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GLOBAL EXISTENCE AND BOUNDEDNESS IN A FULLY PARABOLIC 2D ATTRACTION-REPULSION SYSTEM: CHEMOTAXIS-DOMINANT CASE

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Abstract. This paper deals with the fully parabolic attraction-repulsion system in the two dimensional setting. Although the critical mass guaranteeing global existence of solutions was established for simplified cases, it is still open for the fully parabolic case. In this paper we construct a Lyapunov functional corresponding to the fully parabolic case and derive the critical constant.

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1 Introduction

Consider the following fully parabolic system:

$$\begin{cases} u_t = d_1\Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) & \text{in } \Omega\times(0,\infty), \\ v_t = d_2\Delta v + \alpha u - \beta v & \text{in } \Omega\times(0,\infty), \\ w_t = d_3\Delta w + \gamma u - \delta w & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0, \quad w(\cdot,0) = w_0 & \text{in } \Omega, \end{cases} \tag{1}$$

in a bounded domain $\Omega\subset\mathbb{R}^2$ with smooth boundary $\partial\Omega$. The parameters $\chi,\xi,\alpha,\beta,\gamma,\delta,d_i$ $(i=1,2,3)$ are positive. As to the initial data (u_0,v_0,w_0) we assume that

$$\begin{cases} u_0\in C^0(\overline{\Omega}), \quad u_0\ge 0 \quad \text{in } \overline{\Omega}, \\ v_0\in W^{1,\infty}(\Omega), \quad v_0\ge 0 \quad \text{in } \overline{\Omega}, \\ w_0\in W^{1,\infty}(\Omega) \quad u_0\ge 0 \quad \text{in } \overline{\Omega}. \end{cases} \tag{2}$$

When $\xi=0$ the system (1) is the well-known Keller–Segel system, which is introduced to describe a biological phenomenon *chemotaxis* in [10]. There are many mathematical researches on the system (see surveys [7, 1]). Our system (1) describes the aggregation of microglia in Alzheimer's disease in [13] or the quorum sensing effect in the chemotactic movement in [17].

Based on previous results, our system seems to be classified to three cases:

- $\chi\alpha-\xi\gamma>0$ (attraction-dominant case);
- $\chi\alpha-\xi\gamma<0$ (repulsion-dominant case);
- $\chi\alpha-\xi\gamma=0$ (balanced case).

Actually for the simplified system (the second and the third equation are replaced by elliptic equations, or only the third equation is replaced), the picture seems to be clear. In the attraction-dominant case, the mass critical phenomenon occurs in the two dimensional setting: when $d_1=d_2=d_3=1$, small mass $\|u_0\|_{L^1(\Omega)}<\frac{4\pi}{\chi\alpha-\xi\gamma}$ ($\|u_0\|_{L^1(\Omega)}<\frac{8\pi}{\chi\alpha-\xi\gamma}$ for radial case) implies global existence and boundedness of solutions and also a finite time blowup solution with large mass $\|u_0\|_{L^1(\Omega)}>\frac{8\pi}{\chi\alpha-\xi\gamma}$ is constructed ([18, 4, 11, 9]). On the other hand, solutions exist globally in time independently the magnitude of mass and spacial dimensions in the repulsion-dominant case ([18]).

However, only few results on the fully parabolic system are available. In the attraction-dominant case, global existence and boundedness of solutions for sufficiently small data in the two dimensional setting are established in [4], but the critical number is not clear. Moreover, there is no result on blowup phenomenon. Also, in the repulsion-dominant case, global existence and boundedness of solutions in the lower dimensional setting $(n\le 3)$ are established in [8]. However there is no result for higher dimensions. Indeed even the repulsion system ($\chi = 0$) has many open problems (see [2]). As to the balanced case, global existence is established in the lower dimensional setting in [12]. Recently the relationship of our system and the indirect signal substances chemotaxis system became clear and it was proved that a critical phenomenon occurs in the four dimensional setting in [5, 6].

In this paper we focus on the fully parabolic system in the attraction-dominant case ($\chi\alpha - \xi\gamma > 0$). Since the previous work [4] applies the moment method, which was introduced in [14], this method cannot be extend to the fully parabolic case. Moreover in [9, 16] the authors construct Lyapunov functional to the simplified case and derive the critical constant. However their construction of functional depends on the fact that the lower equations are elliptic. Therefore we reconstruct Lyapunov functional to (1) and derive the critical constant guaranteeing global existence of solutions.

The first result looks a natural extension of the global existence criterion of the Keller-Segel system.

Theorem 1.1. *Let $d_2 = d_3, \beta = \delta$. Assume that $\chi\alpha - \xi\gamma > 0$. If*

$$\int_\Omega u_0 < \frac{4\pi d_1 d_2}{\chi\alpha - \xi\gamma}$$

then the solution (u, v, w) of (1) exists globally in time. Moreover the solution remains bounded:

$$\sup_{t>0} \Big(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{1,\infty}(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)} \Big) < \infty.$$

In the radially symmetric setting the critical number can be relaxed.

Theorem 1.2. *Let $d_2 = d_3, \beta = \delta, R > 0$. Assume that $\chi\alpha - \xi\gamma > 0$. Suppose that $\Omega = \{x \,|\, |x| < R\}$ and (u_0, v_0, w_0) be radially symmetric. If*

$$\int_\Omega u_0 < \frac{8\pi d_1 d_2}{\chi\alpha - \xi\gamma}$$

then the solution (u, v, w) of (1) exists globally in time. Moreover the solution remains bounded:

$$\sup_{t>0} \Big(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{1,\infty}(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)} \Big) < \infty.$$

Without the restriction $\beta = \delta$, we can establish global existence of solutions.

Theorem 1.3. *Let $d_2 = d_3$. Assume that $\chi\alpha - \xi\gamma > 0$. If*

$$\int_\Omega u_0 < \frac{4\pi d_1 d_2}{\chi\alpha - \xi\gamma}$$

then the solution (u, v, w) of (1) exists globally in time.

The radially symmetric version is the following:

Theorem 1.4. *Let $d_2 = d_3, R > 0$. Assume that $\chi\alpha - \xi\gamma > 0$. Suppose that $\Omega = \{x \mid |x| < R\}$ and (u_0, v_0, w_0) be radially symmetric. If*

$$\int_\Omega u_0 < \frac{8\pi d_1 d_2}{\chi\alpha - \xi\gamma}$$

then the solution (u, v, w) of (1) exists globally in time.

Finally let us give a remark on the attraction-dominant case with $d_2 = d_3, \beta = \delta$.

Remark 1.5. Let $d_2 = d_3, \beta = \delta$. Assume that $\chi\alpha - \xi\gamma > 0$. Setting $V := \chi v - \xi w$, then (u, V) satisfies

$$\begin{cases} u_t = d_1 \Delta u - \nabla \cdot (u \nabla V) & \text{in } \Omega \times (0, \infty), \\ V_t = d_2 \Delta V + (\alpha\chi - \gamma\xi) u - \beta V & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0, \quad V(\cdot, 0) = \chi v_0 - \xi w_0 & \text{in } \Omega. \end{cases}$$

If we assume $\chi v_0 - \xi w_0 \geq 0$ then the above system is the Keller-Segel system. Therefore in the two dimensional setting the mass critical phenomenon occurs by the constant $\frac{4\pi d_1 d_2}{\chi\alpha - \xi\gamma}$. Moreover we can construct a finite time blowup solution in the higher dimensional setting independently the magnitude of mass ([19]).

2 Lyapunov functional

We first recall local existence of classical solutions to (1). The following proposition is established in [18, Lemma 3.1].

Proposition 2.1. *There exist $T_{\max} \in (0, \infty]$ and exactly one triplet (u, v, w) of positive functions from $C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \cap C^0([0, T_{\max}); C^0(\overline{\Omega}))$ that solves (1) in the classical sense. Also, the solution (u, v, w) satisfies the mass conservation*

$$\int_\Omega u(t) = \int_\Omega u_0 \quad \textit{for all } t \in (0, T_{\max}). \tag{3}$$

Moreover, if $T_{\max} < \infty$, then

$$\lim_{t \nearrow T_{\max}} \left(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{1,\infty}(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)} \right) = \infty.$$

We construct a Lyapunov functional to (1).

Lemma 2.2. *Let $d_2 = d_3, \beta = \delta$. Assume that $D := \chi\alpha - \xi\gamma > 0$. Let (u, v, w) be a solution of (1) in $\Omega \times (0, T)$. The following identity holds*

$$\frac{d}{dt} \mathcal{F}(u, v, w) + \mathcal{D}(u, v, w) = 0,$$

where

$$\mathcal{F}(u,v,w) = d_1 \int_\Omega u \log u - \int_\Omega u(\chi v - \xi w) + \frac{d_2}{2D} \int_\Omega |\nabla(\chi v - \xi w)|^2 + \frac{\beta}{2D} \int_\Omega |\chi v - \xi w|^2,$$
$$\mathcal{D}(u,v,w) = \frac{1}{D} \int_\Omega |(\chi v - \xi w)_t|^2 + \int_\Omega u |\nabla(d_1 \log u - (\chi v - \xi w))|^2.$$

Proof. Testing the first equation of (1) by $d_1 \log u - (\chi v - \xi w)$ we have

$$\int_\Omega u_t(d_1 \log u - (\chi v - \xi w)) = -\int_\Omega u|\nabla(d_1 \log u - (\chi v - \xi w))|^2.$$

Here the mass conservation law (3) implies

$$\frac{d}{dt}\int_\Omega u \log u = \int_\Omega u_t \log u + \int_\Omega u_t = \int_\Omega u_t \log u,$$

so

$$\begin{aligned}
&\frac{d}{dt}\left(d_1 \int_\Omega u \log u - \int_\Omega u(\chi v - \xi w) \right) \\
&= \int_\Omega u_t(d_1 \log u - (\chi v - \xi w)) - \int_\Omega u(\chi v - \xi w)_t \\
&= -\int_\Omega u|\nabla(d_1 \log u - (\chi v - \xi w))|^2 - \int_\Omega u(\chi v - \xi w)_t. \qquad (4)
\end{aligned}$$

On the other hand, by the assumption $d_2 = d_3, \beta = \delta$, the second and third equations imply

$$(\chi v - \xi w)_t = d_2 \Delta(\chi v - \xi w) - \beta(\chi v - \xi w) + (\chi\alpha - \xi\gamma)u, \qquad (5)$$

and then we see that

$$\begin{aligned}
&\int_\Omega u(\chi v - \xi w)_t \\
&= \frac{1}{D}\left(\int_\Omega |(\chi v - \xi w)_t|^2 - d_2 \int_\Omega \Delta(\chi v - \xi w)(\chi v - \xi w)_t + \beta \int_\Omega (\chi v - \xi w)(\chi v - \xi w)_t \right) \\
&= \frac{1}{D}\int_\Omega |(\chi v - \xi w)_t|^2 + \frac{1}{2D}\frac{d}{dt}\left(d_2 \int_\Omega |\nabla(\chi v - \xi w)|^2 + \beta \int_\Omega |\chi v - \xi w|^2 \right). \qquad (6)
\end{aligned}$$

Therefore combining (4) and (6) we conclude that

$$\begin{aligned}
&\frac{d}{dt}\left(d_1 \int_\Omega u \log u - \int_\Omega u(\chi v - \xi w) + \frac{d_2}{2D}\int_\Omega |\nabla(\chi v - \xi w)|^2 + \frac{\beta}{2D}\int_\Omega |\chi v - \xi w|^2 \right) \\
&+ \frac{1}{D}\int_\Omega |(\chi v - \xi w)_t|^2 + \int_\Omega u|\nabla(d_1 \log u - (\chi v - \xi w))|^2 = 0,
\end{aligned}$$

which is the desired inequality. □

3 Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1 and Theorem 1.2. We can proceed the well-known classical way in the context of study of the Keller-Segel system (see [15]), that is, by applying the Trudinger-Moser inequality ([3] and [15, Theorem 2.1]) to Lyapunov functional we will derive an useful energy estimate: if $\int_\Omega u_0 < \frac{4\pi d_1 d_2}{\chi\alpha-\xi\gamma}$ or $\int_\Omega u_0 < \frac{8\pi d_1 d_2}{\chi\alpha-\xi\gamma}$ for radially symmetric setting, then there exists some $C>0$ such that

$$\int_\Omega u\log u + \int_\Omega |\nabla(\chi v-\xi w)|^2 + \int_\Omega |\chi v-\xi w|^2 \le C,$$

and we handle with regularity estimates. Differently from the Keller-Segel system, in our problem the following equation plays an important role:

$$(\chi v-\xi w)_t = d_2\Delta(\chi v-\xi w) - \beta(\chi v-\xi w) + (\chi\alpha-\xi\gamma)u,$$

which implies L^1 estimate from the mass conservation: there exists some $C>0$ satisfying

$$\|\chi v-\xi w\|_{L^1(\Omega)} \le C.$$

Since the positivity of $\chi v-\xi w$ is not valid now, we need a small but natural modification to apply the classical way in [15]. Indeed, Lemma 2.2 implies that there exists some $C>0$ such that

$$\int_\Omega u\log u - \frac{1}{d_1}\int_\Omega u|\chi v-\xi w| + \frac{d_2}{2d_1 D}\int_\Omega |\nabla(\chi v-\xi w)|^2 \le C.$$

By introducing small $\delta>0$, which will be chosen later, the LHS of the above inequality is written as

$$\begin{aligned}&\int_\Omega u\log u - \left(\frac{1}{d_1}+\delta\right)\int_\Omega u|\chi v-\xi w| + \delta\int_\Omega u|\chi v-\xi w| + \frac{d_2}{2d_1D}\int_\Omega |\nabla(\chi v-\xi w)|^2\\ &\quad= -\int_\Omega u\log\frac{e^{(\frac{1}{d_1}+\delta)|\chi v-\xi w|}}{u} + \delta\int_\Omega u|\chi v-\xi w| + \frac{d_2}{2d_1D}\int_\Omega |\nabla(\chi v-\xi w)|^2.\end{aligned}$$

By Jensen's inequality we arrive at

$$\begin{aligned}-\int_\Omega u\log\frac{e^{(\frac{1}{d_1}+\delta)|\chi v-\xi w|}}{u} &= M\int_\Omega\left(-\log\frac{e^{(\frac{1}{d_1}+\delta)|\chi v-\xi w|}}{u}\right)\cdot\frac{u}{M}\\ &\ge -M\log\left(\int_\Omega \frac{e^{(\frac{1}{d_1}+\delta)|\chi v-\xi w|}}{u}\cdot\frac{u}{M}\right)\\ &= -M\log\int_\Omega e^{(\frac{1}{d_1}+\delta)|\chi v-\xi w|} + M\log M,\end{aligned}$$

where $M=\int_\Omega u(t)=\int_\Omega u_0$. Now by applying the Trudinger-Moser inequality, for any $\varepsilon>0$ there exists some $C_\varepsilon>0$ such that

$$\begin{aligned}&\int_\Omega e^{(\frac{1}{d_1}+\delta)|\chi v-\xi w|}\\ &\le C_\varepsilon\exp\left\{\left(\frac{1}{2\pi^*}+\varepsilon\right)\left(\frac{1}{d_1}+\delta\right)^2\|\nabla(\chi v-\xi w)\|^2_{L^2(\Omega)} + \frac{2(\frac{1}{d_1}+\delta)}{|\Omega|}\|\chi v-\xi w\|_{L^1(\Omega)}\right\},\end{aligned}$$

where $\pi^* := 4\pi$ for general setting; $:= 8\pi$ for radially symmetry setting. Thus we obtain

$$-\int_\Omega u \log \frac{e^{(\frac{1}{d_1}+\delta)|\chi v - \xi w|}}{u} \geq -M \left(\frac{1}{2\pi^*} + \varepsilon\right)\left(\frac{1}{d_1}+\delta\right)^2 \|\nabla(\chi v - \xi w)\|^2_{L^2(\Omega)} + C$$

with some constant C. Combining above and Lyapunov functional, we infer that

$$\left\{\frac{d_2}{2d_1 D} - M\left(\frac{1}{2\pi^*}+\varepsilon\right)\left(\frac{1}{d_1}+\delta\right)^2\right\}\int_\Omega |\nabla(\chi v - \xi w)|^2 + \delta \int_\Omega u|\chi v - \xi w| \leq C$$

with some $C > 0$. Thanks to the assumption on mass, we can choose small ε and δ satisfying

$$\frac{d_2}{2d_1 D} - M\left(\frac{1}{2\pi^*}+\varepsilon\right)\left(\frac{1}{d_1}+\delta\right)^2 > 0,$$

and then conclude the proof. □

4 Proof of Theorem 1.3 and Theorem 1.4

In this section we consider the case with $\beta \neq \delta$. Differently from the previous case, the key equation (5) breaks down. Hence we need a minor modification and unfortunately cannot construct a Lyapunov functional. However the growth of the functional will be controlled and it will help us to obtain global existence.

Lemma 4.1. *Let $d_2 = d_3$. Assume that $D := \chi\alpha - \xi\gamma > 0$. Let (u, v, w) be a solution of (1) in $\Omega \times (0, T)$. The following identity holds*

$$\frac{d}{dt}\mathcal{F}'(u,v,w) + \mathcal{D}'(u,v,w) \leq \frac{\xi^2(\delta-\beta)^2}{2D}\int_\Omega w^2,$$

where

$$\mathcal{F}'(u,v,w) = d_1\int_\Omega u\log u - \int_\Omega u(\chi v - \xi w) + \frac{d_2}{2D}\int_\Omega |\nabla(\chi v - \xi w)|^2 + \frac{\beta}{2D}\int_\Omega |\chi v - \xi w|^2,$$

$$\mathcal{D}'(u,v,w) = \frac{1}{2D}\int_\Omega |(\chi v - \xi w)_t|^2 + \int_\Omega u|\nabla(d_1 \log u - (\chi v - \xi w))|^2.$$

Proof. Since

$$-\chi\beta v + \xi\delta w = -\beta(\chi v - \xi w) + \xi(\delta - \beta)w,$$

the second and third equations imply

$$(\chi v - \xi w)_t = d_2\Delta(\chi v - \xi w) - \beta(\chi v - \xi w) + (\chi\alpha - \xi\gamma)u + \xi(\delta-\beta)w.$$

Hence proceeding the similar way as in Section 2 we see that

$$\begin{aligned}
&\int_\Omega u(\chi v - \xi w)_t \\
&= \frac{1}{D}\int_\Omega |(\chi v - \xi w)_t|^2 + \frac{1}{2D}\frac{d}{dt}\left(d_2\int_\Omega |\nabla(\chi v - \xi w)|^2 + \beta\int_\Omega |\chi v - \xi w|^2\right) \\
&\quad - \frac{\xi(\delta-\beta)}{D}\int_\Omega w(\chi v - \xi w)_t,
\end{aligned}$$

and then we conclude that

$$\begin{aligned}
&\frac{d}{dt}\left(d_1\int_\Omega u\log u-\int_\Omega u(\chi v-\xi w)+\frac{d_2}{2D}\int_\Omega|\nabla(\chi v-\xi w)|^2+\frac{\beta}{2D}\int_\Omega|\chi v-\xi w|^2\right)\\
&+\frac{1}{D}\int_\Omega|(\chi v-\xi w)_t|^2+\int_\Omega u|\nabla(d_1\log u-(\chi v-\xi w))|^2\\
=&\frac{\xi(\delta-\beta)}{D}\int_\Omega w(\chi v-\xi w)_t\\
\le&\frac{1}{2D}\int_\Omega|(\chi v-\xi w)_t|^2+\frac{\xi^2(\delta-\beta)^2}{2D}\int_\Omega w^2.
\end{aligned}$$

which is the desired inequality. □

Proof of Theorem 1.3 and Theorem 1.4. In view of the standard semigroup theory and the mass conservation law (3), the growth of the functional is controlled:

$$\begin{aligned}
\mathcal{F}'(u,v,w)(t)&\le\mathcal{F}'(u,v,w)(0)+\int_0^t\frac{\xi^2(\delta-\beta)^2}{2D}\int_\Omega w^2\\
&\le\mathcal{F}'(u,v,w)(0)+Ct\qquad t>0,
\end{aligned}$$

with some $C>0$, which depends on the initial data. Therefore on any interval $[0,T]$ $(T>0)$ the functional is bounded and then we obtain the energy estimate on any fixed interval $[0,T]$. Therefore by the same way as in Section 3 global existence of solutions is guaranteed. □

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