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LINEAR PROGRAMMING AND APPLICATION IN THE STOCKS BALANCING OF A MANUFACTURE

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Abstract.

This work presents a study on Linear Optimization and its application in the stock balancing for annual production planning and control of a furniture manufacture. The proposed problem is mathematically modeled using the so-called Combined Problem, which connects two important optimization problems: lot sizing and stock cutting problems. The case study, developed in this paper, consists in collecting data from a small-scale furniture plant, located in Cornélio Procópio city, in the state of Paraná, Brazil. The main objective is to plan the production of different types of furniture over an annual period, considering stocks of pieces and of final products. Simplex Method and *Branch and Bound* Algorithm are applied in order to obtain the solutions to the proposed scenarios of production, providing support to the decision make and minimizing the global costs of production. The results indicate that the proposed model, considering stocks balancing, provides the production planning and the minimum production costs in comparison with the production that answer demand by each period.

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1 Introduction

The intensification of technological advances and economic aspects encourage competitiveness between industries, which need to innovate their products and to automate their processes, in order to make the production more efficient and to acquire competitive advantage. In this sense, the optimization is required in the industrial sector, since it allows the production planning in a known period of time.

The administration of the production system, in an industry, is responsible for the planning and control of transforming raw materials into final products [5]. This system is called *production planning and control* and it consists in supporting the control of various activities that compose the plans of production, so that the pre-established production programs can be answered within the deadlines and demanded quantities [6]. Coupled with the administration of production, optimization methods for the control and production planning becomes an essential tool for industrial advancement. Optimization methods aim to solve problems quantitatively, using mathematical and statistical methods, and it helps making the best decisions, by determining the objectives and constraints under which to operate [2].

Linear optimization is a tool that can help to decide the best production plan for an industry, in order to achieve a determined goal, by using linear equations. One advantage of the linear optimization model is the efficiency of the solution algorithms, which allow them to be implemented in spreadsheets on personal computers [10]. The Simplex Method is the more recommended in the literature for resolution of linear programming problems [2, 8].

The large number of surveyed papers shows the increased interest of researchers in lot sizing problems [11]. In Karimi et. al. (2003), single-level lot sizing problems, their variants and solution approaches are considered. Authors discuss these lot sizing problems using exact and heuristic approaches in order to find solutions.

An overview of recent developments about modeling deterministic single-level dynamic lot sizing problems is given by Jans and Degraeve (2008). The focus is on the modeling of various industrial extensions [12]. More recent, Brahimi et. al. (2017) present an updated and extended survey of Single-Item Lot-Sizing Problems with focus on publications from 2004 to 2016. Exact and heuristic solution procedures are studied. According to authors, almost 300 papers on the single item lot sizing problem are surveyed [3].

Stock cutting problem has been also studied extensively in the literature. The environmental situation of the planet makes the use of natural resources be moderate and conscious. United Nations conferences on climate change have occurred and discussions about how to preserve the environment are promoted. Thus, to plan the production of a furniture industry, whose raw material is wood, in order to minimize costs and waste of materials in the cutting process, contributes to reduce the waste of raw materials and the environmental impact of the cutting down of trees. In this sense, the study of Danwé et. al. (2012) [7] aims to solve the problem of optimizing the production on the basis of the commercial value of the cuts, since the loss of raw materials in wood cutting industries has reached high.

The one-dimensional integer cutting stock problem is also studied in Poldi and Arenales (2009) [15], which consists of cutting a set of available stock lengths in order to produce smaller ordered items. There are several stock lengths available in limited quantities. Some heuristic methods are proposed in order to obtain an integer solution and compared with others.

Integrated lot sizing and cutting stock problem is approached by Vanzela et. al. (2013) [17]. The goal is to capture the dependency that exists between two important decisions in the production process, in order to economize raw materials and also reduce production and inventory costs. A column generation technique is used to solve a linear relaxation of the proposed model. In Alem and Morabito (2013) [1], an integrated production planning and cutting stock problem is also investigated. The production costs contained in the manufacturing process and the products demands are not known. Robust optimization models were proposed, since there are uncertainties to be considered.

Recently, Bressan et. al. (2017) [4] applied the Combined Problem in order to minimize production costs of a manufacture, considering two production planning scenarios: first, costs and demands of the final products are constants; second, costs and demands vary over periods of planning. However, [4] does not consider that there is stock from one period to another, i.e., all pieces and final products are empty at the end of a period. Therefore, the main contribution of this proposed study in the literature is to consider that there are stock of pieces and/or final products in each planning period, reflecting the reality of manufacturing in study more precisely. Moreover, the production annual planning of a furniture manufacture contributes to reduce the waste of raw materials in the cutting process.

In this context, the annual production planning of a small-scale furniture plant, located in Cornélio Procópio city, in the state of Paraná, Brazil, by considering stocks balancing of pieces and final products is presented. The main goal is to optimize the production in order to minimize the global costs of production.

The problem is formulated using the *Combined Problem* [5, 9], which connects two important and known optimization problems: *lot sizing* and *stock cutting* problems. The lot sizing problem consists in planning the quantity of final products to be produced in each period over a finite time horizon, in order to meet the demand and minimize production costs and stock costs. The stock cutting problem consists in the optimization of the process of cutting of boards (wood) into smaller parts, to compose the final products. Therefore, the solution of the Combined Problem indicates the quantity of final products to be produced, minimizing global costs of production, preparation and stocking, and the number of boards to be cut in order to compose final products [5, 9].

The Simplex Method is applied in order to obtain the solution of the proposed problems and to help in decision make. For comparison purposes, the *Branch and Bound* algorithm is also applied to the problem for guarantee of obtaining integer solutions. Numerical solutions are obtaining with computational support of LINDO (*"Linear Software Interactive and Discrete Optimizer"*) software, to run both Simplex and Branch and Bound algorithms.

2 The Combined Problem

The lot sizing problem and the cutting stock problem have already been defined in the preview section. In the process of cutting a board into smaller pieces, raw material loss tends to be reduced if the cuts are rearranged in a convenient way on the board. Therefore, some final products could be produced in advance in order to minimize losses. However, the stock costs can suggest the opposite [4]. The Combined Problem deals with this decision problem, joining the lot sizing and cutting stock problems. The solution of the combined problem suggests the number of final products to be produced in each planning horizon period in order to minimize production, preparation and storage costs (lot sizing) and the number of boards to be cut, as well as cutting patterns to compose final products (cutting stock) [9].

One approach to the combined problem disregards preparation costs and relaxes the integrality of the variables representing the number of boards cut in a given pattern, which requires a large demand. This approach can be applied in the furniture industry where wooden boards must be cut to produce items. It is considered that there is only one type of board [5].

The mathematical model is formulated according to [9] and it is described as equations (1) to (5).

$$\min \sum_{i=1}^{M}\sum_{t=1}^{T}(c_{it}x_{it} + h_{it}e_{it}) + \sum_{j=1}^{N}\sum_{t=1}^{T} cpy_{jt} + \sum_{p=1}^{P}\sum_{t=1}^{T} hp_{pt}ep_{pt} \tag{1}$$

subject to:

$$x_{it} + e_{i,t-1} - e_{it} = d_{it} \tag{2}$$

$$\sum_{j=1}^{N} a_{pj}y_{jt} + ep_{p,t-1} - ep_{pt} = \sum_{i=1}^{M} r_{pi}x_{it} \quad \forall \ t = 1, \ldots, T \tag{3}$$

$$\sum_{j=1}^{N} v_j y_{jt} \leq u_t \tag{4}$$

$$x_{it}, e_{it}, y_{jt}, ep_{pt} \geq 0 \tag{5}$$

The indexes, parameters and variables are described below.

Indexes:
t=1, ..., T number of periods of time.
p=1, ..., P number of different pieces to be cut.
j=1, ..., N number of different cutting pattern.
i=1, ..., M number of different final products.

Parameters:
c_{it}: production cost of the final product i in the period t.
h_{it}: stocking cost of the final product i in the period t.
hp_{pt}: stocking cost of the piece p in the period t.
d_{it}: demand of the final product i in the period t.
r_{pi}: number of pieces p required to make product i.
v_j: time to cut a board using cutting pattern j.
a_{pj}: number of piece p cut using cutting pattern j.
u_t: maximum time of saw capacity.
cp: cost of the board to be cut.

Decision Variables:
x_{it}: number of final product i produced in the period t.
e_{it}: number of final product i stocked in the end of the period t.
ep_{pt}: number of piece p stocked in the end of the period t.
y_{jt}: number of boards cut using cutting pattern j in the period t.

The restrictions (2) are related to the stock balancing in relation to final products, which ensures that the demand for items from each period will be met. The restrictions (3) are related to stock balancing in relation to pieces, which ensures that the demand for pieces will be met. These restrictions are those that couple the lot sizing and stock cutting problems, since both include the x_{it} variables, which define the lots size and y_{jt}, which define the number of boards cut using a defined cutting pattern. The restrictions (4) are related to saw capacity, since the time spent in the cutting process of the boards in the various cutting patterns does not exceed the maximum operating time. Finally, (5) represents non-negativity conditions. More details can be seen in [4, 5].

3 Algorithms for Linear Programming Problems

According to [2, 4, 8], the Simplex Method is recommended in order to provide real optimal solution, if it exists, of Linear Programming Problems. Simplex Method is an iterative and numerical process developed in 1947 by the North American mathematician George B. Dantzig. Based on Simplex Method, the Branch and Bound algorithm ensures that solutions are integer numbers. Simplex and Branch and Bound algorithms are described below in order to obtain the optimal solution for the proposed problem involving stock balancing.

3.1 Simplex Algorithm

A linear programming problem can be written in the matricial notation, as:

$$\begin{aligned} &\text{Minimize} && \boldsymbol{cx} \\ &\text{subject to} && \boldsymbol{Ax}=\boldsymbol{b} \\ & && \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A}$ is an $m \times n$ matrix with rank m; $\boldsymbol{c}$ is the $1 \times n$ vector of costs; $\boldsymbol{x}$ is an $n \times 1$ vector of variables and $\boldsymbol{b}$ is an $m \times 1$ constants.

After possibly rearranging the columns of $\boldsymbol{A}$, let $\boldsymbol{A} = [\boldsymbol{B}, \boldsymbol{N}]$ where $\boldsymbol{B}$ (basic matrix) is an $m \times m$ invertible matrix and $\boldsymbol{N}$ (nonbasic matrix) is $m \times (n-m)$ matrix. Similarly, the basic solution vector $\boldsymbol{x}$ is also partitioned as $\boldsymbol{x}_B$ (basic variables) and $\boldsymbol{x}_N$ (nonbasic variables). The Simplex algorithm for minimization problems is described as follows [2].

3.2 Branch and Bound Algorithm

Many real-world problems could be modeled as linear programs except that some or all of the variables are constrained to be integers. Such problems are called integer programming problems [16], in which the integer variables can be any nonnegative integer. One technique for solving problems in this class is called Branch and Bound Algorithm.

The algorithm starts with the following approach: first ignore the constraint that the components of $\boldsymbol{x}$ be integers, solve the resulting linear programming problem, and hope that the solution vector has all integer components. Since hopes are almost always unfulfilled, and so a backup strategy is needed [16].

The linear programming problem obtained by dropping the integrality constraint is called the LP-relaxation. By applying simplex method, let x^* be the optimal solution obtained (a real value) for a specific decision variable. Consider that real value x^*C is between two integer values: x_{min} and x_{max}. The method considers two cases separately: Let P1 denote the linear programming problem obtained by adding the constraint $x^* \leq x_{min}$ to the LP-relaxation, and let P2 denote the problem obtained by including the other possibility, $x^* \geq x_{max}$. Note that we are starting to develop a tree of linear programming subproblems. This tree is called the enumeration tree.

In the branch and bound method, each tree node corresponds to a subproblem defined as a mixed integer bound-constrained maximization problem. The relaxed subproblem associated with a subproblem is defined as the subproblem itself but without its integrality constraints.

A node is fathomed in the search tree if its subproblem is infeasible (it can be trivially checked), the optimal solution of its relaxed subproblem satisfies the integrality constraints (and, therefore, there is no further need for branching) and the optimal value of its relaxed subproblem, that is a upper bound on the optimal value of its (non-relaxed) subproblem, is worse than or equal to the current incumbent solution value [14].

Consider that a node $S_j, j = 0, \cdots, N$ is selected to be solved. The first step is to solve the relaxation. Let $S_j^*, j = 0, \cdots, N$ be the solution vector of the relaxed subproblem. The solution of each subproblem provides a upper bound for the subproblems in the descent nodes of the tree. This process continues until the upper bound exceeds the current lower bound, the subproblem is infeasible, or the solution provides integer values for the integer variables. The integer solution (at the tree node) gives lower bounds to the optimal integer solution. The Branch and Bound algorithm for minimization problems is described in Algorithm 2.

Algorithm 1: Simplex

```
/* INITIALIZATION STEP                                                  */
```

1 Choose a starting basic feasible solution with basis $\boldsymbol{B}$.

```
/* MAIN STEP                                                            */
```

1. Solve the system $\boldsymbol{B}\boldsymbol{x}_B = \boldsymbol{x}$ (with unique solution $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\ \boldsymbol{b} = \bar{b}$).

   Let $\boldsymbol{x}_B = \bar{b}$, $\boldsymbol{x}_N = 0$ and $z = \boldsymbol{c}_B\ \boldsymbol{x}_B$.

2. Solve the system $\boldsymbol{wB} = \boldsymbol{c}_B$ (with unique solution $\boldsymbol{w} = \boldsymbol{c}_B\ \boldsymbol{B}^{-1}$).

   The vector $\boldsymbol{w}$ is the vector of *simplex multipliers*, since its components are the multipliers of the rows of $\boldsymbol{A}$ that are added to the objective function in order to bring it into canonical form.

   Calculate $z_j - c_j = \boldsymbol{w}\boldsymbol{a}_j - c_j$ for all nonbasic variables. Let

   $$z_k - c_k = \max_{j \in J}\{z_j - c_j\}$$

   where J is the current set of indices associated with the nonbasic variables. If $z_k - c_k \leq 0$, then stop with the current basic feasible solution as an optimal solution. Otherwise, go to Step 3 with x_k as the entering variable. (This strategy for selecting an entering variable is known as Dantzig's Rule.)

3. Solve the system $\boldsymbol{B}\ \boldsymbol{y}_k = \boldsymbol{a}_k$ (with unique solution $\boldsymbol{y}_k = \boldsymbol{B}^{-1}\ \boldsymbol{a}_k$). If $\boldsymbol{y}_k \leq 0$, then stop with the conclusion that the optimal solution is unbounded along the ray

   $$\left\{ \begin{bmatrix} \bar{b} \\ 0 \end{bmatrix} + x_k \begin{bmatrix} -y_k \\ e_k \end{bmatrix} : x_k \geq 0 \right\}$$

   where e_k is an $(n - m)$ vector of zeros except for a 1 at the kth position. If $y_k \nleq 0$, go to Step 4.

4. Let x_k enter the basis. The index r of the blocking variable,x_{B_r}, which leaves the basis is determined by the following minimum ratio test:

   $$\frac{\bar{b_r}}{y_{rk}} = \min_{1 \leqslant i \leqslant m} \left\{ \frac{\bar{b_i}}{y_{ik}} : y_{ik} > 0 \right\}.$$

   Update the basis $\boldsymbol{B}$ where $\boldsymbol{a}_k$ replaces $\boldsymbol{a}_{B_r}$, update the index set J, and repeat Step 1.

4 Case Study

In this section, an application of the combined problem is developed so that the annual production of a furniture manufacture can be planned and controlled. The small-scale

Algorithm 2: Branch and Bound

```
Input: active node, tree level
Output: bestInt
Make active node the father node;
Execute Relaxed subproblem node to solve the active node;
Save the best found solution as fatherSolution;
/* Pruning by optimality                                  */
if fatherSolution is integer then
    if fatherSolution > bestInt then
        bestInt = fatherSolution;
    end
end
/* Pruning by bounds                                      */
if fatherSolution is not integer and fatherSolution ≤ bestInt then
    bestInt = fatherSolution;
end
Branch the active node and create two subproblems S1 and S2;
Branch and Bound(S1, tree level+1, bestInt);
Branch and Bound(S2, tree level+1, bestInt);
return bestInt = best integer solution.
```

furniture plant is located in Cornélio Procópio city, in the state of Paraná, Brazil. The numerical parameters are collected from the industry data history, in the year 2017. The products to be considered are the 3 most demanded final products: table, chair and stool. There is variation in the pieces stock costs, final products stock costs, as well as there is variation in demands for final products.

In the first production scenario, the stock of pieces in the previous periods is considered. In the second production scenario, the stock of final products in the previous periods is considered and, in the third scenario, there are both stocks of pieces and final products.

The mathematical formulations of the proposed productions problems of the furniture factory, consider that demands and stock costs vary monthly in the annual planning, considering the variation in market value during the year. The monetary units, indicated by $, are related to the currency in Brazil.

The indexes used in this study, according to the Combined Problem formulation, are described as follows:
$t = 12$ time periods (months);
$p = 6$ pieces;
$j = 4$ board cutting patterns;
$i = 3$ final products.

Therefore, the combined problem can be formulated as follows:

$$\min \sum_{i=1}^{3}\sum_{t=1}^{12} c_{it}x_{it} + h_{it}e_{it} + \sum_{j=1}^{4}\sum_{t=1}^{12} cpy_{jt} + \sum_{p=1}^{6}\sum_{t=1}^{12} hp_{pt}ep_{pt}$$

$$x_{it} + e_{i,t-1} - e_{it} = dit;\;\; t = 1, ...12; i = 1, 2, 3 \text{ (final products stock balancing)}$$

$$\sum_{j=1}^{4} a_{pj}y_{jt} + ep_{p,t-1} - ep_{pt} = \sum_{i=1}^{3} r_{pi}x_{it}; t = 1, ..., 12 \;\; \text{(pieces stock balancing)}$$

$$\sum_{j=1}^{4} v_j y_{jt} \leq u_t; t = 1, ..., 12; j = 1, 2, 3 \;\; \text{(saw capacity)}$$

$$x_{it}, e_{it}, y_{jt}, ep_{pt} \geq 0$$

The types of pieces to be cut, in order to compose the final products, are:
Type 1 = table cover;
Type 2 = foot of the table;
Type 3 = table support;
Type 4 = seat/backrest;
Type 5 = foot of the chair and stool;
Type 6 = chair/stool support.

The cutting patterns are the ways in which the boards are cut into smaller pieces and can be seen in Table 1.

Table 1: Board cutting patterns

Cutting Pattern	p=1	p=2	p=3	p=4	p=5	p=6	Cutting time
j=1	2	0	0	0	0	0	5 s
j=2	1	50	0	0	0	100	350 s
j=3	0	0	10	18	30	0	148 s
j=4	0	10	13	12	103	0	197 s

The constants parameters provided from the manufacture are: c_{1t} = \$255, represents the table production cost, c_{2t} = \$80, the chair production cost, c_{3t} = \$60 the stool production cost, cp = \$137.07 the board cost and the number of pieces are: r_{11}=1, r_{21}= 4, r_{31}=2, r_{42}=2, r_{52}=4, r_{62}=2, r_{43}=1, r_{53}=4 and r_{63}=2.

The final products stocking costs parameters, for each planning period, are described in Table 2.

On the other hand, the pieces stocking costs parameters, for each planning period, are shown in Table 3.

Final products demands are obtained considering the variations over periods of planning. Table 4 presents the used numerical parameters.

After the Combined Problem be solved, the optimal solution should indicates the quantities of final products to be produced in each period of planning, in order to reach

Table 2: Final products stocking costs.

	Table	Chair	Stool
$t = 1$	\$5	\$3	\$2
$t = 2$	\$5	\$3	\$2
$t = 3$	\$5	\$3	\$2
$t = 4$	\$5	\$3	\$2
$t = 5$	\$6	\$3	\$2
$t = 6$	\$6	\$3.5	\$2
$t = 7$	\$6	\$3.5	\$2.5
$t = 8$	\$6	\$3.5	\$2.5
$t = 9$	\$6	\$3.5	\$2.5
$t = 10$	\$6.5	\$3.8	\$2.5
$t = 11$	\$6.5	\$3.8	\$2.7
$t = 12$	\$6.5	\$3.8	\$2.7

Table 3: Pieces stocking costs.

	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$t = 1$	\$0.8	\$0.1	\$0.12	\$0.15	\$0.05	\$0.09
$t = 2$	\$0.8	\$0.1	\$0.12	\$0.15	\$0.05	\$0.09
$t = 3$	\$0.8	\$0.1	\$0.12	\$0.15	\$0.05	\$0.09
$t = 4$	\$0.8	\$0.1	\$0.12	\$0.15	\$0.05	\$0.09
$t = 5$	\$1	\$0.1	\$0.12	\$0.15	\$0.05	\$0.09
$t = 6$	\$1	\$0.12	\$0.14	\$0.15	\$0.06	\$0.09
$t = 7$	\$1	\$0.12	\$0.14	\$0.16	\$0.06	\$0.11
$t = 8$	\$1	\$0.12	\$0.14	\$0.16	\$0.06	\$0.11
$t = 9$	\$1	\$0.12	\$0.14	\$0.16	\$0.06	\$0.11
$t = 10$	\$1.2	\$0.13	\$0.15	\$0.16	\$0.07	\$0.11
$t = 11$	\$1.2	\$0.13	\$0.15	\$0.17	\$0.07	\$0.12
$t = 12$	\$1.2	\$0.13	\$0.15	\$0.17	\$0.07	\$0.12

the minimum production costs. Thus, with computational support of LINDO (*"Linear Software Interactive and Discrete Optimizer"*) software, for comparison purposes, the Simplex Method was run (real solutions) and so the Branch and Bound Algorithm, which guarantees entire solutions.

4.1 Results

In the first production planning scenario, pieces stocking generated in the previous period $(t-1)$ are considered, as described in Table 5.

Simplex Method presents the minimum cost \$33600.75. Optimal solution suggests anticipating the production of chairs, since the decision variable is $x_{21} = 204$, generating stock, and postponing table and stool production for the second period: x_{12}=42 e x_{32}=77. For comparison purposes, the *Branch-and-Bound* algorithm is run, which guarantees en-

Table 4: Final products demands by period.

	Table	Chair	Stool
$t = 1$	1	4	2
$t = 2$	1	4	2
$t = 3$	2	10	3
$t = 4$	3	14	6
$t = 5$	3	14	6
$t = 6$	3	14	6
$t = 7$	4	20	8
$t = 8$	3	14	6
$t = 9$	3	14	6
$t = 10$	6	30	10
$t = 11$	6	30	10
$t = 12$	7	36	12

Table 5: Pieces stocking in the previous production periods.

	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$t = 0$	0	0	0	0	0	0
$t = 1$	0	0	0	0	0	0
$t = 2$	0	5	2	0	1	2
$t = 3$	0	2	1	0	2	3
$t = 4$	1	3	1	2	5	2
$t = 5$	2	2	1	3	6	5
$t = 6$	1	6	5	3	2	4
$t = 7$	2	4	0	2	1	3
$t = 8$	1	3	2	1	2	1
$t = 9$	1	2	3	4	3	2
$t = 10$	2	5	4	2	1	4
$t = 11$	1	2	3	2	2	1

tire solution for linear programming problems. Since Simplex has already presented entire solution, the algorithm also converges to the same solution.

In the second production planning scenario, final products stocking generated in the previous period $(t - 1)$ are considered, as described in Table 6.

Considering this case, Simplex Method presents the minimum cost \$32106.59. Optimal solution suggests anticipating the production of chairs for the first period, since the decision variable is $x_{21} = 198$ and postponing table and stool production for the second period: $x_{12} = 39$ e $x_{32} = 72$. For comparison purposes, the *Branch-and-Bound* algorithm is run again, and since Simplex has already presented entire solution, the algorithm also converges to the same solution.

Finally, in the third production planning scenario, both final products and pieces stocking are considered. This scenario is very common in the daily of furniture factories. The number of pieces in stock in the previous period are the ones presented in Table

Table 6: Final products stocking in the previous production periods

	$i = 1$	$i = 2$	$i = 3$
$t = 0$	0	0	0
$t = 1$	0	2	0
$t = 2$	1	0	2
$t = 3$	0	0	1
$t = 4$	0	1	0
$t = 5$	1	0	0
$t = 6$	0	2	1
$t = 7$	0	0	0
$t = 8$	0	0	0
$t = 9$	0	1	1
$t = 10$	1	0	0
$t = 11$	0	0	0

5 and the number of final products in stock are the ones presented in Table 6. Thus, these data are applied simultaneously in the mathematical model. Simplex Method is run and it presents the minimum cost \$31972.24. Optimal solution suggests anticipating the production, generation stocks, since the decision variables are $x_{21} = 198$, $x_{31} = 72$ and $x_{13} = 39$, which represents, respectively, the production of chairs in the first planning period, the production of stools in the first planning period and the production of tables in the third planning period. Also, for comparison purposes, the *Branch-and-Bound* algorithm is run, and it also converges to the same entire solution.

4.2 Sensibility Analysis

According to [2], it is important to study the effect on optimal solutions for the problem to variations in certain data, without having to resolve the problem from scratch for each run. Suppose that the simplex method produces an optimal basis B. It will be described how to make use of the optimality conditions in order to find a new optimal solution, if some of the problem data change, without resolving the problem from scratch. In particular, the following variations in the problem will be considered: change in the coefficients of objective function and change in the righthand side vector (constants). More details of sensibility analysis can be seen in [2, 16].

Table 7 shows the sensibility analysis for objective coefficient ranges, by considering the *pieces stocking*. First column of the table shows the decision variables; second column shows the current coefficient of objective function; third and fourth columns show, respectively, the allowable increase and decrease ranges in which the basis is unchanged.

For example, decision variable x_{21} can increase to $80 + 3.631 = 83.631$ and the basis is unchanged. If this case happens, the current value of the objective function \$33600.75, will be added to $3.631 \times 204 = 740.72$. Therefore, the new objective function will be \$34341.47. Also considering pieces stocking, Table 8 shows the current righthand side of the constraints and the allowable increase and decrease ranges in which the basis is unchanged.

Table 7: Sensibility analysis for the objective coefficient ranges - pieces stocking

Variable	Current coefficient	Allowable increase	Allowable decrease
x_{12}	255	0	262.72
x_{21}	80	3.631	90.26
x_{32}	60	0	68.35
y_{22}	137.07	0	137.07

Table 8: Sensibility analysis for the righthand side ranges - pieces stocking

Current coefficient	Allowable increase	Allowable decrease
42	1147.44	42
204	430.29	204
77	491.76	77
-148	3442.32	2149
12960	infinity	7978.87

For example, the first constraint, whose righthand side is 42, this value can increase to $42 + 1147.44 = 1189.44$ or decrease to $42 - 42 = 0$ that the basis is unchanged.

Similarly, Table 9 shows the sensibility analysis for objective coefficient ranges, by considering the *final products stocking*. As before, table shows the allowable increase and decrease ranges in which the basis is unchanged.

For example, decision variable x_{21} can increase to 83.631 and the basis is unchanged. If this happens, the current value of the objective function \$32106.59, will be added to $3.631 \times 198 = 718.94$. Therefore, the new objective function will be \$32825.53. Also considering final products stocking, Table 10 shows the current righthand side of the constraints and the allowable increase and decrease ranges in which the basis is unchanged.

For example, the first constraint, whose righthand side is 39, it can increase to $39 + 1128.77 = 1167.77$ or decrease to $39 - 39 = 0$ that the basis is unchanged.

Finally, Table 11 shows the sensibility analysis for objective coefficient ranges, by considering the *final products and pieces stocking*.

For example, decision variable x_{21} can increase to 83.631 and the basis is unchanged. If this happens, the current value of the objective function \$31972.24, will be added to 718.94 and the new objective function will be \$32691.18. Table 12 shows the current

Table 9: Sensibility analysis for the objective coefficient ranges - final products stocking

Variable	Current coefficient	Allowable increase	Allowable decrease
x_{12}	255	0	262.72
x_{21}	80	3.631	90.26
x_{32}	60	0	68.35
y_{22}	137.07	0	137.07

Table 10: Sensibility analysis for the righthand side ranges - final products stocking

Current coefficient	Allowable increase	Allowable decrease
39	1128.77	39
198	423.29	198
72	483.76	72
0	3386.32	2205
12960	infinity	7849.07

Table 11: Sensibility analysis for the objective coefficient ranges - final products and pieces stocking

Variable	Current coefficient	Allowable increase	Allowable decrease
x_{12}	255	0	262.72
x_{21}	80	3.631	90.26
x_{32}	60	0	68.35
y_{22}	137.07	0	137.07

righthand side of the constraints and the allowable increase and decrease ranges in which the basis is unchanged.

For example, the first constraint, whose righthand side is 39, it can increase to $39 + 1178.11 = 1217.11$ or decrease to $39 - 39 = 0$ that the basis is unchanged.

4.3 Discussion

The optimal solutions obtained by the optimization models with stock balancing are compared with the solutions obtained by the furniture manufacture, which meets the demand for each period of production. The production planning proposed by optimization models suggests anticipating the production of some final products, generating stocks, and postponing the production of others. By the other hand, the manufacture in study produces the required demand of each period. Table 13 shows the comparison between the minimum costs obtained from the objective functions (Production Planed) and from

Table 12: Sensibility analysis for the righthand side ranges - final products and pieces stocking

Current coefficient	Allowable increase	Allowable decrease
39	1178.11	39
198	441.79	198
72	504.91	72
-148	3534.31	2057
12960	infinity	8192.12

Table 13: Comparison between values of the objective functions

Stock Balancing	Production Planed	Production by period	Percentage of profit
Pieces	33600.75	35101.55	4.28%
Final Products	32106.59	35079.44	8.47%
Final Products and Pieces	31972.24	35054.05	8.8%

the production by period, considering the 3 cases of stock balancing.

Thus, last column of Table 13 shows the percentage of profit provided by optimal solutions, when these are compared with the production cost that satisfies the demand in each time period (without optimization models).

In relation to the number of boards to be cut, the optimal solutions show that, considering pieces stocking, the number of boards cut is $y_{22} = 14.24$. Considering final products stocking, $y_{22} = 14.61$ and, finally, considering final products and pieces stocking, the number of boards cut is $y_{21} = 13.62$. In practical situations, a strategy can be applied to round fractional variables to assume integer values, in order to obtain a viable solution [13]. Since the number of boards in a solution must be integer, an optimal solution must use at least one board. Thus, if the solution obtained by rounding is not optimal, at most one more board is used than the optimal solution [13]. Comparing the number of boards to be cut with the manufacture in study, the production by period usually cut 15.5 boards, which demonstrantes that optimization models and production planning provide economy of raw material.

5 Conclusions

The production planning and control is essential for the industries that need to organize their operations in optimized way, so that they can maintain and improve the financial and commercial aspects, in order to stand out in relation to other companies.

Linear Programming methods can help in decision make, since the optimal solutions suggest the number of products that must be produced and when, in order to provide the minimum production global costs and optimize production steps of an industry. Among the exact methods for linear programming, the literature recommend the Simplex Method and, to obtain integer solutions, the algorithm Branch and Bound [2, 16].

The Combined Problem is still little explored in the literature, but the relevance of the stock cutting and lot sizing combination in different situations elects this problem as an important issue to be researched, since this optimization model has been minimizing the production global costs.

The proposed production planing of a furniture factory enables small industries to have contact with the solutions presented by mathematical programming methods of production and be able to control and plan the items production, in order to minimize the cost and waste of materials; in this case study, boards to be cut.

Therefore, in this case study, results suggest that to produce in the initial periods of planning and storing the items in stock is more advantageous than the production of demand per period or postpone the production to the final periods. It can be observed that the contribution of optimization methods application for industries, since the production of items, particularly in small industries, usually is done by period, without an effective planning that considers the anticipation of manufacture, for example. In this study, based on history of demands, when the production is anticipated, generating stock, industry can obtain economy of raw material and consequent increase in profits.

As future works, we intend to consider more than one type of board to be cut, with different dimensions, in such a way that they can be stacked to be cut on the same cutting pattern.

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