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NEW ESTIMATIONS FOR DISCRETE STURM–LIOUVILLE PROBLEMS

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Abstract. In this paper, discrete Sturm–Liouville problem with potential function $q(n)$ is considered. Representations of solutions having potential function $q(n)$ in their kernels are obtained. From this point of view, we acquire asymptotic formulas for eigenfunctions and behaviors of eigenfunctions for the problems are analyzed and illustrated by graphics and tables. We find the eigenfunctions corresponding some eigenvalues. Also, we show the number of eigenvalues increases as n increases.

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1 Introduction

Sturm–Liouville problem has been developed firstly in a number of articles published in 1836 and 1837. Sturm–Liouville operator is referred to as one dimensional Schrödinger operator and the function $q(t)$ is defined a potential. It is known that the spectral characteristics are spectra, spectral functions, scattering data, norming constants, etc. Representation of solution of Sturm–Liouville problem must include potential function $q\left(t\right)$ in its kernel and asymptotic formulas for eigenfunctions and eigenvalues must be found as Levitan and Sargsjan's study [22] to be obtained these spectral characteristics. This problem has been studied by some authors [25], [26], [27], [1], [2], [9] and [8], [24], [7]. The Sturm–Liouville problem has an important position in mathematical physics.

Difference equations are the discrete analog of differential equations. Its actual development regarding theoretical was begun to be compared with differential equations. Difference equations have many application areas which are problems of physics, mathematics and engineering, vibrating string, economy, population, actuary and logistics, network theory, random walk, etc. As a natural result of comparing the difference equations differential equations, the theory of linear difference equations has begun to appear in a similar way to the theory of differential equations. Elementary theory of linear difference equations was improved by De Moivre, Euler, Lagrange, Laplace et al. [18]. Hereafter, Peterson-Kelley [20], Agarwal [4], Jirari [19], Bender-Orszag [16], Lakshmikantham and Trigiante [21], Mickens [23].

Especially, in recent years, Sturm–Liouville difference equations have seen a great interest and a lot of studies have been published, but the theory of this subject remains in need of improvement. [13], [14], [15], [10], [11], [3], [12], [1], [9] investigated spectral analysis of second order difference equations and operators. Peterson-Kelley [20], Agarwal [4], Bender [16], Bereketoglu [17] made mention of this subject in their studies, and Jirari [19] studied specifically singular Sturm–Liouville difference equations in his study. Atkinson [5, 6] studied discrete and continuous boundary value problems in his book.

In this study, representations of solutions having potential function $q(n)$ in their kernels and asymptotic estimations for eigenfunctions are obtained for discrete Sturm–Liouville problems in a similar way of the study of Levitan and Sargsjan [22]. They obtained the representation of the solution of Sturm–Liouville initial value problems having potential in its kernel for differential equations and asymptotic formulas for eigenfunctions and eigenvalues and so they found the spectral characteristics. In a similar vein, we try to obtain the representations of solutions having potential function $q(n)$ in their kernels and asymptotic formulas of discrete Sturm–Liouville problems and starting from this point of view, we will obtain the spectral characteristics for this problem in our next studies.

The following problem $(1)-(2)$,

$$\begin{aligned} \Delta^2 x\left(n-1\right)+q\left(n\right)x\left(n\right)+\lambda x\left(n\right) &= 0, n=a,...,b \qquad (1) \\ x\left(a-1\right)+hx\left(a\right) &= 0, \qquad (2) \end{aligned}$$

is called discrete Sturm–Liouville problem, where a, b are finite integers with $a \geq 0$, $a \leq b$, h is a real number, Δ is the forward difference operator, $\Delta x\left(n\right)=x\left(n+1\right)-x\left(n\right)$, λ is the spectral parameter, $q\left(n\right)$ is a real valued potential function for $n \in \left[a,b\right]$, n is a finite integer.

Fundamental theorems and definitions are given in Section 2. Representations of solutions having potential function $q(n)$ in their kernels with two different initial conditions are given and behaviors of eigenfunctions for the problems are analyzed and illustrated by graphics and tables. We find the eigenvalues corresponding some eigenfunctions. Moreover, we show the number of eigenvalues increases as n increases in Section 3 and asymptotic behavior of eigenfunctions are given in Section 4.

2 Preliminaries

Definition 1. [20] The matrix of Casoratian is given by

$$w(n) = \begin{pmatrix} x_1(n) & x_2(n) & \ldots & x_r(n) \\ x_1(n+1) & x_2(n+1) & \ldots & x_r(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(n+r-1) & x_2(n+r-1) & \ldots & x_r(n+r-1) \end{pmatrix}$$

where $x_1(n), x_2(n), ..., x_r(n)$ are solution functions. The determinant of $w(n)$

$$W(n) = \det w(n)$$

is named Casoratian.

Theorem 2. [19] *Assume r and u are solutions of discrete Sturm–Liouville equation (1). Then, for $a \leq n \leq b$*

$$\begin{aligned} W[r,u](n) &= [r(n)\Delta u(n-1) - u(n)\Delta r(n-1)] \\ &= -[r(n)u(n-1) - r(n-1)u(n)] \end{aligned} \tag{3}$$

is a constant (Particularly equal to $W[r,u](a)$).

Definition 3. [19] Let us express discrete Sturm–Liouville equation (1) by means of the operator L,

$$Lx(n) = -\lambda x(n), \ n \in [a,b], \tag{4}$$

with initial conditions

$$\cos\alpha x(a) - \sin\alpha(\nabla x(a)) = 0, \tag{5}$$

where $0 < \alpha$, ∇ is the backward difference operator, $\nabla x(n) = x(n) - x(n-1)$, are equivalent to

$$x(a-1) + (\cot\alpha - 1)x(a) = 0, \tag{6}$$

in other words,

$$x(a-1) + hx(a) = 0, \tag{7}$$

where h is a real number. Initial value problem $(4)-(7)$ is named discrete Sturm–Liouville problem.

Theorem 4. [20](***Summation by parts formula***) *If $m < n$, then the following property is valid*

$$\sum_{k=m}^{n-1} x(k)\, \Delta y(k) = [x(k)\, y(k)]_m^n - \sum_{k=m}^{n-1} \Delta x(k)\, y(k+1).$$

Theorem 5. [20] *If y_n is an indefinite sum of x_n, then*

$$\sum_{k=m}^{n-1} y(k) = x(n) - x(m). \tag{8}$$

3 Main Results

In this study, we obtain representation of solutions having potential function $q(n)$ in their kernels as follows,

$$\Delta^2 x(n-1) + q(n)\, x(n) + \lambda x(n) = 0, n = a, ..., b \tag{9}$$

with initial conditions,

$$x(a-1) + hx(a) = 0, \tag{10}$$

where a, b are finite integers with $a \geq 0$, $a \leq b$, h is a real number, Δ is the forward difference operator, $\Delta x(n) = x(n+1) - x(n)$, λ is the spectral parameter, $q(n)$ is a real valued potential function for $n \in [a, b]$, n is a finite integer.
A self-adjoint difference operator L (9) is noted by,

$$Lx(n) = \Delta^2 x(n-1) + q(n)\, x(n) = -\lambda x(n).$$

In $\ell^2(a, b)$, the Hilbert space of sequences of complex numbers $x(a), ..., x(b)$ with the inner product,

$$\langle x(n), y(n) \rangle = \sum_{n=a}^{b} x(n)\, y(n),$$

for every $x \in D_{L,}$ let's define as follows

$$D_L = \left\{ x(n) \in \ell^2(a, b) : Lx(n) \in \ell^2(a, b),\ x(0) = -h,\ x(1) = 1 \right\}.$$

Hence, equation (9) can be written as below:

$$Lx(n) = -\lambda x(n).$$

Theorem 6. *Let us define discrete Sturm–Liouville problem as follows:*

$$Lx(n) = -\lambda x(n), \tag{11}$$

$$x(0) = -h, x(1) = 1, \tag{12}$$

then discrete Sturm–Liouville problem $(11)-(12)$ *has a unique solution for* $x(n)$ *as*

$$\begin{aligned} x(n,\lambda) &= \left(\frac{2-h\left(2q(0)-2+\lambda+\sqrt{\lambda(\lambda-4)}\right)}{2\sqrt{\lambda(\lambda-4)}}\right)\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^n \\ &+\left(\frac{-2+h\left(2q(0)-2+\lambda-\sqrt{\lambda(\lambda-4)}\right)}{2\sqrt{\lambda(\lambda-4)}}\right)\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^n \\ &-\sum_{i=0}^{n}\frac{q(i)x(i)\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^i}{\sqrt{\lambda(\lambda-4)}}\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^n \\ &+\sum_{i=0}^{n}\frac{q(i)x(i)\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^i}{\sqrt{\lambda(\lambda-4)}}\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^n, \end{aligned} \tag{13}$$

where $\sum_{i=0}^{-1}.=0.$

Proof. We prove the theorem in two steps. At the first step, we prove that how the solution is obtained and at the second step, we prove that the result holds for the equation.

Step1. Let $x_1(n)$ and $x_2(n)$ are homogeneous linearly independent solutions for part of (9), then it is easily found [20]

$$x_h(n) = c_1x_1(n) + c_2x_2(n), \tag{14}$$

$$x_h(n) = c_1\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^n + c_2\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^n. \tag{15}$$

By means of variation of parameters method [20],[16], we get

$$x_p(n) = c_1(n)x_1(n) + c_2(n)x_2(n), \tag{16}$$

$$\begin{aligned} \Delta x_p(n-1) &= \Delta c_1(n-1)x_1(n) + \Delta c_2(n-1)x_2(n) \\ &+c_1(n-1)\Delta x_1(n-1) + c_2(n-1)\Delta x_2(n-1). \end{aligned}$$

We eliminate first two terms, so that

$$\Delta c_1(n-1)x_1(n) + \Delta c_2(n-1)x_2(n) = 0, \tag{17}$$

next, we obtain,

$$\Delta x_p(n-1) = c_1(n-1)\Delta x_1(n-1) + c_2(n-1)\Delta x_2(n-1), \tag{18}$$

and if we use Δ operator to (17), replace into (11), collect terms involving $c_1(n)$ and $c_2(n)$, we have

$$\begin{aligned} &c_1(n-1)\Delta^2 x_1(n-1) + \Delta x_1(n)\Delta c_1(n-1) + \\ &c_2(n-1)\Delta^2 x_2(n-1) + \Delta x_2(n)\Delta c_2(n-1) \\ &+\lambda\left[c_1(n)x_1(n) + c_2(n)x_2(n)\right] = -q(n)x(n). \end{aligned} \tag{19}$$

We can write $c_1(n-1)\,x_1(n)+c_2(n-1)\,x_2(n)$ instead of bracketed expression from (17) and collect terms involving $c_1(n-1)$ and $c_2(n-1)$, so

$$\begin{aligned} & c_1(n-1)\left[\Delta^2 x_1(n-1)+\lambda x_1(n)\right]+c_2(n-1)\left[\Delta^2 x_2(n-1)+\lambda x_2(n)\right] \\ & +\Delta x_1(n)\,\Delta c_1(n-1)+\Delta x_2(n)\,\Delta c_2(n-1) \;=\; -\,q(n)\,x(n)\,, \end{aligned}$$

and since $x_1(n)$ and $x_2(n)$ satisfy homogeneous part of (11), first two terms equal to zero,

$$\Delta c_1(n-1)\,\Delta x_1(n)+\Delta c_2(n-1)\,\Delta x_2(n)=-q(n)\,x(n)\,. \tag{20}$$

Hence, we find system from (17) and (20) and its solution by Cramer's rule,

$$\begin{aligned} c_1(n-1) &= \sum_{i=0}^{n-1}\frac{q(i)\,x(i)\,x_2(i)}{W(x_1(i),x_2(i))}, \\ c_2(n-1) &= -\sum_{i=0}^{n-1}\frac{q(i)\,x(i)\,x_1(i)}{W(x_1(i),x_2(i))}. \end{aligned}$$

Finally, we obtain the general solution

$$\begin{aligned} x(n,\lambda) &= c_1\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^n+c_2\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^n \\ & \quad +\sum_{i=0}^{n}\frac{q(i)x(i)\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^i}{W(x_1(i),x_2(i))}\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^n \\ & \quad +\sum_{i=0}^{n}\frac{-q(i)x(i)\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^i}{W(x_1(i),x_2(i))}\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^n. \end{aligned}$$

Here, W is a constant by Theorem 2,

$$W(x_1(i),x_2(i))=-\sqrt{\lambda(\lambda-4)}.$$

We can find the representation of the solution of the discrete S-L problem having potential function $q(n)$ in its kernel by using initial conditions (12), then we get,

$$\begin{aligned} x(n,\lambda) &= \left(\frac{2-h\left(2q(0)-2+\lambda+\sqrt{\lambda(\lambda-4)}\right)}{2\sqrt{\lambda(\lambda-4)}}\right)\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^n \\ & +\left(\frac{-2+h\left(2q(0)-2+\lambda-\sqrt{\lambda(\lambda-4)}\right)}{2\sqrt{\lambda(\lambda-4)}}\right)\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^n \\ & -\sum_{i=0}^{n}\frac{q(i)x(i)\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^i}{\sqrt{\lambda(\lambda-4)}}\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^n \\ & +\sum_{i=0}^{n}\frac{q(i)x(i)\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^i}{\sqrt{\lambda(\lambda-4)}}\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^n. \end{aligned} \tag{21}$$

Step2. Now, we show that (21) holds for discrete S-L problem (11) – (12). From (11)

$$q(n)\,x(n) = -\Delta^2 x(n-1) - \lambda x(n). \tag{22}$$

Firstly, let us take last two terms in (21) and write equality (22) in place of $q(i)\,x(i)$. Hence, we obtain

$$\begin{aligned} -\sum_{i=0}^{n} \tfrac{\left[-\Delta^2 x(i-1)-\lambda x(i)\right]x_2(i)}{\sqrt{\lambda(\lambda-4)}} x_1(n) &= \sum_{i=0}^{n} \tfrac{\Delta^2 x(i-1) x_2(i)}{\sqrt{\lambda(\lambda-4)}} x_1(n) \\ &\quad + \sum_{i=0}^{n} \tfrac{\lambda x(i) x_2(i)}{\sqrt{\lambda(\lambda-4)}} x_1(n), \end{aligned} \tag{23}$$

$$\begin{aligned} \sum_{i=0}^{n} \tfrac{\left[-\Delta^2 x(i-1)-\lambda x(i)\right]x_1(i)}{\sqrt{\lambda(\lambda-4)}} x_2(n) &= -\sum_{i=0}^{n} \tfrac{\Delta^2 x(i-1) x_1(i)}{\sqrt{\lambda(\lambda-4)}} x_2(n) \\ &\quad - \sum_{i=0}^{n} \tfrac{\lambda x(i) x_1(i)}{\sqrt{\lambda(\lambda-4)}} x_2(n) \end{aligned} \tag{24}$$

Secondly, applying summation by parts method to the first terms at therhs of equation (23) twice and (24) by Theorem 4, we have

$$\begin{aligned} \sum_{i=0}^{n} \tfrac{x_2(i)\Delta^2 x(i-1)}{\sqrt{\lambda(\lambda-4)}} x_1(n) &= [x_2(n+1)\,\Delta x(n) - x_2(0)\,\Delta x(-1) \\ &\quad -\Delta x_2(n+1)\,x(n+1) + \Delta x_2(0)\,x(0) \\ &\quad +\sum_{i=0}^{n} x(i)\,\Delta^2 x_2(i-1) - \Delta^2 x_2(-1)\,x(0)] \tfrac{x_1(n)}{\sqrt{\lambda(\lambda-4)}}, \end{aligned} \tag{25}$$

$$\begin{aligned} -\sum_{i=0}^{n} \tfrac{x_1(i)\Delta^2 x(i-1)}{\sqrt{\lambda(\lambda-4)}} x_2(n) &= -[x_1(n+1)\,\Delta x(n) - x_1(0)\,\Delta x(-1) \\ &\quad -\Delta x_1(n+1)\,x(n+1) + \Delta x_1(0)\,x(0) \\ &\quad +\sum_{i=0}^{n} x(i)\,\Delta^2 x_1(i-1) - \Delta^2 x_1(-1)\,x(0)] \tfrac{x_2(n)}{\sqrt{\lambda(\lambda-4)}}. \end{aligned} \tag{26}$$

If we substitute (25) and (26) into (23) and (24) respectively, then we find that

$$\begin{aligned} -\sum_{i=0}^{n} \tfrac{q(i)x(i)x_2(i)}{\sqrt{\lambda(\lambda-4)}} x_1(n) + \sum_{i=0}^{n} \tfrac{q(i)x(i)x_1(i)}{\sqrt{\lambda(\lambda-4)}} x_2(n) &= [x_2(n+1)\,\Delta x(n) \\ -x_2(0)\,\Delta x(-1) - \Delta x_2(n+1)\,x(n+1) + \Delta x_2(0)\,x(0) \\ -\Delta^2 x_2(-1)\,x(0)] \frac{x_1(n)}{\sqrt{\lambda(\lambda-4)}} - [x_1(n+1)\,\Delta x(n) - x_1(0)\,\Delta x(-1) \\ -\Delta x_1(n+1)\,x(n+1) + \Delta x_1(0)\,x(0) - \Delta^2 x_1(-1)\,x(0)] \frac{x_2(n)}{\sqrt{\lambda(\lambda-4)}}. \end{aligned} \tag{27}$$

and

$$\sum_{i=0}^{n} \frac{\left[-\Delta^2 x(i-1)-\lambda x(i)\right] x_1(i)}{\sqrt{\lambda(\lambda-4)}} x_2(n) = -[x_1(n+1)\Delta x(n) - x_1(0)\Delta x(-1) - \Delta x_1(n+1)x(n+1) + \Delta x_1(0)x(0) - \Delta^2 x_1(-1)x(0)]\frac{x_2(n)}{\sqrt{\lambda(\lambda-4)}} - \sum_{i=0}^{n} \frac{\left[\Delta^2 x_1(i-1)+\lambda x_1(i)\right] x(i)}{\sqrt{\lambda(\lambda-4)}} x_2(n). \tag{28}$$

Since x_1 and x_2 satisfy the homogeneous part of (11), sum expressions at the right hand side of (27) and (28) equal to zero.
Finally, if we substitute (27) – (28) into (25) – (26) respectively and add (25) and (26), then we have

$$-\sum_{i=0}^{n} \frac{q(i)x(i)x_2(i)}{\sqrt{\lambda(\lambda-4)}} x_1(n) + \sum_{i=0}^{n} \frac{q(i)x(i)x_1(i)}{\sqrt{\lambda(\lambda-4)}} x_2(n) = [x_2(n+1)\Delta x(n) - x_2(0)\Delta x(-1) - \Delta x_2(n+1)x(n+1) + \Delta x_2(0)x(0) - \Delta^2 x_2(-1)x(0)]\frac{x_1(n)}{\sqrt{\lambda(\lambda-4)}} - [x_1(n+1)\Delta x(n) - x_1(0)\Delta x(-1) - \Delta x_1(n+1)x(n+1) + \Delta x_1(0)x(0) - \Delta^2 x_1(-1)x(0)]\frac{x_2(n)}{\sqrt{\lambda(\lambda-4)}}. \tag{29}$$

If we expand right hand side of (29), collect terms involving $x(n)$, $x(n+1)$, $x_1(n)$ and $x_2(n)$ and perform necessary operations, then we get

$$\begin{aligned}
&-\sum_{i=0}^{n} \frac{q(i)x(i)x_2(i)}{\sqrt{\lambda(\lambda-4)}} x_1(n) + \sum_{i=0}^{n} \frac{q(i)x(i)x_1(i)}{\sqrt{\lambda(\lambda-4)}} x_2(n) = \\
&\frac{1}{\sqrt{\lambda(\lambda-4)}}[x(n)\sqrt{\lambda(\lambda-4)} + x_1(n)(x_2(0)x(-1) - x_2(-1)x(0)) \\
&+x_2(n)(x_1(-1)x(0) - x_1(0)x(-1))] = \frac{1}{\sqrt{\lambda(\lambda-4)}}[x(n)\sqrt{\lambda(\lambda-4)} \\
&-\sum_{i=0}^{n} \frac{q(i)x(i)x_2(i)}{\sqrt{\lambda(\lambda-4)}} x_1(n) + \sum_{i=0}^{n} \frac{q(i)x(i)x_1(i)}{\sqrt{\lambda(\lambda-4)}} x_2(n) \\
&-\left(\frac{2-h\left(2q(0)-2+\lambda+\sqrt{\lambda(\lambda-4)}\right)}{2\sqrt{\lambda(\lambda-4)}}\right) x_1(n)\sqrt{\lambda(\lambda-4)} \\
&-\left(\frac{-2+h\left(2q(0)-2+\lambda-\sqrt{\lambda(\lambda-4)}\right)}{2\sqrt{\lambda(\lambda-4)}}\right) x_2(n)\sqrt{\lambda(\lambda-4)}] \\
= \;& x(n) - \left(\frac{2-h\left(2q(0)-2+\lambda+\sqrt{\lambda(\lambda-4)}\right)}{2\sqrt{\lambda(\lambda-4)}}\right) x_1(n) \\
&-\left(\frac{-2+h\left(2q(0)-2+\lambda-\sqrt{\lambda(\lambda-4)}\right)}{2\sqrt{\lambda(\lambda-4)}}\right) x_2(n),
\end{aligned} \tag{30}$$

and then

$$x(n) = \left(\frac{2-h\left(2q(0)-2+\lambda+\sqrt{\lambda(\lambda-4)}\right)}{2\sqrt{\lambda(\lambda-4)}}\right)x_1(n) + \left(\frac{-2+h\left(2q(0)-2+\lambda-\sqrt{\lambda(\lambda-4)}\right)}{2\sqrt{\lambda(\lambda-4)}}\right)x_2(n)$$
$$-\sum_{i=0}^{n}\frac{q(i)x(i)x_2(i)}{\sqrt{\lambda(\lambda-4)}}x_1(n) + \sum_{i=0}^{n}\frac{q(i)x(i)x_1(i)}{\sqrt{\lambda(\lambda-4)}}x_2(n).$$

Hence, the proof completes. □

Theorem 7. *Let's define discrete Sturm–Liouville problem as follows:*

$$Ly(n) = -\lambda y(n), \tag{31}$$

$$y(0) = 1, y(1) = 0, \tag{32}$$

then Sturm–Liouville problem (31) − (32) *has a unique solution for* $y(n)$ *as*

$$\begin{aligned} y(n,\lambda) = & \left(1 - \frac{1}{\sqrt{\lambda(\lambda-4)}}\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2} + \frac{q(0)}{\sqrt{\lambda(\lambda-4)}} - q(0)\right)\right)\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^n \\ & + \left(-\frac{1}{\sqrt{\lambda(\lambda-4)}}\left(-1 + \frac{-\lambda-\sqrt{\lambda(\lambda-4)}}{2} + q(0)\right)\right)\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^n \\ & - \sum_{i=0}^{n}\frac{q(i)y(i)\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^i}{\sqrt{\lambda(\lambda-4)}}\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^n \\ & + \sum_{i=0}^{n}\frac{q(i)y(i)\left(\frac{2-\lambda+\sqrt{\lambda(\lambda-4)}}{2}\right)^i}{\sqrt{\lambda(\lambda-4)}}\left(\frac{2-\lambda-\sqrt{\lambda(\lambda-4)}}{2}\right)^n, \end{aligned} \tag{33}$$

where $\sum_{i=0}^{-1}. = 0.$

Proof. Similar arguments used in the proof of Theorem 6 are applied. □

3.1 Asymptotic Estimations for Discrete Sturm–Liouville Problems

In this section, we present the asymptotic estimations for eigenfunctions of the problem. Let's take discrete S-L problem (11) − (12), then we can give the following Theorem.

Theorem 8. *Sturm–Liouville problem* (11) − (12) *has the estimate*

$$x(n) = O\left(e^{|t|n}\right), n \in \mathbb{Z}^+,$$

where $|\lambda - 2| < 2$, $\{x(n)\}$ *is a complex sequence,* $\sum_{i=0}^{\infty} i\,|q(i)| < \infty$.

Proof. Since $|\lambda - 2| < 2$, let

$$\lambda = 2 - 2\cos\theta. \tag{34}$$

Let $x(n) = e^{|t|n} f(n)$. Then from(22) we obtain

$$\begin{aligned} f(n) &= \left\{ -h\cos n\theta + \left(\frac{1 + q(0)\,h + h\cos\theta}{\sin\theta} \right) \sin n\theta \right\} e^{-|t|n} \\ &+ \frac{1}{\sin\theta} \sum_{k=0}^{n} q(k)\, f(k)\, e^{-|t|(n-k)} \sin(n-k)\,\theta. \end{aligned} \tag{35}$$

Let $\nu = \max\limits_{0 \le n \le b} |f(n)|$. Then it follows from the last relation that

$$\nu \le 1 + \frac{|q(0)| + 1}{|\sin\theta|} + \frac{\mu}{\sin\theta} \sum_{k=0}^{n} |q(k)|, \tag{36}$$

and so

$$\nu \le \frac{2|h| + 1 + |q(0)|\,|h|}{|\sin\theta| - \sum\limits_{k=0}^{n} |q(k)|}, \tag{37}$$

let's assume that the denominator is positive. This requires the case $|\sin\theta| > \sum\limits_{k=0}^{n} |q(k)|$, and thus the proof completes.

□

Theorem 9. *Discrete Sturm–Liouville problem* $(31)-(32)$ *has the estimate*

$$y(n) = O\left(e^{|t|n}\right), n \in \mathbb{Z}^{+},$$

where $|\ \lambda - 2\ | < 2$, $\{y(n)\}$ *is a complex sequence,* $\sum\limits_{i=0}^{\infty} i\,|q(i)| < \infty$.

Proof. Similar arguments used in the proof of Theorem 8 are applied. □

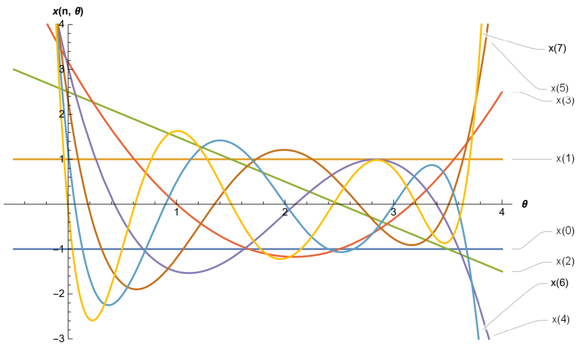

Figure 1: Eigenfunctions for the problem (12) -(13),$q(n) = \frac{1}{n+1}$

$x(7)$	$\lambda_1 = -0.0037$	$\lambda_2 = 0.6182$	$\lambda_3 = 1.505$	$\lambda_4 = 2.445$	$\lambda_5 = 3.209$	$\lambda_6 = 3.638$
$x(6)$	$\lambda_1 = 0.06111$	$\lambda_2 = 0.9003$	$\lambda_3 = 2$	$\lambda_4 = 3.008$	$\lambda_5 = 3.582$	
$x(5)$	$\lambda_1 = 0.1802$	$\lambda_2 = 1.352$	$\lambda_3 = 2.675$	$\lambda_4 = 3.509$		
$x(4)$	$\lambda_1 = 0.4273$	$\lambda_2 = 2.099$	$\lambda_3 = 3.393$			
$x(3)$	$\lambda_1 = 1$	$\lambda_2 = 3.167$				
$x(2)$	$\lambda_1 = 2.5$					

Table 1: Eigenfunctions and corresponding eigenvalues for the problem (12)-(13),$q(n) = \frac{1}{n+1}$

n	$x(n), q(n) = \frac{1}{n+1}$	$x(n), q(n) = \frac{1}{\sqrt{n+1}}$
0	-1	-1
1	1	1
2	1.5	1.29289
3	$1.11022 * 10^{-16}$	-0.45355
4	-1.4999	-1.51967
5	-1.2	-0.38649
6	0.4999	1.29096
7	1.62857	1.18952
8	0.925	-0.52200
9	-0.8063	-1.53752
10	-1.65071	-0.52931
11	-0.6943	1.16780
12	1.01427	1.35999

Table 2: Eigenfunctions correspond to $\lambda = 1$

n	$x(n), q(n) = \frac{1}{n+1}$	$x(n), q(n) = \frac{1}{\sqrt{n+1}}$
0	-1	-1
1	1	1
2	-0.5	-0.70710
3	-0.33	0.11535
4	0.916	0.53407
5	-0.766	-0.88827
6	-0.022	0.71683
7	0.79206	-0.09950
8	-0.86884	-0.58215
9	0.17332	0.87571
10	0.67819	-0.57047
11	-0.91317	-0.13322
12	0.31107	0.74216

Table 3: Eigenfunctions correspond to $\lambda = 3$

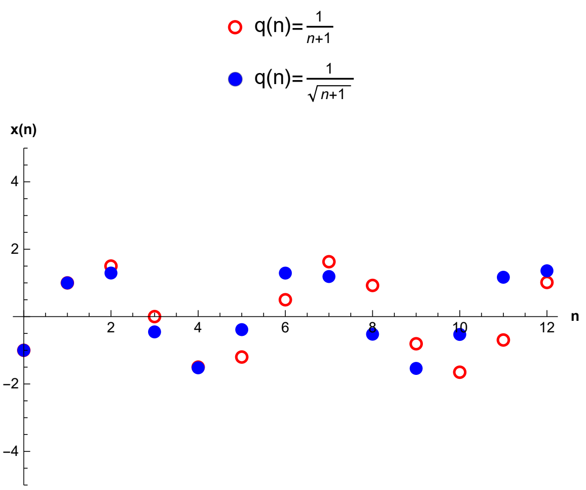

Figure 2: Comparison of data in Table2

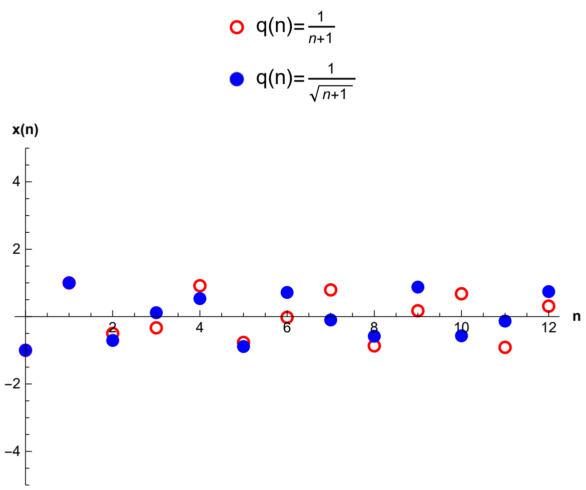

Figure 3: Comparison of data in Table3

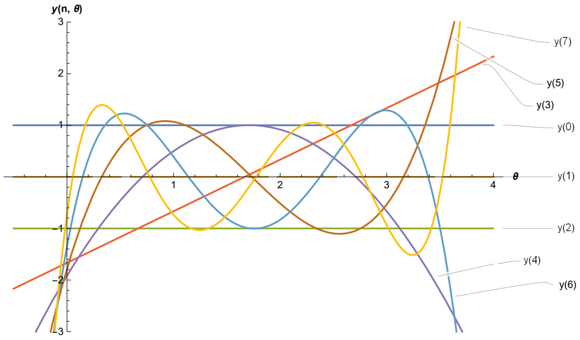

Figure 4: Eigenfunctions for the problem (32)-(33),$q(n) = \frac{1}{n+1}$

$y(7)$	$\lambda_1 = 0.0551$	$\lambda_2 = 0.7767$	$\lambda_3 = 1.777$	$\lambda_4 = 2.776$	$\lambda_5 = 3.527$
$y(6)$	$\lambda_1 = 0.1503,$	$\lambda_2 = 1.138$	$\lambda_3 = 2.376$	$\lambda_4 = 3.387$	
$y(5)$	$\lambda_1 = 0.3273,$	$\lambda_2 = 1.732$	$\lambda_3 = 3.16$		
$y(4)$	$\lambda_1 = 0.708$	$\lambda_2 = 2.706$			
$y(3)$	$\lambda_1 = 1.663$				

Table 4: Eigenfunctions for the problem (32)-(33),$q(n) = \frac{1}{n+1}$

n	$y(n), q(n) = \frac{1}{n+1}$	$y(n), q(n) = \frac{1}{\sqrt{n+1}}$
0	1	1
1	0	0
2	-1	-1
3	0.5	-0.80755
4	1.06667	0.347863
10	0.866111	0.902663
13	-0.948732	-1.05191
14	-0.904103	-0.696555
15	0.104902	0.489407
20	-0.775604	-0.0471693
23	0.715857	-0.302822
25	-1.07611	-0.57186

Table 5: Eigenfunctions correspond to $\lambda = 1$

n	$y(n), q(n) = \frac{1}{n+1}$	$y(n), q(n) = \frac{1}{\sqrt{n+1}}$
0	1	1
1	0	0
2	-1	-1
3	0.3333	0.57735
4	0.916667	0.711325
10	-0.679263	0.339744
13	-0.827776	-0.750299
14	-0.555092	0.805282
15	0.864783	0.542376
20	0.405462	-1.03312
23	0.953096	-0.353119
25	-0.966021	0.539181

Table 6: Eigenfunctions correspond to $\lambda = 2$

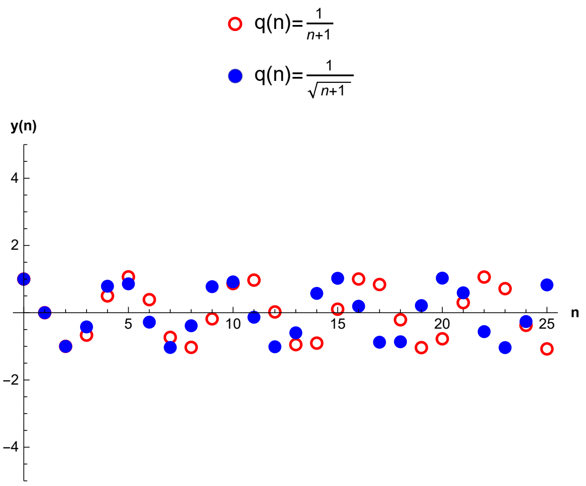

Figure 5: Comparison of data in Table5

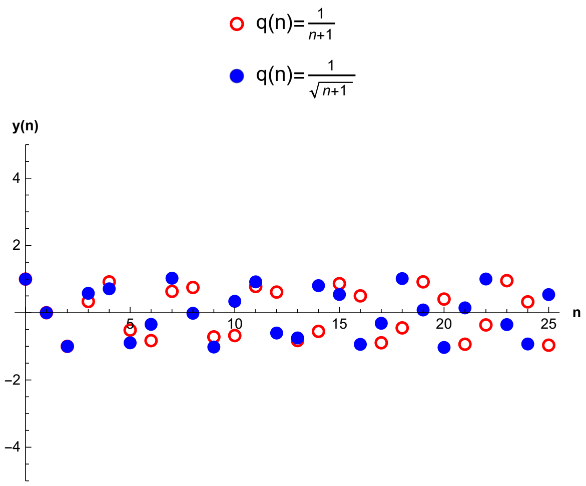

Figure 6: Comparison of data in Table6

4 Conclusion

Potential function $q(n)$ is rather important for Sturm–Liouville problems and its spectral theory for physical meaning, comment, and applications. If the potential function $q(n)$ is taken as a variable, more accurate and more functional results may be obtained for the spectral theory of the problem. In this study, we consider the problem with sum the representations of the solutions of the Sturm–Liouville problem having potential function $q(n)$ in their kernels. From this point of view, we acquire asymptotic formulas for eigenfunctions. Our results are discrete analog of the results of [22] in differential equations (p.5-6).

We observe that oscillation increases as n increases shown in $Fig1$ and $Fig4$, accordingly points intercepting the apsis increase and so the zeros, namely eigenvalues, increase and from this point of view, we find the eigenvalues corresponding some eigenfunctions. It has been already known that Sturm–Liouville difference equations have an nth degree polynomial solution, so in this study, we show the number of eigenvalues increases as n increases, also for the first time we observe the behaviors of eigenfunctions and eigenvalues and we find the eigenvalues under the kernel having potential function $q(n)$.

Moreover, behaviors of eigenfunctions for the problems $(11)-(12)$ and $(31)-(32)$ are analyzed and illustrated by graphics and tables. Firstly, we show the behaviors of eigenvalues while $q(n) = \frac{1}{n+1}$ in $Fig1$, we find the eigenfunctions corresponding these eigenvalues in $Table1$. Then, we analyze the behaviors of eigenfunctions for the specific eigenvalues $\lambda = 1$ and $\lambda = 3$ in $Table2$, $Table3$ and $Fig2$ and $Fig3$. We analyze the similar properties for the problem $(31)-(32)$ in $Table4$, $Table5$, $Table6$, $Fig4$, $Fig5$ and $Fig6$.

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