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GAKKOTOSHO
TOKYO JAPAN

LONG TIME BEHAVIOR OF A TWO FLUID MODEL

Tej-eddine Ghoul
Department of Mathematics
New York University Abu Dhabi*
(E-mail: teg6@nyu.edu)

and

Slim Ibrahim and Shengyi Shen
Department of Mathematics and Statistics
University of Victoria†
Pacific Institute for the Mathematical Sciences
(E-mail: ibrahim@math.uvic.ca and shengyis@uvic.ca)

Abstract. We consider a two fluid model which describes the motion of two charged particles in a strict neutral incompressible plasma. In this paper we mainly study the stability of the solution around zero given that the initial data is small and has sufficient regularity. In this paper we show that our system is a system of regularity-loss and the L^2 norm of lower derivatives of the solution decays with a rate.

*Saadiyat Island, P.O. Box 129188, Abu Dhabi, United Arab Emirates
†3800 Finnerty Road (Ring Road) Victoria, B.C. Canada V8P 5C2, Canada
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1 Introduction

A plasma is an ionized gas in which the molecules are ionized into negatived charged electrons and positive charged ions. The outer sphere of the Sun can be considered as a plasma. A plasma can also be produced in neon signs, controlled nuclear fusion. Particularly in nuclear fusion, the plasma physics plays a key role for simulation and control of the plasma. A way to study plasma is to consider a plasma as an 'electrically conducting fluid'. As we know in classic fluid like water, the frequent collisions of particles make them flow as a whole part. While in a plasma, the collisions of ions and electrons although infrequent, the magnetic field make ions and electrons move together so that the plasma behaves like fluid. The plasma dynamics was first given by Braginskii [2] 1965. Consider a full ionized plasma consisting of two species of particles, i.e. ions and electrons. We denote $v_\pm, \rho_\pm$ the velocities and densities of ion and electron, and (E, B) the electro-magnetic field. Then the whole system in $\mathbb{R}^3$ is the following. We also refer to [9] for a derivation.

$$\begin{cases} \partial_t \rho_- + \nabla \cdot (\rho_- v_-) = 0 \\ \partial_t \rho_+ + \nabla \cdot (\rho_+ v_+) = 0 \\ \rho_- \partial_t v_- = \nu_- \Delta v_- - \rho_- v_- \cdot \nabla v_- - \beta_-(E + v_- \times B) - R - \nabla p_- \\ \rho_+ \partial_t v_+ = \nu_+ \Delta v_+ - \rho_+ v_+ \cdot \nabla v_+ + \beta_+(E + v_+ \times B) + R - \nabla p_+ \\ \partial_t E = \dfrac{1}{\varepsilon_0 \mu_0} \nabla \times B - \dfrac{1}{\varepsilon_0}(\beta_+ v_+ - \beta_- v_-) \\ \partial_t B = -\nabla \times E \\ p_\pm n_\pm^{-\gamma} = constant \\ R := -\alpha(v_+ - v_-) \\ \mathrm{div} B = 0, \quad \mathrm{div} E = \dfrac{1}{\varepsilon_0}(\beta_+ - \beta_-). \end{cases} \tag{1}$$

Here $\rho_\pm = m_\pm n_\pm$, $\beta_- = en_-$ and $\beta_+ = en_+ Z$. The physical meaning of these parameters are

- p: the pressure;
- $m_\pm$: mass of ion and electro;
- $n_\pm$: number density of ion and electro;
- e: the elementary charge;
- Z: charge number of ion;
- $\nu_\pm$: kinetic viscosities of ion and electro;
- ε_0, μ_0: vacuum dielectric constant and permeability;
- α: a positive coefficient for momentum change between ions and electrons. More specifically, $\alpha = \nu_{ei} n_- m_- = \nu_{ie} n_+ m_+$ and ν_{ei}, ν_{ie} here are fixed coefficients;

- γ: a constant depends on the heat flux assumption and the isotropy of the energy distribution. For example, for isothermal plasma, the temperature is fixed and $\gamma = 1$.

The first two equations of (1) are the continuity equations for fluids. The third and fourth ones represent the momentum balance. The rest are Maxwell system and equations of state. In physical application, usually the fifth equation in (1) is written in an approximate way

$$\partial_t E = \frac{1}{\varepsilon_0 \mu_0} \nabla \times B - \frac{\beta_-}{\varepsilon_0}(v_+ - v_-),$$

where $\beta_- \simeq \beta_+$ is used since most of the plasma is quasi-neutral. However $\operatorname{div} E = \frac{1}{\varepsilon_0}(\beta_+ - \beta_-)$ is kept due to the fact that in experiment even very small changes in $\operatorname{div} E$ will lead to an observable changes in electromagnetic field. This is the so-called plasma approximation (see [9]). The system has 16 unknowns $(n_\pm, v_\pm, E, B, p_\pm)$ and 18 equations. Since the last two equations $\operatorname{div} B = 0$, $\operatorname{div} E = \frac{1}{\varepsilon_0}(\beta_+ - \beta_-)$ can be derived from fifth and sixth equation in (1), the system (1) actually has 16 equations. So that this system is closed.

In this paper, we mainly consider the incompressible isothermal neutral plasma so that the two continuity equations simply become $\operatorname{div} v_\pm = 0$ and the whole system rereads

$$\begin{cases} \rho_- \partial_t v_- = \nu_- \Delta v_- - \rho_- v_- \cdot \nabla v_- - \beta(E + v_- \times B) - R - \nabla p_- \\ \rho_+ \partial_t v_+ = \nu_+ \Delta v_+ - \rho_+ v_+ \cdot \nabla v_+ + \beta(E + v_+ \times B) + R - \nabla p_+ \\ \partial_t E = \dfrac{1}{\varepsilon_0 \mu_0} \nabla \times B - \dfrac{\beta}{\varepsilon_0}(v_+ - v_-) \\ \partial_t B = -\nabla \times E \\ R := -\alpha(v_+ - v_-) \\ \operatorname{div} v_- = \operatorname{div} v_+ = \operatorname{div} B = 0, \quad \operatorname{div} E = 0, \end{cases} \tag{2}$$

where $\beta = en_- = en_+ Z$ meaning that the plasma is strictly neutral.
A typical example of incompressible plasma is the outer core of the earth. The motion of ions and electrons around the core of the earth creates the strong magnetic field around the earth, protecting lives from high energy particles coming from the sun.

Mathematical analysis of system (2) started with the work of Giga-Yoshida [8] who considered a three-dimensional bounded domain with no-slip and perfectly conductive boundary condition. They proved the unique local solvability as well as global-in-time solvability when the initial data is small and whose magnetic effect is small compared with velocity. Recently, Giga, Ibrahim, Shen and Yoneda [7] improved Giga-Yoshida's result. They proved wellposedness in 2D and existence of global weak solution in 3D as well as global wellposedness with small initial data in 3D. It is important here to note that in this result, no additional constraint on the size of the initial electromagnetic field was imposed. Moreover the regularity of the initial data is lower compared to [7].

Remark 1.1. Applying the scaling $\tilde{E} = \sqrt{\frac{\varepsilon_0}{2}} E, \tilde{B} = \sqrt{\frac{1}{2\mu_0}} B$ and setting $\varepsilon = \frac{1}{\beta\sqrt{2\mu_0}}, \nu_\pm =$

$\nu, \rho_\pm = 1$ our system becomes

$$\begin{cases} \partial_t v_- = \nu \Delta v_- - v_- \cdot \nabla v_- - \dfrac{1}{\varepsilon}(c\tilde{E} + v_- \times \tilde{B}) - R - \nabla p_- \\ \partial_t v_+ = \nu \Delta v_+ - v_+ \cdot \nabla v_+ + \dfrac{1}{\varepsilon}(c\tilde{E} + v_+ \times \tilde{B}) + R - \nabla p_+ \\ \dfrac{1}{c}\partial_t \tilde{E} = \nabla \times \tilde{B} - \dfrac{1}{2\varepsilon}(v_+ - v_-) \\ \dfrac{1}{c}\partial_t \tilde{B} = -\nabla \times \tilde{E} \\ R := -\dfrac{1}{2\sigma\varepsilon^2}(v_+ - v_-) \\ \mathrm{div} v_- = \mathrm{div} v_+ = \mathrm{div}\tilde{B} = 0, \quad \mathrm{div}\tilde{E} = 0, \end{cases} \tag{3}$$

where $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ is the speed of light, σ is the electrical conductivity.
In 2015, Arsénio, Ibrahim and Masmoudi [1] proved that the solution of (3) actually converges to the standard MHD system under the relaxing limit $\lim_{c\to\infty} c^2\varepsilon = \infty$, or equivalently $\varepsilon_0\mu_0 \to 0$, $\frac{1}{\varepsilon_0\mu_0 n_- e\sqrt{2\mu_0}} \to \infty$. For example, for the out core of the earth one has $n_- \simeq 6 * 10^{30}$. So that

$$\frac{1}{\varepsilon_0\mu_0 n_- e\sqrt{2\mu_0}} \simeq 7 * 10^7,$$

the relaxing limit makes sense physically.

Remark 1.2. Formally, the classic one fluid Navier-Stokes-Maxwell system,

$$\begin{cases} \partial_t v + v \cdot \nabla v - \nu \Delta u + \nabla p = j \times B, \\ \partial_t E - \nabla \times B = -j, \\ \partial_t B + \nabla \times E = 0, \\ \sigma(E + u \times B) = \eta j, \\ \nabla \cdot v = \nabla \cdot B = 0. \end{cases} \tag{4}$$

can be derived from our system (2)(see [12]). Indeed, if we define the bulk velocity by $v = \frac{\rho_- v_- + \rho_+ v_+}{\rho}$ where $\rho = \rho_+ + \rho_-$ and current density by $j = \beta(v_+ - v_-)$, then adding the two velocities equations in our original system (2) and dividing by ρ gives

$$\begin{aligned} \partial_t v = {}& \frac{\nu_- + \nu_+}{\rho}\Delta v + \frac{\rho_-\nu_+ - \rho_+\nu_-}{\beta\rho^2}\Delta j \\ & - \frac{1}{\rho}(\rho_- v_- \cdot \nabla v_- + \rho_+ v_+ \cdot \nabla v_+) + \frac{1}{\rho} j \times B - \nabla \frac{p_- + p_+}{\rho}. \end{aligned}$$

Since $m_-/m_+ \ll 1$, we have $\rho \approx \rho_+, \rho_-/\rho_+ \approx 0$ so that formally

$$\begin{aligned} & \frac{\rho_-\nu_+ - \rho_+\nu_-}{\beta\rho^2}\Delta j \approx 0, \\ & \frac{1}{\rho}(\rho_- v_- \cdot \nabla v_- + \rho_+ v_+ \cdot \nabla v_+) = v \cdot \nabla v + \frac{\rho_+\rho_-}{\beta^2\rho^2} j \cdot \nabla j \approx v \cdot \nabla v, \end{aligned}$$

which yield the first equation in (4) after normalizing some constants to 1.
Next we drive Ohm's law formally (the fourth equation in (4)). Since $v_- = v - \frac{\rho_+ j}{(\rho_+ + \rho_-)\beta} \approx v - \frac{j}{\beta}$, the first equation in (2) can be written as

$$\frac{m_-}{e}\partial_t v - \frac{m_-}{e^2 n_-}\partial_t j + \frac{1}{en_-} v_- \cdot \nabla v_- = -\frac{1}{en_-}\nabla p_- - (E + (v - \frac{j}{en_-}) \times B) + \frac{\alpha}{e^2 n_-^2} j.$$

Using the approximation that $\frac{m_-}{e}, \frac{m_-}{e^2 n_-}, \frac{1}{en_-} \approx 0$ to eliminate the unnecessary terms and letting $\eta = \frac{\alpha}{e^2 n_-^2}$ gives the Ohm's law.
We emphasize that the derivation above is quite formal and refer to [12].

In this paper, we focus on the time decay or stability of the zero solution to (2). In the case of Navier-Stokes equations, we refer to the pioneer works of Schonbek [14] and [15] where the author derived the optimal decay rate of solutions to 2D and 3D Navier-Stokes system. In [14], she established that for 2D Navier-Stokes equation, if the initial data $u_0 \in H^s \cap L^1$ (no smallness condition), then the solution u satisfies $\|D^k u\|_{L^2}^2 \lesssim (1+t)^{-1/2-k/2}$, where D^k is any spatial derivative with order k. Furthermoer if the average of u_0 is zero, i.e. $\int u_0 dx = 0$ then the lower bound holds $\|u_0\|_{L^2} \gtrsim (1+t)^{-1/2}$. In the later work [15], a better result is given: if $\int u_0 dx = 0$ and $u_0 \in L^1 \cap H^1$ then it holds that $\|u_0\|_{L^2}^2 \approx (1+t)^{-d/2+1}$, $d = 2, 3$. When $d = 2$, u is the classic solution. For $d = 3$, u is a suitable Leray-Hopf solution in the sense of Caffarelli-Kohn-Nirenberg [3]. The idea is to decompose the frequency space into two time depended subset, then obtain a first order differential inequality for the H^k norm of the solution. The difficulty here is mainly the low frequency part which was overcame by taking advantages of the linear system of Navier-Stokes equation.

For our system, one can observe that (2) is damped Navier-Stokes equations coupled with Maxwell equations. Due to this coupling, the whole linear system requires more regularity on initial data to get our desired decay result (see Lemma 3.1). Roughly speaking, the solution of the linear system in Fourier side satisfies

$$\hat{U}(t,\xi) \lesssim e^{-\rho(\xi)t}\hat{U}_0(\xi),$$

where $U = (v_-, v_+, E, B)$ and $\rho(\xi) \approx |\xi|^2$ for $|\xi| \leq 1$, $\rho(\xi) \approx \frac{1}{|\xi|^2}$ for $|\xi| \geq 1$. So that at the linear level one has

$$\|D^k U\|_{L^2} \lesssim (1+t)^{-3/4-k/2}\|U_0\|_{L^2} + (1+t)^{-l/2}\|D^{k+l}U_0\|_{L^2}.$$

To get such an estimate, one can rewrite the system in Fourier side and obtain a Fourier muliplier matrix. In gerneral, it is easy to explicitly compute the eigenvalues of this Fourier muliplier matrix and thus obtain the linear decay. For example, this is the case in [16], [17] and [18]. However in our case, the two velocity equations and the non-normalized physical constants make such calculation very complicated and hard. To overcome such complex computation, we introduce a Lyapunov function that gives in a more systemetic way the linear decay. Such an idea can be implicitly found, for example, in [4] and [5], for other models.
The bad behavior of the high frequency part requires the extra regularity on initial data to

get the time decay. Thus our model is a system of regularity-loss type. There are plenty of works studying the decay property of equations of regularity-loss type, for example the work of Hosono and Kawashima [10] on some nonlinear hyperbolic-elliptic equation, Houari [13] on a nonlinear Bresse system. Another well-know system of regularity-loss type is one fluid compressible Euler-Maxwell system. We refer [16] and [17] for details. Recently Xu and Cao [18] proved the decay of one fluid compressible Navier-Stokes-Maxwell system. The time weighted energy method is a key tool to prove the decay for regularity-loss type system. Here we choose a nonlinear hyperbolic-elliptic system from [10] to briefly introduce and motivate the time weighted energy method. The system reads

$$\begin{cases} \partial_t u + \partial_x (u^2/2) + \partial_x q = 0, \\ \partial_x^4 q - \partial_x^2 q + q + \partial_x u = 0, \end{cases} \tag{5}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. One can easily solve the linear system in Fourier side for u: $\hat{u}(t,\xi) = e^{-\rho(\xi)t}\hat{u}_0$, where $\rho(\xi) = \frac{\xi^2}{1+\xi^2+\xi^4}$. Then the linear solution u satisfies

$$\|D^k u\|_{L^2} \lesssim (1+t)^{-1/4-k/2}\|u_0\|_{L^2} + (1+t)^{-l/2}\|D^{k+l}u_0\|_{L^2}.$$

Hence system (5) is also a system of regularity-loss type. Now back to the nonlinear system, we want to obtain the decay of k-th derivative of nonlinear solution u. If we applied D^k to (5) and the classic energy method, one gets with initial data $u_0 \in H^s$

$$\|u\|_{H^s}^2 + 2\int_0^t \|q\|_{H^{s+2}}^2 d\tau \lesssim \|U_0\|_{H^s}^2 + \int_0^t \|\partial_x u(\tau)\|_{L^\infty} \|\partial_x u(\tau)\|_{H^{s-1}}^2 d\tau.$$

To control the nonlinearity, we need to control the term $\int_0^t \|\partial_x u(\tau)\|_{H^{s-1}}^2 d\tau$. Usually this can be done by the help of dissipative term in second equation of (5). However, the dissipative term only gives us

$$\int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau \lesssim \int_0^t \|q(\tau)\|_{H^{s+2}}^2 d\tau,$$

which can not control the nonlinearity due to the loss of regularity.
To overcome this difficult of regularity-loss, when applying the classic H^k energy method, instead of multiplying by $D^k u$, we multiply by $(1+t)^\alpha D^k u$. This will give us

$$\begin{aligned} &(1+t)^\alpha \|\partial_x^k u\|_{L^2}^2 + 2\int_0^t (1+\tau)^\alpha \|\partial_x^k q(\tau)\|_{H^2}^2 d\tau \\ &\lesssim \|\partial_x^k u_0\|_{L^2}^2 + \alpha \int_0^t (1+\tau)^{\alpha-1} \|\partial_x^k u(\tau)\|_{L^2}^2 d\tau \\ &+ \int_0^t (1+\tau)^\alpha \|\partial_x u(\tau)\|_{L^\infty} \|\partial_x^k u(\tau)\|_{L^2}^2 d\tau. \end{aligned}$$

If we choose $\alpha < 0$, then the term $\alpha \int_0^t (1+\tau)^{\alpha-1} \|\partial_x^t u(\tau)\|_{L^2}^2 d\tau$ is like an artificial dissipative term and is good enough to control the nonlinearity if $(1+t)\|\partial_x u\|_{L^\infty}$ is small. For system (5), we refer to [10] for the detailed discussion.
Our first result is about the existence of smooth solution of system (2).

Theorem 1.1. Let $s \geq 3$ be an integer and the initial data of system (2) $U_0 = (v_{-,0}, v_{+,0}, E_0, B_0) \in H^s$. Then there exists a constant $\delta > 0$ such that if $\|U_0\|_{H^s} < \delta$, the Cauchy problem of (2) has a unique solution and satisfies that for any $T > 0, U \in C([0,T]; H^s)$. Furthermoer, $v_\pm \in C^1([0,T); H^{s-2}), E, B \in C^1([0,T); H^{s-1})$.

After that we apply the time weighted energy method and get the following decay result.

Theorem 1.2. Let $s \geq 7$ and $U_0 \in L^1 \cap H^s$ the initial data of system (2). Then there exists a constant $\delta > 0$ such that if $\|U_0\|_{L^1 \cap H^s} \leq \delta$ the unique solution given by Theorem 1.1 satisfies the following decay property,

$$\|D^k U\|_{H^{s-2k-3}} \lesssim (1+t)^{-3/4-k/2},$$

for all integers $0 \leq k \leq [(s-1)/2] - 1$.

Remark 1.3. Taking $v_{\pm,0} = v_0, E = B = 0$ then the above decay result reads

$$\|D^k v\|_{H^{s-2k-3}} \lesssim (1+t)^{-3/4-k/2},$$

and thus one recovers Schonbek's result (see [14] and [15]) for small k.

Remark 1.4. Unlike the Euler-Maxwell system ([16]), there is no uniformly time decay if the initial data is only in H^s. The presence of the dissipation term requires that $U_0 \in L^1$ to get the uniform decay. See for example [14]

Remark 1.5. This remark compares our result with the decay result of classic one fluid Navier-Stokes-Maxwell system (4). Ghoul, Ibrahim and Said-Houari [6] showed that for s big enough and small initial data $U_0 \in L^1 \cap H^s$, it holds that $\|D^k U\|_{L^2} \lesssim (1+t)^{-3/4-k/2}$ for $0 \leq k \leq s$. They use Lyapunov functional method to prove the decay of (E, B) in linear level which actually behaves better than the solutions to hyperbolic system. So that the whole system (4) is not regularity-loss type.

We use the following notation through out the whole paper.

- $\hat{\cdot}$, Fourier transform in space;
- $\|\cdot\|_{H^s}$, the H^s inhomogeneous norm defined by

$$\|u\|_{H^s}^2 := \int_{\xi \in \mathbb{R}^3} (1+|\xi|^2)^s |\hat{u}|^2 d\xi.$$

- $f \lesssim g$, there is a universal constant C such that $f \leq Cg$.

The paper is organized as following. In next section, we proved existence by energy method. Then we do linear estimate to show our system is a regularity-loss type system. And the proof of Theorem 1.2 follows in last section.

2 proof of Theorem 1.1

By the similar fixed point argument as in [7], one can easily construct a unique solution in $U \in C([0,T];H^s)$ with T small enough. We will prove that $U \in C([0,T];H^s)$ for any $T>0$ so that the global existence holds true. Then using the equation, it is clear that

$$\partial_t v_\pm \in C([0,T];H^{s-2}) \quad , \quad \partial_t E, \partial_t B \in C([0,T];H^{s-1}).$$

Now, we prove that $U \in C([0,T];H^s)$ for any $T>0$.
Recall that $U=(v_-,v_+,E,B), v=(v_-,v_+)$. Fixing $s>0$, we define the following norms:

$$\begin{aligned}
&N_0(t) = \sup_{0\le\tau\le t} \|U\|_{H^s}\\
&D_0^2(t) = \int_0^t \|Dv\|_{H^s}^2 + \|v_- - v_+\|_{H^s}^2 + \|E\|_{H^{s-1}}^2 + \|DB\|_{H^{s-2}}^2 d\tau\\
&W_0(t) = \sup_{0\le\tau\le t} \|U\|_{W^{1,\infty}}, \quad J_0^2(t) = \int_0^t \|v\|_{L^\infty}^2 d\tau.
\end{aligned} \tag{6}$$

It is sufficient to prove the uniform bound of $N_0(t)$ and $D_0(t)$. Applying D^l to (2) we have

$$\begin{cases}
\rho_-\partial_t D^l v_- - \nu_-\Delta D^l v_- + \rho_- v_-\cdot\nabla D^l v_- + \beta(D^l E + D^l v_- \times B) + \alpha(D^l v_- - D^l v_+)\\
\qquad = -\rho_-[D^l(v_-\cdot\nabla v_-) - v_-\cdot\nabla D^l v_-] - \beta[D^l(v_-\times B) - D^l v_- \times B] + \nabla D^l p_-,\\
\rho_+\partial_t D^l v_+ - \nu_+\Delta D^l v_+ - \rho_+ v_+\cdot\nabla D^l v_+ - \beta(D^l E + D^l v_+ \times B) - \alpha(D^l v_- - D^l v_+)\\
\qquad = -\rho_+[D^l(v_+\cdot\nabla v_+) - v_+\cdot\nabla D^l v_+] + \beta[D^l(v_+\times B) - D^l v_+ \times B] + \nabla D^l p_-,\\
\partial_t D^l E = \dfrac{1}{\varepsilon_0\mu_0}\nabla\times D^l B - \dfrac{\beta}{\varepsilon_0} D^l(v_+ - v_-),\\
\partial_t D^l B = -\nabla\times D^l E.
\end{cases} \tag{7}$$

Next, we will apply standard energy estimate in H^l. Multiplying the first equation by $D^l v_-$, second equation by $D^l v_+$, third equation by $\varepsilon_0 D^l E$, and the last equation by $\frac{1}{\mu_0}D^l B$ then integrating in space and adding together yields.

$$\begin{aligned}
&\frac{\rho_-}{2}\frac{d}{dt}\|D^l v_-\|_{L^2}^2 + \frac{\rho_+}{2}\frac{d}{dt}\|D^l v_+\|_{L^2}^2 + \frac{\varepsilon_0}{2}\frac{d}{dt}\|D^l E\|_{L^2}^2 + \frac{d}{dt}\frac{1}{2\mu_0}\|D^l B\|_{L^2}^2\\
&\quad + \nu_-\|\nabla D^l v_-\|_{L^2}^2 + \nu_+\|\nabla D^l v_+\|_{L^2}^2 + \alpha\|D^l v_- - D^l v_+\|_{L^2}^2\\
&= -\rho_-\left(D^l(v_-\cdot\nabla v_-) - v_-\cdot\nabla D^l v_-, D^l v_-\right) - \rho_+\left(D^l(v_+\cdot\nabla v_+) - v_+\cdot\nabla D^l v_+, D^l v_+\right)\\
&\quad - \beta\left(D^l(v_-\times B) - D^l v_-\times B, D^l v_-\right) + \beta\left(D^l(v_+\times B) - D^l v_+\times B, D^l v_+\right),
\end{aligned} \tag{8}$$

where $(\cdot,\cdot)$ is the L^2 inner product. Applying the following lemma (we refer to Lemma 3.4 in [11]) to the right hand side of (8),

Lemma 2.1. For $l>0$, it holds that

$$\begin{aligned}
&\|D^l(ab)\|_{L^2} \lesssim \|a\|_{L^\infty}\|D^l b\|_{L^2} + \|D^l a\|_{L^2}\|b\|_{L^\infty}\\
&\|D^l(a\cdot\nabla b) - a\cdot\nabla D^l b\|_{L^2} \lesssim \|D^l a\|_{L^2}\|\nabla b\|_{L^\infty} + \|\nabla a\|_{L^\infty}\|D^l b\|_{L^2}
\end{aligned}$$

we have the classic H^l energy estimate,

$$\begin{aligned}
&\frac{\rho_-}{2}\frac{d}{dt}\|D^l v_-\|_{L^2}^2 + \frac{\rho_+}{2}\frac{d}{dt}\|D^l v_+\|_{L^2}^2 + \frac{\varepsilon_0}{2}\frac{d}{dt}\|D^l E\|_{L^2}^2 + \frac{d}{dt}\frac{1}{2\mu_0}\|D^l B\|_{L^2}^2 \\
&\quad + \nu_-\|\nabla D^l v_-\|_{L^2}^2 + \nu_+\|\nabla D^l v_+\|_{L^2}^2 + \alpha\|D^l v_- - D^l v_+\|_{L^2}^2 \\
&\lesssim \|\nabla v_\pm\|_{L^\infty}\|D^l v_\pm\|_{L^2}^2 + \|B\|_{L^\infty}\|D^l v_\pm\|_{L^2}^2 + \|v_\pm\|_{L^\infty}\|D^l v_\pm\|_{L^2}\|D^l B\|_{L^2},
\end{aligned} \tag{9}$$

where $1 \leq l \leq s$. When $l = 0$, (9) becomes the classic energy estimate in which the right hand side of (8) vanishes.
Therefore summing (9) from $l = 0$ to s then integrating in time give us the first ápriori energy estimate

$$N_0(t) + \int_0^t \|Dv\|_{H^s}^2 + \|v_- - v_+\|_{H^s}^2 d\tau \lesssim \|U_0\|_{H^s}^2 + W_0(t)D_0^2(t) + N_0(t)J_0(t)D_0(t). \tag{10}$$

Now we linearize (2) around the trivial solution $U = 0$. Namely, rewrite this system as

$$\begin{cases}
\rho_-\partial_t v_- - \nu_-\Delta v_- + \beta E + \alpha(v_- - v_+) + \nabla p_- = -\rho_- v_- \cdot \nabla v_- - v_- \times B, \\
\rho_+\partial_t v_+ - \nu_+\Delta v_+ - \beta E - \alpha(v_- - v_+) + \nabla p_+ = -\rho_+ v_+ \cdot \nabla v_+ + v_+ \times B, \\
\partial_t E - \dfrac{1}{\varepsilon_0\mu_0}\nabla \times B + \dfrac{\beta}{\varepsilon_0}(v_+ - v_-) = 0, \\
\partial_t B + \nabla \times E = 0,
\end{cases} \tag{11}$$

To estimate the dissipation term $D_0(t)$, apply D^l to (11), multiply the first three equations by $\frac{1}{\rho_-}D^l E, -\frac{1}{\rho_+}D^l E, D^l(v_- - v_+)$ respectively and summing them yields

$$\begin{aligned}
&\partial_t \left(D^l(v_- - v_+), D^l E\right) + \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})\|D^l E\|_{L^2}^2 = \\
&\quad (\frac{\nu_-}{\rho_-}\Delta D^l v_- - \frac{\nu_+}{\rho_+}\Delta D^l v_+, D^l E) \\
&\quad - \left(\frac{\alpha}{\rho_-} + \frac{\alpha}{\rho_+}\right)\left(D^l(v_- - v_+), D^l E\right) + \frac{\beta}{\varepsilon_0}\|D^l(v_- - v_+)\|_{L^2}^2 \\
&\quad + \frac{1}{\varepsilon_0\mu_0}\left(\nabla \times D^l B, D^l(v_- - v_+)\right) + f_1 + f_2,
\end{aligned} \tag{12}$$

where,

$$\begin{aligned}
f_1 &= -\left(D^l(\mathrm{div}(v_- \otimes v_-) - \mathrm{div}(v_+ \otimes v_+)), D^l E\right), \\
f_2 &= -\left(\frac{1}{\rho_-}D^l(v_- \times B) + \frac{1}{\rho_+}D^l(v_+ \times B), D^l E\right).
\end{aligned}$$

By Lemma 2.1, f_1, f_2 can be estimated as the following,

$$\begin{aligned}
f_1 &\lesssim \|v_\pm\|_{L^\infty}\|D^{l+1}v_\pm\|_{L^2}\|D^l E\|_{L^2}, \\
f_2 &\lesssim (\|D^l v_\pm\|_{L^2}\|B\|_{L^\infty} + \|v_\pm\|_{L^\infty}\|D^l B\|_{L^2})\|D^l E\|_{L^2}.
\end{aligned}$$

Applying Cauchy-Schwarz and Young's inequality to the other terms on the right hand side of (12), then plugging the estimates of f_1, f_2 into (12) yields that for $l = 0$,

$$\begin{aligned} &\partial_t \left((v_- - v_+), E\right) + \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})\|E\|_{L^2}^2 \\ &\lesssim C_\varepsilon \|D_0^2 v_\pm\|_{L^2}^2 + C_\varepsilon \|(v_- - v_+)\|_{L^2}^2 + \varepsilon\|E\|_{L^2}^2 + \varepsilon\|DB\|_{L^2}^2 \\ &\quad + \|v_\pm\|_{L^\infty}\|Dv_\pm\|_{L^2}\|E\|_{L^2} \\ &\quad + (\|v_\pm\|_{L^2}\|B\|_{L^\infty} + \|v_\pm\|_{L^\infty}\|B\|_{L^2})\|E\|_{L^2}. \end{aligned} \tag{13}$$

For $l \geq 1$, using $\big(\nabla \times D^l B, D^l(v_- - v_+)\big) = \big(D^l B, \nabla \times D^l(v_- - v_+)\big)$, we get

$$\begin{aligned} &\partial_t \left(D^l(v_- - v_+), D^l E\right) + \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})\|D^l E\|_{L^2}^2 \\ &\lesssim C_\varepsilon \|D^{l+2} v_\pm\|_{L^2}^2 + C_\varepsilon \|D^l(v_- - v_+)\|_{H^1}^2 + \varepsilon\|D^l E\|_{L^2}^2 + \varepsilon\|D^l B\|_{L^2}^2 \\ &\quad + \|v_\pm\|_{L^\infty}\|D^{l+1} v_\pm\|_{L^2}\|D^l E\|_{L^2} \\ &\quad + (\|D^l v_\pm\|_{L^2}\|B\|_{L^\infty} + \|v_\pm\|_{L^\infty}\|D^l B\|_{L^2})\|D^l E\|_{L^2}. \end{aligned} \tag{14}$$

Remark 2.1. Here we deal with the case $l = 0$ and $l > 0$ in sightly different ways. Noting that we will integrate in time at last and the L^2 norms of derivatives of B should be controlled by the dissipation norm $D_0(t)$, the terms $\|B\|_{L^2}$ and $\|D^s B\|_{L^2}$ can not appear on the right hand side of (13) and (14).

By choosing $\varepsilon < \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})$, integrating (13), (14) in time and summing from $l = 0$ to $l = s - 1$ yields

$$\begin{aligned} \int_0^t \|E\|_{H^{s-1}}^2 d\tau \lesssim\ & \|U\|_{H^s}^2 + \int_0^t \|Dv\|_{H^s}^2 + \|v_- - v_+\|_{H^s}^2 d\tau \\ & + \|U_0\|_{H^s}^2 + \varepsilon \int_0^t \|DB\|_{H^{s-2}}^2 d\tau \\ & + \int_0^t \sum_{l=0}^{s-1} \|v_\pm\|_{L^\infty}\|D^{l+1} v_\pm\|_{L^2}\|D^l E\|_{L^2} d\tau \\ & + \int_0^t \sum_{l=0}^{s-1} (\|D^l v_\pm\|_{L^2}\|B\|_{L^\infty} + \|v_\pm\|_{L^\infty}\|D^l B\|_{L^2})\|D^l E\|_{L^2} d\tau \end{aligned} \tag{15}$$

By (10) and

$$\begin{aligned} &\int_0^t \sum_{l=0}^{s-1} (\|v_\pm\|_{L^\infty}\|D^{l+1} v_\pm\|_{L^2} + \|D^l v_\pm\|_{L^2}\|B\|_{L^\infty} + \|v_\pm\|_{L^\infty}\|D^l B\|_{L^2})\|D^l E\|_{L^2} \\ &\lesssim N_0(t) J_0(t) D_0(t), \end{aligned}$$

the inequality (15) becomes

$$\int_0^t \|E\|_{H^{s-1}}^2 d\tau \lesssim \|U_0\|_{H^s}^2 + \varepsilon \int_0^t \|DB\|_{H^{s-2}}^2 d\tau + W_0(t) D_0^2(t) + N_0(t) J_0(t) D_0(t). \tag{16}$$

For the dissipation estimate of B we only use maxwell equations in (11). Applying $\nabla\times D^l$ to third and fourth equations in (11) yields

$$\begin{cases}\partial_t\nabla\times D^lE+\dfrac{1}{\varepsilon_0\mu_0}\Delta D^lB+\dfrac{\beta}{\varepsilon_0}\nabla\times D^l(v_--v_+)=0,\\ \partial_t\nabla\times D^lB-\Delta D^lE=0.\end{cases}\tag{17}$$

Multiply the first equation by D^lB and the second one by D^lE then integrate in space and add together yielding

$$\begin{aligned}&\partial_t(\nabla\times D^lE,D^lB)-\frac{1}{\varepsilon_0\mu_0}\|D^{l+1}B\|_{L^2}^2+\\&\|D^{l+1}E\|_{L^2}^2+\frac{\beta}{\varepsilon_0}\big(D^l(v_--v_+),\nabla\times D^lB\big)=0.\end{aligned}$$

Applying Cauchy-Schwarz inequality the above inequality becomes

$$\begin{aligned}\|D^{l+1}B\|_{L^2}^2\lesssim\varepsilon\|D^{l+1}B\|_{L^2}^2+C_\varepsilon\|D^l(v_--v_+)\|_{L^2}^2+\|D^{l+1}E\|_{L^2}^2\\+\partial_t(\nabla\times D^lE,D^lB).\end{aligned}\tag{18}$$

Choosing ε small, integrating (18) in time and summing from $l=0$ to $l=s-2$ give

$$\begin{aligned}\int_0^t\|DB\|_{H^{s-2}}^2d\tau\lesssim\|U\|_{H^s}^2+\|U_0\|_{H^s}^2+\int_0^t\|E\|_{H^{s-1}}^2d\tau\\+\int_0^t\|v_--v_+\|_{H^s}^2d\tau.\end{aligned}\tag{19}$$

Again using (10) and (16), (19) becomes

$$\int_0^t\|DB\|_{H^{s-2}}^2d\tau\lesssim\|U_0\|_{H^s}^2+W_0(t)D_0^2(t)+N_0(t)J_0(t)D_0(t).\tag{20}$$

By a suitable linear combination of (10),(16) and (20) and choosing small ε in (16) we have

$$N_0^2(t)+D_0^2(t)\lesssim\|U_0\|_{L^2}^2+W_0(t)D_0^2(t)+N_0(t)J_0(t)D_0(t).\tag{21}$$

For $s\geq3$, it holds that $W_0(t)\lesssim N_0(t), J_0(t)\lesssim D_0(t)$. Therefore, together with (21) there exists a small constant $\delta>0$ such that if $\|U_0\|_{H^s}\leq\delta$, then $N_0(t)+D_0(t)\lesssim\delta$ uniformly in time. This ends the proof of Theorem 1.1.

Remark 2.2. For $s\geq3$, it is clear that $W_0(t)\lesssim N_0(t)$. To prove $J_0(t)\lesssim D_0(t)$, it is sufficient to prove $\|v\|_{L^\infty}^2\lesssim\|v\|_{\dot H^1}\|v\|_{\dot H^2}$. Indeed, we have

$$\begin{aligned}|v|=&\left|\int_{\xi\in\mathbb{R}^3}e^{ix\cdot\xi}\hat v d\xi\right|=\left|\int_{|\xi|\leq\lambda}+\int_{|\xi|\geq\lambda}e^{ix\cdot\xi}\hat v d\xi\right|\\\leq&\int_{|\xi|\leq\lambda}|\xi||\hat v|\frac{d\xi}{|\xi|}+\int_{|\xi|\geq\lambda}|\xi|^2|\hat v|\frac{d\xi}{|\xi|^2}\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\int_{|\xi|\leq\lambda} |\xi|^2|\hat{v}|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{|\xi|\leq\lambda} \frac{1}{|\xi|^2} d\xi\right)^{\frac{1}{2}} \\
&+ \left(\int_{|\xi|\geq\lambda} |\xi|^4|\hat{v}|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{|\xi|\geq\lambda} \frac{1}{|\xi|^4} d\xi\right)^{\frac{1}{2}} \\
&\lesssim \lambda^{1/2}\|v\|_{\dot{H}^1} + \lambda^{-1/2}\|v\|_{\dot{H}^2}.
\end{aligned}
$$

Optimizing in λ leads to $\lambda = \frac{\|v\|_{\dot{H}^2}}{\|v\|_{\dot{H}^1}}$. Therefore the desired estimate holds.

3 Linear estimate

To get the decay result, we state the linear decay first. We can rewrite the system (2) as $\partial_t U = \mathcal{L}U + \mathcal{N}(U)$, where $\mathcal{L}$ is the linear operator. We take advantages of a special Lyapunov function to show the linear estimate which can avoid the complexity of the calculation of the solution to the whole system. The linear estimate is the following.

Lemma 3.1. The solution to linear system of (2): $\partial_t U = \mathcal{L}U$, where

$$
\mathcal{L} = \left\{ \begin{array}{cccc}
\nu_-\Delta - \frac{\alpha}{\rho_-} & \frac{\alpha}{\rho_-} & -\frac{\beta}{\rho_-} & 0 \\
\frac{\alpha}{\rho_+} & \nu_+\Delta - \frac{\alpha}{\rho_+} & \frac{\beta}{\rho_+} & 0 \\
\frac{\beta}{\varepsilon_0} & -\frac{\beta}{\varepsilon_0} & 0 & \frac{1}{\varepsilon_0\mu_0}\nabla\times \\
0 & 0 & -\nabla\times & 0,
\end{array} \right\}
$$

satisfies that for all integers $k \geq 0, l \geq 0$

$$
\|D^k(e^{t\mathcal{L}}U_0)\|_{L^2}^2 \lesssim (1+t)^{-3/2-k}\|U_0\|_{L^1}^2 + (1+t)^{-l}\|D^{k+l}U_0\|_{L^2}^2.
$$

Proof. We rewrite the linear system of (2) in Fourier side,

$$
\begin{cases}
\rho_-\partial_t\hat{v}_- = -\nu_-|\xi|^2\hat{v}_- - \beta\hat{E} + \alpha(\hat{v}_+ - \hat{v}_-) \\
\rho_+\partial_t\hat{v}_+ = -\nu_+|\xi|^2\hat{v}_+ + \beta\hat{E} - \alpha(\hat{v}_+ - \hat{v}_-) \\
\partial_t\hat{E} = c_1 i\xi\times\hat{B} - c_2\beta(\hat{v}_+ - \hat{v}_-) \\
\partial_t\hat{B} = -i\xi\times\hat{E} \\
\xi\cdot\hat{v}_\pm = \xi\cdot\hat{B} = \xi\cdot\hat{E} = 0,
\end{cases} \tag{22}
$$

where $c_1 = \frac{1}{\varepsilon_0\mu_0}, c_2 = 1/\varepsilon_0$ and define the energy

$$
\hat{\mathcal{E}} := \frac{1}{2}\rho_-|\hat{v}_-|^2 + \frac{1}{2}\rho_+|\hat{v}_+|^2 + \frac{1}{2c_2}|\hat{E}|^2 + \frac{c_1}{2c_2}|\hat{B}|^2.
$$

We immediately have the following energy balance.

$$
\frac{d}{dt}\hat{\mathcal{E}} = -|\xi|^2(\nu_-|\hat{v}_-|^2 + \nu_+|\hat{v}_+|^2) - \alpha|\hat{v}_- - \hat{v}_+|^2. \tag{23}
$$

Define the Lyapunov function

$$L(\xi,t) := \gamma(1+|\xi|^2)\hat{\mathcal{E}} + F,$$

where

$$F := -\frac{1}{1+|\xi|^2} < i\xi\times\hat{B},\hat{E} > + < \hat{v}_- - \hat{v}_+,\hat{E} >,$$

γ is a constant which will be determined later. Now we calculate time derivative of L. Noting that $i\xi\times(i\xi\times\hat{E}) = |\xi|^2\hat{E}$ and $|\xi\times\hat{B}|^2 = |\xi|^2|\hat{B}|^2$ under the divergence free condition of E, B, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(-\frac{1}{1+|\xi|^2} < i\xi\times\hat{B},\hat{E} >\Big)\\
=&\frac{|\xi|^2}{1+|\xi|^2}|\hat{E}|^2 - \gamma_2 c_1\frac{|\xi|^2}{1+|\xi|^2}|\hat{B}|^2\\
&+c_2\frac{1}{1+|\xi|^2} < i\xi\times\hat{B},\beta(\hat{v}_+-\hat{v}_-) >,
\end{aligned}\tag{24}$$

and

$$\begin{aligned}
&\frac{d}{dt}(< \hat{v}_- - \hat{v}_+,\hat{E} >)\\
=&|\xi|^2 < \frac{\nu_+}{\rho_+}\hat{v}_+ - \frac{\nu_-}{\rho_-}\hat{v}_-,\hat{E} > -\beta\Big(\frac{1}{\rho_-}+\frac{1}{\rho_+}\Big)|\hat{E}|^2\\
&+\alpha\Big(\frac{1}{\rho_-}+\frac{1}{\rho_+}\Big) < \hat{v}_+ - \hat{v}_-,\hat{E} >\\
&-c_1 < \hat{v}_+ - \hat{v}_-, i\xi\times\hat{B} > + c_2 < \hat{v}_+ - \hat{v}_-,\beta(\hat{v}_+ - \hat{v}_-) > .
\end{aligned}\tag{25}$$

Taking (23) into account we have

$$\begin{aligned}
\frac{d}{dt}L =& -\gamma|\xi|^2(1+|\xi|^2)(\nu_-|\hat{v}_-|^2+\nu_+|\hat{v}_+|^2) - \beta\Big(\frac{1}{\rho_-}+\frac{1}{\rho_+}\Big)|\hat{E}|^2\\
&-c_1\frac{|\xi|^2}{1+|\xi|^2}|\hat{B}|^2 - \alpha\gamma(1+|\xi|^2)|\hat{v}_- - \hat{v}_+|^2\\
&+\frac{|\xi|^2}{1+|\xi|^2}|\hat{E}|^2 + c_2\frac{1}{1+|\xi|^2} < i\xi\times\hat{B},\beta(\hat{v}_+-\hat{v}_-) >\\
&+|\xi|^2 < \frac{\nu_+}{\rho_+}\hat{v}_+ - \frac{\nu_-}{\rho_-}\hat{v}_-,\hat{E} > +\alpha\Big(\frac{1}{\rho_-}+\frac{1}{\rho_+}\Big) < \hat{v}_+ - \hat{v}_-,\hat{E} >\\
&-c_1 < \hat{v}_+ - \hat{v}_-, i\xi\times\hat{B} > + c_2 < \hat{v}_+ - \hat{v}_-,\beta(\hat{v}_+ - \hat{v}_-) > .
\end{aligned}$$

Cauchy-Schwarz inequality and Young's inequality yields

$$\begin{aligned}
\frac{d}{dt}L \leq & -\gamma|\xi|^2(1+|\xi|^2)(\nu_-|\hat{v}_-|^2+\nu_+|\hat{v}_+|^2)-\beta(\frac{1}{\rho_-}+\frac{1}{\rho_+})|\hat{E}|^2 \\
& -c_1\frac{|\xi|^2}{1+|\xi|^2}|\hat{B}|^2-\alpha\gamma(1+|\xi|^2)|\hat{v}_--\hat{v}_+|^2 \\
& +|\hat{E}|^2+\varepsilon c_2\frac{|\xi|^2}{1+|\xi|^2}|\hat{B}|^2+c(\varepsilon)c_2\beta^2\frac{1}{1+|\xi|^2}|\hat{v}_+-\hat{v}_-|^2 \\
& +c(\varepsilon)|\xi|^4\left(\left(\frac{\nu_+}{\rho_+}\right)^2|\hat{v}_+|^2+\left(\frac{\nu_-}{\rho_-}\right)^2|\hat{v}_-|^2\right)+\varepsilon|\hat{E}|^2 \\
& +c(\varepsilon)\alpha^2\left(\frac{1}{\rho_-}+\frac{1}{\rho_+}\right)^2|\hat{v}_+-\hat{v}_-|^2+\varepsilon|\hat{E}|^2 \\
& +c(\varepsilon)c_1^2(1+|\xi|^2)|\hat{v}_+-\hat{v}_-|^2+\varepsilon\frac{|\xi|^2}{1+|\xi|^2}|\hat{B}|^2+c_2\beta|\hat{v}_+-\hat{v}_-|^2 \\
\leq & -(\gamma-c(\varepsilon,c_2,\beta,\rho_\pm,\nu_\pm))\,|\xi|^2(1+|\xi|^2)(\nu_-|\hat{v}_-|^2+\nu_+|\hat{v}_+|^2) \\
& -\left(\frac{\beta}{\rho_-}+\frac{\beta}{\rho_+}-1-2\varepsilon\right)|\hat{E}|^2-(c_1-\varepsilon c_2-\varepsilon)\frac{|\xi|^2}{1+|\xi|^2}|\hat{B}|^2 \\
& -(\alpha\gamma-c(\varepsilon,\alpha,\rho_\pm,c_1,c_2))\,(1+|\xi|^2)|\hat{v}_--\hat{v}_+|^2.
\end{aligned} \tag{26}$$

By choosing ε small enough, we have

$$\left(\frac{\beta}{\rho_-}+\frac{\beta}{\rho_+}-1-2\varepsilon\right)>0, \quad (c_1-\varepsilon c_2-\varepsilon)>0.$$

After ε is fixed, we choose γ big enough so that

$$(\gamma-c(\varepsilon,c_2,\beta_\pm,\rho_\pm,\nu_\pm))>0, \quad (\alpha\gamma-c(\varepsilon,\alpha,\rho_\pm,c_1,c_2))>0.$$

Therefore, there exists a positive constant d_1 such that

$$\begin{aligned}
\frac{d}{dt}L \leq & -d_1|\xi|^2(1+|\xi|^2)(\frac{1}{2}\rho_-|\hat{v}_-|^2+\frac{1}{2}\rho_+|\hat{v}_+|^2) \\
& -d_1\frac{3}{4c_2}|\hat{E}|^2-d_1\frac{|\xi|^2}{1+|\xi|^2}\frac{c_1}{c_2}|\hat{B}|^2.
\end{aligned} \tag{27}$$

Because

$$\frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4}\leq|\xi|^2(1+|\xi|^2), \quad \frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4}\leq\frac{3}{2}, \quad \frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4}\leq\frac{2|\xi|^2}{1+|\xi|^2},$$

(27) implies

$$\begin{aligned}
\frac{d}{dt}L = & -d_1\frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4}(\frac{1}{2}\rho_-|\hat{v}_-|^2+\frac{1}{2}\rho_+|\hat{v}_+|^2) \\
& -d_1\frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4}(\frac{1}{2c_2}|\hat{E}|^2+\frac{c_1}{2c_2}|\hat{B}|^2) \\
\leq & -d_1\frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4}\hat{\mathcal{E}}.
\end{aligned} \tag{28}$$

On the other hand, since

$$|F| \lesssim (|\hat{v}_-|^2 + |\hat{v}_+|^2 + |\hat{E}|^2 + |\hat{B}|^2),$$

there exists d_2, d_3 such that

$$d_2(1+|\xi|^2)\hat{\mathcal{E}} \le L \le d_3(1+|\xi|^2)\hat{\mathcal{E}}. \tag{29}$$

Furthermore, we could notice that d_3 can be chosen as large as it can be, Plugging (29) into (28) implies

$$\frac{d}{dt}L \le -\frac{d_1}{d_3}\frac{|\xi|^2}{(1+|\xi|^4)}L.$$

Gronwall's Lemma gives us, for $t \ge 0$,

$$L(\xi,t) \le L(\xi,0)e^{-\frac{d_1}{d_3}\frac{|\xi|^2}{(1+|\xi|^4)}t}.$$

Again, thanks to (29), we end up with

$$\hat{\mathcal{E}}(\xi,t) \le \frac{d_3}{d_2}\hat{\mathcal{E}}(\xi,0)e^{-\frac{d_1}{d_3}\frac{|\xi|^2}{(1+|\xi|^4)}t}. \tag{30}$$

By (30), for any $k \ge 0$, it holds that

$$\begin{aligned}\|D^kU\|_{L^2}^2 &\lesssim \int_{\mathbb{R}^3} |\hat{U}_0|^2|\xi|^{2k}e^{-\frac{d_1}{d_3}\rho(|\xi|)t}d\xi \\ &\lesssim \int_{|\xi|\le 1} + \int_{|\xi|>1} |\hat{U}_0|^2|\xi|^k e^{-\frac{d_1}{d_3}\rho(|\xi|)t}d\xi.\end{aligned}$$

Estimating the low and high frequency parts of the above inequality will prove our theorem.
For $|\xi| \le 1$, $\rho(|\xi|) \ge \frac{|\xi|^2}{2}$. Therefore, for any $k > 0$

$$\begin{aligned}\int_{|\xi|\le 1} |\hat{U}_0|^2|\xi|^{2k}e^{-\frac{d_1}{d_3}\rho(|\xi|)t}d\xi &\le \|\hat{U}_0\|_{L^\infty}^2 \int_{|\xi|\le 1} |\xi|^{2k}e^{-\frac{d_1}{d_3}\rho(|\xi|)t}d\xi \\ &\lesssim c(k)(1+t)^{-3/2-k}\|U_0\|_{L^1}^2\end{aligned} \tag{31}$$

For $|\xi| > 1$, $\rho(|\xi|) \ge \frac{1}{2|\xi|^2}$. Therefore, for any $l > 0$,

$$\begin{aligned}\int_{|\xi|>1} |\hat{U}|^2|\xi|^{2k}e^{-\frac{d_1}{d_3}\rho(|\xi|)t}d\xi &\le \int_{|\xi|\ge 1} |\hat{U}|^2|\xi|^{2k}e^{-\frac{d_1}{d_3}\frac{1}{2|\xi|^2}t}d\xi \\ &\le \sup_{|\xi|\ge 1}(e^{-\frac{d_1}{d_3}}\frac{t}{2|\xi|^2}|\xi|^{-2l}) \int_{|\xi|>1} |\hat{U}|^2|\xi|^{2k+2l}d\xi \\ &\le c(l)(1+t)^{-l}\|U\|_{\dot{H}^{k+l}}^2\end{aligned} \tag{32}$$

Put (32) and (31) together and we finish the proof. $\square$

4 Proof of Theorem 1.2

To prove the decay result, we introduce the following time weighted norm and the corresponding dissipation norm.

$$
\begin{aligned}
M(t) &= \sum_{k=0}^{[(s-1)/2]-1} \sup_{0\le\tau\le t}(1+\tau)^{3/4+k/2}\|D^kU\|_{H^{s-2k-3}},\\
N^2(t) &= \sum_{k=0}^{[s/2]} \sup_{0\le\tau\le t}(1+\tau)^{k}\|D^kU\|^2_{H^{s-2k}},\\
D^2(t) &= \sum_{k=0}^{[s/2]} \int_0^t (1+\tau)^k\left(\|D^{k+1}v\|^2_{H^{s-2k}} + \|D^k(v_- - v_+)\|^2_{H^{k-2k}}\right)d\tau\\
&\quad+\sum_{k=0}^{[s/2]-1}\int_0^t (1+\tau)^k\left(\|D^kE\|^2_{H^{s-2k-1}}+\|D^{k+1}B\|^2_{H^{s-2k-2}}\right)d\tau,
\end{aligned}
\tag{33}
$$

where $U=(v_-,v_+,E,B), v=(v_-,v_+)$.
The goal is to bound $M(t)$ uniformly in time when the initial data is small enough. Using Duhamel principle and linear estimate, one could have a self control estimate of $M(t)$ provided that $N(t)$ is bounded (see Lemma 4.4). The boundedness of $N(t)$ can be derived through time weighted energy method as well as $D(t)$.
The proof is based on several lemmas.

Lemma 4.1. Let $s\ge 7$, $0\le k\le [s/2]$. We have

$$
\begin{aligned}
&(1+t)^k\|D^kU\|^2_{H^{s-2k}} + \int_0^t(1+\tau)^k\left(\|D^{k+1}v\|^2_{H^{s-2k}}+\|D^k(v_- - v_+)\|^2_{H^{s-2k}}\right)d\tau\\
&\lesssim \|U_0\|^2_{H^s} + k\int_0^t(1+\tau)^{k-1}\|D^kU\|^2_{H^{s-2k}}d\tau + (N(t)+M(t))D^2(t).
\end{aligned}
$$

Proof. Multiplying (9) by $(1+t)^k$ and integrating in time then summing over $k\le l\le s-k$ yields

$$
\begin{aligned}
&(1+t)^k\|D^kU\|^2_{H^{s-2k}} + \int_0^t(1+\tau)^k\left(\|D^{k+1}v\|^2_{H^{s-2k}}+\|D^k(v_- - v_+)\|^2_{H^{s-2k}}\right)d\tau\\
&\lesssim \|U_0\|^2_{H^s} + k\int_0^t(1+\tau)^{k-1}\|D^kU\|^2_{H^{s-2k}}d\tau + T_1+T_2,
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
T_1 &= \int_0^t (1+\tau)^k\|Dv\|_{L^\infty}\sum_{l=k}^{s-k}\|D^lv\|^2_{L^2}d\tau,\\
T_2 &= \int_0^t (1+\tau)^k\sum_{l=k}^{s-k}\left(\|B\|_{L^\infty}\|D^lv\|^2_{L^2}+\|v\|_{L^\infty}\|D^lv\|_{L^2}\|D^lB\|_{L^2}\right)d\tau.
\end{aligned}
$$

Noting that when $l = 0$, (9) becomes the classic energy identity

$$\begin{aligned}&\frac{\rho_-}{2}\frac{d}{dt}\|v_-\|_{L^2}^2 + \frac{\rho_+}{2}\frac{d}{dt}\|v_+\|_{L^2}^2 + \frac{\varepsilon_0}{2}\frac{d}{dt}\|E\|_{L^2}^2 + \frac{d}{dt}\frac{1}{2\mu_0}\|B\|_{L^2}^2 \\ &\quad + \nu_-\|\nabla v_-\|_{L^2}^2 + \nu_+\|\nabla v_+\|_{L^2}^2 + \alpha\|v_- - v_+\|_{L^2}^2 = 0.\end{aligned}$$

So that when $k = 0$,

$$\begin{aligned}T_1 &= \int_0^t \|Dv\|_{L^\infty}\|Dv\|_{H^{s-1}}^2 d\tau \lesssim W_0(t)D_0^2(t) \lesssim N(t)D^2(t), \\ T_2 &= \int_0^t \|B\|_{L^\infty}\|Dv\|_{H^{s-1}}^2 + \|v\|_{L^\infty}\|Dv\|_{H^{s-1}}\|DB\|_{H^{s-1}} d\tau \\ &\lesssim W_0(t)D_0^2(t) + N_0(t)J_0(t)D_0(t) \lesssim N(t)D^2(t).\end{aligned}$$

For $1 \le k \le [s/2]$, we estimate T_1, T_2 as

$$\begin{aligned}T_1 &= \int_0^t (1+\tau)^k \|Dv\|_{L^\infty}\|D^k v\|_{H^{s-2k}}^2 d\tau \\ &\lesssim \sup_{0\le\tau\le t}\{(1+\tau)\|Dv\|_{L^\infty}\}\int_0^t (1+\tau)^{k-1}\|D^k v\|_{H^{s-2k}}^2 d\tau \\ &\lesssim M(t)D^2(t), \\ T_2 &\lesssim \int_0^t (1+\tau)^k \left(\|B\|_{L^\infty}\|D^k v\|_{H^{s-2k}}^2 + \|v\|_{L^\infty}\|D^k v\|_{H^{s-2k}}\|D^l B\|_{H^{s-2k}}\right) d\tau \\ &\lesssim M(t)D^2(t) + M(t)\int_0^t (1+\tau)^{k-1}\left(\|D^k v\|_{H^{s-2k}}\|D^k B\|_{H^{s-2k}}\right) d\tau \\ &\lesssim M(t)D^2(t),\end{aligned}$$

where we use $\sup_{0\le\tau\le t}(1+\tau)\|U\|_{W^{1,\infty}} \lesssim M(t)$ for $s \ge 7$.
Substituting the estimates of T_1, T_2 into (34) proves our lemma. □

Now comes the estimate of the dissipation of E and B. More precisely, we have

Lemma 4.2. Let $s \ge 7, 0 \le k \le [s/2] - 1$. It holds that

$$\begin{aligned}&\int_0^t (1+\tau)^k \left(\|D^k E\|_{H^{s-2k-1}}^2 + \|D^{k+1}B\|_{H^{s-2k-2}}^2\right) d\tau \\ \lesssim&\|U_0\|_{H^s}^2 + k\int_0^t (1+\tau)^{k-1}\|D^k U\|_{H^{s-2k}}^2 d\tau + (N(t)+M(t))D^2(t).\end{aligned}$$

Proof. The dissipation estimate for $v_\pm, v_- - v_+$ are done in the previous lemma. Let us first do estimate on E. Like the proof of Theorem 1.1 (see Remark 2.1), we deal with the case $l = k$ and $l > k$ in different ways. Fixing $0 \le k \le [s/2] - 1$, by (12) for $l = k$, we have

$$\partial_t\left(D^k(v_- - v_+), D^kE\right) + \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})\|D^kE\|_{L^2}^2$$
$$\lesssim C_\varepsilon\|D^{k+2}v\|_{L^2}^2 + C_\varepsilon\|D^k(v_- - v_+)\|_{L^2}^2 + \varepsilon\|D^kE\|_{L^2}^2 + \varepsilon\|D^{k+1}B\|_{L^2}^2 \tag{35}$$
$$+\|v\|_{L^\infty}\|D^{k+1}v\|_{L^2}\|D^kE\|_{L^2} + \|B\|_{L^\infty}\|D^kv\|_{L^2}\|D^kE\|_{L^2}$$
$$+\|v\|_{L^\infty}\|D^kB\|_{L^2}\|D^kE\|_{L^2}.$$

For $l > k$ we have

$$\partial_t\left(D^l(v_- - v_+), D^lE\right) + \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})\|D^lE\|_{L^2}^2$$
$$\lesssim C_\varepsilon\|D^{l+2}v\|_{L^2}^2 + C_\varepsilon\|D^l(v_- - v_+)\|_{H^1}^2 + \varepsilon\|D^lE\|_{L^2}^2 + \varepsilon\|D^lB\|_{L^2}^2 \tag{36}$$
$$+\|v\|_{L^\infty}\|D^{l+1}v\|_{L^2}\|D^lE\|_{L^2} + \|B\|_{L^\infty}\|D^lv\|_{L^2}\|D^lE\|_{L^2}$$
$$+\|v\|_{L^\infty}\|D^lB\|_{L^2}\|D^lE\|_{L^2}.$$

Multiplying (35) and (36) by $(1+t)^k$, integrating in time and summing over $k \le l \le s-k-1$ yields

$$\int_0^t(1+\tau)^k\|D^kE\|_{H^{s-2k-1}}^2d\tau$$
$$\lesssim\|U_0\|_{H^s}^2 + (1+t)^k\|D^kU\|_{H^{s-2k-1}}^2 + k\int_0^t(1+\tau)^{k-1}\|D^kU\|_{H^{s-2k-1}}^2d\tau$$
$$+C_\varepsilon\int_0^t(1+\tau)^k\left(\|D^{k+2}v\|_{H^{2-2k-1}}^2 + \|D^k(v_- - v_+)\|_{H^{s-2k}}^2\right)d\tau \tag{37}$$
$$+\varepsilon\int_0^t(1+t)^k\|D^{k+1}B\|_{H^{s-2k-2}}^2d\tau + S,$$

where

$$S = \sum_{l=k}^{s-k-1}\int_0^t(1+\tau)^k\left(\|v\|_{L^\infty}\|D^{l+1}v\|_{L^2}\|D^lE\|_{L^2} + \|B\|_{L^\infty}\|D^lv\|_{L^2}\|D^lE\|_{L^2}\right.$$
$$\left.+\|v\|_{L^\infty}\|D^lB\|_{L^2}\|D^lE\|_{L^2}\right)d\tau.$$

By Lemma 4.1, the inequality (37) becomes

$$\int_0^t(1+\tau)^k\|D^kE\|_{H^{s-2k-1}}^2d\tau \lesssim\|U_0\|_{H^s}^2 + k\int_0^t(1+\tau)^{k-1}\|D^kU\|_{H^{s-2k}}^2d\tau$$
$$+\varepsilon\int_0^t(1+t)^k\|D^{k+1}B\|_{H^{s-2k-2}}^2d\tau + S \tag{38}$$
$$+(N(t)+M(t))D^2(t),$$

Like the proof of previous lemma, we estimate the remaining term S by the case $k=0$ and $k \ge 1$. When $k=0$, one has

$$S \lesssim \int_0^t\|v\|_{L^\infty}\|Dv\|_{H^{s-1}}\|E\|_{H^{s-1}} + \|B\|_{L^\infty}\|v\|_{H^{s-1}}\|E\|_{H^{s-1}}$$

$$
\begin{aligned}
&+ \|v\|_{L^\infty}\|B\|_{H^{s-1}}\|E\|_{H^{s-1}} d\tau \\
\lesssim & N_0(t)J_0(t)D_0(t) \lesssim N(t)D^2(t).
\end{aligned}
$$

For $1 \le k \le [s/2]-1$, we have

$$
\begin{aligned}
S \ \lesssim \ & \int_0^t (1+\tau)^k \big(\|v\|_{L^\infty}\|D^{k+1}v\|_{H^{s-2k-1}}\|D^kE\|_{H^{s-2k-1}} \\
& + \|B\|_\infty \|D^k v\|_{H^{s-2k-1}}\|D^kE\|_{H^{s-2k-1}} \\
& + \|v\|_{L^\infty}\|D^kB\|_{H^{s-2k-1}}\|D^kE\|_{H^{s-2k-1}} \big) \, d\tau \\
\lesssim \ & M(t)\int_0^t (1+\tau)^{k-1}\|D^kE\|_{H^{s-2k-1}} \left(\|D^kv\|_{H^{s-2k}} + \|D^kB\|_{H^{s-2k-1}} \right) d\tau \\
\lesssim \ & M(t)D^2(t).
\end{aligned}
$$

Therefore, plugging the estimate of S into (38) yields for $0 \le k \le [s/2]-1$

$$
\begin{aligned}
\int_0^t (1+\tau)^k \|D^kE\|^2_{H^{s-2k-1}} d\tau \lesssim & \|U_0\|^2_{H^s} + k\int_0^t (1+\tau)^{k-1}\|D^kU\|^2_{H^{s-2k}} d\tau \\
& + \varepsilon \int_0^t (1+t)^k \|D^{k+1}B\|^2_{H^{s-2k-2}} d\tau \\
& + (N(t)+M(t))D^2(t).
\end{aligned} \tag{39}
$$

For dissipation estimate on B, multiplying (18) by $(1+t)^k$, integrating in time and summing l from k to $s-k-2$ yield

$$
\begin{aligned}
& \int_0^t \|D^{k+1}B\|^2_{H^{s-2k-1}} d\tau \\
\lesssim & \|U_0\|^2_{H^s} + (1+t)^k \|D^kU\|^2_{H^{s-2k}} \\
& + k\int_0^t (1+\tau)^{k-1} \left(\|D^{k+1}E\|^2_{H^{s-2k-2}} + \|D^kB\|^2_{H^{s-2k-2}} \right) d\tau \\
& + \int_0^t (1+\tau)^k \left(C_\varepsilon \|D^k(v_+ - v_+)\|^2_{H^{s-2k-2}} + \|D^{k+1}E\|^2_{H^{s-2k-2}} d\tau \right).
\end{aligned} \tag{40}
$$

By Lemma 4.1 inequality (40) becomes

$$
\begin{aligned}
\int_0^t \|D^{k+1}B\|^2_{H^{s-2k-1}} d\tau \lesssim & \|U_0\|^2_{H^s} + k\int_0^t (1+\tau)^{k-1}\|D^kU\|^2_{H^{s-2k}} d\tau \\
& + (N(t)+M(t))D^2(t).
\end{aligned} \tag{41}
$$

Choosing ε small enough and a suitable linear combination of (39) and (41) prove this lemma. $\square$

Using Lemma 4.1 and Lemma 4.2 we can derive the following key inequality that shows the self control of $N(t)$ and $D(t)$.

Lemma 4.3. Let $s \ge 7$. Then it holds that

$$
N^2(t) + D^2(t) \lesssim \|U_0\|^2_{H^s} + (N(t)+M(t))D^2(t).
$$

Proof. It is sufficient to prove that for $0 \le k \le [s/2]-1$,

$$\begin{aligned}&(1+t)^k\|D^kU\|_{H^{s-2k}}^2+\int_0^t(1+\tau)^k\left(\|D^{k+1}v\|_{H^{s-2k}}^2+\|D^k(v_+-v_+)\|_{H^{s-2k}}^2\right.\\ &\qquad\qquad\qquad\qquad\left.\|D^kE\|_{H^{s-2k-1}}^2+\|D^{k+1}B\|_{H^{s-2k-2}}^2\right)d\tau\\ &\lesssim\|U_0\|_{H^s}^2+(N(t)+D(t))\,D^2(t),\end{aligned}\tag{42}$$

and for $k=[s/2]$,

$$\begin{aligned}&(1+t)^k\|D^kU\|_{H^{s-2k}}^2+\int_0^t(1+\tau)^k\left(\|D^{k+1}v\|_{H^{s-2k}}^2+\|D^k(v_+-v_+)\|_{H^{s-2k}}^2\right.\\ &\lesssim\|U_0\|_{H^s}^2+(N(t)+D(t))\,D^2(t).\end{aligned}\tag{43}$$

The proof can be done easily using an induction argument. Clearly (42) is true for $k=0$. Assume that (42) is true for $k=l-1, 1\le l\le[s/2]-1$. Then for $k=l$,

$$\begin{aligned}&(1+t)^l\|D^lU\|_{H^{s-2l}}^2+\int_0^t(1+\tau)^l\left(\|D^{l+1}v\|_{H^{s-2l}}^2+\|D^l(v_+-v_+)\|_{H^{s-2l}}^2\right.\\ &\qquad\qquad\qquad\qquad\left.\|D^lE\|_{H^{s-2l-1}}^2+\|D^{l+1}B\|_{H^{s-2l-2}}^2\right)d\tau\\ &\lesssim\|U_0\|_{H^s}^2+(N(t)+M(t))\,D^2(t)+l\int_0^t(1+\tau)^{l-1}\|D^lU\|_{H^{s-2l}}^2d\tau.\end{aligned}$$

Since

$$\begin{aligned}&\int_0^t(1+\tau)^{l-1}\|D^lU\|_{H^{s-2l}}^2d\tau\\ \lesssim&\int_0^t(1+\tau)^{l-1}\left(\|D^lv\|_{H^{s-2(l-1)}}^2+\|D^{l-1}(v_--v_+)\|_{H^{s-2(l-1)}}^2\right.\\ &\left.+\|D^{l-1}E\|_{H^{s-2(l-1)-1}}^2+\|D^lB\|_{H^{s-2l}}^2\right)d\tau\\ \lesssim&\|U_0\|_{H^s}^2+(N(t)+M(t))\,D^2(t),\end{aligned}$$

inequality (42) holds for $k=l$.
When $k=[s/2]$, inequality (43) holds for the same reason. □

To proof our main theorem, we still need another inequality that controls $M(t)$.

Lemma 4.4. Let $s\ge3$. Then it holds that

$$M(t)\lesssim\|U_0\|_{L^1\cap H^s}+M^2(t)+M(t)N(t).$$

Proof. By Duhamel principle,

$$U=e^{t\mathcal{L}}U_0+\int_0^te^{(t-\tau)\mathcal{L}}\mathcal{P}\mathcal{N}(U(\tau))d\tau,\tag{44}$$

Where $\mathcal{P}$ is Leray projection. Fixing $0\le k\le[\frac{s-1}{2}]-1$, for $0\le m\le s-2k-3$ applying D^{k+m} to (44) and taking L^2 norm yields

$$\|D^{k+m}U\|_{L^2}\lesssim\|D^{k+m}e^{t\mathcal{L}}U_0\|_{L^2}+R_1(m)+R_2(m),\tag{45}$$

where,

$$
\begin{aligned}
R_1 = & \int_0^{t/2} \|D^{k+m+1} e^{(t-\tau)\mathcal{L}} (v_- \otimes v_-, v_+ \otimes v_+, 0, 0)^T(\tau)\|_{L^2} d\tau \\
& + \int_{t/2}^{t} \|D^{m+1} e^{(t-\tau)\mathcal{L}} D^k (v_- \otimes v_-, v_+ \otimes v_+, 0, 0)^T(\tau)\|_{L^2} d\tau \\
& := R_{11} + R_{12}, \\
R_2 = & \int_0^{t/2} \|D^{k+m} e^{(t-\tau)\mathcal{L}} (-v_- \times B, v_+ \times B, 0, 0)^T(\tau)\|_{L^2} d\tau \\
& + \int_{t/2}^{t} \|D^{m} e^{(t-\tau)\mathcal{L}} D^k (-v_- \times B, v_+ \times B, 0, 0)^T(\tau)\|_{L^2} d\tau \\
& := R_{21} + R_{22}.
\end{aligned}
$$

For the estimate of the linear part, applying Lemma 3.1 by replacing k by $k+m$ and l by $k+2$ the summing in m over $0 \leq m \leq s-2k-3$ yields

$$
\begin{aligned}
& \|D^k e^{t\mathcal{L}} U_0\|_{H^{s-2k-3}} \sim \sum_{m=0}^{s-2k-3} \|D^{k+m} e^{t\mathcal{L}} U_0\|_{L^2} \\
\lesssim & \sum_{m=0}^{s-2k-3} (1+t)^{-3/4-(k+m)/2} \|U_0\|_{L^1} + (1+t)^{-1-k/2} \|D^{2k+m+2} U\|_{L^2} \\
\lesssim & (1+t)^{-3/4-k/2} \|U_0\|_{L^1 \cap H^s}.
\end{aligned}
\tag{46}
$$

To estimate R_1 and R_2, we separate the time integral into two parts: from 0 to $t/2$ and $t/2$ to t, as we showed in the definition of R_1 and R_2. The decay of the first part comes from the linear estimate while the decay of the second part is due to the definition of our weighted norms.

For R_{11}, applying the linear estimate Lemma 3.1 with $k = k+m+1, l = k+2$ and summing m over $0 \leq m \leq s-2k-3$ we have

$$\begin{aligned}
&\sum_{m=0}^{s-2k-3} R_{11}(m) \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-5/4-(k+m)/2} \|v \otimes v\|_{L^1} d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-1-k/2} \|D^{2k+m+3}(v \otimes v)\|_{L^2} d\tau \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-5/4-(k+m)/2} \|v\|_{L^2}^2 d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-1-k/2} \|D^{2k+m+3} v\|_{L^2} \|v\|_{L^\infty} d\tau \\
\lesssim &M^2(t) \int_0^{t/2} (1+t-\tau)^{-5/4-k/2} (1+\tau)^{-3/2} d\tau \\
+ &\sup_{0 \leq \tau \leq t/2} \{(1+\tau)\|v\|_{L^\infty}\} N(t) \int_0^{t/2} (1+t-\tau)^{-1-k/2} (1+\tau)^{-1} d\tau \\
\lesssim &(1+t)^{-3/4-k/2} M^2(t) + (1+t)^{-3/4-k/2} M(t) N(t) (1+t)^{-1/4} \ln(1+t) \\
\lesssim &(1+t)^{-3/4-k/2} \left(M^2(t) + M(t)N(t) \right),
\end{aligned}$$

where we use Lemma 2.1 in the second step. Noting that in above inequality, we need $2k+m+3 \leq s$ for all $0 \leq m \leq s-2k-3$ and this is where the restriction on k comes from.

For R_{12} we do the similar procedure as we did for R_{11}. When applying the linear estimate, we set $k=m+1, l=1$ in Lemma 3.1 which leads to

$$
\begin{aligned}
&\sum_{m=0}^{s-2k-3} R_{12}(m) \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-5/4-m/2} \|D^k(v\otimes v)\|_{L^1} d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2} \|D^{k+m+2}(v\otimes v)\|_{L^2} d\tau \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-5/4-m-2} \|D^k v\|_{L^2}\|v\|_{L^2} d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2} \|D^{k+m+2} v\|_{L^2}\|v\|_{L^\infty} d\tau \\
\lesssim &M^2(t) \sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-5/4-m/2}(1+\tau)^{-3/2-k/2} d\tau \\
+ &\sup_{t/2\le\tau\le t}\{(1+\tau)\|v\|_{L^\infty}\} \sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2}(1+\tau)^{-1} \|D^{k+m+1} v\|_{H^1} d\tau \\
\lesssim &(1+t)^{-3/4-k/2} M^2(t) \\
&+ M(t)N(t) \int_{t/2}^{t} (1+t-\tau)^{-1/2}(1+\tau)^{-1}(1+\tau)^{-(k+1)/2} d\tau \\
\lesssim &(1+t)^{-3/4-k/2} M^2(t) \\
&+ M(t)N(t) \int_{t/2}^{t} (1+t-\tau)^{-1/2}(1+\tau)^{-3/4}(1+\tau)^{-k/2-3/4} d\tau \\
\lesssim &(1+t)^{-3/4-k/2} \left(M^2(t) + M(t)N(t)\right).
\end{aligned}
$$

Together with the estimate of R_{11}, R_{12}, we have

$$
\sum_{m=0}^{s-2k-3} R_1(m) \lesssim (1+t)^{-3/4-k/2} \left(M^2(t) + M(t)N(t)\right). \tag{47}
$$

We estimate R_2 in a similar way. For R_{21} applying the linear estimate Lemma 3.1 with $k=k+m, l=k+2$ and summing m over $0\le m\le s-2k-3$ yield

$$
\begin{aligned}
&\sum_{m=0}^{s-2k-3} R_{21}(m) \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_{0}^{t/2} (1+t-\tau)^{-3/4-(k+m)/2} \|v\times B\|_{L^1} d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_{0}^{t/2} (1+t-\tau)^{-1-k/2} \|D^{2k+m+2}(v\times B)\|_{L^2} d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-3/4-(k+m)/2} \|U\|_{L^2}^2 d\tau \\
&+ \sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-1-k/2} \|D^{2k+m+2}U\|_{L^2} \|U\|_{L^\infty} d\tau \\
&\lesssim M^2(t) \int_0^{t/2} (1+t-\tau)^{-3/4-k/2} (1+\tau)^{-3/2} d\tau \\
&+ \sup_{0\le\tau\le t/2} \{(1+\tau)\|U\|_{L^\infty}\} N(t) \int_0^{t/2} (1+t-\tau)^{-1-k/2} (1+\tau)^{-1} d\tau \\
&\lesssim (1+t)^{-3/4-k/2} M^2(t) + (1+t)^{-3/4-k/2} M(t)N(t)(1+t)^{-1/4} \ln(1+t) \\
&\lesssim (1+t)^{-3/4-k/2} \left(M^2(t) + M(t)N(t)\right).
\end{aligned}
$$

For R_{21}, just like the way we estimate R_{12} but choosing $k = m, l = 1$ when applying Lemma 3.1 we get

$$
\begin{aligned}
&\sum_{m=0}^{s-2k-3} R_{22}(m) \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-3/4-m/2} \|D^k(v\times B)\|_{L^1} d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2} \|D^{k+m+1}(v\times B)\|_{L^2} d\tau \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-3/4-m-2} \|D^k U\|_{L^2} \|U\|_{L^2} d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2} \|D^{k+m+1}U\|_{L^2} \|U\|_{L^\infty} d\tau \\
\lesssim &M^2(t) \sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-3/4-m/2} (1+\tau)^{-3/2-k/2} d\tau \\
+ &\sup_{t/2\le\tau\le t} \{(1+\tau)\|U\|_{L^\infty}\} \sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2} (1+\tau)^{-1} \|D^{k+m+1}v\|_{H^1} d\tau \\
\lesssim &M^2(t) \int_{t/2}^{t} (1+t-\tau)^{-3/2} (1+\tau)^{-3/4-k/2} d\tau \\
+ &M(t)N(t) \int_{t/2}^{t} (1+t-\tau)^{-1/2} (1+\tau)^{-1} (1+\tau)^{-(k+1)/2} d\tau \\
\lesssim &(1+t)^{-3/4-k/2} \left(M^2(t) + M(t)N(t)\right).
\end{aligned}
$$

So that we have

$$
R_2 \lesssim (1+t)^{-3/4-k/2} \left(M^2(t) + M(t)N(t)\right). \tag{48}
$$

The lemma is proved when we put (46), (47) and (48) together. □

Now we have all the ingredients to prove Theorem 1.2. By Lemma 4.3 and Lemma 4.4, we have for $s \geq 7$

$$N^2(t) + D^2(t) \lesssim \|U_0\|_{H^s}^2 + (N(t) + M(t))\, D^2(t),$$
$$M(t) \lesssim \|U_0\|_{L^1 \cap H^s} + M^2(t) + M(t)N(t).$$

Let $Y = N(t) + M(t) + D(t)$. So that the above inequalities imply that

$$Y^2(t) \lesssim \|U_0\|_{H^s}^2 + Y^3(t) + Y^4(t).$$

Thus by choosing $\|U_0\|_{L^1 \cap H^s} \leq \delta$ with δ small enough, one has

$$Y = M(t) + N(t) + D(t) \lesssim \delta,$$

for all $t \geq 0$ which proves the theorem.
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