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EXACT CONTROLLABILITY OF A STRONGLY NONLINEAR WAVE EQUATION WITH BOUNDARY CONTROLS

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Abstract. The exact controllability of a strongly nonlinear wave equation in bounded domains of $R^n, n = 2, 3$ with a boundary control is established. Techniques of the theory of variational inequalities are used.

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1 Introduction

The purpose of the paper is to establish the exact controllability of a strongly nonlinear wave equation in bounded regions of R^n with boundary controls.

Let $\{u_0, u_1\}$ be the initial values and let $\{v_0, v_1\}$ be the target values with both in a subset of $H^1(\Omega) \times L^2(\Omega)$, one wishes to find (i) a control χ acting on a portion of the boundary $\partial\Omega$ and (ii) a weak solution w of the problem

$$\begin{aligned} w'' - \Delta w + \mid w \mid^{p-2} w \quad &= \quad f \text{ in } \Omega \times (0,T), \\ w = 0 \text{ on } (\partial\Omega/\Gamma) \times (0,T) \quad &, \quad \nu.\nabla w = \chi \text{ on } \Gamma \times (0,T), \\ \{w, w'\} \mid_{t=0} = \{u_0, u_1\} \quad &, \quad \{w, w'\} \mid_{t=T} = \{v_0, v_1\} \end{aligned} \tag{1.1}$$

The exact controllability of the linear wave equation has been the subject of investigations in the late 80's following the introduction of the HUM's method of J.L.Lions [11]. More recently, Carleman estimates were used to establish the exact controllability of the wave equation with a potential by L.Baudoin,M de Buhanand and S.Ervedoza [2], by I.Lasiecka,R.Triggiani and P.F. Rao [6] for a plate equation on a Riemann manifold with energy level terms .

For nonlinear wave equations with Dirichlet-type controls, the problem has been studied by E.Zuazua [18,19] using a variant of HUM's method. The nonlinear term is asymptotically linear and in C^1 if the initial and target spaces are in $H^1(\Omega)\times L^2(\Omega)$, has at most a linear growth if the spaces are in $H^\gamma(\Omega)\times H^{-\gamma}(\Omega)$ for $[0,1], \gamma \neq 1/2$. The exact controllability for semi linear wave equations has been investigated by C.Bardos,G.Lebeau and J.Rauch[1], X.Fu,J.Yong and X.Zhang [4], L.Hu [3],G.Leugering,T.Li and Y.Wang [5],T.T Li and B.P.Rao [9], T.T.Li,B.P.Rao and P.F.Yao [10], L.Li and X.Zhang [7],R,Triggiani [15],P.F.Yao [16], X.Zhang [17] and others. In the works on semi-linear wave equations of [18 ,19] , the first step is the application of Lions's result on the exact controllability of a *linear* wave equation with Dirichlet-type controls. The solution is shown to be in $L^2(0,T;H^{1/2}(\Omega))$ with the control in $L^2(0,T;L^2(\Gamma))$ and appropriate initial and target values.The lack of space regularity has been a major obstacle to the study of the exact controllability of wave equations with a nonlinear polynomial growth.

The approach taken in this paper is different in several respects : (i) the study of the exact controllability for a linear wave equation with an *internal instead of boundary controls*, (ii) the use of techniques of the theory of variational inequalities to show the existence of a special T-periodic solution with a control on the solution which turns out to be the key to the establishment of the exact controllability with an internal control ,(iii)the use of the time-reversibility of the wave equation.

Similar ideas have been used by the writer in Ton [14] to study the exact controllability of nonlinear wave and Schrodinger equations and for T-periodic solutions of nonlinear wave equations in [13].

Notations are given in Section 2. The exact controllability of the linear wave equation with an internal control with mixed Neumann-Dirichlet boundary conditions is established in Section 3. A nonlinear problem is considered in Section 4 and the main result of the paper is proved in Section 5.

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2 Notations

Let Ω be a bounded open subset of R^n with a smooth boundary and let Γ be a non-empty closed subset of $\partial\Omega$. Let $\mathcal{H}_0^1(\Omega)$ be the Hilbert space

$$\mathcal{H}_0^1(\Omega) = \big\{u : u \in H^1(\Omega), u = 0 \text{ on } \partial\Omega/\Gamma\big\}.$$

We have

$$\mathcal{H}_0^1(\Omega) \subset L^2(\Omega) \subset \mathcal{H}^{-1}(\Omega).$$

Throughout the paper the pairing between various spaces are all denoted by $(., .)$.

Let J be the duality mapping of $L^2(0,T;L^2(\Omega))$ into $L^2(0,T;L^2(\Omega))$ associated with the gauge function $\Phi(r) = r$. Then J is a hemi-continuous monotone operator and

$$\begin{aligned}\int_0^T (Ju,u)dt &= \|Ju\|_{L^2(0,T;L^2(\Omega))}\|u\|_{L^2(0,T;L^2(\Omega))} \\ &= \|u\|^2_{L^2(0,T;L^2(\Omega))}\end{aligned}$$

Let K be the set

$$K = \big\{u : u \in L^2(0,T;L^2(\Omega)), \int_0^T u(.,t)dt = 0\big\}$$

K is a closed convex subset of $L^2(0,T;L^2(\Omega))$. We denote by β the penalty function associated with the closed convex subset K of $L^2(0,T;L^2(\Omega))$. It is defined by

$$\beta(u) = J(u - P_K u)$$

where P_K is the projection of $\mathcal{H}(\Omega)$ onto K. We know that β is a monotone hemi-continuous mapping of $L^2(0,T;L^2(\Omega))$ into $L^2(0,T;L^2(\Omega))$.

3 The linear case

In this section we shall establish the exact controllability of the linear wave equation with an interior control. The main result of the section is the following theorem.

Theorem 3.1. *Let* $\{y_0,y_1\}$ *and* $\{z_0,z_1\}$ *be in* $\mathcal{H}_0^1(\Omega)\cap H^2(\Omega)\}\times H^2(\Omega)$ *with*

$$\nu.\nabla y_0 = \nu.\nabla z_0 = \nu.\nabla y_1 = \nu.\nabla z_1 = 0 \ \ on\ \Gamma.$$

There exists

(i) an internal control $\theta = \tilde{\theta} + \hat{\theta}$ *with*

$$\{\tilde{\theta}, \hat{\theta}\} \in C^1(0,T;L^2(\Omega)) \times H^{-1}(\Omega)$$

(ii) a weak solution α *of the problem*

$$\begin{array}{rcl} \alpha'' - \Delta\alpha + \theta &=& 0 \ \ in\ \Omega \times (0,T), \\ \alpha = 0 \ \ on\ (\partial\Omega/\Gamma) \times (0,T) &,& \nu.\nabla\alpha = 0 \ \ on\ \Gamma \times (0,T) \\ \{\alpha, \alpha'\} \mid_{t=0} = \{y_0, y_1\} &,& \{\alpha, \alpha'\} \mid_{t=T} = \{z_0, z_1\} \end{array} \tag{3.1}$$

Moreover

$$\begin{array}{rl} \|\alpha\|_{L^\infty(0,T;\mathcal{H}_0^1(\Omega))} & + \ \ \|\alpha'\|_{L^\infty(0,T;H^1(\Omega))} + \|\alpha''\|_{L^\infty(0,T;L^2(\Omega))} \\ & + \ \ \|\hat{\theta}\|_{C^1(0,T;L^2(\Omega))} + \|\tilde{\theta}\|_{H^{-1}(\Omega)} \\ & \leq \ \ C\big\{\|y_0\|_{\mathcal{H}_0^1(\Omega)\cap H^2(\Omega)} + \|y_1\|_{H^2(\Omega)} + \|z_0\|_{\mathcal{H}_0^1(\Omega)\cap H^2(\Omega)} \\ & + \ \ \|z_1\|_{H^2(\Omega)}\big\} \end{array}$$

The proof of the theorem will be carried out in two steps. First we shall use the penalty function and a regularization to study the problem

$$\begin{array}{rcl} \hat{\alpha}'' - \Delta\hat{\alpha} + \hat{\theta} &=& 0 \ \text{ in } \Omega \times (0,T), \\ \hat{\alpha} = 0 \ \text{ on } (\partial\Omega\Gamma) \times (0,T) &,& \nu.\nabla\hat{\alpha} = 0 \text{ on } \Gamma \times (0,T), \\ \{\hat{\alpha}, \hat{\alpha}'\} \mid_{t=0} = \{y_0, \hat{y}_1\} &,& \hat{\alpha}(\,.,T) = y_0 \end{array} \tag{3.2}$$

In Step 2, we note that if the given data $\hat{y}_1 = 0$ then we have a special T-periodic solution as $\hat{\alpha}'(\,.,T) = 0 = \hat{\alpha}'(\,.,0)$ and use that crucial fact to construct the internal control $\tilde{\theta}$

Step 1. Let $0 < \varepsilon$ and consider the problem

$$\begin{array}{rcl} \alpha_\varepsilon'' - \varepsilon\Delta\alpha_\varepsilon' - \Delta\alpha_\varepsilon &+& \varepsilon^{-1}\beta(\alpha_\varepsilon') = 0 \ \text{ in } \Omega \times (0,T), \\ \alpha_\varepsilon = 0 \text{ on } (\partial\Omega/\Gamma) \times (0,T) &,& \nu.\nabla\alpha_\varepsilon = 0 \text{ on } \Gamma \times (0,T) \\ \alpha_\varepsilon(\,.,0) = y_0 &,& \alpha_\varepsilon'(\,.,0) = \hat{y}_1 \end{array} \tag{3.3}$$

Lemma 3.1. *Let* $\{y_0, \hat{y}_1\}$ *be in* $\mathcal{H}_0^1(\Omega) \times H^1(\Omega)$, *then there exists a weak solution* α_ε *of (3.3). Moreover*

$$
\begin{aligned}
\|\alpha'_\varepsilon(t)\|^2_{L^2(\Omega)} &+ \|\nabla\alpha_\varepsilon(t)\|^2_{L^2(\Omega)} + 2\varepsilon\|\nabla\alpha_\varepsilon\|^2_{L^2(0,t;L^2(\Omega))} \\
&+ 2\varepsilon^{-1}\int_0^t (\beta(\alpha'_\varepsilon)\,,\,,\alpha'_\varepsilon)ds \\
&\le \|\nabla y_0\|^2_{L^2(\Omega)} + \varepsilon\|\nabla\hat{y}_1\|^2_{L^2(\Omega)} + \|\hat{y}_1\|^2_{L^2(\Omega)}
\end{aligned}
$$

The constant C is independent of ε.

Proof. Let $\{\varphi_j\}$ be an orthonormal basis of $\mathcal{H}^1_0(\Omega)$ and consider the Galerkin approximating system

$$
\begin{aligned}
(\alpha''_n, \varphi_j) + \varepsilon(\nabla\alpha'_n, \nabla\varphi_j) &+ (\nabla\alpha_n,\, \nabla\varphi_j) + \varepsilon^{-1}(\beta(\alpha'_n), \varphi_j) = 0 \\
\alpha_n(\,.,0) = y_{0,n} \quad &, \quad \alpha'(\,.,0) = y_{1,n}\ ;\ \ j = 1,..,n
\end{aligned}
$$

The existence of a solution with the stated estimate is easy to establish and we shall not reproduce it □

Lemma 3.2. *Suppose all the hypotheses of Lemma 3.1 are satisfied and let α_ε be as in Lemma 3.1. Then there is a subsequence such that*

$$\{\alpha_\varepsilon, \alpha'_\varepsilon\} \to \{\hat{\alpha}, \hat{\alpha}'\}$$

in

$$\{[L^\infty(0,T;\mathcal{H}^1_0(\Omega))]_{weak^*} \cap C(0,T;L^2(\Omega))\} \times [L^\infty(0,T;L^2(\Omega))]_{weak^*}$$

Moreover

$$\|\hat{\alpha}'(t)\|^2_{L^2(\Omega)} + \|\nabla\hat{\alpha}(t)\|^2_{L^2(\Omega)} \le \|\hat{y}_1\|^2_{L^2(\Omega)} + \|\nabla y_0\|^2_{L^2(\Omega)}$$

and

$$\hat{\alpha}(\,.,0) = y_0 = \hat{\alpha}(\,.,T).$$

Proof. 1) We have only to show that $\hat{\alpha}(\,.,0) = y_0 = \hat{\alpha}_\varepsilon$ as all the other assertions follow from Lemma 3.1. We get

$$
\begin{aligned}
\|\beta(\alpha'_\varepsilon)\|_{L^2(0,T;L^2(\Omega))} &= \|J(\alpha'_\varepsilon - P_K\alpha'_\varepsilon)\|_{L^2(0,T;L^2(\Omega))} \\
&= \|\alpha'_\varepsilon - P_K\alpha'_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \\
&\le 2\|\alpha'_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \\
&\le C
\end{aligned}
$$

with C independent of ε. Thus,

$$\beta(\alpha'_\varepsilon) \to \chi \text{ in } [L^2(0,T;L^2(\Omega))]_{weak}.$$

On the other hand we have from the equation

$$\varepsilon\big\{\alpha''_\varepsilon - \varepsilon\Delta\alpha'_\varepsilon - \Delta\alpha_\varepsilon\big\} \to 0$$

in $\mathcal{D}'(0,T;H^{-1}(\Omega))$. Hence $\chi = 0$.
2) From the estimate of Lemma 3.1 we have

$$\int_0^T (\beta(\alpha'_\varepsilon)\,.,\alpha'_\varepsilon)dt \to 0$$

as $\varepsilon \to 0$. . Hence

$$\int_0^T (-\beta(v'),\alpha'_\varepsilon - v')dt \geq 0 \ \ \forall v' \in L^2(0,T;L^2(\Omega)).$$

With $v' = \hat{\alpha}' + \lambda u'$ for u' in $L^2(0,T;L^2(\Omega))$ we obtain

$$\int_0^T (\beta(\alpha'_\varepsilon + \lambda u'),u')dt \leq 0$$

Let $\lambda \to 0$ and since β is hemi continuous, we deduce that

$$\int_0^T (\beta(\hat{\alpha}'),u')dt \leq 0 \ \ \forall u' \in L^2(0,T;L^2(\Omega))$$

With $-u'$ instead of u' and we have

$$\int_0^T (\beta(\hat{\alpha}'),u')dt \geq 0 \ \forall\ u' \in L^2(0,T;\mathcal{H}_0^1(\Omega))$$

It follows that $\beta(\hat{\alpha}') = 0$ i.e.

$$\hat{\alpha}(\,.,0) = y_0 = \hat{\alpha}(\,.,T).$$

Lemma 3.3. *Suppose all the hypotheses of Lemma 3.1. Then there exists a time-independent $\hat{\theta}$ in $\mathcal{H}^{-1}(\Omega)$ and $\hat{\alpha}$ is a solution of the problem*

$$\begin{array}{rcl} \hat{\alpha}'' - \Delta\hat{\alpha} + \hat{\theta} & = & 0 \ \mathit{in}\ \Omega\times(0,T), \\ \hat{\alpha} = 0 \ \ \mathit{on}\ (\partial\Omega/\Gamma)\times(0,T) & , & \nu.\nabla\hat{\alpha} = 0 \ \mathit{on}\ \Gamma\times(0,T), \\ \hat{\alpha}(\,.,0) = y_0 = \hat{\alpha}(\,.,T) & , & \hat{\alpha}'(\,.,0) = \hat{y}_1. \end{array} \quad (3.4)$$

Furthermore

$$\int_0^T (\hat{\theta}, v)dt = \lim_{\varepsilon\to 0} \varepsilon^{-1} \int_0^T (\beta(\alpha'_\varepsilon), v)dt \ \forall v \in C_0^\infty(0, T; \mathcal{H}_0^1(\Omega)).$$

Proof. 1) Let φ be in $C_0^\infty(0, T; \mathcal{H}_0^1(\Omega))$ then φ' is in K and therefore $\beta(\varphi') = 0$. We have

$$\begin{aligned} \int_0^T (\beta(\alpha'_\varepsilon) - \beta(\varphi'), \alpha'_\varepsilon - \varphi')dt &= \int_0^T (\beta(\alpha'_\varepsilon), \alpha'_\varepsilon - \varphi')dt \\ &\geq 0 \ \forall \varphi \in C_0^\infty(0, T; \mathcal{H}_0^1(\Omega)) \end{aligned}$$

It follows from (3.3) that

$$\begin{aligned} \int_0^T (\alpha''_\varepsilon, \alpha'_\varepsilon - \varphi')dt &+ \varepsilon \int_0^T (\nabla \alpha'_\varepsilon, \nabla(\alpha'_\varepsilon - \varphi'))dt \\ &+ \int_0^T (\nabla \alpha_\varepsilon, \nabla(\alpha'_\varepsilon - \varphi'))dt \\ &\leq 0 \quad \forall \varphi \in C_0^\infty(0, T; \mathcal{H}_0^1(\Omega)). \end{aligned}$$

We rewrite it as

$$E(\alpha_\varepsilon) \leq \Phi(\alpha_\varepsilon, \varphi)$$

with

$$\begin{aligned} E(\alpha_\varepsilon) &= \frac{1}{2}\|\alpha'_\varepsilon(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|y_1\|^2_{L^2(\Omega)} + \varepsilon\|\nabla\alpha'_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \frac{1}{2}\|\nabla\alpha_\varepsilon(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla y_0\|^2_{L^2(\Omega)} \end{aligned}$$

and

$$\Phi(\alpha_\varepsilon, \varphi) = \int_0^T (\alpha''_\varepsilon, \varphi')dt + \varepsilon \int_0^T (\nabla\alpha'_\varepsilon, \nabla\varphi')dt + \int_0^T (\nabla\alpha_\varepsilon, \nabla\varphi')dt$$

Changing φ into $-\varphi$ and we obtain

$$\Phi(\alpha_\varepsilon, \varphi) \leq -E(\alpha_\varepsilon)$$

Combining the two inequalities and we get

$$E(\alpha_\varepsilon) \le \Phi(\alpha_\varepsilon, \varphi) \le -E(\alpha_\varepsilon) \;\; \forall \varphi \in C_0^\infty(0, T; \mathcal{H}_0^1(\Omega))$$

Let $\lambda > 0$ and consider φ/λ instead of φ and we have

$$\lambda E(\alpha_\varepsilon) \le \Phi(\alpha_\varepsilon, \varphi) \le -\lambda E(\alpha_\varepsilon)$$

Hence

$$\Phi(\alpha_\varepsilon, \varphi) = 0 \;\; \forall \; \varphi \in C_0^\infty(0, T; \mathcal{H}_0^1(\Omega))$$

Let $\varepsilon \to 0$ and we get

$$\Phi(\hat{\alpha}, \varphi) = 0 \;\; \forall \varphi \in C_0^\infty(0, T; \mathcal{H}_0^1(\Omega))$$

i.e.

$$\{\hat{\alpha}'' - \Delta\hat{\alpha}\}' = 0 \;\text{ in } \mathcal{D}'(0, T; \mathcal{H}^{-1}(\Omega)).$$

2) Hence there exists a time-independent $\hat{\theta}$ in $\mathcal{H}^{-1}(\Omega)$ such that

$$\hat{\alpha}'' - \Delta\hat{\alpha} + \hat{\theta} = 0$$

and

$$\begin{aligned} \int_0^T (\hat{\alpha}'', \varphi) dt \quad &+ \quad \int_0^T (\nabla\hat{\alpha}, \nabla\varphi) dt \\ &= - \quad \int_0^T (\hat{\theta}, \varphi) dt \;\; \forall \varphi \in L^2(0, T; \mathcal{H}_0^1(\Omega)) \end{aligned} \tag{3.5}$$

Let $\varepsilon \to 0$ in (3.3) and we get

$$\begin{aligned} \int_0^T (\hat{\alpha}'', \varphi) dt \;\; &+ \;\; \int_0^T (\nabla\hat{\alpha}, \nabla\varphi) dt \\ &= \;\; -\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^T (\beta(\alpha_\varepsilon') \,, \varphi) dt \;\; \forall \varphi \in L^2(0, T; \mathcal{H}_0^1(\Omega)) \end{aligned} \tag{3.6}$$

It follows from (3.5)-(3.6) that

$$\int_0^T (\hat{\theta}, \varphi) dt = \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^T (\beta(\alpha_\varepsilon'), \varphi) \tag{3.7}$$

for all $\varphi \in L^2(0, T; \mathcal{H}_0^1(\Omega))$

3) We now show that the control $\hat{\theta}$ given by (3.7) is unique. Indeed if γ is another element of $H^{-1}(\Omega)$ such that

$$\hat{\alpha}'' - \Delta\hat{\alpha} = -\gamma$$

then

$$\int_0^T (\hat{\theta}, \varphi)dt = \int_0^T (\gamma, \varphi)dt = \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^T (\beta(\alpha'_\varepsilon), \varphi)dt$$

for all $\varphi \in L^2(0,T;\mathcal{H}_0^1(\Omega))$

Therefore $\hat{\theta} = \gamma$ □

Lemma 3.4. *Let $\hat{\theta}$ be the time-independent control of Lemma 3.3 then*

$$\|\hat{\theta}\|_{H^{-1}(\Omega)} \leq C\big\{\|y_0\|_{H_0^1(\Omega)} + \|\hat{y}_1\|_{L^2(\Omega)}\big\}$$

Proof. Let ζ be a $C_0^\infty(0,T)$ function and let v be in $\mathcal{H}_0^1(\Omega)$. From (3.5) we get

$$-(\hat{\theta}, v)\int_0^T \zeta dt = \int_0^T (\hat{\alpha}, v\zeta'')dt - \int_0^T (\nabla\hat{\alpha}, \zeta\nabla v)dt$$

Hence

$$\begin{aligned} |\ (\hat{\theta}, v)\ | &\leq \|\hat{\alpha}\|_{\mathcal{H}_0^1(\Omega)}\|v\|_{\mathcal{H}_0^1(\Omega)} \\ &\leq C\big\{\|y_0\|_{\mathcal{H}_0^1(\Omega)} + \|\hat{y}_1\|_{L^2(\Omega)}\big\}\|v\|_{\mathcal{H}_0^1(\Omega)} \ \ \forall v \in \mathcal{H}_0^1(\Omega) \end{aligned}$$

and the lemma is proved □

Lemma 3.5. *Let $\{y_0, \hat{y}_1\}$ be in $\{H_0^1(\Omega) \cap H^2(\Omega)\} \times H^1(\Omega)$ with*

$$\nu.\nabla y_0 = \nu.\nabla z_0 = \nu.\nabla z_1 = 0 \ \ on \ \ \Gamma$$

and let $\hat{\alpha}$ be as in Lemmas 3.3-3.4, then

$$\{\hat{\alpha}, \hat{\alpha}', \hat{\alpha}''\}$$

in

$$L^\infty(0,T;\mathcal{H}_0^1(\Omega)) \times L^\infty(0,T;H^1(\Omega)) \times L^\infty(0,T;L^2(\Omega))$$

Moreover

$$\|\nabla\hat{\alpha}(\,.t)\|^2_{L^2(\Omega)} + \|\hat{\alpha}'(t)\|^2_{L^2(\Omega)} \leq \|\nabla y_0\|^2_{L^2(\Omega)} + \|\hat{y}_1\|^2_{L^2(\Omega)}$$

Furthermore if $\hat{y}_1 = 0$ then $\hat{\alpha}'(\,.,0) = 0 = \hat{\alpha}'(\,.,T)$.

Proof.
The estimate is an immediate consequence of that of Lemma 3.1.

1) Let d_h be the usual time difference quotient. Since θ is time-independent we get from (3.4)

$$\begin{array}{rcl} (d_h\hat{\alpha})'' - \Delta(d_h\hat{\alpha}) & = & 0 \text{ in } \Omega\times(0,T), \\ d_h\hat{\alpha} = 0 \text{ on } (\partial\Omega/\Gamma)\times(0,T) & , & \nu.\nabla d_h\alpha = 0 \text{ on } \Gamma\times(0,T) \\ (d_h\hat{\alpha})(\,.,0) = h\hat{y}_1 & , & (d_h\hat{\alpha})'(\,.,0) = h\Delta y_0 \end{array}$$

We have

$$\|d_h\hat{\alpha}\|_{L^\infty(0,T;H_0^1(\Omega))} + \|(d_h\hat{\alpha})'\|_{L^\infty(0,T;L^2(\Omega))} \leq C\big\{\|h\hat{y}_1\|_{H^1(\Omega)} + \|h\Delta y_0\|_{L^2(\Omega)}\big\}$$

It follows that $\{\hat{\alpha}', \hat{\alpha}''\}$ is in $L^\infty(0,T;H^1(\Omega))\times L^\infty(0,T;L^2(\Omega))$

3) It is clear from the estimate that if $\hat{y}_1 = 0$ then $\hat{\alpha}'(\,.,T) = 0$ as $\hat{\alpha}(\,.,0) = y_0 = \alpha(\,.,T)$ □

Step 2. We shall now construct the inner control F and establish the exact controllability of (3.1).

Proof of Theorem 3.1. Let

$$\begin{array}{rcl} \tilde{\alpha}(\,.,t) & = & t^2T^{-2}\{z_0 - y_0\} - t(t-T)T^{-1}y_1 \\ & - & t^2(t-T)T^{-2}\{-z_1 - y_1 + 2T^{-1}[z_0 - y_0]\} \end{array} \tag{3.8}$$

Then

$$\tilde{\alpha}(\,.,0) = 0,\ \tilde{\alpha}(\,.,T) = z_0 - y_0,\ \tilde{\alpha}'(\,.,0) = y_1,\ \tilde{\alpha}'(\,.,T) = z_1$$

with

$$\begin{array}{rcl} \tilde{\alpha}'(\,.,t) & = & 2tT^{-2}\{z_0 - y_0\} - (2t-T)T^{-1}y_1 \\ & - & (3t^2 - 2tT)T^{-2}\{-z_1 - y_1 + 2T^{-1}[z_0 - y_0]\} \end{array}$$

Set

$$\alpha = \hat{\alpha} + \tilde{\alpha} \text{ with } \hat{\alpha} \text{ as in Lemma 3.5 with } \hat{y}_1 = 0$$

then

$$\begin{array}{c} \alpha(\,.,0) = y_0,\ \alpha(\,.,T) = y_0 + z_0 - y_0 = z_0, \\ \alpha'(\,.,0) = \hat{\alpha}'(\,.,0) + \tilde{\alpha}'(\,.,0) = 0 + y_1 = y_1, \\ \alpha'(\,.,T) = \hat{\alpha}'(\,.,T) + \tilde{\alpha}'(\,.,T) = 0 + z_1 = z_1 \end{array}$$

With

$$\nu.\nabla y_0 = \nu.\nabla z_0 = \nu.\nabla z_1 = 0 \text{ on } \Gamma$$

it is clear that

$$\nu.\nabla \alpha = 0 \text{ on } \Gamma \times (0,T) \text{as } y_1 = 0.$$

Set

$$\tilde{\theta} = -\{\tilde{\alpha}'' - \Delta\tilde{\alpha}\} \tag{3.9}$$

It is trivial to check that α is a solution of (3.1) with all the stated properties □

4 A nonlinear problem

In this section we shall consider a nonlinear problem before establishing the exact controllability for the nonlinear wave equations. The main result of the section is the following theorem.

Theorem 4.1. *Let $\{u_0, u_1, f\}$ be in $\mathcal{H}_0^1(\Omega) \times L^2(\Omega) \times L^2(0,T;L^2(\Omega))$ and let $\{\alpha, \hat{\theta}, \tilde{\theta}\}$ be as in Theorem 3.1.*

Suppose that $2 \leq p \leq 2(n-1)/(n-2)$ with $2 \leq p < \infty$ if $n = 2$. There exists
(i) a time-independent control g in $\mathcal{H}^{-1}(\Omega)$
(ii) a weak solution w of the problem

$$\begin{aligned} w'' - \Delta w + \mid w + \alpha \mid^{p-2} (w+\alpha) &= f + \hat{\theta} + \tilde{\theta} \text{ in } \Omega \times (0,T), \\ w = 0 \text{ on } (\partial\Omega/\Gamma) \times (0,T) \quad &, \quad \nu.\nabla w = -g \text{ on } \Gamma \times (0,T), \\ w(\,.,0) = u_0 - y_0 = w(\,.,T) \quad &, \quad w'(\,.,0) = u_1 - y_1 \end{aligned} \tag{4.1}$$

with $\{w, w'\}$ in $L^\infty(0,T;\mathcal{H}_0^1(\Omega)) \times L^\infty(0,T;L^2(\Omega))$.

We shall proceed as in Section 3 with some minor differences. In this section ,J is the duality mapping of $L^2(0,T;L^2(\Gamma)$ into $L^2(0,T;L^2(\Gamma))$ with gauge function $\Phi(r) = r$ and β is the e penalty function associated with the closed convex subset K_* of $L^2(0,T;L^2(\Gamma))$ with

$$K_* = \left\{u \mid_\Gamma : u \in K, \|u\|_{L^2(0,T;L^2(\Omega\cup\Gamma))} \leq C\right\}$$

for a large positive constant C.

First we consider the problem

$$\begin{aligned} w''_{\varepsilon,\eta} - \varepsilon\Delta w'_{\varepsilon,\eta} - \Delta w_{\varepsilon,\eta} &+ \mid w_{\varepsilon,\eta} + \alpha \mid^{p-2} (w_{\varepsilon,\eta} + \alpha) = f + \hat{\theta} + \tilde{\theta} \\ w_{\varepsilon,\eta} &= 0 \text{ on } (\partial\Omega/\Gamma) \times (0,T) \\ \nu.\nabla(\varepsilon w'_{\varepsilon,\eta} + w_{\varepsilon,\eta}) &= -\eta^{-1}\beta(w'_{\varepsilon,\eta}) \text{ on } \Gamma \times (0,T), \\ w_{\varepsilon,\eta}(\,.,0) = u_0 - y_0 \quad &, \quad w'_{\varepsilon,\eta}(\,.,0) = u_1 - y_1 \end{aligned} \tag{4.2}$$

Lemma 4.1. *Suppose all the hypotheses of Theorem 4.1 are satisfied. There exists a solution $w_{\varepsilon,\eta}$ of (4.2) and*

$$\begin{aligned} \|w'_{\varepsilon,\eta}(t)\|^2_{L^2(\Omega)} &+ 2\varepsilon\|\nabla w'_{\varepsilon,\eta}\|^2_{L^2(0,t;L^2(\Omega))} + \|\nabla w_{\varepsilon,\eta}(t)\|^2_{L^2(\Omega)} \\ &+ \frac{2}{p}\|(w_{\varepsilon,\eta}+\alpha)(t)\|^p_{L^p(\Omega)} + 2\eta^{-1}\int_0^t\int_\Gamma \beta(w'_{\varepsilon,\eta}).w'_{\varepsilon,\eta}d\sigma dt \\ &\le \|u_1 - y_1\|^2_{L^2(\Omega)} + \|\nabla(u_0 - y_0)\|^2_{L^2(\Omega)} + \frac{2}{p}\|u_0\|^p_{L^p(\Omega)} \\ &+ \int_0^t (f - \hat{\theta} - \tilde{\theta}, w'_{\varepsilon,\eta})ds + \frac{p-1}{p}\int_0^t \|(w_{\varepsilon,\eta}+\alpha)(s)\|^p_{L^p(\Omega)}ds \\ &+ \frac{1}{p}\|\alpha'\|^p_{L^\infty(0,T;\mathcal{H}^1_0(\Omega))} \end{aligned}$$

Proof. Let φ_j be an orthonormal basis of $\mathcal{H}^1_0(\Omega)$ and consider the Galerkin approximation system

$$\begin{aligned} (w''_n, \varphi_j) + \varepsilon(\nabla w'_n, \nabla\varphi_j) &+ (\nabla w_n, \nabla\varphi_j) + (\mid w_n + \alpha \mid^{p-2} (w_n + \alpha), \varphi_j) \\ &+ \eta^{-1}\int_\Gamma \beta(w').\varphi_j d\sigma = (f + \hat{\theta} + \tilde{\theta}, \varphi_j), \ j \le n \end{aligned}$$

with

$$\{w_n(\,.,0), w'_n(\,.,0)\} = \{w_{n,0}, w_{n,1}\} \to \{u_0 - y_0, u_1 - y_1\} \text{ in } \mathcal{H}^1_0(\Omega) \times L^2(\Omega)$$

The existence of a solution is known and we have

$$\begin{aligned} \|w'_n(t)\|^2_{L^2(\Omega)} &+ 2\varepsilon\|\nabla w'\|^2_{L^2(0,t;L^2(\Omega)} + \|\nabla w_n(t)\|^2_{L^2(\Omega)} \\ &+ \frac{2}{p}\|(w_n+\alpha)(t)\|^p_{L^p(\Omega)} - \int^t (\mid w_n + \alpha \mid (w_n + \alpha), \alpha')dt \\ &+ 2\eta^{-1}\int_0^t\int_\Gamma \beta(w'_n).w'_n d\sigma ds = \int_0^t (f + \hat{\theta} + \tilde{\theta}, w'_n)ds \\ &+ \|u_1 - y_1\|^2_{L^2(\Omega)} + \|\nabla(u_0 - y_0)\|^2_{L^2(\Omega)} + \frac{2}{p}\|u_0\|^p_{L^p(\Omega)} \end{aligned}$$

Since $2 \leq p \leq 2n/(n-2)$ we have

$$\mathcal{H}_0^1(\Omega) \subset L^p(\Omega).$$

With α' in $L^\infty(0,T;\mathcal{H}_0^1(\Omega))$ we get

$$\begin{aligned} \left| \int_0^t (\mid w_{\varepsilon,\eta} + \alpha \mid^{p-2} (w_{\varepsilon,\eta} + \alpha), \alpha')ds \right| &\leq \frac{p-1}{p} \int_0^t \|w_{\varepsilon,\eta} + \alpha\|^p_{L^p(\Omega)} ds \\ &+ \frac{1}{p} \|\alpha'\|^p_{L^\infty(0,T;\mathcal{H}_0^1(\Omega))}. \end{aligned}$$

Hence

$$\begin{aligned} \|w_n'(t)\|^2_{L^2(\Omega)} &+ 2\varepsilon \|\nabla w_n'\|^2_{L^2(0,t:L^2(\Omega))} + \|\nabla w_n(t)\|^2_{L^2(\Omega)} \\ &+ \frac{2}{p} \|(w_n + \alpha)(t)\|^p_{L^p(\Omega)} + 2\eta^{-1} \int_0^t \int_\Gamma \beta(w_n') .w_n' d\sigma ds \\ &\leq \frac{p-1}{p} \int_0^t \|(w_n + \alpha)(s)\|^p_{L^p(\Omega)} ds + \|u_1 - y_1\|^2_{L^2(\Omega)} + \|\nabla (u_0 - y_0)\|^2_{L^2(\Omega)} \\ &+ \frac{2}{p} \|u_0\|^p_{L^p(\Omega)} + \int_0^t (f - \hat{\theta} - \tilde{\theta}, w_n')ds + \frac{1}{p} \|\alpha'\|^p_{L^\infty(0,T;\mathcal{H}_0^1(\Omega))} \end{aligned}$$

Taking the Gronwall's lemma into account, then let $n \to \infty$ and we get the stated estimate $\Box$

Lemma 4.2. *Let $w_{\varepsilon,\eta}$ be as in Lemma 4.1.Then there exists a subsequence such that*

$$\{w_{\varepsilon,\eta}, w'_{\varepsilon,\eta}\} \to \{w_\varepsilon, w'_\varepsilon\}$$

in

$$[L^\infty(0,T;\mathcal{H}_0^1(\Omega))]_{weak^*} \times [L^\infty(0,T;L^2(\Omega))]_{weak^*}$$

with $\beta(w'_\varepsilon) = 0$ i.e. $w_\varepsilon(.,0) = u_0 - y_0 = w_\varepsilon(.,T)$.
Moreover

$$\begin{aligned} \|w'_\varepsilon(t)\|^2_{L^2(\Omega)} &+ 2\varepsilon \|\nabla w'_\varepsilon\|^2_{L^2(,t;L^2(\Omega))} + \frac{2}{p} \|(w_\varepsilon + \alpha)(t)\|^p_{L^p(\Omega))} \\ &\leq \|u_1 - y_1\|^2_{L^2(\Omega)} + \|\nabla (u_0 - y_0)\|^2_{L^2(\Omega)} + \frac{2}{p} \|u_0\|^p_{L^p(\Omega)} \\ &+ \int_0^t (f + \tilde{\theta}, w'_\varepsilon)ds + (\hat{\theta}, w'_\varepsilon(t) - u_1) + \frac{1}{p} \|\alpha'\|^p_{L^\infty(0,T;\mathcal{H}_0^1(\Omega))} \end{aligned}$$

Proof. 1) The first assertion of the lemma is an immediate consequence of the estimate of Lemma 4.1. We have

$$\begin{aligned}\|\beta(w'_{\varepsilon,\eta})\|_{L^2(0,T;L^2(\Gamma))} &= \|J(w'_{\varepsilon,\eta} - P_{K_*}w'_{\varepsilon,\eta})\|_{L^2(0,T;L^2(\Gamma))} \\ &= \|w'_{\varepsilon,\eta} - P_{K_*}w'_{\varepsilon,\eta}\|_{L^2(0,T;L^2(\Gamma)} \\ &\leq 2\|w'_{\varepsilon,\eta}\|^2_{L^2(0,T;H^1(\Omega))} \\ &\leq 2\varepsilon^{-1}C\end{aligned}$$

Hence

$$\beta(w'_{\varepsilon,\eta}) \to \chi_\varepsilon \text{ in } [L^2(0,T;L^2(\Gamma))]_{weak}.$$

On the other hand from the equation we get

$$\int_0^T \int_\Gamma \beta(w'_{\varepsilon,\eta})\varphi d\sigma dt \to 0 \ \ \forall\varphi \in L^2(0,T;\mathcal{H}^1_0(\Omega))$$

Therefore $\chi_\varepsilon \mid_\Gamma = 0$.

2) With $\varphi \in L^2(0,T;H^1_0(\Omega))$ we have

$$\begin{aligned}(w''_{\varepsilon,\eta},\varphi) &+ \varepsilon(\nabla w'_{\varepsilon,\eta},\nabla\varphi) + (\nabla w_{\varepsilon,\eta},\nabla\varphi) \\ &+ (\mid w_{\varepsilon,\eta}+\alpha\mid^{p-2}(w_{\varepsilon,\eta}+\alpha),\varphi) = (f+\tilde{\theta}+\hat{\theta},\varphi)\end{aligned}$$

The boundary term disappears as $\varphi = 0$ on Γ. Thus,

$$\|w''_{\varepsilon,\eta}\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_\varepsilon$$

and

$$w'_{\varepsilon,\eta} \to w'_\varepsilon \text{ in } L^2(0,T;L^2(\Gamma)) \cap L^2(0,T;H^\gamma(\Omega)),\ \forall\gamma, 0<\gamma<1.$$

We deduce that

$$\beta(w'_{\varepsilon,\eta}) \to \beta(w'_\varepsilon) = \chi_\varepsilon = 0 \text{ in } [L^2(0,T;L^2(\Gamma))]_{weak}$$

Therefore $w'_\varepsilon \in K$, i.e $w_\varepsilon(\,.,0) = w_\varepsilon(\,.,T)$.

3) Let φ be in $C^\infty_0(0,T;\mathcal{H}^1_0(\Omega))$ with

$$\|\varphi'\|_{L^2(0,T;H^1(\Omega))} \leq R$$

then $\varphi' \in K$ and $\varphi'_{|\Gamma} \in K_*$. We have

$$\begin{aligned}\int_0^T \int_\Gamma \{\beta(w'_{\varepsilon,\eta}) - \beta(\varphi')\}\{w'_{\varepsilon,\eta} - \varphi'\} d\sigma dt &= \int_0^T \int_\Gamma \beta(w'_{\varepsilon,\eta})\{w'_{\varepsilon,\eta} - \varphi'\} d\sigma dt \\ &\geq 0\end{aligned}$$

It now follows from (4.2) that

$$E(w_{\varepsilon,\eta}) \leq \Phi(w_{\varepsilon,\eta}, \varphi)\ \forall \varphi \in C_0^\infty(0,T;\mathcal{H}_0^1(\Omega)).$$

with

$$\begin{aligned}E(w_{\varepsilon,\eta}) &= \frac{1}{2}\|w'_{\varepsilon,\eta}(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_1 - y_1\|^2_{L^2(\Omega)} + \varepsilon\|\nabla w'_{\varepsilon,\eta}\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \frac{1}{2}\|\nabla w_{\varepsilon,\eta}(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla(u_0 - y_0)\|^2_{L^2(\Omega)} + \frac{1}{p}\|w_{\varepsilon,\eta}(T) + \alpha(T)\|^p_{L^p(\Omega)} \\ &- \frac{1}{p}\|u_0\|^p_{L^p(\Omega)} - \int_0^T (f - \tilde{\theta}, w'_\varepsilon) dt \\ &- \int_0^T (\mid w_{\varepsilon,\eta} + \alpha \mid^{p-2} (w_{\varepsilon,\eta} + \alpha), \alpha') dt + (\hat{\theta}, w'_{\varepsilon,\eta}(T)) - (\hat{\theta}, u_1 - y_1)\end{aligned}$$

and

$$\begin{aligned}\Phi(w_{\varepsilon,\eta}, \varphi) &= \int_0^T (w''_{\varepsilon,\eta}, \varphi') dt + \varepsilon \int_0^T (\nabla w'_{\varepsilon,\eta}, \nabla\varphi') dt + \int_0^T (\nabla w_{\varepsilon,\eta}, \nabla\varphi') dt \\ &+ \int_0^T (\mid w_{\varepsilon,\eta} + \alpha \mid^{p-2} (w_{\varepsilon,\eta} + \alpha), \varphi' - \int_0^T (f + \tilde{\theta} + \hat{\theta}, \varphi') dt\end{aligned}$$

for all φ in$C_0^\infty(0,T;\mathcal{H}_0^1(\Omega))$ with

$$\|\varphi'\|_{L^2(0,T;H^1(\Omega))} \leq R$$

Changing φ into $-\varphi$ and we obtain

$$\Phi(w_{\varepsilon,\eta}, \varphi) \leq -E(w_{\varepsilon,\eta})$$

Therefore

$$E(w_{\varepsilon,\eta}) \leq \Phi(\varepsilon, \eta) \leq -E(w_{\varepsilon,\eta})$$

for all φ in $C_0^\infty(0,T;\mathcal{H}_0^1(\Omega))$ with

$$\|\varphi'\|_{L^2(0,T;H^1(\Omega))} \leq R$$

Let $R \to \infty$ and we get

$$E(w_{\varepsilon,\eta}) \leq \Phi(w_{\varepsilon,\eta},\varphi) \leq -E(w_{\varepsilon,\eta}) \ \forall \varphi \in C_0^\infty(0,T;\mathcal{H}_0^1(\Omega)).$$

With $\lambda > 0$ and $\lambda^{-1}\varphi$ instead of φ , we obtain by combining the two cases

$$\lambda E(w_{\varepsilon,\eta}) \leq \Phi(w_{\varepsilon,\eta},\varphi) \leq -\lambda E(w_{\varepsilon,\eta})$$

Hence

$$\Phi(w_{\varepsilon,\eta},\varphi) = 0 \ \forall \varphi \in C_0^\infty(0,GT;\mathcal{H}_0^1(\Omega))$$

Let $\eta \to 0$ and a simple argument gives

$$\Phi(w_\varepsilon,\varphi) = 0 \ \ \forall \varphi \in C_0^\infty(0,T;\mathcal{H}_0^1(\Omega))$$

i.e.

$$\{w_\varepsilon'' - \varepsilon\Delta w_\varepsilon' - \Delta w_\varepsilon + \mid w_\varepsilon + \alpha \mid^{p-2} (w_\varepsilon + \alpha) - f - \tilde{\theta} - \hat{\theta}\}' = 0 \text{ in } \mathcal{D}'(0,T;\mathcal{H}_0^{-1}(\Omega))$$

Thus there exists g_ε in $\mathcal{H}^{-1}(\Omega)$ such that

$$w_\varepsilon'' - \varepsilon\Delta w_\varepsilon' - \Delta w_\varepsilon + \mid w_\varepsilon + \alpha \mid^{p-2} (w_\varepsilon + \alpha) = f - g_\varepsilon + \tilde{\theta} + \hat{\theta} \tag{4.3}$$

We now show that g_ε has its support in Γ and that it is uniquely defined. From (4.2)-(4.3) we deduce that

$$\int_0^T (g_\varepsilon,\varphi)dt = \lim_{\eta\to 0} \eta^{-1} \int_0^T \int_\Gamma \beta(w_{\varepsilon,\eta}') \,.\varphi d\sigma dt \ \ \forall \varphi \in L^2(0,T;\mathcal{H}_0^1(\Omega))$$

It is clear from the above equation that the support of g_ε is in Γ.
Suppose that h_ε in $\mathcal{H}^{-1}(\Omega)$ is such that

$$w_\varepsilon'' - \varepsilon\Delta w_\varepsilon' - \Delta w_\varepsilon + \mid w_\varepsilon + \alpha \mid^{p-2} (w_\varepsilon + \alpha) = f - h_\varepsilon + \tilde{\theta} + \hat{\theta}$$

then

$$\int_0^T (g_\varepsilon,\varphi)dt = \int_0^T (h_\varepsilon,\varphi)dt = \lim_{\eta\to 0} \eta^{-1} \int_0^T \int_\Gamma \beta(w_{\varepsilon,\eta}') \,.\varphi d\sigma dt$$

for all φ in $L^2(0,T;\mathcal{H}_0^1(\Omega))$. Hence $g_\varepsilon = h_\varepsilon$ $\square$

Lemma 4.3. *Let g_ε be as in Lemma 4.2 then*

$$\|g_\varepsilon\|_{\mathcal{H}^{-1}(\Omega)} \le C\{\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)}\}$$

C is a constant independent of ε.

Proof Let ζ be in $C_0^\infty(0,T)$ and let v be in $\mathcal{H}_0^1(\Omega)$ then we have

$$\begin{aligned}(g_\varepsilon, v)\int_0^T \zeta dt &= -\int_0^T (w_\varepsilon', \zeta' v)dt - \varepsilon\int_0^T (\nabla w_\varepsilon, \zeta'\nabla v)dt \\ &+ \int_0^T (\nabla w_\varepsilon, \zeta\nabla v)dt + \int_0^T (| w_\varepsilon + \alpha |^{p-2} (w_\varepsilon + \alpha), \zeta v)dt \\ &= \int_0^T (f + \tilde{\theta}, \zeta v)dt.\end{aligned}$$

Hence

$$\begin{aligned}| (g_\varepsilon, v) | &\le C\big\{\|w_\varepsilon'\|_{L^2(0,T;L^2(\Omega))} + \|\nabla w_\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \|w_\varepsilon + \alpha\|^{p-1}_{L^\infty(0,T;L^p(\Omega))} \\ &+ \|f\|_{L^2(0,T;L^2(\Omega))} + \|\tilde{\theta}\|_{L^2(0,T;L^2(\Omega)}\big\}\|v\|_{\mathcal{H}_0^1(\Omega)}\big\}\end{aligned}$$

Taking into account the estimate of Lemma 4.2 and of Theorem 3.1 , we get the stated result □

Proof of Theorem 4.1 1) Let $\{w_\varepsilon, g_\varepsilon\}$ be as in Lemmas 4.2-4.3. Then there exists a subsequence such that

$$\{w_\varepsilon, w_\varepsilon', g_\varepsilon\} \to \{w, w', g\}$$

in

$$[L^\infty(0,T;\mathcal{H}_0^1(\Omega))]_{weak^*} \times [L^\infty(0,T;L^2(\Omega))]_{weak^*} \times [\mathcal{H}^{-1}(\Omega)]_{weak}.$$

Furthermore

$$\{w_\varepsilon, w_\varepsilon'\} \to \{w, w'\} \text{ in } C(0,T;L^2(\Omega)) \times C(0,T;\mathcal{H}^{-1}(\Omega)).$$

It follows that $w(\,.,0) = u_0 - y_0 = w(\,.,T)$ and $w'(\,.,0) = u_1 - y_1$.
A standard argument gives

$$| w_\varepsilon + \alpha |^{p-2} (w_\varepsilon + \alpha) \to | w + \alpha |^{p-2} (w + \alpha) \text{ in}[L^q(0,T;L^q(\Omega))]_{weak}$$

2) With

$$(g, \varphi) = \lim_{\varepsilon\to 0}(g_\varepsilon, \varphi)\ \forall\varphi \in \mathcal{H}_0^1(\Omega)$$

and support of g_ε in Γ, it is clear that the support of g is also in Γ.
It is now trivial to check that $\{w, g\}$ is a solution of (4.1) □

5 Exact controllability

In this section we shall establish the exact controllability of a strongly nonlinear wave equation with Dirichlet boundary controls. The main result of the paper is the following theorem

Theorem 5.1. *Let f be in $L^2(0,T;L^2(\Omega))$ and let $\{u_0,u_1\}$, $\{v_0,v_1\}$ be in $\{\mathcal{H}_0^1(\Omega)\cap H^2(\Omega)\}\times H^1(\Omega)$. Suppose that $2\le p\le 2(n-1)/(n-2)$ with $2\le p<\infty$ if $n=2$. Then there exists*
(i) a time independent control χ in $\mathcal{H}^{-1}(\Omega))$, support on Γ
(ii) a weak solution w of the problem

$$\begin{aligned} w''-\Delta w+\mid w\mid^{p-2}w \quad &= \quad f \text{ in } \Omega\times(0,T),\\ w=0 \text{ on } (\partial\Omega/\Gamma)\times(0,T) \quad &, \quad \nu.\nabla w=\chi \text{ on } \Gamma\times(0,T),\\ \{w,w'\}\mid_{t=0}=\{u_0,u_1\} \quad &, \quad \{w,w'\}\mid_{t=T}=\{v_0,v_1\} \end{aligned} \tag{5.1}$$

Moreover $\{w,w'\}\in L^\infty(0,T;\mathcal{H}_0^1(\Omega))\times L^\infty(0,T;L^2(\Omega))$

Let α be as in Theorem 3.1 with $y_1=0$ and denote by α_* the extension of α to $(-T,0)$ as an even function

$$\alpha_*(\,.,t)=\alpha(\,.,t) \text{ for } t\in[0,T],\ \alpha_*(\,.,t)=\alpha(\,.,-t) \text{ for } t\in[-T,0]$$

Since $\alpha'(\,.,0)=y_1=0$ we have $\alpha_*'(\,.,0)=0.$ and α_* is in $L^\infty(-T,T;H^1(\Omega))$. Similarly we denote by $f_*,\ \theta_*$ the extension of $f,\ \theta=\hat\theta+\tilde\theta$ to $(-T,0)$ as even functions.

First we consider the nonlinear problem

$$\begin{aligned} \hat w''-\Delta\hat w+\mid\hat w+\alpha_*\mid^{p-2}(\hat w+\alpha_*) \quad &= \quad f_*+\theta_* \text{ in } (-T,0)\times\Omega,\\ \hat w=0 \text{ on } (\partial\Omega/\Gamma)\times(-T,0) \quad &, \quad \nu.\nabla\hat w=-g \text{ on } \Gamma\times(-T,0),\\ \hat w(\,.,-T)=u_0-y_0=\hat w(\,.,0) \quad &, \quad \hat w'(\,.,-T)=-(v_1-z_1) \end{aligned} \tag{5.2}$$

Lemma 5.1. *Let u_0,v_0,z_0,α be as in Theorem 4.1 with $y_1=0$. Then there exists*
(i) a time-independent control g in $\mathcal{H}^{-1}(\Omega)$ with support in Γ,
(ii) a weak solution $\hat w$ of (5.2) with $\{\hat w,\hat w'\}$ in

$$L^\infty(-T,0;\mathcal{H}_0^1(\Omega))\times L^\infty(-T,0;L^2(\Omega)).$$

Proof. It is Theorem 4.1 when we start at $-T$ instead of 0 □

Let

$$\tilde w(\,.,s)=\hat w(\,.,t) \text{ for } s=-t,\ -T<t<0$$

Then

$$\frac{\partial^2 \tilde{w}}{\partial s^2} = \hat{w}'',\ \Delta\tilde{w} = \Delta\hat{w}$$

and (5.2) becomes

$$\begin{aligned} \tilde{w}''(\,.,s) - \Delta\tilde{w} + \mid \tilde{w} + \alpha_* \mid^{p-2} (\tilde{w} + \alpha_*) &= f_* + \theta_* \text{ in } (0,T) \times \Omega, \\ \tilde{w} = 0 \text{ on } (\partial\Omega) \times (0,T) \quad &, \quad \nu.\nabla\tilde{w} = -g \text{ on } \Gamma \times (0,T), \\ \tilde{w}(\,.,T) = u_0 - y_0 = \tilde{w}(\,.,0) \quad &, \quad \tilde{w}'(\,.,T) = v_1 - z_1 \end{aligned} \tag{5.3}$$

Set

$$\begin{aligned} w_*(\,.,t) &= \hat{w}(\,.,t) \text{ for } -T < t \leq 0 \\ &= \tilde{w}(\,.,t) = \hat{w}(\,.,-t) \text{ for } 0 \leq T. \end{aligned}$$

From (5.2)-(5.3) we get

$$\begin{aligned} w_*'' - \Delta w_* + \mid w_* + \alpha_* \mid^{p-2} (w_* + \alpha_*) &= f_* + \theta_* \text{ on } (-T,T) \times \Omega, \\ w_* = 0 \text{ on } (\partial\Omega/\Gamma) \times \Omega \quad &, \quad \nu.\nabla w_* = -g \text{ on } (-T,T) \times \Gamma \\ w_*(\,.,-T) = u_0 - y_0 &= w_*(\,.,0) \\ w_*'(\,.,-T) = -(v_1 - z_1) \quad &, \quad w_*'(\,.,T) = v_1 - z_1 \end{aligned} \tag{5.4}$$

We now consider the initial boundary value problem

$$\begin{aligned} v'' - \Delta v + \mid v + \alpha_* \mid^{p-2} (v + \alpha_*) &= f_* + \theta_* \text{ in } (-T,T) \times \Omega, \\ v = 0 \text{ on } (\partial\Omega/\Gamma) \times (0,T) \quad &, \quad \nu.\nabla v = -g \text{ on } \Gamma \times (-T,T) \\ v(\,.,-T) = u_0 - y_0 \quad &, \quad v'(\,.,-T) = -(v_1 - z_1) \end{aligned} \tag{5.5}$$

Lemma 5.2. *Suppose all the hypotheses of Theorem 5.1 are satisfied then there exists a unique solution v of (5.5) with $\{v, v', v''\}$ in*

$$L^\infty(-T,T;\mathcal{H}_0^1(\Omega)) \times L^\infty(-T,T;L^2(\Omega)) \times L^2(-T,T;H^{-1}(\Omega))$$

Proof. The proof of the existence of a solution is standard and we shall not reproduce it.

Suppose that $\tilde{v}$ and $\hat{v}$ are two solutions of (5.5) and set

$$v_* = \tilde{v} - \hat{v} = \{\tilde{v} + \alpha_*\} - \{\hat{v} + \alpha_*\}$$

We have as in Lions [5], p.15 (1.55)

$$
\begin{aligned}
&\mid (\mid \tilde{v}+\alpha_* \mid^{p-2} (\tilde{v}+\alpha_*) - \mid \hat{v}+\alpha_* \mid^{p-2} (\hat{v}+\alpha_*),\, v_*') \mid \\
&\leq C \sup\{\|\tilde{v}+\alpha_*\|^{p-2}_{L^\infty(0,T;L^n(\Omega))}, \|\hat{v}+\alpha_*\|^{p-2}_{L^\infty(0,T;L^n(\Omega))}\} \\
&\times \|(\tilde{v}+\alpha_*)(t) - (\hat{v}+\alpha_*)(t)\|_{L^q(\Omega)} \|v'(t)\|_{L^2(\Omega)} \\
&\leq C\{\|\tilde{v}+\alpha_*\|_{L^\infty(0,T;H^1(\Omega))} + \|\hat{v}+\alpha_*\|_{L^\infty(0,T;H^1(\Omega))}\} \|v(t)\|_{H^1(\Omega)} \|v_t'\|_{L^2(\Omega)}
\end{aligned}
$$

with $1/q + 1/n + 1/2 = 1$.

Now the same proof as in [12], p.15 shows that the solution is unique .

Proof of Theorem 5.1 1) We deduce from Lemma 5.2 that $v = w_*$ and thus

$$w_*' \in C(-T, T; H^{-1}(\Omega))$$

and since w_* is an even function we have

$$D_t^+ w_*(\,.\,, 0) = -D_t^- w_*(\,.\,, 0)$$

On the other hand w_*' is in $C(-T, T; H^{-1}(\Omega))$, therefore $w_*'(\,.\,, 0) = 0$.
2) We now have from (5.3)

$$
\begin{aligned}
\tilde{w}'' - \Delta\tilde{w} + \mid \tilde{w}+\alpha \mid^{p-2} (\tilde{w}+\alpha) &= f + \theta \text{ on } \Omega \times (0,T), \\
\tilde{w} = 0 \ \text{ on } (\partial\Omega/\Gamma) \times (0,T) \quad &, \quad \nu.\nabla\tilde{w} = -g \ \text{ on } \Gamma \times (0,T), \\
\{\tilde{w}, \tilde{w}'\} \mid_{t=0} = \{u_0 - y_0, 0\} \quad &, \quad \{\tilde{w}, \tilde{w}'\} \mid_{t=T} = \{z_0 - y_0,\, v_1 - z_1\}
\end{aligned}
$$

Set

$$w = \tilde{w} + \alpha$$

with α as in Theorem 3.1. Then

$$
\begin{aligned}
\{w, w'\} \mid_{t=0} &= \{y_0 + u_0 - y_0, 0 + u_1\} = \{u_0, u_1\} \\
\{w, w'\} \mid_{t=T} &= \{y_0 + v_0 - y_0, z_1 + v_1 - z_1\} = \{v_0, v_1\}
\end{aligned}
$$

and taking (3.1) into account we have

$$w'' - \Delta w + \mid w \mid^{p-2} w = f \ \text{ in } \Omega \times (0,T)$$

with

$$w = 0 \text{ on } (\partial\Omega/\Gamma) \times (0,T)$$

and

$$\nabla w \mid_{\Gamma} = \nu.\nabla \tilde{w} \mid_{\Gamma} + \nu.\nabla \alpha \mid_{\Gamma} = -g + 0$$

The theorem is proved □

Remark One may wish to have Dirichlet type of boundary control.Let α be as in Theorem 3.1 but with null Dirichlet boundary conditions instead of mixed Neumann-Dirichlet null boundary conditions and let $w = \tilde{w} + \alpha$, then we have

$$\begin{array}{rcl} w'' - \Delta w + \mid w \mid^{p-2} w & = & f \text{ in } \Omega \times (,T), \\ \{w, w'\} \mid_{t=0} = \{u_0, u_1\} & , & \{w, w'\} \mid_{t=T} = \{v_0, v_1\} \\ w = 0 \text{ on } (\partial\Omega/\Gamma) \times (0,T) & , & w = \chi \text{ on } \Gamma \times (0,T) \end{array}$$

where

$$\chi = \tilde{w} \mid_{\Gamma} + \alpha \mid_{\Gamma} = \tilde{w} \mid_{\Gamma}$$

Since $\tilde{w}$ is in $L^{\infty}(0,T;\mathcal{H}_0^1(\Omega))$, we get χ in $L^{\infty}(0,T;H^{\gamma}(\Gamma))$ with $0 < \gamma < 1/2$. The control is time-dependent.

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