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GALOIS THEORY AND THE UNIFORMIZATION OF RIEMANN SURFACES

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Abstract. The solvability of an algebraic equation is represented by the vanishing of an irreducible polynomial $f(x_0, y)$ over $\mathbb{Q}$ and it is based on the properties of the Galois group of the splitting field. The function of two variables $f(x, y)$ which generates this equation defines a Riemann surface, derived from the Puiseux expansion, with a covering surface that is equivalent to a punctured sphere. The relation between the conformal group of the punctured sphere and the Galois group of the function field on the Riemann surface yields a description of the entire set of symmetries of the integrals over compactified moduli spaces of surfaces with a higher number of punctures.

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1 Introduction

Riemann surfaces can be constructed by the sewing of thrice-punctured spheres, and the conformal group of these punctured spheres contains a symmetric permutation group S_3 that interchanges the punctures. More generally, the Riemann surface constructed from an irreducible polynomial of two variables can be shown to have a group of deck transformations represented by the Galois group of the function field. Since all of the smooth finite-genus Riemann surfaces may be described as algebraic curves, solvability of the equation with the Galois group equal to S_n, resulting from the fixing of the value of one of the variables of a two-variable polynomial, implies that the uniformizing group is projected to the symmetric group on the solution set.

The uniformization of spheres with n punctures is known to introduce functions with special modular properties. For the thrice-punctured sphere, it is the elliptic modular function, and the additional invariance under the permutations of the punctures yields a group generated by two transformations of order 2 adjoined to $\Gamma(2)$. These transformations also generate the mapping class group $\Gamma_{0,[3]}$ for three unmarked points, which is shown to be isomorphic to the quotient of the braid group by equivalence relations for the two generators.

The isomorphism between $\Gamma(1)/\Gamma(2)$ is examined in light of the matrices with negative determinant, and it is found that consistency is restored modulo 2. Once the transformations representing S_3 modulo 2 are allowed to act on punctures on $\mathbb{C}^*$, a sets of three points, which are being permuted, would differ for each generator. Consequently, the inclusion of matrices of negative determinant in the presentation of the permutation group has been found to be necessary.

The punctures of a collection of thrice-punctured spheres can be sewed together to construct a higher-genus surface with fewer punctures. The integrals over these higher-genus surfaces therefore should reflect some of the symmetries of the thrice-punctured sphere including invariance of under the permutation group. This result is found to hold, and the implications for the vestiges of this symmetry are elaborated.

Amongst the surfaces that may be formed by the sewing of the punctures of thrice-punctured spheres are the n-punctured spheres. Translating the problem of defining the uniformizing parameters of higher-genus surfaces to the punctured sphere, a paracompact model of the universal moduli space of punctured spheres is defined. The connection with the absolute Galois group is described.

2 Galois Groups and Irreducible Polynomials

If $f(x, y)$ is an irreducible polynomial of finite degree, suppose that $g(x, y) \in \mathbb{C}(x)[y]$ is the monic irreducible polynomial over $\mathbb{Q}$ with Y as a root, where Y is a primitive element of the splitting field K_f. Fixing y, the equation in x resulting from $f(x, y)$ has a solution in the algebraic numbers. The coefficients, which may be expressed as combinations of the roots, also must belong to a field of algebraic numbers. Therefore, $f(x, y) \in \mathbb{Q}(x)[y]$, where $\mathbb{Q}(x)$ is a field extension of the rational numbers. By the Hilbert irreducibility theorem [13], there exist infinitely many values x_0 such that the polynomial in $\mathbb{Q}(x_0)[y]$ is irreducible over the rational numbers. It is known that Y can be chosen such that the N roots of $g(x, y)$ in K_f are polynomials of degree less than or equal to $N-1$ in Y [18],

$$\begin{aligned} &Y_j = c_{0j} + c_{1j}Y + ... + c_{N-1,j}Y^{N-1} \\ &c_{ij} \in \mathbb{Q}(\alpha_1, ..., \alpha_n)(x). \end{aligned} \tag{1}$$

where $\alpha_1, ..., \alpha_n$ are introduced through a set of conditions on the nonlinear combinations of the coefficients. The algebraicity of the roots is determined therefore by Y. The roles of the roots $Y = Y_1$, Y_2,..., Y_N can be permuted when the Galois group is S_N, and the root which is an algebraic number in K_f may be selected. From the relations between the roots, it follows that all roots of $g(x_0, y)$ are algebraic numbers given by finite radicals if x_0, $\alpha_1, ..., \alpha_n$ and one of the roots are similarly characterized.

The finite-degree algebraic equations define surfaces with a finite number of branch cuts and curves with a finite number of covering sheets. The conclusions regarding algebraic numbers can be restricted to finite algebraic expressions. If one root is a finite radical expression, and $g(x_0, y) \in \mathbb{Q}(x_0)[y]$ where $\mathbb{Q}(x_0)$ is an extension of $\mathbb{Q}$ by finite radicals, then all roots of $g(x_0, y)$ are finite radical expressions if the Galois grouup of the equation $g(x_0, y) = 0$ is S_N. It is necessary to establish the connection between the nature of the polynomial $g(x, y)$ and the roots $Y = Y_1, ..., Y_N$. Given a choice of x_0, Y is a finite radical expression, it may be viewed as a root of an irreducible polynomial $g(x_0, y)$ that belongs to $\mathbb{Q}(\alpha_1, ..., \alpha_n)(x_0)$, where $\alpha_1, ..., \alpha_n$ are Ruffini radicals, and $c_{ij} \in \mathbb{Q}(\alpha_1, ..., \alpha_n)(x)$. For fixed x_0, $c_{ij} \in \mathbb{Q}(\alpha_1, ..., \alpha_n)(x_0)$. If x_0 is an algebraic number that is expressed in terms of a finite number of radicals, $c_{ij} \in \mathbb{Q}(\alpha_1, ..., \alpha_n, x_0)$ is a soluble extension of $\mathbb{Q}$. Therefore, if one root is a Ruffini radical, all of the roots would be Ruffini radicals. The basis for the conclusion stems from the arrangement of the decomposition of the polynomial $g(x, y)$ into linear factors in y. Dividing $g(x, Y)$ by $y - Y$ yields a polynomial of degree $N-1$ in Y. It follows that

$$(Y - Y_2)...(Y - Y_N) = Y^{N-1} - \sum_{j=1}^{N-1} Y_j + ... + (-1)^{N-1} Y_2...Y_N \tag{2}$$

where $\sum_j Y_j = \sum_j (c_{0j} + c_{1j}Y + ... + c_{N-1,j}Y^{N-1})$. The powers of Y which must have a vanishing coefficient are $Y^N, ..., Y^{N(N-1)}$, yielding $N + N(N-2)$ equations for $N(N-1)$ coefficients with values in $\mathbb{Q}(x)$. There exists a nontrivial solution to these equations consisting of nonlinear combinations of the coefficients c_{ij}, since $N(N-2)$ coefficients can be eliminated systematically without introducing any new radicals. The remaining N coefficients can be expressed in terms of Ruffini radicals $\alpha_1, ..., \alpha_n$, because it requires only of finite number of arithmetical operations to transform the nonlinear combinations into linear expressions in the coefficients, Y has been given by a finite set of radicals, and the polynomial ring generated by Ruffini radicals is closed. This polynomial ring differs from the field extensions K_j of $K_0 = \mathbb{Q}(s_1, ..., s_N)$, with $s_1, ..., s_N$ being the symmetric polynomials of the roots $Y_1, ..., Y_N$, where $K_j = K_{j-1}(\alpha_j)$, $\alpha_j^{n_j} \in K_j$, $j = 0, 1, ..., r$ and $K_r = \mathbb{Q}(Y_1, ..., Y_N)$, since a sequential nesting of the radicals $\{\alpha_j\}$ is not required in a general polynomial of $\{\alpha_j\}$. Therefore, interchanging Y with Y_j, $j = 2, ..., N$, it would not be possible to express Y as a polynomial of degree $N-1$ in Y_j unless Y_j is given by a finite arithmetical combination of Ruffini radicals, and $c_{k,1}^{(j)} \in \mathbb{Q}(\alpha_1, ..., \alpha_n, x_0)$, where $Y = c_{0,1}^{(j)} + c_{1,1}^{(j)} Y_j + ... + c_{N-1,1}^{(j)} Y_j^{N-1}$. The nature of the root Y may be deduced from Galois theory. The Galois group of the splitting field of an algebraic equation is S_n generally, although polynomials with discriminants that are perfect squares have this group A_n [13][21]. If there is one proper normal subgroup of S_n with an abelian quotient, this can be used to generate a root that is a Ruffini radical of the equation. Therefore, when this property holds for the Galois group of the equation, all of the roots have finite radical expressions.

The definition of the Riemann surface as an algebraic curve and the uniformization of these surfaces can be reformulated in terms of parameters describing the covering group of the surface constructed from the Puiseux elements of the irreducible polynomial. The permutation group of three elements can be interpreted as a Galois group of a function field on a Riemann surface related to an equation of degree three. The algebraicity of the roots of the equation, derived after the fixing of one of the variables, is confirmed by the finiteness of the number of uniformizing parameters for finite-genus surfaces.

3 Galois Group of the Function Field on a Riemann Surface and the Conformal Group of the Punctured Sphere

It has been demonstrated that the Riemann surface defined by the function $f(x, y)$ may be constructed from the Puiseux elements of the irreducible polynomial $f(x, y)$, which are the triples (a, r, u) with
(i) r is an integer larger than 0
(ii) $u \in \mathbb{C}((t))^*$
(iii) $u = \sum a_m t^m$, $supp\ u = \{m | a_m \neq 0\}$, $d(u) = gcd\ supp(u)$, $gcd(r, d(u)) = 1$
(iv) $f(a + t^r, u) = 0$ in $\mathbb{C}((t))$, $a \in \mathbb{C}$.

The set of Puiseux elements for the function at a shall be denoted by $\mathcal{P}(a, f)$ and let $\mathcal{P}_f$ be the set of points at which Puiseux elements are defined.

If $f(x, y) = c_0 y^n + c_1(x) y^{n-1} + ... + c_n(x)$, let S_+ be the set of zeros of $c_0(x)$, S_- be the set of poles of $c_i(x)$ and D be the union of the set of zeros of $D(f)$ and ∞. Then the Riemann surface $\mathcal{R}$ is equated to

$$\mathcal{R}^* \times I_n \cup \mathcal{P} \times I_n \times (\ell \times I_n \times \{L.R\}) \tag{3}$$

where $S = S_+ \cup S_- \cup D$, where I_n is an index set $\{1, ..., n\}$,

$$\begin{aligned} \ell &= \cup_p \ell_{q,p},\ q \in \mathbb{C}^*,\ q \notin S \\ \ell_{q,p} &= \pi_{q,p}(0,1);\ \pi_{q,p}\ \textit{is a piecewise linear map of}\ [0,1]\ \textit{into}\ \mathbb{C}^* \end{aligned} \tag{4}$$

and $\mathcal{R}^* = \mathcal{C}^*/(\{\mathcal{P}_f \cup \ell\})$.

Each $(z, i) \in \mathcal{R}^* \times I_n$ will be contained in an open set $(D(z, r), i)$, and the union of such sets would form $D(z, r) \subset \mathcal{R}^*$. Since there are permutations of these sets given by equivalences $(z, i, L) \sim (z, \sigma_z(i), R)$, this union also can expressed as $(\Delta_1, i) \cup (\Delta_2, \sigma_{p_1}^{-1}(i)) \cup ...(\Delta_{r+1}, \sigma_{p_r}^{-1}\sigma_{p_{r-1}}^{-1}...\sigma_1^{-1}(i)) \cup ...$, where $\sigma_1...\sigma_{p_n} = 1$ [21][22].

Since the Riemann surface $\mathcal{R}_g$ is constructed by this method, the group of conformal transformations contains $G(K_f/\mathbb{Q}(x))$ [21]. The identification of a subgroup of the conformal group with the Galois group of permutations results from the interchange of the sector $\{\Delta_i\}$, provided that the maps are the identity for identical domains. However, a more general set of maps from Δ_i to Δ_j and p_i to p_j can be described, such that the continuous transformations are not identity mappings. There is a group of conformal transformations of Riemann spheres with n punctures, with the locations of the punctures permuted under the mappings, that includes these transformations.

Since the covering space of a Riemann surface of genus greater than one is $\mathcal{H}$, a presentation of the conformal group may be found. For $\mathcal{C}^* \backslash \{0, 1, \infty\}$, the three generators satisfy $g_1 g_2 g_3 = 1$. If these transformations are required to fix only one of the punctures, the group is conjugate to $\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) \middle| b \equiv c \equiv 0\ mod\ 2 \right\}$. As $\Gamma(1)/\Gamma(2) \simeq S_3$, the permutation group of the three points, the use of the larger group can be considered.

Theorem 3.1. *A union of the sets of generators of $S_3 \rtimes \Gamma(2)$ permuting the points $\{0, 1, \infty\}$ in the thrice-punctured sphere and fixing separately one puncture has four independent basis elements. Factorization by those matrices that do not leave invariant the set of three punctures yields the permutation group S_3, which is isomorphic to the subgroup of the conformal group of the covering space that fixes the set $\{0, 1, \infty\}$. The normalizer of S_3 within $GL(2; \mathbb{R})$ is S_3 and it is included in the modular group $SL(2; \mathbb{Z})$. The subgroup of the conformal group is isomorphic to the Galois group of the function field of the thrice-punctured sphere.*

Proof. The presentation of the generators satisfying the conditions

$$\begin{array}{lll} g_1(0) = 1 & g_1(1) = \infty & g_1(\infty) = 0 \\ g_2(0) = \infty & g_2(1) = 0 & g_2(\infty) = 1 \\ g_3(0) = 0 & g_3(1) = \infty & g_3(\infty) = 1 \\ g_4(0) = \infty & g_4(1) = 1 & g_4(\infty) = 0 \end{array} \tag{5}$$

and $tr\ g_1 = tr\ g_2 = 2$ would be

$$\begin{aligned} g_1 &= \begin{pmatrix} 0 & 2 \\ -2 & 2 \end{pmatrix} \\ g_2 &= \begin{pmatrix} 2 & -2 \\ 2 & 0 \end{pmatrix} \\ g_3 &= \begin{pmatrix} a' & 0 \\ a' & -a' \end{pmatrix} \\ g_4 &= \begin{pmatrix} 0 & b'' \\ b'' & 0 \end{pmatrix}. \end{aligned} \tag{6}$$

Neither g_3 or g_4 can be parabolic elements, and the requirement of a parabolic conformal group must be relaxed to include these generators. Alternatively, since $\frac{1}{2}g_1$ is an elliptic element representing a fractional transformation of order 3 that permutes the three punctures, even though $\left(\frac{1}{2}g_1\right)^2 = -\mathbb{I}$ and g_3 at $a' = 1$ is an element of order 2, it is sufficient to add these matrices to the group conjugate to $\Gamma(2)$, yielding the group

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \right\rangle / \langle \pm\mathbb{I} \rangle. \tag{7}$$

which represent a basis of generators of $S_3 \rtimes \Gamma(2)$. Since the latter two transformations do not leave invariant the set $\{0, 1, \infty\}$, factorization by the group generated by these elements leaves S_3.

When $n = 3$, the normalizer $N_{\Gamma(2)}$ of the group $\Gamma(2)$, which is conjugate to a group with each generator fixing only one puncture, is a proper subset of $SL(2;\mathbb{Z})$ and $N_{\Gamma(2)}/\Gamma(2) \subset \Gamma(1)/\Gamma(2)$. The Galois group $Gal(K_f/\mathbb{Q}(x))$ is not contained in $N_{\Gamma(2)}/\Gamma(2)$ [7] but it is a subgroup of the conformal group of the surface.

It is necessary to consider the subgroup of the conformal group of the covering space which leaves invariant the set of three punctures. Consequently, the generators of this conformal group would only differ by a scale factor from the matrices $g_i,\ i = 1, 2, 3, 4$ that is established by the determinant condition. Within multiplication by $\pm\mathbb{I}$, the matrices in the group

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right\rangle$$

are

$$\mathbb{I}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

and the matrices $\frac{1}{2}g_2$ and g_4 at $b'' = 1$ belong to this set.

These matrices will belong to the normalizer of S_3. The maximal subgroup of $GL(2;\mathbb{R})$, which is a normalizer of S_3 and fixes the three punctures, will be determined. Consider, for example, a similarity transformation with determinant 1 of $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -bd - (a+b)c & b^2 + (a+b)a \\ -d^2 - (c+d)c & bd + (c+d)a \end{pmatrix} \tag{9}$$

If the transformed matrix is set equal to $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$,

$$\begin{aligned} -bd - (a+b)c &= 1 \\ b^2 + (a+b)a &= 0 \\ -d^2 - (c+d)c &= -1 \\ bd + (c+d)a &= 1 \end{aligned} \tag{10}$$

The second condition is satisfied by $a = b = 0$ for real matrices. The only group element in Eq.(5) which could equal a nontrivial conjugation of this matrix is $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$.

Further suppose that the determinant of the $GL(2;\mathbb{R})$ matrix is allowed to be ± 1. Equality with $\pm \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ yields four relations for a $GL(2;\mathbb{R})$ matrix are

$$\begin{aligned} -bd - (a+b)c &= -1 \\ b^2 + (a+b)a &= 1 \\ -d^2 - (c+d)c &= -1 \\ bd + (c+d)a &= 0 \end{aligned} \tag{11}$$

From the second and third relations,

$$\begin{aligned} b &= 1/2\left(-a \pm \sqrt{4 - 3a^2}\right) \\ d &= 1/2\left(-c \pm \sqrt{4 - 3c^2}\right) \end{aligned} \tag{12}$$

The fourth condition then is equivalent to

$$\left(\frac{-a \pm \sqrt{4-3a^2}}{2}\right)\left(\frac{-c \pm \sqrt{4-3c^2}}{2}\right) + \left(c + \frac{-c \pm \sqrt{4-3c^2}}{2}\right) a = 0 \tag{13}$$

or

$$\left(3a \mp \sqrt{4-3a^2}\right) c \pm a\sqrt{4-3a^2} + \sqrt{(4-3a^2)(4-3c^2)} = 0 \tag{14}$$

Dividing by $\sqrt{4-3c^2}$ gives

$$\frac{c}{\sqrt{4-3c^2}} = \frac{\mp a - \sqrt{4-3a^2}}{3a \mp \sqrt{4-3a^2}} \tag{15}$$

Then

$$d = c\ \left(\frac{\pm a - \sqrt{4-3a^2}}{\mp a - \sqrt{4-3a^2}}\right) \tag{16}$$

Finally, it follows from the first condition

$$\begin{aligned} c\left(\frac{-a \pm \sqrt{4-3a^2}}{2}\right)\left(\frac{\pm a - \sqrt{4-3a^2}}{\mp a - \sqrt{4-3a^2}}\right) + \left(\frac{a \pm \sqrt{4-3a^2}}{2}\right) c = 1 \\ c\left(\frac{a \mp \sqrt{4-3a^2}}{2}\right) + c\left(\frac{a \pm \sqrt{4-3a^2}}{2}\right) = 1 \end{aligned} \tag{17}$$

which implies that $c = \frac{1}{a}$. By Eq.(8),

$$\frac{\frac{1}{a}}{\sqrt{4-\frac{3}{a^2}}} = \frac{\mp a - \sqrt{4-3a^2}}{3a \mp \sqrt{4-3a^2}} \tag{18}$$

Squaring this relation gives

$$\frac{1}{4a^2-3} = \frac{4-2a^2 \pm 2a\sqrt{4-3a^2}}{4+6a^2 \mp 6a\sqrt{4-3a^2}} \tag{19}$$

and

$$4+6a^2-(4a^2-3)(4-2a^2) = \pm 8a^3\sqrt{4-3a^2} \tag{20}$$

or

$$\begin{aligned} (a^4-2a^2+2)^2 &= a^6(4-3a^2) \\ (a^4-a^2+1)^2 &= a^4 \end{aligned} \tag{21}$$

Therefore, either $a^4 - 2a^2 + 1 = 0$ or $a^4 + 1 = 0$. The first set of solutions is $a = \pm 1$ while the second equation has complex solutions only, and therefore $c = \pm 1$. From the conditions on the matrix elements, $b = 0$ if $a = 1, \ -1$ and $d = -1$ if $c = 1$ or $d = 1$ if $c = -1$.

The matrices to consider are

$$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{22}$$

The matrices, with determinants equal to -1, produce the required matrix within a factor of $\pm I$, since the inverse must include a negative sign with respect to the matrix multiplication (5).

The effect of the fractional linear transformation represented by the first matrix in Eq.(13) is $x + iy \to \frac{x+iy}{(x-1)+iy} = \frac{x(x-1)-y^2-iy}{(x-1)^2+y^2}$, which has a negative imaginary part. The conformal group of $\mathbb{C}^*\backslash\{0, 1, \infty\}$ will be required to allow transformations from the upper half-plane to the lower half-plane that leave the real line invariant. Because three of the matrices in Eq.(4) have determinant -1, the group $SL(2;\mathbb{R})$ should be enlarged to $SL(2;\mathbb{R}) \cup \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} SL(2;\mathbb{R})$. While the covering space of $\mathbb{C}^*\backslash\{p_1, ..., p_n\}$ is generally chosen to be $\mathcal{H}$ or the unit disk U [14][19], the covering group can be selected such that its action on $\mathcal{H}$ has a range which includes $\bar{\mathcal{H}}$ for some of the elements.

The final permutation of two elements

$g_5 = \begin{pmatrix} a''' & b''' \\ c''' & d''' \end{pmatrix}$ would satisfy

$$g_5(0) = 1 \qquad g_5(1) = 0 \qquad g_5(\infty) = \pm\infty \tag{23}$$

or

$$\frac{b'''}{d'''} = 1 \qquad a''' + d''' = 0 \qquad c''' = 0 \tag{24}$$

with ∞ and $-\infty$ identified. Setting $b''' = 1$ gives the matrix

$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \notin \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right\rangle / \langle \pm\mathbb{I} \rangle \tag{25}$$

which, again, has a determinant equal to -1.

Multiplication of the matrices in Eq.(4) by $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ yields

$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \tag{26}$$

These matrices, however, represent fractional transformations that map $\{0,\ 1,\ \infty\}$ to the set $\{-1,\ -\frac{1}{2},\ 0\}$. Furthermore, multiplication of the elements would close only after generating a much larger group. Finally, the conjugation of $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ by $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix}$, which does not belong to the set of matrices in Eq.(4).

Therefore, the normalizer of S_3 in $SL(2;\mathbb{R}) \cup \left(\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} SL(2;\mathbb{R})\right)$ is S_3, and N_{S_3}/S_3 would not contain the symmetric permutation group. It is sufficient, however, to consider that subgroup of this conformal group that preserves the set $\{0, 1, \infty\}$ within a permutation, which is isomorphic with the Galois group of the function field of the thrice-punctured sphere. $\square$

Since three matrices in Eq.(4) have a determinant equal to -1, it may be investigated whether all presentations of the symmetric permutation group as fractional linear transformations on $\mathbb{C}^*$ are represented by matrices with a determinant equal to -1.

Theorem 3.2. *The presentation of generators of S_3, which are fractional linear transformations on $\mathbb{C}^*$ preserving a set of three points in the finite complex plane, requires matrices with determinants equal to -1 for the permutation of two elements.*

Proof. Because the action of fractional linear transformations on $\mathbb{C}^*\backslash\{0, 1, \infty\}$ can be complicated by the point at ∞, consider instead the locations of the three punctures at $\{-1, 0, 1\}$. The action of the permutation of order 3 is $\left\{ \begin{matrix} -1 & 0 & 1 \\ 0 & 1 & -1 \end{matrix} \right\}$, the conditions on the matrix elements are

$$a_1 = b_1 \qquad b_1 = d_1 \qquad a_1 + b_1 = -(c_1 + d_1) \qquad c_1 = -3a_1$$

and a unit determinant results if $a_1 = \pm\frac{1}{2}$. Let $a_1 = \frac{1}{2}$. Then the matrix is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$. Within multiplication by $\pm\mathbb{I}$, this matrix generates a cyclic group of order 3.

The permutation of two points is given by

$$\left\{ \begin{matrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{matrix} \right\}, \left\{ \begin{matrix} -1 & 0 & 1 \\ -1 & 1 & 0 \end{matrix} \right\}, \left\{ \begin{matrix} -1 & 0 & 1 \\ 1 & 0 & -1 \end{matrix} \right\} \tag{27}$$

The conditions on the matrix representing the first permutation are

$$a_2 = b_2 \qquad b_2 = -d_2 \qquad a_2 + b_2 = c_2 + d_2 \qquad c_2 = 3a_2$$

such that

$$det\left[a_2\begin{pmatrix}1 & 1\\ 3 & -1\end{pmatrix}\right] = -4a_2^2 = -1 \tag{28}$$

if $a_2 = \pm\frac{1}{2}$. When $a_2 = \frac{1}{2}$, the matrix is

are $\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{3}{2} & -\frac{1}{2}\end{pmatrix}$ with a determinant equal to -1. The relations for the second permutation are

$$a_3 = -b_3 \qquad b_3 = d_3 \qquad a_3 - b_3 = -(c_3 - d_3) \qquad c_3 = -3a_3 \tag{29}$$

and

$$det\left[a_3\begin{pmatrix}1 & -1\\ -3 & -1\end{pmatrix}\right] = -4a_3^2 = -1 \tag{30}$$

if $a_3 = \pm\frac{1}{2}$, yielding $\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{3}{2} & -\frac{1}{2}\end{pmatrix}$. The constraints for the third permutation are

$$\begin{aligned} a_4 - b_4 = c_4 - d_4 \qquad b_4 = 0 \qquad a_4 + b_4 = -(c_4 + d_4) \\ c_4 = 0 \\ a_4 = -d_4 \end{aligned} \tag{31}$$

and, again, the matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ has a determinant equal to -1. Each of the permutations of order 2, combined with the permutations of order 3, produces a group of six elements represented by the matrices

$$\mathbb{I},\ \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{3}{2} & \frac{1}{2}\end{pmatrix},\ \begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\\ -\frac{3}{2} & -\frac{1}{2}\end{pmatrix},\ \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{3}{2} & -\frac{1}{2}\end{pmatrix},\ \begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{3}{2} & \frac{1}{2}\end{pmatrix},\ \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{32}$$

within multiplication by $\pm\mathbb{I}$.

By contrast with the points $\{0, 1, \infty\}$, the interchange of any two of the punctures in the set $\{-1, 0, 1\}$ would not alter the presentation of the permutation group. For any three finite real values, the application of an $SL(2;\mathbb{R})$ matrix would be sufficient to transform $\{-1, 0, 1\}$ to these points. Then multiplication of these generators by this matrix again would give a group of order 6, with three elements represented by matrices of determinant -1. □

The differential equation for the universal covering map of $\mathbb{C}^*\backslash\{p_1, ..., p_n\}$ has been given [12]. A presentation for the subgroup of the conformal group of transformations on the n-punctured sphere leaving invariant $p_1, ..., p_n$ can be found by an embedding of the matrices in $SO(2,1)$. Although there are continuous transformations from one sector, Δ_i, to another, Δ_j, these analytic mappings may be retracted to the identity, After identification of the domains, combined with permutation of the points $\{p_i\}$. The discrete group $G(K_f/\mathbb{Q}(x))$ results from a retraction of the group of transformations of the conformal equivalence class of the Riemann surface $\mathcal{R}_g$. Permutation of the fixed points is sufficient for this retraction.

4 The Moduli Spaces of Punctured Spheres and the Symmetric Permutation Group

The elliptic modular function is the conformal universal covering map for the thrice-punctured sphere [1]. The relation between $\Gamma(1) = SL(2;\mathbb{Z})$ and the congruence subgroup $\Gamma(2)$ is evident in the modular invariance of $J(\omega) = \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$ under modular transformations, where $\lambda = \frac{e_1-e_3}{e_2-e_3}$, $e_1 = \mathcal{J}\left(\frac{w_1}{2}\right)$, $e_2 = \mathcal{J}\left(\frac{w_2}{2}\right)$ and $e_3 = \mathcal{J}\left(\frac{w_1+w_2}{2}\right)$, $\omega = \frac{w_2}{w_1}$, invariance of λ with respect to $\Gamma(2)$ and $\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$ under the transformations $\lambda \to \frac{1}{\lambda}$ and $\lambda \to 1-\lambda$, which generates a group of order 6.

Theorem 4.1. *The group S_3 may be generated by two permutations of order 2, which are represented by matrices with determinants equal to -1. The elements also generate the fundamental group of the moduli space of the sphere with three unmarked points which is a quotient of the braid group $\mathcal{B}_3$ by equivalence relations for the two generators.*

Proof. This presentation of the symmetric group is determined by the transformations of order 2

$$\begin{aligned} \sigma_1 : \quad & \lambda \to \frac{1}{\lambda} \\ \sigma_2 : \quad & \lambda \to 1-\lambda \end{aligned} \tag{33}$$

since

$$\begin{aligned} \sigma_1\sigma_2 : \quad & \lambda \to \frac{1}{1-\lambda} \\ \sigma_2\sigma_1 : \quad & \lambda \to \frac{\lambda-1}{\lambda} \\ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 : \quad & \lambda \to \frac{\lambda}{\lambda-1} \end{aligned} \tag{34}$$

The relations for the two generators are

$$\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1 \qquad \sigma_1^2 = 1 \qquad \sigma_2^2 = 1 \tag{35}$$

For the braid group $\mathcal{B}_n$ with generators $\sigma_1, ..., \sigma_{n-1}$, the relations are

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_1\sigma_{i+1} \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \qquad if\ |i-j| \geq 2 \end{aligned} \tag{36}$$

For the braid group $\mathcal{B}_3$, $|i-j| \leq 1$, and the the first relation is

$$\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1 \tag{37}$$

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Given that

$$\begin{aligned} y_i &= \sigma_{i-1}...\sigma_1 \cdot \sigma_1...\sigma_{i-1} \qquad 2 \leq i \leq n \\ z_i &= (\sigma_1...\sigma_{i-1})^i \end{aligned} \tag{38}$$

it follows that

$$y_3 = \sigma_2\sigma_1^2\sigma_2 \qquad z_3 = (\sigma_1\sigma_2)^3 \tag{39}$$

If $y_3 = 1$, it follows from Eq.(24) that

$$z_3 = \sigma_1(\sigma_2\sigma_1\sigma_2)\sigma_1\sigma_2 = \sigma_1(\sigma_1\sigma_2\sigma_1)\sigma_1\sigma_2 = \sigma_1^2(\sigma_2\sigma_1^2\sigma_2) = \sigma_1^2 \tag{40}$$

The equality $z_3 = 1$ is equivalent to $\sigma_1^2 = 1$. Substituting this relation into y_3 gives $\sigma_2^2 = 1$. Therefore, $S_3 \simeq \mathcal{B}_3/\langle y_3 = 1,\ z_3 = 1\rangle$. This space can be identified with $\Gamma_{0,[3]}$, the mapping class group of the sphere with three unmarked points [11][20]. From the transformations representing σ_1, σ_2, $\sigma_1\sigma_2$, $\sigma_2\sigma_1$ and $\sigma_1\sigma_2\sigma_1$, it follows that $\mathbb{I}$, $\sigma_1\sigma_2$ and $\sigma_2\sigma_1$ correspond to matrices with a determinant equal to 1, while $\sigma_1,\ \sigma_2$ and $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ are permutations of two points that are represented by matrices of determinant -1. $\square$

The principal congruence group of level 2 is the kernel of the homomorphism $\mu_2 : PSL(2;\mathbb{Z}) \to PSL(2;\mathbb{Z}_2)$. The only matrices in $PSL(2;\mathbb{Z}_2)$ that have a determinant equal to 1 are $\mathbb{I}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and the six elements modulo 2 are

$$\mathbb{I},\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{41}$$

Again, the determinants of three of the matrices equals -1, although, modulo 2, it can be equated to 1. However, it can be verified that there is no set of three points which are permuted by the second matrix, a coalescence of the three punctures at 0 is required for the third matrix, and the locations of the three punctures for the remaining matrices are

$$\begin{aligned}
&\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}: \; x = \frac{-3 \pm \sqrt{5}}{2}, \; y = \frac{2}{\pm\sqrt{5} - 1}, \; z = \frac{-1 \pm \sqrt{3}i}{2} \\
&\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}: \; x = \frac{1 \pm \sqrt{3}i}{2}, \; y = \frac{3 \pm \sqrt{3}i}{1 \pm \sqrt{3}i}, \; z = \frac{1 \pm \sqrt{5}}{2} \\
&\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}: \; x = 0, \; y = \infty, \; z = \pm 1
\end{aligned} \tag{42}$$

Consequently, the fractional linear transformations cannot fix the same set of points in $\mathbb{C}^*$, and the conclusions of Theorem 3.2 are confirmed.

It follows that for a thrice-punctured sphere, an S_3 permutation symmetry, representing the interchange of the punctures, exists. This group also acts freely on any finite complex homotopic to S^3, and by restriction, S^2 [4]. Furthermore, this symmetry will be included in the group of deck transformations of the covering of the surface.

String theory amplitudes have been given formally by a sum over genus of moduli space integrals defined over all quasiconformal deformations of the metric not related by the identity component of the diffeomorphism group and factored by Weyl rescalings of the two-dimensional metrics on the Riemann surfaces. The action of the S_3 transformation may considered for string theory amplitudes. Since the thrice-punctured spheres are used initially for surfaces of genus $g \geq 2$, it is necessary to introduce vertex operators at zero and one loops. Furthermore, it can be concluded that the invariance under an appropriate S_3 action on the modular parameter integration region is reflected at lower energies in a universality principle for trilinear couplings in the field theory limit.

The genus-g surface can be constructed by the sewing of spheres with three punctures. Consider $2g-2$ spheres with three punctures with $g-1$ components in two separate groups. The first set will be labelled as $\{\Sigma_{0,3}^{(1)}, ..., \Sigma_{0,3}^{(g-1)}\}$, while the second group is $\{\Sigma_{0,3}^{(g)}, ..., \Sigma_{0,3}^{(2g-2)}\}$. Suppose that a handle is placed between two of the punctures on $\Sigma_{0,3}^{(1)}$ and the third puncture is joined to a puncture of $\Sigma_{0,3}^{(2)}$. The process of creating a handle between two punctures is the reverse of that of the pinching of a handle that yields a singular surface in the compactification divisor of moduli space. By connecting the two remaining punctures of $\Sigma_{0,3}^{(2)}$ to punctures on $\Sigma_{0,3}^{(3)}$ and $\Sigma_{0,3}^{(g+1)}$, $\Sigma_{0,3}^{(2)}$ is sewed to two other components of the Riemann surface. This procedure can be iterated with the two remaining punctures of $\Sigma_{0,3}^{(i)}$ being joined to $\Sigma_{0,3}^{(i+1)}$ and $\Sigma_{0,3}^{(g+i-1)}$, $i = 3, ..., g-1$. Then all of the punctures of the first set of components have been sewed, and there are two available punctures on the each of the components in the second set. Since it is not necessary to sew any of these components to other thrice-punctured spheres, handles can be placed on $\Sigma_{0,3}^{(g)}, ..., \Sigma_{0,3}^{(2g-2)}$. It follows that the sewed surface has g handles and no punctures.

This technique can be extended to N-point, g-loop amplitudes represented by the sewing of $2g-2+N$ thrice-punctured spheres. The integration over moduli space would be described by the nature of the handles and the propagators between different components The interchange of two punctures which are replaced by a handle does not affect the amplitude, whereas the interchange of the punctures that are connected to different components would transform the integration to a separate part of moduli space. The action of the symmetric permutation group may be applied to each component $\Sigma_{0,3}$ of the decomposition on which no handles are attached, it must be possible to partition the integration region of the modular parameters at genus g according to an $S_3 \times S_3 \times ... \times S_3 = S_3^{g-2}$ symmetry.

The method of sewing thrice-punctured spheres to construct a genus-g surface begins at $g = 2$. Summing the amplitudes from genus 3, there would be S_3, $S_3 \times S_3$, S_3^3, S_3^4, ... actions on the integration regions for the path integral. The total amplitude would contain an invariance only under the intersection of these symmetries, which would be

$$\cap_{n=3}^{\infty}(S_3)^n = S_3. \tag{43}$$

At genus 1, when the handle is placed between two punctures, the inclusion an additional vertex operators would require a thrice-punctured sphere. The genus 2 surfaces are constructed by connecting two once-punctured tori. Likewise, at tree level, thrice-punctured spheres again occur when $N \geq 3$. The effect of an S_3 transformation on the integral for the one-point, one-loop diagram would be the interchange of the location of the vertex operator with that of the handle in the corner of moduli space defined by the thinning of the handle until a puncture appears. For the higher-point one-loop diagrams, the interchange of vertex operators arises as well. The action of the symmetric permutation group therefore implies a comparison of the coincidences of vertex and handle operators with the coincidences of vertex operators [2]. This possibility, which potentially complicated the analysis of divergences in superstring theory, can be shown to have a regular solution.

It may be recalled that one-point functions determine the stability of the vacuum, two-point amplitudes are related to mass or wavefunction renormalization and three-point amplitudes yield predictions of the couplings in the low-energy field theory. Consequently, the three-point tree-level amplitude would govern the leading-order contribution to the Yukawa couplings in the field theory [25].

It is known that N-point superstring amplitudes vanish for $N < 4$, and therefore, it might appear that the three-point amplitude cannot be used. However, it may be conjectured that the supersymmetry breaking in a phenomenological Lagrangian will render the three-point functions nonvanishing. Therefore, if the formal S_3 invariance in the integral for the superstring amplitude is preserved at lower energies, its effect would be evident in the field-theory limit. This result is confirmed by the universality of the couplings in interactions between photons, intermediate vector bosons and gluons and the fermions in the three different generations.

The symmetric permutation group of three elements also arises in the automorphism group of the standard model. It has been demonstrated that the spinor space of the standard model may be expressed as $\oplus_{i=1}^{3}(\mathbb{C}_i \otimes \mathbb{H}_i \otimes \mathbb{O}_i)$. The automorphism group [3] then equals

$$Aut(\oplus_{i=1}^{3}(\mathbb{C}_i \otimes \mathbb{H}_i \otimes \mathbb{O}_i)) = Aut(\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O})\ wr\ S_3 = G_2 \times SU(2) \times U(1)\ wr\ S_3.$$

Symmetry breaking generates different masses for the quarks and leptons in each of the generations. There are relations between the masses of the charged leptons, however, which reveal an underlying S_3 invariance [16] . It is known that there is an extended model with a larger $S_3 \times S_3$ invariance [8] which can be used as a theoretical basis for the Koide relation. Again, it can be concluded that this effected may be connected to invariances of amplitudes in string theory.

Thrice-punctured spheres may be used construct the moduli spaces of higher-genus surfaces with punctures and the rays in the Hilbert space representing the universal Grassmannian [23]. The energy-momentum tensor of the string can be identified with the action of meromorphic vector fields on the Riemann surface, which generate a Krichever-Novikov algebra at higher genus [15].

Theorem 4.2. *A paracompact model of the universal moduli space of surfaces with n punctures is diffeomorphic to the union $(\mathbb{CP}^1)^{n-3} \cup (\mathbb{CP}^1)^n \cup \cup_{g=2}^{\infty}((\mathbb{CP}^1)^{3g+n-3}/(S_3^{g-1})$ for $n \geq 4$.*

Proof. The moduli space of the four-punctured sphere, $\mathbb{CP}^1 - \{0, 1, \infty\}$, is diffeomorphic to the thrice-punctured sphere and its closed compactification is $\bar{\mathcal{M}}_{0,4} = \mathbb{CP}^1$. Because the moduli space of the five-punctured sphere is $\mathcal{M}_{0,5} = (\mathbb{CP}^1 - \{0, 1, \infty\})^2 - \langle x = y \rangle$, with $x, y \in \mathbb{CP}^1$, and the divisor includes $\mathcal{M}_{0,4}$ [20], the closed compactification could be identified with $(\mathbb{CP}^1)^2$ after filling in the remaining punctures and abbreviating the process of adding the boundary by excluding three extra lines at the divisor. Similarly, since $\mathcal{M}_{0,n} = (\mathbb{CP}^1 - \{0, 1, \infty\})^{n-3} - \Delta$ when $n \geq 5$, where Δ is a set of codimension greater than or equal to one. there would be a transformation from $\bar{\mathcal{M}}_{0,n}$ to $((\mathbb{CP}^1)^{n-3})$. The methods for evaluating accessory parameters for the torus with one puncture and the four-punctured sphere, which have hyperbolic metrics, are known to be equivalent, since there is a covering of $T_{1,1} \simeq U/G$ over $\Sigma_{0,4} \simeq U/\tilde{G}$, with punctures at ∞, e_1, e_2 and e_3 [17], with the Weierstrass function invariant under the interchange of e_1, e_2 and e_3, and $(\mathcal{M}_{1,1})/\mathbb{Z}_4 \simeq \mathcal{M}_{0,[4]} \simeq \mathcal{M}_{0,4}/S_4$. Similarly, the transformation from $\mathcal{M}_{1,2}$ to $\mathcal{M}_{0,5}/S_4$ [20] confirms the factorization by S_4. The two-valued uniformizing function from the four-punctured sphere to the complex plane yields a cancellation of the $\mathbb{Z}_2$ arising from fixing the fifth puncture in $\Sigma_{0,5}$ at one of the two punctures in $\mathcal{M}_{1,2}$. Consequently, the addition of a handle is equivalent to that of three punctures, such that the compactified moduli space is described by a tensor product with $((\mathbb{CP}^1)^3 \times \mathbb{Z}_4)/S_4$. A relation between $\bar{\mathcal{M}}_{g,n}$ and $((\mathbb{CP}^1)^{3g+n-3} \times \mathbb{Z}_4^{g-1})/(S_4)^{g-1}$ would follow from a sewing of disks around the punctures to create the handles and the additional modular parameters resulting from the use of two punctures, with $3g - 3$ moduli for each genus, when $g \geq 2$. The quotients by the powers of the symmetric groups may defined through the action of S_4 on $(\mathbb{CP}^1)^4$, by the permutation of indices in the cross-ratio function which is a holomorphic mapping of ordered four-tuples on $\mathbb{CP}^1$ to $\mathbb{C}\backslash\{0,1\}$, that can be restricted to the action of S_3 on $\mathbb{CP}^3$ [9]. There would exist a transformation from the universal moduli space of n-punctured surfaces with $n \geq 4$ to $(\mathbb{CP}^1)^{n-3} \cup (\mathbb{CP}^1)^n \cup \cup_{g=2}^{\infty}((\mathbb{CP}^1)^{3g+n-3}/(S_3^{g-1})$. $\square$

The total derivative ambiguities in superstring amplitudes [24] can be eliminated at genus one and two [6]. At higher genus, the integral of a total derivative over the covering space of the union of closed compactifications of the moduli spaces would vanish.

The group of quasi-special, symmetric outer automorphisms of the profinite completion $\hat{\Gamma}_{0,4}$ [21] of the fundamental group $\Gamma_{0,4} = F_2 = \langle x, y, z | xyz = 1\rangle$, the free group of two elements consisting of squares of Dehn twists [20], is defined by the action of automorphisms on the two elements x and y. It is known similarly that the automorphism groups $Out^{\#}(\hat{\Gamma}_{0,n})$ for $n \geq 5$ are isomorphic, with $\bar{\mathcal{M}}_{0,5} \simeq (\mathbb{CP}^1)^2 \simeq S^2 \times S^2$, and the outer automorphisms of the profinite completions of the fundamental groups of the moduli spaces $\mathcal{M}_{g,n}$ include the absolute Galois group $G_{\mathbb{Q}}$ [10]. At higher genus, stability of the outer automorphism group $Out^{\#}(\hat{\Gamma}_{g,n})$ can be valid for n larger than a genus-dependent lower bound related to a generalization of the five-value theorem [5].

5 Conclusion

The relation between the roots of an monic irreducible polynomial in two variables $g(x, y)$ at a value x_0 that is an algebraic function of the coefficients, which yields an irreducible polynomial in one variable, $g(x_0, y)$ is sufficient to establish the algebraicity of the roots of the $g(x_0, y)$ given the algebraicity of one of the roots. A group theoretical equivalent of this result has been given. Furthermore, there is a connection between the Galois group of the quotient of the splitting field of an irreducible polynomial $f(x, y)$ and $\bar{Q}(x)$ and the deck transformations of a Riemann surface that can be constructed with punctures at the zeros of the function in a neighbourhood of a point x_0. The conformal group of thrice-punctured sphere is examined and shown to coincide with the Galois group, the permutation group S_3, if the matrix representation of the group of covering transformation is enlarged from $SL(2;\mathbb{R})$ to $SL(2;\mathbb{R}) \cup T \cdot SL(2;\mathbb{R})$, where $det\, T = -1$. More generally, the presentation of matrices representing the action of the fractional linear transformations on $\mathbb{C}^*$ which preserve the set of three punctures is shown to always contain three matrices of determinant -1 within multiplication by $\mathbb{I}$.

The necessity of the matrices of determinant -1 for the presentation of S_3 has implications for the uniformization theorem of Riemann surfaces. While surfaces of genus 2 are known to have the upper half plane $\mathcal{H}$ or the unit disk U as the covering surface, the thrice-punctured sphere, which is conventionally viewed as belonging to the class of surfaces with $\mathcal{H}$ as a covering surface, must be uniformized with an enlargement of the covering group.

The study of moduli spaces of n-punctured spheres becomes nontrivial only when $n \geq 4$. Therefore, the description of $\mathcal{M}_{g,n}$ would be expected to require the moduli spaces of m-punctured spheres, with $4 \leq m \leq n$, even though the thrice-punctured spheres may be sewn together to form the higher-genus surfaces. Based on work relating the moduli spaces of the punctured torus and the four-punctured sphere, the moduli and the accessory parameters related to the addition of handles, in principle, can be evaluated. A method for compactifying the moduli spaces of punctured surfaces at higher genus follows from the description of $\bar{\mathcal{M}}_{1,1}$, the addition of new punctures and the sewing rules. The union of each of the compactified moduli spaces over the genus then furnishes a paracompact model of a universal moduli space.

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