alpha(t).\end{aligned}\tag{3.21}$$

Integrating (3.21) over the time interval $(0,t)$ lead to

$$\sup_{0\le s\le t}\|\nabla\rho\|_{L^4}^4 \le C + C\int \psi^\alpha ds.\tag{3.22}$$

On the other hand, since (ρ, u, P) satisfies the following Stokes system,

$$-\mu\triangle u + \nabla P = -\rho u_t - \rho u\cdot\nabla u - \kappa\mathrm{div}(\nabla\rho\otimes\nabla\rho),$$

applying the standard L^p-estimate yields

$$\begin{aligned}&\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2\\ &\le C\|\rho u_t\|_{L^2}^2 + C\|\rho u\cdot\nabla u\|_{L^2}^2 + C\||\nabla\rho||\nabla^2\rho|\|_{L^2}^2\\ &\le C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{H^1} + C\|\nabla\rho\|_{L^\infty}^2\|\nabla^2\rho\|_{L^2}^2\\ &\le C\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{1}{2}\|\nabla^2 u\|_{L^2}^2 + C\psi^\alpha,\end{aligned}\tag{3.23}$$

which implies that,

$$\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 \le C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\psi^\alpha.\tag{3.24}$$

Substituting (3.24) into (3.19) and choosing ϵ suitably small, one gets

$$\begin{aligned}\|\nabla u\|_{L^2}^2 + \int_0^t\int \rho|u_t|^2dxds &\le C + C\|\nabla\rho\|_{L^4}^4 + C\int_0^t \psi^\alpha ds\\ &\le C + C\int_0^t\psi^\alpha ds,\end{aligned}\tag{3.25}$$

where in the second inequality we have used (3.22). Thus we complete the proof of Lemma 3.2.

□

Remark 2. Here we want to give some comments on the proof of Lemma 3.2. As the same with Y. Liu et. al. [13], this lemma is used to derive the L^∞-estimate on $\|\nabla u\|_{L^2}$. The different part is the treatment of the capillary term $\int \mathrm{div}(\nabla\rho\otimes\nabla\rho)\cdot u_t dx$. In the paper of Y. Liu et. al. [13], they remark from the divergence free property of the velocity that $\int \mathrm{div}(\nabla\rho\otimes\nabla\rho)\cdot u_t dx = \int \triangle\rho\nabla\rho\cdot u_t dx$. Then combining the Hardy-type inequality and Hölder inequality, they complete the estimate in terms of $\psi(t)$ and $\|\nabla u_t\|_{L^2}$. In order to close the estimate, they have to derive the estimate of $\sup\|\sqrt{\rho}u_t\|_{L^2}$ in the next step, therefore the initial value of $\sup \|\sqrt{\rho}u_t\|_{L^2}$ will be involved, to bound this term, the compatibility condition (1.9) is necessary. My way is different, we observe that

$$\int \nabla\rho\otimes\nabla\rho : \nabla u_t dx = \frac{d}{dt}\int \nabla\rho\otimes\nabla\rho : \nabla u dx + 2\int \nabla(u\cdot\nabla\rho)\otimes\nabla\rho : \nabla u dx, \tag{3.26}$$

and to bound the first term of the right-hand side in (3.26), we also derive a new estimate for the density, see (3.22).

Lemma 3.3. *Let (ρ, u, P) and T_1 be as in Lemma 3.2. Then there exists a positive constant $\alpha > 1$, such that for all $t \in (0, T_1]$,*

$$\sup_{0\le s\le t}(s\|\sqrt{\rho}u_s\|_{L^2}^2) + \int_0^t (\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + s\|\nabla^2 u\|_{L^q}^2 + s\|\nabla u_s\|_{L^2}^2)dt \le C\exp\left(C\int_0^t \psi^\alpha ds\right). \tag{3.27}$$

Proof. Differentiating the momentum equations $(1.1)_2$ with respect to t, using the continuity equation $(1.1)_1$, we derive

$$\begin{aligned} &\rho u_{tt} + \rho u\cdot\nabla u_t - \mu\triangle u_t + \nabla P_t \\ =& -\rho_t(u_t + u\cdot\nabla u) - \rho u_t\cdot\nabla u - \kappa\triangle\rho\nabla(u\cdot\nabla\rho) - \kappa\triangle(u\cdot\nabla\rho)\nabla\rho. \end{aligned} \tag{3.28}$$

Multiplying (3.28) by u_t, we get after integration by parts over B_R that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2 dx + \mu\int |\nabla u_t|^2 dx \\ &\le C\int \rho|u|(|\nabla u_t| + |\nabla u|^2 + |u||\nabla^2 u|)dx + C\int \rho|u|^2|\nabla u||\nabla u_t|dx \\ &\quad + C\int \rho|u_t|^2|\nabla u|dx + \int |\triangle\rho||\nabla(u\cdot\nabla\rho)||u_t|dx + |\int \triangle(u\cdot\nabla\rho)(\nabla\rho\cdot u_t)dx| \\ &:= \sum_{i=1}^5 J_i. \end{aligned} \tag{3.29}$$

Now let us estimate the terms on the right hand side of (3.29) one by one. First

$$\begin{aligned} J_1 &\le C\|\rho^{\frac12}u\|_{L^6}\|\rho^{\frac12}u_t\|_{L^2}^{\frac12}\|\rho^{\frac12}u_t\|_{L^6}^{\frac12}(\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^4}^2) \\ &\quad + C\|\rho^{\frac14}u\|_{L^{12}}^2\|\rho^{\frac12}u_t\|_{L^2}^{\frac12}\|\rho^{\frac12}u_t\|_{L^6}^{\frac12}\|\nabla^2 u\|_{L^2} \\ &\le C(1 + \|\nabla u\|_{L^2}^2)\|\rho^{\frac12}u_t\|_{L^2}^{\frac12}(\|\rho^{\frac12}u_t\|_{L^2} + \|\nabla u_t\|_{L^2})^{\frac12} \\ &\quad \times(\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2} + \|\nabla^2 u\|_{L^2}) \\ &\le \frac{\mu}{6}\|\nabla u_t\|_{L^2}^2 + C\psi^\alpha\|\sqrt{\rho}u_t\|_{L^2}^2 + C\psi^\alpha + C(1 + \|\nabla u\|_{L^2}^2)\|\nabla^2 u\|_{L^2}^2. \end{aligned} \tag{3.30}$$

Then, Hölder inequality combined with (3.16) leads to

$$\begin{aligned} J_2 + J_3 &\le C\|\rho^{\frac12}u\|_{L^8}^2\|\nabla u\|_{L^4}\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2}\|\rho^{\frac12}u_t\|_{L^6}^{\frac32}\|\rho^{\frac12}u_t\|_{L^2}^{\frac12} \\ &\le \frac{\mu}{6}\|\nabla u_t\|_{L^2}^2 + C\psi^\alpha\|\sqrt{\rho}u_t\|_{L^2}^2 + C(\psi^\alpha + \|\nabla^2 u\|_{L^2}^2). \end{aligned} \tag{3.31}$$

Next,

$$
\begin{aligned}
J_4 &\le C\int |\nabla^2\rho||u||\triangle\rho||u_t|dx + C\int |\nabla\rho||\nabla u||\triangle\rho||u_t|dx \\
&\le C\|\bar{x}^a\nabla^2\rho\|_{L^q}\|\bar{x}^a\triangle\rho\|_{L^q}\|\bar{x}^{-a}u\|_{L^{q^*}}\|\bar{x}^{-a}u_t\|_{L^{q^*}} \\
&\quad + C\|\nabla\rho\|_{L^\infty}\|\bar{x}^a\triangle\rho\|_{L^q}\|\nabla u\|_{L^2}\|\bar{x}^{-a}u_t\|_{L^{q^*}} \\
&\le C\psi^\alpha(1+\|\nabla u\|_{L^2})(\|\rho^{\frac{1}{2}}u_t\|_{L^2}+\|\nabla u_t\|_{L^2}) + C\psi^\alpha(\|\rho^{\frac{1}{2}}u_t\|_{L^2}+\|\nabla u_t\|_{L^2}) \\
&\le \frac{\mu}{6}\|\nabla u_t\|_{L^2}^2 + C\psi^\alpha\|\sqrt{\rho}u_t\|_{L^2}^2.
\end{aligned}
\tag{3.32}
$$

Finally,

$$
\begin{aligned}
J_5 &= |\int \triangle(u\cdot\nabla\rho)(\nabla\rho\cdot u_t)dx| \\
&\le C\int |\nabla^2\rho|^2|u||u_t|dx + \int |\nabla\rho||\nabla^2\rho||\nabla u||u_t|dx \\
&\quad + \int |\nabla^2\rho||\nabla\rho||u||\nabla u_t|dx + \int |\nabla\rho|^2|\nabla u||\nabla u_t|dx \\
&\le \frac{\mu}{6}\|\nabla u_t\|_{L^2}^2 + C\psi^\alpha\|\sqrt{\rho}u_t\|_{L^2}^2 + C\psi^\alpha.
\end{aligned}
\tag{3.33}
$$

Inserting the estimates (3.30)-(3.33) into (3.29), we get

$$
\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2dx + \mu\int |\nabla u_t|^2dx \le C\psi^\alpha(1+\|\rho^{\frac{1}{2}}u_t\|_{L^2}^2). \tag{3.34}
$$

Multiplying (3.34) by t, using Gronwall inequality, we get

$$
\sup_{0\le s\le t}(s\|\sqrt{\rho}u_s\|_{L^2}^2) + \int_0^t (s\|\nabla u_s\|_{L^2}^2)dt \le C\exp\left(C\int_0^t \psi^\alpha ds\right). \tag{3.35}
$$

Finally, we show that

$$
\begin{aligned}
&\int_0^t \left(\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + \|\nabla P\|_{L^q}^{\frac{q+1}{q}} + s\|\nabla^2 u\|_{L^q}^2 + s\|\nabla P\|_{L^q}^2\right)ds \\
&\le C\exp\left\{C\int_0^t \psi^\alpha(s)ds\right\}.
\end{aligned}
\tag{3.36}
$$

Applying the Stokes estimate and Gagliardo-Nirenberg inequality, we have

$$
\begin{aligned}
\|\nabla^2 u\|_{L^q} + \|\nabla P\|_{L^q} &\le C(\|\rho u_t\|_{L^q} + \|\rho u\cdot\nabla u\|_{L^q} + \||\nabla\rho||\nabla^2\rho|\|_{L^q}) \\
&\le C(\|\rho u_t\|_{L^q} + \|\rho u\|_{L^{2q}}\|\nabla u\|_{L^{2q}} + \|\nabla\rho\|_{L^\infty}\|\nabla^2\rho\|_{L^q}) \\
&\le C\|\rho u_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\|\rho u_t\|_{L^{q^2}}^{\frac{q^2-2q}{q^2-2}} + C\psi^\alpha(1+\|\nabla^2 u\|_{L^2}^{\frac{q-1}{q}}) \\
&\le C(\|\sqrt{\rho}u_t\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\|\nabla u_t\|_{L^2}^{\frac{q^2-2q}{q^2-2}} + \|\sqrt{\rho}u_t\|_{L^2}) + C\psi^\alpha(1+\|\nabla^2 u\|_{L^2}^{\frac{q-1}{q}}),
\end{aligned}
\tag{3.37}
$$

which together with (3.7) and (3.35) implies that

$$
\begin{aligned}
&\int_0^t (\|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + \|\nabla P\|_{L^q}^{\frac{q+1}{q}})ds \\
&\le C\int_0^t s^{-\frac{q+1}{2q}}(s\|\sqrt{\rho}u_t\|_{L^2}^2)^{\frac{q^2-1}{q(q^2-2)}}(s\|\nabla u_t\|_{L^2}^2)^{\frac{(q-2)(q+1)}{2(q^2-2)}}ds \\
&\quad + \int_0^t \|\sqrt{\rho}u_t\|_{L^2}^{\frac{q+1}{q}}ds + C\int_0^t \psi^\alpha(1+\|\nabla^2 u\|_{L^2}^{\frac{q^2-1}{q^2}})ds \\
&\le C\sup_{0\le s\le t}(s\|\sqrt{\rho}u_t\|_{L^2}^2)^{\frac{q^2-1}{q(q^2-2)}}\int_0^t s^{-\frac{q+1}{2q}}(s\|\nabla u_t\|_{L^2}^2)^{\frac{(q-2)(q+1)}{2(q^2-2)}}ds \qquad (3.38)\\
&\quad + C\int_0^t (\psi^\alpha + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2)ds \\
&\le C\exp\left\{C\int_0^t \psi^\alpha(s)ds\right\}\left(1+\int_0^t (s^{-\frac{q^3+q^2-2q-2}{q^3+q^2-2q}} + s\|\nabla u_t\|_{L^2}^2)ds\right) \\
&\le C\exp\left\{C\int_0^t \psi^\alpha(s)ds\right\}.
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t (s\|\nabla^2 u\|_{L^q}^2 + s\|\nabla P\|_{L^q}^2)ds \\
&\le C\int_0^t s\|\sqrt{\rho}u_t\|_{L^2}^2 ds + C\int_0^t (s\|\sqrt{\rho}u_t\|_{L^2}^2)^{\frac{2(q-1)}{q^2-2}}(s\|\nabla u_t\|_{L^2}^2)^{\frac{q^2-2q}{q^2-2}}ds \\
&\quad + C\int_0^t s\psi^\alpha(1+\|\nabla^2 u\|_{L^2}^{\frac{2(q-1)}{q}})ds \qquad (3.39)\\
&\le C\int_0^t s\|\sqrt{\rho}u_t\|_{L^2}^2 ds + C\int_0^t s\|\nabla u_t\|_{L^2}^2 ds + C\int_0^t (\psi^\alpha + s\|\nabla^2 u\|_{L^2}^2)ds \\
&\le C\exp\left\{C\int_0^t \psi^\alpha(s)ds\right\}.
\end{aligned}
$$

Therefore we complete the proof of Lemma 3.3. □

Lemma 3.4. *Let (ρ, u, P) and T_1 be as in Lemma 3.3. Then there exists a positive constant $\alpha > 1$ such that for all $t \in (0, T_1]$,*

$$
\sup_{0\le s\le t}\|\bar{x}^a\rho\|_{L^1\cap H^2\cap W^{2,q}} \le \exp\left\{C\exp\left\{C\int_0^t \psi^\alpha ds\right\}\right\}. \qquad (3.40)
$$

Proof. First, it follows from Sobolev inequality and that for $\delta \in (0,1)$,

$$
\begin{aligned}
\|u\bar{x}^{-\delta}\|_{L^\infty} &\le C(\delta)\left(\|u\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}} + \|\nabla(u\bar{x}^{-\delta})\|_{L^3}\right) \\
&\le C(\delta)\left(\|u\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}} + \|\nabla u\|_{L^3} + \|u\bar{x}^{-\delta}\|_{L^{\frac{4}{\delta}}}\|\bar{x}^{-1}\nabla\bar{x}\|_{L^{\frac{12}{4-3\delta}}}\right) \qquad (3.41)\\
&\le C(\delta)(\psi^\alpha + \|\nabla^2 u\|_{L^2}).
\end{aligned}
$$

Multiplying the continuity equation $(1.1)_1$ by $\bar{x}^a$, after some simple calculation, we get

$$\partial_t(\bar{x}^a\rho) + u\cdot\nabla(\bar{x}^a\rho) - a\bar{x}^a\rho u\cdot\nabla\log\bar{x} = 0. \tag{3.42}$$

To obtain the estimate of first order spatial derivatives of $\bar{x}^a\rho$, we differentiate (3.42) with respect to $x_i, i = 1, 2$:

$$\begin{aligned}\partial_t\partial_{x_i}(\bar{x}^a\rho) &+ u\cdot\nabla\partial_{x_i}(\bar{x}^a\rho) + \partial_{x_i}u\cdot\nabla(\bar{x}^a\rho)\\ &- a\partial_{x_i}(\bar{x}^a\rho)u\cdot\nabla\log\bar{x} - a\bar{x}^a\rho\partial_{x_i}(u\cdot\nabla\log\bar{x}) = 0.\end{aligned} \tag{3.43}$$

Multiplying (3.43) by $r|\partial_i(\bar{x}^a\rho)|^{r-2}\partial_i(\bar{x}^a\rho)$ for $r \in [2, q]$, and integrating the resulting equality over B_R, we get

$$\begin{aligned}\frac{d}{dt}\|\nabla(\bar{x}^a\rho)\|_{L^r} \le\ & C(1 + \|\nabla u\|_{L^\infty} + \|u\cdot\nabla\log\bar{x}\|_{L^\infty})\|\nabla(\bar{x}^a\rho)\|_{L^r}\\ &+ C\|\bar{x}^a\rho\|_{L^\infty}\|\nabla(u\nabla\log\bar{x})\|_{L^r}.\end{aligned} \tag{3.44}$$

To obtain the second order spatial derivatives of $\bar{x}^a\rho$, differentiate the equation (3.43) with respect to $x_j, j = 1, 2$, after some calculation, one has

$$\begin{aligned}&\partial_t\partial_{x_i}\partial_{x_j}(\bar{x}^a\rho) + u\cdot\nabla\partial_{x_i}\partial_{x_j}(\bar{x}^a\rho) + \partial_j u\cdot\nabla(\partial_{x_i}(\bar{x}^a\rho)) + \partial_i u\cdot\nabla(\partial_{x_j}(\bar{x}^a\rho))\\ &+ \partial_{x_i}\partial_{x_j}u\cdot\nabla(\bar{x}^a\rho) - a\partial_{x_i}\partial_{x_j}(\bar{x}^a\rho)u\cdot\nabla\log\bar{x} - a\partial_{x_i}(\bar{x}^a\rho)\partial_{x_j}(u\cdot\nabla\log\bar{x})\\ &- a\partial_{x_j}(\bar{x}^a\rho)\partial_{x_i}(u\cdot\nabla\log\bar{x}) - a(\bar{x}^a\rho)\partial_{x_i}\partial_{x_j}(u\cdot\nabla\log\bar{x}) = 0,\end{aligned} \tag{3.45}$$

multiplying (3.45) by $r|\partial_i\partial_j(\bar{x}^a\rho)|^{r-2}\partial_i\partial_j(\bar{x}^a\rho)$ for $r \in [2, q]$, and integrating the resulting equality over B_R, and using $(1.1)_1$, we derive

$$\begin{aligned}\frac{d}{dt}\|\nabla^2(\bar{x}^a\rho)\|_{L^r} \le\ & C(1 + \|\nabla u\|_{L^\infty} + \|u\cdot\nabla\log\bar{x}\|_{L^\infty})\|\nabla^2(\bar{x}^a\rho)\|_{L^r}\\ &+ C\|\bar{x}^a\rho\|_{L^\infty}\|\nabla^2(u\nabla\log\bar{x})\|_{L^r}\\ &+ C\|\nabla(\bar{x}^a\rho)\|_{L^\infty}(\|\nabla^2 u\|_{L^r} + \|\nabla(u\nabla\log\bar{x})\|_{L^r}),\end{aligned} \tag{3.46}$$

combining it with (3.44), and summing up for $i, j = 1, 2$, leads to

$$\begin{aligned}\frac{d}{dt}\|\nabla(\bar{x}^a\rho)\|_{W^{1,r}} \le\ & C(1 + \|\nabla u\|_{L^\infty} + \|u\cdot\nabla\log\bar{x}\|_{L^\infty})\|\nabla(\bar{x}^a\rho)\|_{W^{1,r}}\\ &+ C\|\bar{x}^a\rho\|_{L^\infty}(\|\nabla(u\nabla\log\bar{x})\|_{L^r} + \|\nabla^2(u\nabla\log\bar{x})\|_{L^r})\\ &+ C\|\nabla(\bar{x}^a\rho)\|_{L^\infty}(\|\nabla^2 u\|_{L^r} + \|\nabla(u\nabla\log\bar{x})\|_{L^r})\\ \le\ & C(\psi^\alpha + \|\nabla^2 u\|_{L^2\cap L^q})(1 + \|\nabla(\bar{x}^a\rho)\|_{W^{1,r}} + \|\nabla(\bar{x}^a\rho)\|_{W^{1,q}})\end{aligned} \tag{3.47}$$

Using (3.7), (3.36), (3.13), (3.44), (3.47), and Gronwall inequality, one thus get (3.40) , therefore we complete the proof of Lemma 3.4. □

Now, we are in a position to give a proof of Proposition 1, which is a direct consequence of Lemmas 3.1-3.4.

Proof of Proposition 1. It follows from (3.4), (3.7), and (3.40) that

$$\psi(t) \le \exp\left\{C\exp\left\{C\int_0^t \psi^\alpha ds\right\}\right\}.$$

Standard arguments yield that for $M := e^{Ce}$ and $T_0 := \min\{T_1, (CM^\alpha)^{-1}\}$,

$$\sup_{0\le t\le T_0} \psi(t) \le M.$$

This combines with (3.24) and (3.27) gives

$$\sup_{0\le t\le T_0} (t\|\nabla^2 u\|_{L^2}^2 + t\|\nabla P\|_{L^2}^2) \le C(M),$$

which together with (3.7), (3.27), (3.40) gives (3.3). Therefore the proof of Proposition 1 is completed. □

4 Local existence and uniqueness of strong solutions

This section is devoting to prove the main result Theorem 1.2 with the aid of the a priori estimates obtained in Section 3.

Let (ρ_0, u_0) be as in Theorem 1.2. Without loss of generality, the initial density ρ_0 is assumed to satisfy

$$\int_{\mathbb{R}^2} \rho_0 dx = 1,$$

which implies that there exists a positive constant N_0 such that

$$\int_{B_{N_0}} \rho_0 dx \ge \frac{3}{4}\int_{\mathbb{R}^2} \rho_0 dx = \frac{3}{4}. \tag{4.1}$$

We construct $\rho_0^R = \hat\rho_0^R + R^{-1}e^{-|x|^2}$, where $0 \le \hat\rho_0^R \in C_0^\infty(\mathbb{R}^2)$ satisfies

$$\begin{cases} \int_{B_{N_0}} \hat\rho_0^R dx \ge \frac{1}{2}, \\ \bar{x}^a \hat\rho_0^R \to \bar{x}^a \rho_0 \qquad \text{in } \ L^1(\mathbb{R}^2) \cap H^2(\mathbb{R}^2) \cap W^{2,q}(\mathbb{R}^2), \text{as } \ R \to \infty. \end{cases} \tag{4.2}$$

Since $\nabla u_0 \in L^2(\mathbb{R}^2)$, choosing $v_i^R \in C_0^\infty(B_R)(i = 1, 2)$ such that

$$\lim_{R\to\infty} \|v_i^R - \partial_i u_0\|_{L^2(\mathbb{R}^2)} = 0, \quad i = 1, 2. \tag{4.3}$$

We consider the unique smooth solution u_0^R of the following elliptic problem:

$$\begin{cases} -\triangle u_0^R + \rho_0^R u_0^R + \nabla P_0^R = \sqrt{\rho_0^R} h^R - \partial_i v_i^R, & \text{in } \ B_R, \\ \text{div} u_0^R = 0, & \text{in } \ B_R, \\ u_0^R = 0, & \text{on } \ \partial B_R, \end{cases} \tag{4.4}$$

where $h^R = (\sqrt{\rho_0}u_0) * j_{1/R}$ with j_δ being the standard mollifying kernel with width δ.

Extending u_0^R to $\mathbb{R}^2$ by defining 0 outside B_R and denoting it by $\tilde{u}_0^R$, we claim that

$$\lim_{R\to\infty}\left(\|\nabla(\tilde{u}_0^R - u_0)\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\rho_0^R}\tilde{u}_0^R - \sqrt{\rho_0}u_0\|_{L^2(\mathbb{R}^2)}\right) = 0. \tag{4.5}$$

In fact, it is easy to find that $\tilde{u}_0^R$ is also a solution of (4.4) in $\mathbb{R}^2$. Multiplying (4.4) by $\tilde{u}_0^R$ and integrating the resulting equation over $\mathbb{R}^2$ lead to

$$\begin{aligned}
&\int_{\mathbb{R}^2}\rho_0^R|\tilde{u}_0^R|^2dx + \int_{\mathbb{R}^2}|\nabla\tilde{u}_0^R|^2dx\\
&\le \|\sqrt{\rho_0^R}\tilde{u}_0^R\|_{L^2(B_R)}\|h^R\|_{L^2(B_R)} + C\|v_i^R\|_{L^2(B_R)}\|\partial_i\tilde{u}_0^R\|_{L^2(B_R)}\\
&\le \frac{1}{2}\|\nabla\tilde{u}_0^R\|^2_{L^2(B_R)} + \frac{1}{2}\int_{\mathbb{R}^2}\rho_0^R|\tilde{u}_0^R|^2dx + C\|h^R\|^2_{L^2(B_R)} + C\|v_i^R\|^2_{L^2(B_R)},
\end{aligned} \tag{4.6}$$

which implies

$$\int_{\mathbb{R}^2}\rho_0^R|\tilde{u}_0^R|^2dx + \int_{\mathbb{R}^2}|\nabla\tilde{u}_0^R|^2dx \le C \tag{4.7}$$

for some C independent of R. This together with (4.2) yields that there exist a subsequence $R_j\to\infty$ and a function $\tilde{u}_0 \in \{\tilde{u}_0 \in H^1_{loc}(\mathbb{R}^2) | \sqrt{\rho_0}\tilde{u}_0 \in L^2(\mathbb{R}^2), \nabla\tilde{u}_0 \in L^2(\mathbb{R}^2)\}$ such that

$$\begin{cases}\sqrt{\rho_0^{R_j}}\tilde{u}_0^{R_j} \rightharpoonup \sqrt{\rho_0}\tilde{u}_0 \quad \text{weakly in} \quad L^2(\mathbb{R}^2),\\ \nabla\tilde{u}_0^{R_j} \rightharpoonup \nabla\tilde{u}_0 \quad \text{weakly in} \quad L^2(\mathbb{R}^2).\end{cases} \tag{4.8}$$

Next we will show

$$\tilde{u}_0 = u_0. \tag{4.9}$$

Indeed, multiplying (3.12) by a test function $\pi \in C_0^\infty(\mathbb{R}^2)$ with $\text{div}\pi = 0$, it holds that

$$\int_{\mathbb{R}^2}\partial_i(\tilde{u}_0^{R_j} - u_0)\cdot\partial_i\pi dx + \int_{\mathbb{R}^2}\sqrt{\rho_0^{R_j}}(\sqrt{\rho_0^{R_j}}\tilde{u}_0^{R_j} - h^{R_j})\cdot\pi dx = 0. \tag{4.10}$$

Let $R_j\to\infty$, it follows from (4.2), (4.3) and (4.8) that

$$\int_{\mathbb{R}^2}\partial_i(\tilde{u}_0 - u_0)\cdot\partial_i\pi dx + \int_{\mathbb{R}^2}\rho_0(\tilde{u}_0 - u_0)\cdot\pi dx = 0, \tag{4.11}$$

which implies (4.9).

Furthermore, multiplying (4.4) by $\tilde{u}_0^R$ and integrating the resulting equation over $\mathbb{R}^2$, by the same arguments as (4.11), we have

$$\lim_{R_j\to\infty}\int_{\mathbb{R}^2}(\rho_0^{R_j}|\tilde{u}_0^{R_j}|^2 + |\nabla\tilde{u}_0^{R_j}|^2)dx = \int_{\mathbb{R}^2}(\rho_0|u_0|^2 + |\nabla u_0|^2)dx,$$

which combined with (4.8) leads to

$$\lim_{R_j\to\infty}\int_{\mathbb{R}^2}|\nabla\tilde{u}_0^{R_j}|^2dx = \int_{\mathbb{R}^2}|\nabla\tilde{u}_0|^2dx, \quad \lim_{R_j\to\infty}\int_{\mathbb{R}^2}\rho_0^{R_j}|\tilde{u}_0^{R_j}|^2dx = \int_{\mathbb{R}^2}\rho_0|\tilde{u}_0|^2dx.$$

This, along with (4.9) and (4.8), gives (4.5).

Hence, by virtue of Lemma 2.1, the initial and boundary value problem (1.1) and (2.2) with the initial data (ρ_0^R, u_0^R) has a classical solution (ρ^R, u^R, P^R) on $B_R \times [0, T_R]$. Moreover, Proposition 1 shows that there exists a T_0 independent of R such that holds for (ρ^R, u^R, P^R).

For simplicity, in what follows, we denote

$$L^p = L^p(\mathbb{R}^2), \quad W^{k,p} = W^{k,p}(\mathbb{R}^2).$$

Extending (ρ^R, u^R, P^R) by zero on $\mathbb{R}^2/B_R$ and denoting it by

$$(\tilde{\rho}^R = \phi_R \rho^R, \tilde{u}^R, \tilde{P}^R)$$

with ϕ_R satisfying (3.8). First, (3.3) leads to

$$\sup_{0\le t\le T_0} \left(\|\sqrt{\tilde{\rho}^R}\tilde{u}^R\|_{L^2} + \|\nabla \tilde{u}^R\|_{L^2}\right) \le \sup_{0\le t\le T_0} \left(\|\sqrt{\rho^R}u^R\|_{L^2(B_R)} + \|\nabla u^R\|_{L^2(B_R)}\right) \le C, \tag{4.12}$$

and

$$\sup_{0\le t\le T_0} \|\bar{x}^a \tilde{\rho}^R\|_{L^1\cap L^\infty} \le C. \tag{4.13}$$

Similarly, it follows from (3.3) that for $q > 2$,

$$\begin{aligned} &\sup_{0\le t\le T_0} t^{\frac{1}{2}} \left(\|\sqrt{\tilde{\rho}^R}\tilde{u}_t^R\|_{L^2} + \|\nabla^2 \tilde{u}^R\|_{L^2}\right) \\ &+ \int_0^{T_0} \left(\|\sqrt{\tilde{\rho}^R}\tilde{u}_t^R\|_{L^2}^2 + \|\nabla^2 \tilde{u}^R\|_{L^2}^2 + \|\nabla^2 \tilde{u}^R\|_{L^q}^{\frac{q+1}{q}}\right) dt \\ &+ \int \left(t\|\nabla^2 \tilde{u}^R\|_{L^q}^2 + t\|\nabla \tilde{u}_t^R\|_{L^2}^2\right) dt \le C. \end{aligned} \tag{4.14}$$

Next, for $p \in [2, q]$, we obtain from (3.3) and (3.40) that

$$\begin{aligned} \sup_{0\le t\le T_0} \|\nabla^2(\bar{x}^a \tilde{\rho}^R)\|_{L^p} &\le C \sup_{0\le t\le T_0} (\|\nabla^2(\bar{x}^a \rho^R)\|_{L^p(B_R)} \\ &\quad + R^{-1}\|\nabla(\bar{x}^a \rho^R)\|_{L^p(B_R)} + R^{-2}\|\bar{x}^a \rho^R\|_{L^p(B_R)}) \\ &\le C \sup_{0\le t\le T_0} \|\bar{x}^a \rho^R\|_{H^2(B_R)\cap W^{2,p}(B_R)}) \le C, \end{aligned} \tag{4.15}$$

which together with (3.41) and (3.3) yields

$$\begin{aligned} \int_0^{T_0} \|\partial_t(\bar{x}\tilde{\rho}^R)\|_{L^p}^2 dt &\le C \int_0^{T_0} \|\bar{x}|u^R||\nabla \rho^R|\|_{L^p(B_R)}^2 dt \\ &\le C \int_0^{T_0} \|\bar{x}^{1-a} u^R\|_{L^\infty}^2 \|\bar{x}^a \nabla \rho^R\|_{L^p(B_R)}^2 dt \\ &\le C. \end{aligned} \tag{4.16}$$

By virtue of the same arguments as those of (3.27) and (3.36) , one gets

$$\sup_{0\le t\le T_0} t^{\frac{1}{2}} \|\nabla \tilde{P}^R\|_{L^2} + \int_0^{T_0} (\|\nabla \tilde{P}^R\|_{L^2}^2 + \|\nabla \tilde{P}^R\|_{L^q}^{\frac{q+1}{q}}) dt \le C. \tag{4.17}$$

With the estimates (4.13)-(4.17) at hand, we find that the sequence $(\tilde{\rho}^R, \tilde{u}^R, \tilde{P}^R)$ converges, up to the extraction of subsequences, to some limit (ρ, u, P) in the weak sence, that is, as $R \to \infty$, we have

$$\bar{x}\tilde{\rho}^R \to \bar{x}\rho, \text{in } C^1(\overline{B_N} \times [0, T_0]), \text{for any } N > 0, \tag{4.18}$$

$$\bar{x}^a\tilde{\rho}^R \rightharpoonup \bar{x}^a\rho, \text{weakly * in } L^\infty(0, T_0; H^2 \cap W^{2,q}), \tag{4.19}$$

$$\sqrt{\tilde{\rho}^R}\tilde{u}^R \rightharpoonup \sqrt{\rho}u, \nabla\tilde{u}^R \rightharpoonup \nabla u, \text{weakly * in } L^\infty(0, T_0; L^2) \tag{4.20}$$

$$\nabla^2\tilde{u}^R \rightharpoonup \nabla^2 u, \nabla\tilde{P}^R \rightharpoonup \nabla P, \text{weakly in } L^{\frac{q+1}{q}}(0, T_0; L^q) \cap L^2(\mathbb{R}^2 \times (0, T_0)), \tag{4.21}$$

$$\sqrt{t}\nabla^2\tilde{u}^R \rightharpoonup \sqrt{t}\nabla^2 u, \text{weak in } L^2(0, T_0; L^q), \text{weak * in } L^\infty(0, T_0; L^2), \tag{4.22}$$

$$\sqrt{t}\sqrt{\tilde{\rho}}\tilde{u}_t^R \rightharpoonup \sqrt{t}\sqrt{\rho}u_t, \sqrt{t}\nabla\tilde{P}^R \rightharpoonup \sqrt{t}\nabla P, \text{weak * in } L^\infty(0, T_0; L^2), \tag{4.23}$$

$$\sqrt{t}\nabla^2\tilde{u}_t^R \rightharpoonup \sqrt{t}\nabla^2 u_t, \text{weak * in } L^2(\mathbb{R}^2 \times (0, T_0)), \tag{4.24}$$

with

$$\bar{x}^a\rho \in L^\infty(0, T_0; L^1), \quad \inf_{0\le t\le T_0} \int_{B_{2N_0}} \rho(x,t)dx \ge \frac{1}{4}. \tag{4.25}$$

Then letteing $R \to \infty$, standard arguments together with (4.18)-(4.25) show that (ρ, u, P) is a strong solution of on $\mathbb{R}^2 \times (0, T_0)$ satisfying (1.7) and (1.8). Indeed, the existence of a pressure P follows immediately from $(1.1)_1$ $(1.1)_3$ and by a classical consideration. The proof of the existence part of Theorem 1.2 is finished.

The final work is only to prove the uniqueness of the strong solution satisfying (1.7) and (1.8). Let (ρ, u, P) and $(\bar{\rho}, \bar{u}, \bar{P})$ be two strong solutions satisfying (1.7) and (1.8) with the same initial data, and denote

$$\Theta := \rho - \bar{\rho}, U := u - \bar{u}.$$

First, subtracting the mass equation satisfied by (ρ, u, P) and $(\bar{\rho}, \bar{u}, \bar{P})$ gives

$$\Theta_t + \bar{u}\cdot\nabla\Theta + U\cdot\nabla\rho = 0. \tag{4.26}$$

Multiplying (4.26) by $2\Theta\bar{x}^{2r}$ for $r \in (1, \tilde{a})$ with $\tilde{a} = \min\{2, a\}$, and integrating by parts yield

$$\begin{aligned}
&\frac{d}{dt}\int |\Theta\bar{x}^r|^2 dx \\
&\le C\|\bar{u}\bar{x}^{-\frac{1}{2}}\|_{L^\infty}\|\Theta\bar{x}^r\|_{L^2} + C\|\Theta\bar{x}^r\|_{L^2}\|U\bar{x}^{-(\tilde{a}-r)}\|_{L^{\frac{2q}{(q-2)(\tilde{a}-r)}}}\|\bar{x}^{\tilde{a}}\nabla\rho\|_{L^{\frac{2q}{q-(q-2)(\tilde{a}-r)}}} \\
&\le C(1 + \|\nabla\bar{u}\|_{W^{1,q}}\|\Theta\bar{x}^r\|_{L^2}^2 + C\|\Theta\bar{x}^r\|_{L^2}(\|\nabla U\|_{L^2} + \|\sqrt{\rho}U\|_{L^2})
\end{aligned} \tag{4.27}$$

due to Sobolev inequality, (1.8), (3.16), (3.41). This combined with Gronswall inequality shows that for all $0 \le t \le T_0$,

$$\|\Theta\bar{x}^r\|_{L^2} \le C\int_0^t (\|\nabla U\|_{L^2} + \|\sqrt{\rho}U\|_{L^2})ds. \tag{4.28}$$

Next, taking the gradient in (4.26), multiplying the resulting equation by $\nabla\Theta$, and integrating over the $\mathbb{R}^2$, we get

$$\frac{1}{2}\frac{d}{dt}\int|\nabla\Theta|^2dx+\int(\nabla\Theta\cdot\nabla\bar{u})\cdot\nabla\Theta dx+\int(\nabla\rho\cdot\nabla U)\cdot\nabla\Theta dx+\int(\nabla^2\rho\cdot U)\cdot\nabla\Theta dx=0. \tag{4.29}$$

Observe that

$$-\int\triangle\Theta\nabla\rho\cdot Udx=\int\nabla\Theta\cdot(\nabla\rho\cdot U)dx=\int\nabla\Theta\cdot(\nabla^2\rho\cdot U)dx+\int\nabla\Theta\cdot(\nabla\rho\cdot\nabla U)dx. \tag{4.30}$$

Next, subtracting the momentum equation satisfied by (ρ,u,P) and $(\bar{\rho},\bar{u},\bar{P})$ leads to

$$\rho U_t+\rho u\cdot\nabla U-\mu\triangle U=-\rho U\cdot\nabla\bar{u}-\Theta(\bar{u}_t+\bar{u}\cdot\nabla\bar{u})-\nabla(P-\bar{P})+\kappa\triangle\Theta\nabla\rho+\kappa\triangle\bar{\rho}\nabla\Theta. \tag{4.31}$$

Multiplying by U, integration by parts and combine with (4.29) yield

$$\begin{aligned}&\frac{d}{dt}\int(\frac{1}{2}\rho|U|^2+\frac{\kappa}{2}|\nabla\Theta|^2)dx+\int\frac{\mu}{2}|\nabla U|^2dx\\&=\int-\rho U\cdot\nabla\bar{u}\cdot U-\Theta(\bar{u}_t+\bar{u}\cdot\nabla\bar{u})\cdot U-\kappa\triangle\bar{\rho}\nabla\Theta\cdot U-\kappa(\nabla\Theta\cdot\bar{u})\cdot\nabla\Theta dx\\&\le C\|\nabla\bar{u}\|_{L^\infty}\int(\rho|U|^2+|\nabla\Theta|^2)dx+C\int|\Theta||U|(|\bar{u}_t|+|\bar{u}||\nabla\bar{u}|)dx\\&\quad+C\int|\triangle\bar{\rho}||\nabla\Theta||U|dx.\end{aligned} \tag{4.32}$$

To finish the proof, we estimate the last two terms on the right hand side of (4.32). First,

$$\begin{aligned}\int|\Theta||U|(|\bar{u}_t|+|\bar{u}||\nabla\bar{u}|)dx&\le C\|\Theta\bar{x}^r\|_{L^2}\|U\bar{x}^{-r/2}\|_{L^4}(\|\bar{u}_t\bar{x}^{-r/2}\|_{L^4}+\|\nabla\bar{u}\|_{L^\infty}\|\bar{u}\bar{x}^{-r/2}\|_{L^4})\\&\le C(\epsilon)(\|\sqrt{\bar{\rho}}\bar{u}_t\|_{L^2}^2+\|\nabla\bar{u}_t\|_{L^2}^2+\|\nabla\bar{u}\|_{L^\infty}^2)\|\Theta\bar{x}^r\|_{L^2}^2\\&\quad+\epsilon(\|\sqrt{\rho}U\|_{L^2}^2+\|\nabla U\|_{L^2}^2)\\&\le C(\epsilon)(1+t\|\nabla\bar{u}_t\|_{L^2}^2+t\|\nabla^2\bar{u}\|_{L^q}^2)\int_0^t(\|\nabla U\|_{L^2}^2+\|\sqrt{\rho}U\|_{L^2}^2)ds\\&\quad+\epsilon(\|\sqrt{\rho}U\|_{L^2}^2+\|\nabla U\|_{L^2}^2),\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}\int|\triangle\bar{\rho}||\nabla\Theta||U|dx&\le C\|\bar{x}^r\triangle\rho\|_{L^q}\|U\bar{x}^{-r/2}\|_{L^{q^*}}\|\nabla\Theta\|_{L^2}\\&\le C(\|\sqrt{\rho}U\|_{L^2}^2+\|\nabla U\|_{L^2}^2)\|\bar{x}^r\triangle\rho\|_{L^q}\|\nabla\Theta\|_{L^2}\\&\le\epsilon(\|\sqrt{\rho}U\|_{L^2}^2+\|\nabla U\|_{L^2}^2)+C(\epsilon)\|\bar{x}^r\triangle\rho\|_{L^q}^2\|\nabla\Theta\|_{L^2}^2.\end{aligned} \tag{4.34}$$

Denoting

$$G(t):=\|\sqrt{\rho}U\|_{L^2}^2+\int_0^t(\|\nabla U\|_{L^2}^2+\|\sqrt{\rho}U\|_{L^2}^2)ds,$$

then substituting the above into (4.32) and choosing ϵ suitably small lead to

$$G'(t) \leq C(1 + \|\bar{x}^r \triangle \rho\|_{L^q}^2 + \|\nabla \bar{u}\|_{L^\infty} + t\|\nabla \bar{u}_t\|_{L^2}^2 + t\|\nabla^2 u\|_{L^q}^2)G(t),$$

which together with Gronwall inequality and (1.7) implies that $G(t) = 0$. Hence, $U(x,t) = 0$ for almost everywhere $(x,t) \in \mathbb{R}^2 \times (0,T)$. Finally, one can deduce from (4.28) that $\Theta = 0$ for almost everywhere $(x,t) \in \mathbb{R}^2 \times (0,T)$. The proof of Theorem 1.2 is completed.

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