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VARIATIONAL AND QUASI-VARIATIONAL INEQUALITIES OF NAVIER-STOKES TYPE WITH VELOCITY CONSTRAINTS

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Abstract. In this paper we deal with parabolic variational inequalities of Navier-Stokes type with time-dependent constraints on velocity fields, including gradient constraint case. One of the objectives of this paper is to propose a weak variational formulation for variational inequalities of Navier-Stokes type and to solve them by applying the compactness theorem, which was recently developed by the authors (cf. [22]).

Another objective is to approach to a class of quasi-variational inequalities associated with Stefan/Navier-Stokes problems in which we are taking into account the freezing effect of materials in fluids. As is easily understood, the phase change from liquid into solid gives a great influence to the velocity field in the fluid. For instance, in the mussy region, the velocity of the fluid is constrained by some obstacle caused by moving solid. We shall challenge to the mathematical modeling of Stefan/Navier-Stokes problem as a quasi-variational inequality and solve it as an application of parabolic variational inequalities of Navier-Stokes type.

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1. Introduction

In this paper we study parabolic variational inequalities of Navier-Stokes type with velocity constraint of the form

$$
\begin{aligned}
&\boldsymbol{v}(t) \in K(t), \quad 0 < t < T, \quad \boldsymbol{v}(0) = \boldsymbol{v}_0; \\
&\int_Q \{\boldsymbol{v}_t \cdot (\boldsymbol{v} - \boldsymbol{\xi}) + \nu \nabla \boldsymbol{v} \cdot \nabla (\boldsymbol{v} - \boldsymbol{\xi}) + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \cdot (\boldsymbol{v} - \boldsymbol{\xi})\} dxdt \\
&\qquad \le \int_Q \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{\xi}) dxdt, \quad \forall \boldsymbol{\xi} \text{ with } \boldsymbol{\xi}(t) \in K(t), \quad 0 < t < T,
\end{aligned}
\tag{1.1}
$$

where Ω is bounded domain in $\mathbf{R}^3$ and $Q := \Omega \times (0, T)$, $0 < T < \infty$, and $K(t)$ is a prescribed constraint set in the 3-dimensional solenoidal function space $\boldsymbol{W}^{1,2}_{0,\sigma}(\Omega)$; a positive constant ν, an initial datum $\boldsymbol{v}_0$ and a source term $\boldsymbol{g}$ are given.

We mainly consider the following two cases as $K(t)$:

$$
K(t) = K^1(\gamma; t) := \{\boldsymbol{z} \in \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega) \mid |\boldsymbol{z}| \le \gamma(\cdot, t) \text{ a.e. on } \Omega\}, \ 0 \le t \le T, \tag{1.2}
$$

and

$$
K(t) = K^2(\gamma; t) := \{\boldsymbol{z} \in \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega) \mid |\nabla \boldsymbol{z}| \le \gamma(\cdot, t) \text{ a.e. on } \Omega\}, \ 0 \le t \le T, \tag{1.3}
$$

where γ is a nonnegative continuous function on $\overline{Q}$ and permitted to take ∞ somewhere in $\overline{Q}$; note that the continuity of γ should be understood in the extended sense. Since it is difficult to expect the differentiability of the solution $\boldsymbol{v}$ in time, we shall discuss the problem (1.1) in a framework of weak variational inequalities.

There are so many nonlinear dynamical systems in our real world whose mechanism are still not clear from the theoretical point of view. For instance, we meet very interesting phase transition phenomena in fluids, such as biofilm growth and melting ice in the sea or lake. Our experiments suggest that in general, biofilm growth is improved in fluids, but its mechanism makes complex by taking account of freezing or melting effect in fluids. It is a very important task for us to provide some realistic mathematical models to such a complex system. To this end the authors started in [20, 21, 22] the development of mathematical tools describing various aspects of fluid dynamics; one of them is the establishment of the theory on variational inequalities of Navier-Stokes type with velocity constraints (1.2) and (1.3).

In each of (1.2) and (1.3), there are two cases of γ, the non-degenerate case and degenerate case:

(Non-degenerate case) $\quad c_* \le \gamma(x, t) \le \infty$ on $\overline{Q}$ for a positive constant c_*.

(Degenerate case) $\quad 0 \le \gamma(x, t) \le \infty$ on $\overline{Q}$ and γ vanishes somewhere in Q.

These cases are separately treated, because in the degenerate case we need an extended use of Helmholtz decomposition in the solenoidal function spaces for the construction of a weak solution (see section 4), but in the non-degenerate case we do not need it and the treatment is much easier. So far as the case of gradient constraint (1.3) is concerned, the variational inequality of Navier-Stokes type (1.1) is a new problem and there has not been any existence result on it in our knowledge. We shall prove it in the non-degenerate case (see section 3).

In the degenerate case the existence proof of a weak solution of the problem (1.1) with $K(t) = K^1(\gamma; t)$ was given in the author's paper [20] in which the crucial step is how to show the strong convergence of regular approximate solutions in $L^2(Q)^3$. This is quite important to handle well the convergence of nonlinear terms in approximate variational inequalities for which we needed the Helmholtz decomposition of solenoidal functions. But, after the publication of the author's original paper, a gap was found by the authors in the usage of the Helmholtz decomposition. We shall make the correction for the gap in section 4 of this paper under a slight additional assumption on the obstacle function γ that

$$\gamma \text{ is Lipschitz continuous in a neighborhood of } \{(x,t) \in Q \mid \gamma(x,t) = 0\}. \tag{1.4}$$

The result in [20] was used in the paper [21] on the biofilm growth problem, so such an additional assumption as (1.4) should be required in [21], too.

The second aim of this paper is to give some applications of the results obtained in this paper to the Stefan/Navier-Stokes problem, which is a coupled system of the enthalpy formulation of Stefan problem with convection,

$$\begin{aligned} &w_t - \Delta\beta(w) + \boldsymbol{v} \cdot \nabla w = h(x,t), \quad (x,t) \in Q, \\ &w(\cdot, 0) = w_0 \text{ in } \Omega, \quad \frac{\partial \beta(w)}{\partial n} + n_0 \beta(w) = 0 \text{ on } \Sigma := \partial\Omega \times (0,T), \end{aligned} \tag{1.5}$$

and the variational inequality of Navier-Stokes type,

$$\begin{aligned} &\boldsymbol{v}(t) \in K^i(\gamma(w^{\varepsilon_0}); t), \quad 0 < t < T, \quad \boldsymbol{v}(0) = \boldsymbol{v}_0; \\ &\int_Q \{\boldsymbol{v}_t \cdot (\boldsymbol{v} - \boldsymbol{\xi}) + \nu \nabla \boldsymbol{v} \cdot \nabla(\boldsymbol{v} - \boldsymbol{\xi}) + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \cdot (\boldsymbol{v} - \boldsymbol{\xi})\} dxdt \\ &\qquad \le \int_Q \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{\xi}) dxdt, \quad \forall \boldsymbol{\xi} \text{ with } \boldsymbol{\xi}(t) \in K^i(\gamma(w^{\varepsilon_0}); t), \quad 0 < t < T, \end{aligned} \tag{1.6}$$

where $i = 1,\ 2$, w_0 is an initial datum, h is a given source and $\beta(\cdot)$ is a Lipschitz continuous and increasing function on $\mathbf{R}$ such that

$$\beta(r) = 0,\ \forall r \in [0,1], \quad \beta'(r) > 0,\ \forall r > 1 \text{ or } r < 0, \quad \lim_{|r| \to \infty} \frac{\beta(r)}{r} > 0,$$

and $\gamma(\cdot)$ is continuous function from $\mathbf{R}$ into $[0, \infty]$ such that

$$\begin{aligned} &\gamma(r) = 0,\ \forall r \le 0, \quad \gamma \text{ is strictly increasing on } [0,1), \\ &\gamma(r) \uparrow \infty \text{ as } r \uparrow 1, \quad \gamma(r) = \infty,\ \forall r \ge 1, \\ &\gamma(r) \text{ is Lipschitz continuous near } r = 0. \end{aligned}$$

We denote by $w^{\varepsilon_0}(x,t)$ the spatial average of $w(x,t)$, namely

$$w^{\varepsilon_0}(x,t) = [\rho_{\varepsilon_0} * w(\cdot, t)](x) := \int_\Omega \rho_{\varepsilon_0}(x-y) w(y,t) dy, \quad \forall x \in \Omega,$$

where $\rho_{\varepsilon_0}(\cdot)$ is the usual mollifier on $\mathbf{R}^3$ with support in $|x| \le \varepsilon_0$. Throughout this paper, the parameter $\varepsilon_0 > 0$ is fixed, although it is close to 0, and we do not consider the limit $\varepsilon_0 \downarrow 0$. In section 5 we shall prove the existence of a weak solution $\{w, \boldsymbol{v}\}$ to problem (1.5)-(1.6).

Especially, the gradient obstacle problem is a new challenge to the Navier-Stokes variational inequalities. From various different motivations it has been studied by many researchers so far (cf. [4, 5, 6, 7, 16, 18, 31, 32]), but most of cases were treated in the non-degenerate case. It would be expected to generalize to the degenerate case from some serious physical/mechanical motivations in order to make more realistic modelings in nonlinear phenomena in fluids (cf. [1, 14, 29, 30]).

(**Notation**)

Let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary $\Gamma := \partial\Omega$, $Q := \Omega \times (0,T)$, $0 < T < \infty$ and $\Sigma := \Gamma \times (0,T)$, and denote by $|\cdot|_X$ the norm in various function spaces X built on Ω. We consider the usual solenoidal function spaces:

$$\begin{aligned}
&\boldsymbol{\mathcal{D}}_\sigma(\Omega) := \{\boldsymbol{v} = (v^{(1)}, v^{(2)}, v^{(3)}) \in \mathcal{D}(\Omega)^3 \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\},\\
&\boldsymbol{H}_\sigma(\Omega) := \text{the closure of } \boldsymbol{\mathcal{D}}_\sigma(\Omega) \text{ in } L^2(\Omega)^3, \text{ with norm } |\cdot|_{0,2},\\
&\boldsymbol{V}_\sigma(\Omega) := \text{the closure of } \boldsymbol{\mathcal{D}}_\sigma(\Omega) \text{ in } H^1_0(\Omega)^3, \text{ with norm } |\cdot|_{1,2},\\
&\boldsymbol{W}_\sigma(\Omega) := \text{the closure of } \boldsymbol{\mathcal{D}}_\sigma(\Omega) \text{ in } W^{1,4}_0(\Omega)^3,\\
&\qquad\qquad \text{in the case of (1.2), with norm } |\cdot|_{1,4};\\
&\boldsymbol{W}_\sigma(\Omega) := \text{the closure of } \boldsymbol{\mathcal{D}}_\sigma(\Omega) \text{ in } W^{2,4}_0(\Omega)^3,\\
&\qquad\qquad \text{in the case of (1.3); with norm } |\cdot|_{2,4};
\end{aligned}$$

in these spaces the norms are given as usual by: for $\boldsymbol{v} := (v^{(1)}, v^{(2)}, v^{(3)})$

$$|\boldsymbol{v}|_{0,2} := \left\{\sum_{k=1}^{3} \int_\Omega |v^{(k)}|^2 dx\right\}^{\frac{1}{2}}, \quad |\boldsymbol{v}|_{1,2} := \left\{\sum_{k=1}^{3} \int_\Omega |\nabla v^{(k)}|^2 dx\right\}^{\frac{1}{2}}$$

and

$$|\boldsymbol{v}|_{1,4} := \left\{\sum_{k=1}^{3} \int_\Omega |\nabla v^{(k)}|^4 dx\right\}^{\frac{1}{4}}, \quad |\boldsymbol{v}|_{2,4} := \left\{|\boldsymbol{v}|_{1,4}^4 + \sum_{k,j,\ell=1}^{3} \int_\Omega \left|\frac{\partial^2 v^{(k)}}{\partial x_j \partial x_\ell}\right|^4 dx\right\}^{\frac{1}{4}}.$$

For simplicity we denote the dual spaces of $\boldsymbol{V}_\sigma(\Omega)$ and $\boldsymbol{W}_\sigma(\Omega)$ by $\boldsymbol{V}^*_\sigma(\Omega)$ and $\boldsymbol{W}^*_\sigma(\Omega)$, respectively, which are equipped with their dual norms. Also, we denote the inner product in $\boldsymbol{H}_\sigma(\Omega)$ by $(\cdot,\cdot)_\sigma$ and the duality between $\boldsymbol{V}^*_\sigma(\Omega)$ and $\boldsymbol{V}_\sigma(\Omega)$ by $\langle\cdot,\cdot\rangle_\sigma$, namely for $\boldsymbol{v}_i = (v_i^{(1)}, v_i^{(2)}, v_i^{(3)})$, $i = 1,2$,

$$(\boldsymbol{v}_1, \boldsymbol{v}_2)_\sigma := \sum_{k=1}^{3} \int_\Omega v_1^{(k)} v_2^{(k)} dx, \qquad \langle \boldsymbol{F}\boldsymbol{v}_1, \boldsymbol{v}_2\rangle_\sigma = \sum_{k=1}^{3} \int_\Omega \nabla v_1^{(k)} \cdot \nabla v_2^{(k)} dx,$$

where $\boldsymbol{F}$ denotes the duality mapping from $\boldsymbol{V}_\sigma(\Omega)$ onto $\boldsymbol{V}^*(\Omega)$. Then, by identifying the dual of $\boldsymbol{H}_\sigma(\Omega)$ with itself, we have:

$$\boldsymbol{V}_\sigma(\Omega) \subset \boldsymbol{H}_\sigma(\Omega) \subset \boldsymbol{V}^*_\sigma(\Omega), \quad \boldsymbol{W}_\sigma(\Omega) \subset C(\overline{\Omega})^3 \ (C^1(\overline{\Omega})^3 \text{ in the case (1.3)});$$

and all these embeddings are compact.

By the way we introduce the usual simplified notation in the theory of Navier-Stokes equations:

$$\int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\cdot\boldsymbol{z}dx := \sum_{k,j=1}^{3}\int_\Omega v^{(k)}\frac{\partial w^{(j)}}{\partial x_k}z^{(j)}dx$$

for $\boldsymbol{v} := (v^{(1)}, v^{(2)}, v^{(3)})$, $\boldsymbol{w} := (w^{(1)}, w^{(2)}, w^{(3)})$ and $\boldsymbol{z} := (z^{(1)}, z^{(2)}, z^{(3)})$.

Since Ω is fixed throughout this paper, the spaces $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$, $\boldsymbol{V}_\sigma(\Omega)$, $\boldsymbol{H}_\sigma(\Omega)$ and $\boldsymbol{W}_\sigma(\Omega)$ are simply denoted by $\boldsymbol{\mathcal{D}}_\sigma$, $\boldsymbol{V}_\sigma$, $\boldsymbol{H}_\sigma$ and $\boldsymbol{W}_\sigma$, respectively. When these types of spaces are built on other open set Ω' in $\mathbf{R}^3$, we write them as $\boldsymbol{\mathcal{D}}_\sigma(\Omega')$, $\boldsymbol{V}_\sigma(\Omega')$, $\boldsymbol{H}_\sigma(\Omega')$ and $\boldsymbol{W}_\sigma(\Omega')$. The notation $(\cdot,\cdot)_\sigma$ is commonly used for the inner product in $\boldsymbol{H}_\sigma$ or in $\boldsymbol{H}_\sigma(\Omega')$ as well as $\langle\cdot,\cdot\rangle_\sigma$ for the duality between $\boldsymbol{W}_\sigma^*$ and $\boldsymbol{W}_\sigma$ or $\boldsymbol{W}_\sigma^*(\Omega')$ and $\boldsymbol{W}_\sigma(\Omega')$ in case of no confusion.

For the general knowledge about solenoidal function spaces and Navier-Stokes equation, we refer to the monographs [19, 34].

2. Time-derivative under constraint and compactness theorem

2.1. A compactness theorem

In this section we recall some results in [22] with the following setup:

(h1) H is a Hilbert space, and its dual H^* is identified with H.

(h2) V is a reflexive Banach space which is dense and compactly embedded in H, therefore we have $V \subset H \subset V^*$ with compact embeddings.

(h3) W is another reflexive and separable Banach space which is continuously embedded in V and dense in H; since $H \subset W^*$, we have

$$V \subset H \subset W^* \quad \text{with dense and compact embeddings.}$$

(h4) V, V^*, W and W^* are strictly convex.

(h5) The numbers: $p > 1$, $p' := \frac{p}{p-1}$, and $T > 0$ are fixed.

We begin with the definition of total variation, which refers here to the time variable. For any function $w : [0,T] \to W^*$, the **total variation** of w, denoted by $\mathrm{Var}_{W^*}(w)$, is defined by

$$\mathrm{Var}_{W^*}(w) := \sup_{\substack{\eta\in C_0^1(0,T;W),\\ |\eta|_{L^\infty(0,T;W)}\le 1}} \int_0^T \langle w, \eta'\rangle_{W^*,W}dt.$$

We refer to [10; Appendice 2] or [15; Chapter 5] for the fundamental properties of total variation functions. Let us now define the set which will be the point of our interest in this section.

The next lemma is concerned with the compactness property of functions having bounded total variation from $[0,T]$ into W^*.

Lemma 2.1. *Let M be any positive number and set*

$$\mathcal{X}(M) := \left\{ u \in L^p(0,T;H) \;\middle|\; \begin{array}{l} |u|_{L^p(0,T;V)} \le M, \\ |u|_{L^\infty(0,T;H)} \le M, \\ \mathrm{Var}_{W^*}(u) \le M \end{array} \right\}.$$

Then we have:

(1) Given any sequence $\{u_n\}$ in $\mathcal{X}(M)$, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in \mathcal{X}(M)$ such that

$$u_{n_k}(t) \to u(t) \text{ weakly in } H,\ \forall t \in [0,T]\ (\text{as } k \to \infty).$$

Hence, $u_{n_k}(t) \to u(t)$ in W^* for every $t \in [0,T]$.

(2) $\mathcal{X}(M)$ is compact and convex in $L^q(0,T;H)$ for every $q \in [1,\infty)$.

See [22; Lemma 3.3] for the proof of Lemma 2.1.

Definition 2.1. Given $\kappa > 0$, $M_0 > 0$ and $u_0 \in H$, consider the set $Z_p(\kappa, M_0, u_0)$ in $L^p(0,T;V) \cap L^\infty(0,T;H)$ given by: $u \in Z_p(\kappa, M_0, u_0)$ if and only if $|u|_{L^p(0,T;V)} \le M_0$, $|u|_{L^\infty(0,T;H)} \le M_0$ and there exists $f \in L^{p'}(0,T;V^*)$ such that

$$\int_0^T \langle f, u\rangle dt \le M_0,\ |f|_{L^1(0,T;W^*)} \le M_0$$

and

$$\int_0^T \langle \eta' - f, u - \eta\rangle dt \le \frac{1}{2}|u_0 - \eta(0)|_H^2, \tag{2.1}$$

$$\forall \eta \in L^p(0,T;V) \text{ with } \eta' \in L^{p'}(0,T;V^*),\ \eta(t) \in \kappa B_W(0),\ \forall t \in [0,T],$$

where $B_W(0)$ is the closed unit ball in W with center at the origin and $\langle \cdot,\cdot\rangle = \langle \cdot,\cdot\rangle_{V^*,V}$.

Remark 2.1. In Definition 2.1 the variational inequality (2.1) relates f to the time derivative of u, taking into account the convex constraint $\kappa B_W(0)$. This is explored as follows. We note for now that if $f = u'$ and $u(0) = u_0$, then (2.1) holds for any test function η. Indeed, for any u, $\eta \in L^p(0,T;V)$ with u', $\eta' \in L^{p'}(0,T;V^*)$ with $u(0) = u_0$ we have by integration by parts

$$\int_0^T \langle \eta' - u', u - \eta\rangle dt = \frac{1}{2}|u(0) - \eta(0)|_H^2 - \frac{1}{2}|u(T) - \eta(T)|_H^2 \le \frac{1}{2}|u_0 - \eta(0)|_H^2.$$

Thus, given $u \in L^p(0,T;V) \cap L^\infty(0,T;H)$ and $u_0 \in H$, the set of all f satisfying (2.1) includes u', provided u' exists in $L^{p'}(0,T;V^*)$ and $u(0) = u_0$. However, in general, it is an extremely large set; note that in the definition of $Z_p(\kappa, M_0, u_0)$, any differentiability of u in time is not required.

Lemma 2.2. *Let $Z_p(\kappa, M_0, u_0)$ be the set given by Definition 2.1. Then there is a positive constant C^* such that*

$$\mathrm{Var}_{W^*}(u) \le C^*, \quad \forall u \in Z_p(\kappa, M_0, u_0). \tag{2.2}$$

Moreover, we can take $M_0 + \frac{1}{\kappa} M_0 + \frac{1}{2\kappa}|u_0|_H^2$ as the constant C^*

The uniform estimate (2.2) of total variation for $Z_p(\kappa, M_0, u_0)$ is directly obtained from variational inequality (2.1). See [20; Lemma 3.2] for the proof of Lemma 2.2. Combining both of the above lemmas, we arrive at our compactness theorem.

Theorem 2.1 (cf. [20: Theorem 3.1]) *Let $\kappa > 0$, $M_0 > 0$ be any numbers and u_0 be any element in $\overline{K(0)}$. Then the set $Z_p(\kappa, M_0, u_0)$ is relatively compact in $L^p(0, T; H)$. Moreover, the convex closure of $Z_p(\kappa, \kappa, M_0)$, denoted by $\overline{\mathrm{conv}}[Z_p(\kappa, M_0, u_0)]$, in $L^p(0, T; V)$ is bounded in $L^p(0, T; V)$ and compact in $L^p(0, T; H)$.*

Here we compare Theorem 2.1 with the Aubin compactness theorem [3] (or [27; Chapter 1]), saying that for any number $M_0 > 0$ the set

$$\{u \mid |u|_{L^p(0,T;V)} \le M_0,\ |u'|_{L^q(0,T;W^*)} \le M_0\},\ 1 < p < \infty,\ 1 < q < \infty,$$

is relatively compact in $L^p(0, T; H)$. We can say in rough that our compactness theorem is the one obtained by replacing "$|u'|_{L^q(0,T;W^*)} \le M_0$" by the total variation estimate "$\mathrm{Var}_{W^*}(u) \le M_0$" in the Aubin compactness theorem.

Remark 2.2. A compactness theorem of the Aubin type was extended to various directions, for instance [13] and [23], and further to a quite general setup [33].

2.2. Time-derivative under convex constraints

We assume here **(h1)**, **(h2)**, **(h4)** except for W and **(h5)**: we will not be using the space W here. Again, for the sake of simplicity of notation, we write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_{V^*,V}$.

As V^* is strictly convex, the duality mapping F from V into V^*, associated with the gauge function $r \to |r|^{p-1}$, namely $F : V \to V^*$ is the subdifferential of $u \to \frac{1}{p}|u|_V^p$, is singlevalued and demicontinuous from V into V^*.

Definition 2.2. Let $\{K(t)\}_{t\in[0,T]}$ be a family of non-empty, closed and convex sets in V such that there are functions $\alpha \in W^{1,2}(0, T)$ and $\beta \in W^{1,1}(0, T)$ satisfying the following property: for any $s, t \in [0, T]$ and any $z \in K(s)$ there is $\tilde{z} \in K(t)$ such that

$$|\tilde{z} - z|_H \le |\alpha(t) - \alpha(s)|(1 + |z|_V^{\frac{p}{2}}),\quad |\tilde{z}|_V^p - |z|_V^p \le |\beta(t) - \beta(s)|(1 + |z|_V^p). \tag{2.3}$$

We denote by $\Phi(\alpha, \beta)$ the set of all such families $\{K(t)\}$, and put

$$\Phi_S := \bigcup_{\alpha\in W^{1,2}(0,T),\ \beta\in W^{1,1}(0,T)} \Phi(\alpha, \beta),$$

which is called the **strong class of time-dependent convex sets**.

Given $\{K(t)\} \in \Phi_S$, we consider the following time-dependent convex function on H:

$$\varphi_K^t(z) := \begin{cases} \dfrac{1}{p}|z|_V^p + I_{K(t)}(z), & \text{if } z \in K(t), \\ \infty, & \text{otherwise,} \end{cases}$$

where $I_{K(t)}(\cdot)$ is the indicator function of $K(t)$ on H. For each $t \in [0,T]$, $\varphi_K^t(\cdot)$ is proper, l.s.c. and strictly convex on H and on V. By the general theory on nonlinear evolution equations generated by time-dependent subdifferentials, see [24], condition (2.3) is a sufficient condition in order that for any $f \in L^2(0,T;H)$ and $u_0 \in \overline{K(0)}$ (the closure of $K(0)$ in H), the Cauchy problem

$$u'(t) + \partial\varphi_K^t(u(t)) \ni f(t), \quad u(0) = u_0, \text{ in } H,$$

admits a unique solution u such that $u \in C([0,T];H) \cap L^p(0,T;V)$ with $u(0) = u_0$, $t^{\frac{1}{2}}u' \in L^2(0,T;H)$ and $t \to t\varphi_K^t(u(t))$ is bounded on $(0,T]$ and absolutely continuous on any compact interval in $(0,T]$.

Next, taking constraints of obstacle type into account, we introduce a weak class of time-dependent convex sets. To this end, we first recall a notion of convergence of time–dependent convex sets introduced in [17]. This convergence is defined by means of admissible geometrical perturbations: we choose them to be homothetic and parallel transformations. To set this up, we define a perturbation operator $\mathcal{F}_\varepsilon(t) : V \to V$, which can be a sum of an expansion / contraction with an ε–dependent modulus (close to 1) and an (ε, t)–dependent (small) parallel transformation. Roughly, the sets will be considered to be close one to the other if after this kind of perturbation, at any time moment, they 'fit' one to the other. Note that we do not include rotations in our perturbation operator in order to avoid complexity which would be irrelevant from the point of view of applications. However, this can be done; see [26].

Definition 2.3. Let c_0 be a fixed constant and σ_0 be a fixed function in $C([0,T];V)$ with $\sigma_0' \in L^{p'}(0,T;V^*)$. Associated with these c_0 and σ_0, for any small positive number ε, the mapping $\mathcal{F}_\varepsilon : [0,T] \times V \to V$ is defined by

$$\mathcal{F}_\varepsilon(t)z = (1 + \varepsilon c_0)z + \varepsilon\sigma_0(t), \ \forall t \in [0,T], \ \forall z \in V. \tag{2.4}$$

Let $\{K(t)\}_{t\in[0,T]}$ be a family of non-empty, closed and convex sets in V and $\{K_n(t)\}_{t\in[0,T]}$ a sequence of such families. We say that $\{K_n(t)\}$ **converges to** $\{K(t)\}$ as $n \to \infty$, which is denoted by

$$K_n(t) \Longrightarrow K(t) \quad \text{on } [0,T] \text{ (as } n \to \infty),$$

if for any $\varepsilon \in (0, \varepsilon_1]$ $(0 < \varepsilon_1 < 1)$ there is a positive integer N_ε satisfying

$$\mathcal{F}_\varepsilon(t)(K_n(t)) \subset K(t), \quad \mathcal{F}_\varepsilon(t)(K(t)) \subset K_n(t), \quad \forall t \in [0,T], \quad \forall n \geq N_\varepsilon.$$

Note that this notion of convergence depends on the choice of the perturbation operator $\mathcal{F}_\varepsilon(t)$, which depends itself on the constant c_0 and on the function σ_0. The operator's form defines the perturbations that we allow, and that we can further restrict by choosing concrete c_0 and σ_0. As we are going to see in the examples, it is often enough to take them as equal to 0 or ± 1. We are now ready to define the weak class of constraints, which is the closure of the strong class with respect to this convergence.

Definition 2.4. $\{K(t)\} \in \Phi_W$, the **weak class of time-dependent convex sets**, if and only if the following two conditions are satisfied:

(a) $K(t)$ is a closed and convex set in V for all $t \in [0,T]$,

(b) there exists a sequence $\{\{K_n(t)\}\}_{n\in\mathbf{N}} \subset \Phi_S$ such that $K_n(t) \Longrightarrow K(t)$ on $[0,T]$, as $n \to \infty$, according to Definition 2.3.

We give typical examples of $\{K(t)\}$ in the weak class Φ_W.

Example 2.1. Let Ω be a bounded smooth domain in $\mathbf{R}^3$ and $Q := \Omega \times (0,T)$. Let

$$H := \boldsymbol{H}_\sigma(\Omega), \quad V := \boldsymbol{V}_\sigma(\Omega),$$

Moreover, let $\gamma := \gamma(x,t) \in C(\overline{Q})$ such that

$$\gamma \geq c_* \ \text{ on } \overline{Q}$$

for a positive constant c_* with $0 < c_* < 1$, and choose a sequence $\{\gamma_n\}$ in $C^2(\overline{Q})$ such that $\gamma_n \geq c_*$ and $\gamma_n \to \gamma$ in $C(\overline{Q})$. Now, constraint sets $K(t)$ and $K_n(t)$ are defined by

$$K^1(t) := \{\boldsymbol{z} \in \boldsymbol{V}_\sigma \mid |\boldsymbol{z}(x)| \leq \gamma(x,t) \text{ for a.e. } x \in \Omega\}, \ \forall t \in [0,T],$$

and

$$K_n^1(t) := \{\boldsymbol{z} \in \boldsymbol{V}_\sigma \mid |\boldsymbol{z}(x)| \leq \gamma_n(x,t) \text{ for a.e. } x \in \Omega\}, \ \forall t \in [0,T].$$

Given $\varepsilon > 0$, take a positive integer N_ε so that

$$|\gamma_n - \gamma| \leq \varepsilon c_* \ \text{ on } \overline{Q}, \ \forall n \geq N_\varepsilon. \tag{2.5}$$

In this case, with the choice of $c_0 = -\frac{1}{c_*}$ and $\sigma \equiv 0$, consider the mapping $\mathcal{F}_\varepsilon(t)$ of the form $\mathcal{F}_\varepsilon(t)\boldsymbol{z} = (1 - \frac{\varepsilon}{c_*})\boldsymbol{z}$, which maps V into itself for all small $\varepsilon > 0$. Then we have:

(i) We show that $\{K_n(t)\} \in \Phi_S$. Fix n and note that $\gamma_n \in C^2(\overline{Q})$. Therefore it is possible to take a partition $0 = t_0 < t_1 < t_2 < \cdots < t_k(n) = T$ of $[0,T]$ so that

$$|\gamma_n(s) - \gamma_n(t)|_{C(\overline{\Omega})} \leq L(\gamma_n)|s-t| < c_*, \ \forall s, \ t \in [t_{k-1}, t_k], \ k = 1, 2, \cdots, k(n), \tag{2.6}$$

where $L(\gamma_n)$ is the Lipschitz constant of γ_n. Now, given $\boldsymbol{z} \in K_n^1(s)$ and $s, \ t \in [t_{k-1}, t_k]$, the function $\tilde{\boldsymbol{z}} = (1 - \frac{|\gamma_n(s)-\gamma_n(t)|_{C(\overline{\Omega})}}{c_*})\boldsymbol{z}$ satisfies by (2.6)

$$\begin{aligned}
|\tilde{\boldsymbol{z}}(x)| &= \left(1 - \frac{|\gamma_n(s) - \gamma_n(t)|_{C(\overline{\Omega})}}{c_*}\right) |\boldsymbol{z}(x)| \\
&\leq \left(1 - \frac{|\gamma_n(s) - \gamma_n(t)|_{C(\overline{\Omega})}}{c_*}\right) \gamma_n(x,s) \\
&\leq \gamma_n(x,s) - |\gamma_n(s) - \gamma_n(t)|_{C(\overline{\Omega})} \\
&\leq \gamma_n(x,s) - |\gamma_n(x,s) - \gamma_n(x,t)| \leq \gamma_n(x,t)
\end{aligned}$$

Thus $\tilde{\boldsymbol{z}} \in K_n^1(t)$ and

$$|\tilde{\boldsymbol{z}} - \boldsymbol{z}|_{0,2} = \frac{|\gamma_n(s) - \gamma_n(t)|_{C(\overline{\Omega})}}{c_*}|\boldsymbol{z}|_{0,2} \leq \frac{L(\gamma_n)}{c_*}|s-t||\boldsymbol{z}|_{0,2}.$$

Generally, for any $s,\ t \in [0,T]$ with $s < t$ and $\boldsymbol{z} \in K_n^1(s)$, by repeating the above argument we can find $\tilde{\boldsymbol{z}} \in K_n^1(t)$ such that

$$|\boldsymbol{z} - \tilde{\boldsymbol{z}}|_{0,2} \leq L_n|s-t|(1 + |\boldsymbol{z}|_{0,2}),$$

for a positive constant L_n depending only on n. Also, we have $|\nabla\tilde{\boldsymbol{z}}|_{0,2}^2 \leq |\nabla\boldsymbol{z}|_{0,2}^2$, too. Therefore $\{K_n^1(t)\}$ belongs to $\Phi(\alpha,\beta)$ with $p = 2$, $\alpha(t) = L_n t$ and $\beta(t) = 0$. Thus $\{K_n^1(t)\} \in \Phi_S$.

(ii) $\{K^1(t)\} \in \Phi_W$. In fact, for any $\boldsymbol{z}_n \in K_n^1(t)$, we have by (2.5)

$$\begin{aligned} |\mathcal{F}_\varepsilon \boldsymbol{z}_n| &= \left(1 - \frac{\varepsilon}{c_*}\right)|\boldsymbol{z}_n| \\ &\leq \left(1 - \frac{\varepsilon}{c_*}\right)\gamma_n(\cdot,t) \leq \left(1 - \frac{\varepsilon}{c_*}\right)(\gamma(\cdot,t) + \varepsilon c_*) \\ &\leq \gamma(\cdot,t) + \varepsilon(c_* - 1 - \varepsilon) \leq \gamma(\cdot,t) \text{ a.e. on } \Omega, \end{aligned}$$

which implies $\mathcal{F}_\varepsilon(K_n^1(t)) \subset K^1(t)$. Similarly, $\mathcal{F}_\varepsilon(K^1(t)) \subset K_n^1(t)$. Hence $K_n^1(t) \Longrightarrow K^1(t)$ on $[0,T]$, and thus $\{K^1(t)\} \in \Phi_W$.

Example 2.2. Under the same situation as in Example 2.1, let us consider the gradient constraint case:

$$K^2(t) := \{\boldsymbol{z} \in \boldsymbol{V}_\sigma \mid |\nabla\boldsymbol{z}| \leq \gamma(\cdot,t) \text{ a.e. on } \Omega\}, \quad 0 \leq t \leq T.$$

and

$$K_n^2(t) := \{\boldsymbol{z} \in \boldsymbol{V}_\sigma \mid |\nabla\boldsymbol{z}| \leq \gamma_n(\cdot,t) \text{ a.e. on } \Omega\}, \quad 0 \leq t \leq T.$$

Then we see just in the same way as Example 2.1 that $\{K_n^2(t)\} \in \Phi_S$ and $K_n^2(t) \Longrightarrow K^2(t)$ on $[0,T]$ (as $n \to \infty$). Hence $\{K^2(t)\} \in \Phi_W$.

Next, we introduce the time-derivative under constraint $\{K(t)\} \in \Phi_W$. Put

$$\mathcal{K} := \{v \in L^p(0,T;V) \mid v(t) \in K(t) \text{ for a.e. } t \in [0,T]\}$$

and

$$\mathcal{K}_0 := \{\eta \in \mathcal{K} \mid \eta' \in L^{p'}(0,T;V^*)\}.$$

Definition 2.5. Let $\{K(t)\} \in \Phi_W$ and $u_0 \in \overline{K(0)}$. Then we define an operator L_{u_0} whose graph is given as follows: $f \in L_{u_0}u$ if and only if

$$u \in \mathcal{K},\ f \in L^{p'}(0,T;V^*), \quad \int_0^T \langle \eta' - f, u - \eta\rangle dt \leq \frac{1}{2}|u_0 - \eta(0)|_H^2, \quad \forall \eta \in \mathcal{K}_0.$$

In the next theorems we mention some of important properties of L_{u_0}.

Theorem 2.2. *Let* $\{K(t)\} \in \Phi_W$ *and* $u_0 \in \overline{K(0)}$. *Then,* L_{u_0} *is maximal monotone from* $D(L_{u_0}) \subset L^p(0,T;V)$ *into* $L^{p'}(0,T;V^*)$, *and the domain* $D(L_{u_0})$ *is included in the set* $\{u \in C([0,T];H) \cap \mathcal{K} \mid u(0) = u_0\}$.

The characterization and fundamental properties of the mapping L_{u_0} are given in the following theorem.

Theorem 2.3. *Let* $\{K(t)\} \in \Phi_W$. *Then we have:*

(1) Let $u_0 \in \overline{K(0)}$. *Then* $f \in L_{u_0}u$ *if and only if there are* $\{\{K_n(t)\}\} \subset \Phi_S$, $\{u_n\} \subset L^p(0,T;V)$ *with* $u_n \in \mathcal{K}_n := \{v \in L^p(0,T;V) \mid v(t) \in K_n(t)$ *for a.e* $t \in [0,T]\}$ *and* $u_n' \in L^{p'}(0,T;V^*)$, $\{f_n\} \subset L^{p'}(0,T;V^*)$ *such that*

$$K_n(t) \Longrightarrow K(t) \text{ on } [0,T],$$

$$u_n \to u \text{ in } C([0,T];H) \textit{ and weakly in } L^p(0,T;V),$$

$$f_n \to f \textit{ weakly in } L^{p'}(0,T;V^*),$$

$$\int_0^T \langle u_n' - f_n, u_n - v\rangle dt \le 0, \ \forall v \in \mathcal{K}_n, \ \forall n,$$

$$\limsup_{n\to\infty} \int_{t_1}^{t_2} \langle f_n, u_n\rangle dt \le \int_{t_1}^{t_2} \langle f, u\rangle dt, \ \forall t_1, \ t_2 \textit{ with } 0 \le t_1 \le t_2 \le T.$$

(2) Let $u_0 \in \overline{K(0)}$ *and* $f \in L_{u_0}u$. *Then, for any* $t_1, \ t_2 \in [0,T]$ *with* $t_1 \le t_2$,

$$\int_{t_1}^{t_2} \langle \eta' - f, u - \eta\rangle dt + \frac{1}{2}|u(t_2) - \eta(t_2)|_H^2 \le \frac{1}{2}|u(t_1) - \eta(t_2)|_H^2, \quad \forall \eta \in \mathcal{K}_0.$$

(3) Let $u_{i0} \in \overline{K(0)}$, *and* $f_i \in L_{u_{i0}}u_i$ *for* $i = 1, 2$. *Then, for any* $t_1, \ t_2 \in [0,T]$ *with* $t_1 \le t_2$,

$$\frac{1}{2}|u_1(t_2) - u_2(t_2)|_H^2 \le \frac{1}{2}|u_1(t_1) - u_2(t_2)|_H^2 + \int_{t_1}^{t_2} \langle f_1 - f_2, u_1 - u_2\rangle dt. \tag{2.7}$$

We refer to [22; Theorems 5.1 and 5.2] for the precise proof of the above theorems.

Remark 2.3. If $0 \in K(t)$ for all $t \in [0,T]$, then the relation $f \in L_{u_0}u$ gives

$$\frac{1}{2}|u(s)|_H^2 \le \frac{1}{2}|u_0|_H^2 + \int_0^s \langle f, u\rangle dt, \quad \forall s \in [0,T];$$

in fact, noting $0 \in L_0 0$ and using (2.7), we get the above inequality.

Once the maximal monotonicity of L_{u_0} is proved, it is quite useful for the weak solvability of parabolic variational inequalities with time-dependent constraint $K(t)$. In fact, for any coercive maximal monotone or pseudomonotne operator $A : D(A) = L^p(0,T;V) \to L^{p'}(0,T;V^*)$ and $f \in L^{p'}(0,T;V^*)$ we see by the general theory on nonlinear operators of monotone type (cf. [25; section 5]) the functional inclusion

$$L_{u_0}u + Au \ni f \quad \text{in } L^{p'}(0,T;V^*) \tag{2.8}$$

admits a solution u.

3. Variational inequalities of Navier-Stokes type

3.1. Weak formulation of variational inequalities of Navier-Stokes type

We are given a nonnegative function $\gamma = \gamma(x,t)$ on $\overline{Q}$ as an obstacle function such that $0 \le \gamma(x,t) \le \infty$ for all $(x,t) \in \overline{Q}$. For simplicity we use the following notation: for any constant $c \in [0,\infty]$,

$$\overline{Q}(\gamma = c) := \{(x,t) \in \overline{Q} \mid \gamma(x,t) = c\},$$

$$\overline{Q}(\gamma \ge c) := \{(x,t) \in \overline{Q} \mid \gamma(x,t) \ge c\},\ \overline{Q}(\gamma \le c) := \{(x,t) \in \overline{Q} \mid \gamma(x,t) \le c\},$$

and similarly $\overline{Q}(\gamma > c)$ and $\overline{Q}(\gamma < c)$ are defined. Besides, for the set $\hat{Q} := \Omega \times [0,T]$, $\hat{Q}(\gamma = c)$, $\hat{Q}(\gamma \ge c)$, etc. are similarly defined, too.

We assume that γ is continuous from $\overline{Q}$ into $[0,\infty]$, namely,

$$\begin{cases} \text{the set } \overline{Q}(\gamma = \infty) \text{ is closed in } \overline{Q}, \\ \forall \kappa \in (0,\infty),\ \gamma \text{ is continuous on } \overline{Q}(\gamma \le \kappa), \\ \forall M \in (0,\infty), \text{there is an open set } U_M\ (\subset \mathbf{R}^4) \text{ containing } \overline{Q}(\gamma = \infty) \\ \quad \text{such that } \gamma \ge M \text{ on } U_M \cap \overline{Q}. \end{cases} \tag{3.1}$$

It is easily seen that (3.1) is equivalent to the continuity on $\overline{Q}$ in the usual sense, of the function

$$\alpha(x,t) := \begin{cases} \dfrac{\gamma(x,t)}{1+\gamma(x,t)}, & \text{if } 0 \le \gamma(x,t) < \infty, \\ 1. & \text{if } \gamma(x,t) = \infty, \end{cases}$$

We are now ready to formulate our problem with the constraint sets $K(t)$ given by

$$K^1(\gamma;t) := \{\boldsymbol{z} \in \boldsymbol{V}_\sigma \mid |\boldsymbol{z}| \le \gamma(\cdot,t) \text{ a.e. on } \Omega\},\quad 0 \le t \le T,$$

or

$$K^2(\gamma;t) := \{\boldsymbol{z} \in \boldsymbol{V}_\sigma \mid |\nabla \boldsymbol{z}| \le \gamma(\cdot,t) \text{ a.e. on } \Omega\},\quad 0 \le t \le T,$$

and the classes of test functions

$$\mathcal{K}^i(\gamma) := \{\boldsymbol{\xi} \in L^2(0,T;\boldsymbol{V}_\sigma) \mid \boldsymbol{\xi}(t) \in K^i(\gamma;t),\ \text{a.e. } t \in (0,T)\},\ i = 1,2,$$

$$\mathcal{K}^1_0(\gamma) := \left\{ \boldsymbol{\xi} \in C^1([0,T];\boldsymbol{W}_\sigma) \;\middle|\; \begin{array}{l} \boldsymbol{\xi} \in K^1(\gamma;t),\ \forall t \in [0,T], \\ \text{supp}(|\boldsymbol{\xi}|) \subset \hat{Q}(\gamma > 0), \end{array} \right\}$$

and

$$\mathcal{K}_0^2(\gamma) := \left\{ \boldsymbol{\xi} \in C^1([0,T]; \boldsymbol{W}_\sigma) \;\middle|\; \begin{array}{l} \boldsymbol{\xi} \in K^2(\gamma; t),\ \forall t \in [0,T], \\ \mathrm{supp}(|\nabla \boldsymbol{\xi}|) \subset \hat{Q}(\gamma > 0) \end{array} \right\}$$

and $\mathrm{supp}(|\boldsymbol{\xi}|)$ and $\mathrm{supp}(|\nabla\boldsymbol{\xi}|)$ denote the supports of $|\boldsymbol{\xi}|$ and $|\nabla\boldsymbol{\xi}|$, respectively.

Definition 3.1. For given data

$$\nu > 0 \text{ (constant)},\ \boldsymbol{g} \in L^2(0,T; \boldsymbol{H}_\sigma),\ \boldsymbol{v}_0 \in \boldsymbol{H}_\sigma,$$

our problem, referred to $NS^i(\gamma; \boldsymbol{g}, \boldsymbol{v}_0)$, is to find a function $\boldsymbol{v} := (v^{(1)}, v^{(2)}, v^{(3)})$ from $[0,T]$ into $\boldsymbol{H}_\sigma$ satisfying the following (i) and (ii).

(i) $\boldsymbol{v}(0) = \boldsymbol{v}_0$ in $\boldsymbol{H}_\sigma$, and $t \mapsto (\boldsymbol{v}(t), \boldsymbol{\xi}(t))_\sigma$ is of bounded variation on $[0,T]$ for all $\boldsymbol{\xi} \in \mathcal{K}_0^i(\gamma)$.

(ii) $\sup_{t\in[0,T]} |\boldsymbol{v}(t)|_{0,2} < \infty$, $\boldsymbol{v} \in \mathcal{K}^i(\gamma)$ and

$$\begin{aligned} &\int_0^t (\boldsymbol{\xi}'(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\xi}(\tau))_\sigma d\tau + \nu \int_0^t \langle \boldsymbol{F}\boldsymbol{v}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\xi}(\tau)\rangle_\sigma d\tau \\ &\quad + \int_0^t \int_\Omega (\boldsymbol{v}(x,\tau)\cdot\nabla)\boldsymbol{v}(x,\tau)\cdot(\boldsymbol{v}(x,\tau) - \boldsymbol{\xi}(x,\tau))dxd\tau + \frac{1}{2}|\boldsymbol{v}(t) - \boldsymbol{\xi}(t)|_{0,2}^2 \\ &\qquad \le \int_0^t (\boldsymbol{g}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\xi}(\tau))_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|_{0,2}^2,\ \forall t \in [0,T],\ \forall \boldsymbol{\xi} \in \mathcal{K}_0^i(\gamma). \end{aligned} \tag{3.2}$$

Such a function $\boldsymbol{v}$ is called a weak solution of $NS^i(\gamma; \boldsymbol{g}, \boldsymbol{v}_0),\ i = 1, 2$.

Remark 3.1. In the non-degenerate case of γ, namely $\gamma \ge c_*$ (> 0) on $\overline{Q}$, $\mathcal{K}_0^i(\gamma)$ is simply described as

$$\mathcal{K}_0^i(\gamma) := \{\boldsymbol{v} \in C^1([0,T]; \boldsymbol{W}_\sigma(\Omega)) \mid \boldsymbol{v} \in K^i(\gamma; t),\ \forall t \in [0,T]\}. \tag{3.3}$$

Hence, (i) of Definition 3.1 implies that $t \mapsto (\boldsymbol{v}(t), \boldsymbol{\xi}(t))_\sigma$ is of bounded variation on $[0,T]$ for all $\boldsymbol{\xi} \in C^1([0,T]; \boldsymbol{W}_\sigma)$. Similary, in the degenerate case of γ, it implies in the case $i = 1$ that $t \mapsto (\boldsymbol{v}(t), \boldsymbol{\xi}(t))_\sigma$ is of bounded variation on $[0,T]$ for all $\boldsymbol{\xi} \in C^1([0,T]; \boldsymbol{W}_\sigma)$ with $\mathrm{supp}(|\boldsymbol{\xi}|) \subset \hat{Q}(\gamma > 0)$.

We note in Definition 3.1 that $\boldsymbol{v}$ is defined for *every* $t \in [0,T]$ according to the given $\boldsymbol{v}_0$, even if we do not require it to be continuous in time: our definition permits jumps in time, including the initial time $t = 0$. What we will prove, is that $\boldsymbol{v}$ is a limit of continuous approximate solutions.

The first main result of this paper is stated as follows.

Theorem 3.1 (Non-degenerate case). *In addition to (3.1), assume that*

$$\gamma \ge c_* \text{ on } \overline{Q} \text{ for a positive constant } c_*. \tag{3.4}$$

Let $i = 1$ or 2. Let $\boldsymbol{g} \in L^2(0,T;\boldsymbol{H}_\sigma)$ and $\boldsymbol{v}_0 \in K^i(\gamma;0)$ and

$$\begin{cases} |\boldsymbol{v}_0| \in L^\infty(\Omega) & \text{in the case of } i = 1, \\ |\nabla \boldsymbol{v}_0| \in L^\infty(\Omega) & \text{in the case of } i = 2. \end{cases} \tag{3.5}$$

Then there exists at least one weak solution of $NS^i(\gamma;\boldsymbol{g},\boldsymbol{v}_0)$; namely

- *(i) $\boldsymbol{v}(0) = \boldsymbol{v}_0$ in $\boldsymbol{H}_\sigma$, and $t \mapsto (\boldsymbol{v}(t),\boldsymbol{\xi}(t))_\sigma$ is of bounded variation on $[0,T]$ for all $\boldsymbol{\xi} \in C^1([0,T];\boldsymbol{W}_\sigma)$,*
- *(ii) $\sup_{t\in[0,T]} |\boldsymbol{v}(t)|_{0,2} < \infty$, $\boldsymbol{v} \in \boldsymbol{\mathcal{K}}^i(\gamma)$ and (3.2) holds.*

As to the weak solvability of $NS^1(\gamma;\boldsymbol{g},\boldsymbol{v}_0)$ in the degenerate case we have:

Theorem 3.2 (Degenerate case). *Assume (3.1) holds, $\boldsymbol{g} \in L^2(0,T;\boldsymbol{H}_\sigma)$ and $\boldsymbol{v}_0 \in K^1(\gamma;0) \cap L^\infty(\Omega)^3$ with*

$$\mathrm{supp}(|\boldsymbol{v}_0|) \subset \{x \in \Omega \mid \gamma(x,0) > 0\}.$$

Moreover assume that for each $t \in [0,T]$ and all small $\kappa > 0$, $\gamma(x,t)$ is Lipschitz continuous on $\overline{\Omega}(\gamma(\cdot,t) \le \kappa) := \{x \in \overline{\Omega} \mid \gamma(x,t) \le \kappa\}$, namely

$$|\gamma(x,t) - \gamma(x',t)| \le L_\gamma(t,\kappa)|x - x'|, \quad \forall x,\ \forall x' \in \overline{\Omega}(\gamma(\cdot,t) \le \kappa), \tag{3.6}$$

where $L_\gamma(t,\kappa)$ is a positive constant depending on t,κ. Then $NS^1(\gamma;\boldsymbol{g},\boldsymbol{v}_0)$ has at least one weak solution $\boldsymbol{v}$ in the sense of Definition 3.1.

In the degenerate case of γ, we need the Helmholtz decomposition of solenoidal functions (cf. [19]) in the construction of weak solution of $NS^1(\gamma;\boldsymbol{g},\boldsymbol{v}_0)$; regarding problem $NS^2(\gamma;\boldsymbol{g},\boldsymbol{v}_0)$ the degenerate case is still open question except some special cases of γ.

Remark 3.2. The degenerate case of γ, namely $\gamma = 0$ somewhere in Q, problem $NS^1(\gamma;\boldsymbol{g},\boldsymbol{u}_0)$ was earlier discussed without condition (3.6) in the statement of the main result ([20; Theorem 1.1]). However, after the publication of this result, unfortunately a gap was found in the proof by the authors. In this paper, we shall make the precise correction in section 4 of this paper.

3.2. Approximation of $NS^i(\gamma;\boldsymbol{g},\boldsymbol{v}_0)$, $i = 1,2$

Our main theorems will be proved in two steps of

- Approximation of $NS^i(\gamma;\boldsymbol{g},\boldsymbol{v}_0)$,
- Convergence of approximate solutions.

We begin with the approximation of γ given by

$$\gamma_{\delta,N}(r) := (\gamma(r) \vee \delta) \wedge N,\ r \in \mathbf{R},\ 0 < \delta < 1,\ N > 0,$$

where $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$ for any real numbers a, b. Clearly, $\gamma_{\delta,N}$ is everywhere bounded, strictly positive and continuous on $\overline{Q}$, and $\gamma_{\delta,N} \to \gamma$ as $\delta \downarrow 0$ and $N \uparrow \infty$ in the sense that

$$\begin{cases} \forall M > 0,\ \exists \delta_M > 0,\ \exists N_M,\ \exists M'_M > 0 \text{ such that} \\ \qquad\qquad \overline{Q}(\gamma > M) \subset \overline{Q}(\gamma_{\delta,N} > M'_M),\ \forall \delta \in (0, \delta_M),\ \forall N > N_M, \\ \forall k > 0,\ \gamma_{\delta,N} \to \gamma \text{ uniformly on } \overline{Q}(\gamma \le k) \text{ as } \delta \to 0 \text{ and } N \to \infty. \end{cases} \tag{3.7}$$

The approximate problem $NS^i(\gamma_{\delta,N}; \boldsymbol{g}, \boldsymbol{v}_0)$ to $NS^i(\gamma; \boldsymbol{g}, \boldsymbol{v}_0)$ is formulated as follows: Find a function $\boldsymbol{v}$ such that

$$\boldsymbol{v} \in C([0,T]; \boldsymbol{H}_\sigma) \cap \boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}), \quad \boldsymbol{v}(0) = \boldsymbol{v}_0,$$

and

$$\begin{aligned} &\int_0^t \langle \boldsymbol{\xi}', \boldsymbol{v} - \boldsymbol{\xi} \rangle_\sigma d\tau + \nu \int_0^t \langle \boldsymbol{F}\boldsymbol{v}, \boldsymbol{v} - \boldsymbol{\xi} \rangle_\sigma d\tau \\ &\qquad + \int_0^t \int_\Omega (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \cdot (\boldsymbol{v} - \boldsymbol{\xi}) dx d\tau + \frac{1}{2} |\boldsymbol{v}(t) - \boldsymbol{\xi}(t)|_{0,2}^2 \\ &\le \int_0^t \langle \boldsymbol{g}, \boldsymbol{v} - \boldsymbol{\xi} \rangle_\sigma d\tau + \frac{1}{2} |\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|_{0,2}^2, \quad \forall \boldsymbol{\xi} \in \boldsymbol{\mathcal{K}}_0^i(\gamma_{\delta,N}), \forall t \in [0,T], \end{aligned} \tag{3.8}$$

where

$$\boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}) := \{\boldsymbol{\xi} \in L^2(0,T; \boldsymbol{V}_\sigma) \mid \boldsymbol{\xi}(t) \in K^i(\gamma_{\delta,N}; t),\ \text{a.e. } t \in [0,T]\},$$
$$\boldsymbol{\mathcal{K}}_0^i(\gamma_{\delta,N}) := \{\boldsymbol{\xi} \in C^1([0,T]; \boldsymbol{W}_\sigma) \mid \boldsymbol{\xi}(t) \in K^i(\gamma_{\delta,N}; t),\ \forall t \in [0,T]\}$$

with

$$K^1(\gamma_{\delta,N}; t) := \{\boldsymbol{z} \in \boldsymbol{V}_\sigma \mid |\boldsymbol{z}| \le \gamma_{\delta,N}(\cdot, t) \text{ a.e. on } \Omega\},\ \forall t \in [0,T],$$
$$K^2(\gamma_{\delta,N}; t) := \{\boldsymbol{z} \in \boldsymbol{V}_\sigma \mid |\nabla \boldsymbol{z}| \le \gamma_{\delta,N}(\cdot, t) \text{ a.e. on } \Omega\},\ \forall t \in [0,T].$$

As was seen in Examples 2.1 and 2.2 these classes $\{K^i(\gamma_{\delta,N}; t)\}$, $i = 1, 2$, belong to the weak class Φ_W, with $p = 2$, of time-dependent convex sets in $\boldsymbol{V}_\sigma$ and the time derivative $\boldsymbol{L}_{\boldsymbol{v}_0}^i(\delta, N)$ associated with constraints $\{K^i(\gamma_{\delta,N}; t)\}$ is well defined by Theorem 2.2. By definition, $\boldsymbol{f} \in \boldsymbol{L}_{\boldsymbol{v}_0}^i(\delta, N)\boldsymbol{v}$ if and only if $\boldsymbol{f} \in L^2(0,T; \boldsymbol{V}_\sigma^*)$, $\boldsymbol{v} \in \boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N})$ and

$$\int_0^T \langle \boldsymbol{\xi}' - \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{\xi} \rangle_\sigma dt \le \frac{1}{2} |\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|_{0,2}^2, \quad \forall \boldsymbol{\xi} \in \boldsymbol{\mathcal{K}}_0^i(\gamma_{\delta,N}). \tag{3.9}$$

Since $\gamma_{\delta,N} \le N$ on $\mathbf{R}$, there is a positive constant C_N, depending only on N, such that

$$K^i(\gamma_{\delta,N}; t) \subset K^* := \{\boldsymbol{z} \in \boldsymbol{V}_\sigma \mid |\boldsymbol{z}| \le C_N \text{ a.e. on } \Omega\},\ \forall t \in [0,T],\ i = 1,\ 2.$$

With this set K^*, we introduce a mapping $\boldsymbol{G}(\cdot,\cdot) : K^* \times \boldsymbol{V}_\sigma \to \boldsymbol{V}_\sigma^*$ defined by

$$\langle \boldsymbol{G}(\boldsymbol{w}, \boldsymbol{v}), \boldsymbol{z} \rangle_\sigma := \int_\Omega (\boldsymbol{w} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{z} dx,\ \forall \boldsymbol{w} \in K^*,\ \forall \boldsymbol{v},\ \boldsymbol{z} \in \boldsymbol{V}_\sigma.$$

For any fixed $\boldsymbol{w} \in K^*$, the mapping $\boldsymbol{z} \mapsto \boldsymbol{G}(\boldsymbol{w}, \boldsymbol{z})$ is bounded, linear and monotone from $\boldsymbol{V}_\sigma$ into $\boldsymbol{V}_\sigma^*$, because

$$\langle \boldsymbol{G}(\boldsymbol{w}, \boldsymbol{z}), \boldsymbol{z} \rangle_\sigma = 0, \quad \forall \boldsymbol{z} \in \boldsymbol{V}_\sigma, \tag{3.10}$$

by the divergencefreeness of $\boldsymbol{w}$.

Proposition 3.1. *Let* $\boldsymbol{g} \in L^2(0,T;\boldsymbol{H}_\sigma)$, $\boldsymbol{v}_0 \in K^i(\gamma,0)$ *and assume (3.5) holds for the initial datum* $\boldsymbol{v}_0$. *Then, for any small positive* $\delta > 0$ *and large* $N > 0$ *there is one and only one solution* $\boldsymbol{v}$ *of* $NS^i(\gamma_{\delta,N};\boldsymbol{g},\boldsymbol{v}_0)$ *and it is given by the solution of*

$$\boldsymbol{g} \in \boldsymbol{L}^i_{\boldsymbol{v}_0}(\delta,N)\boldsymbol{v} + \nu \boldsymbol{F}\boldsymbol{v} + \boldsymbol{G}(\boldsymbol{v},\boldsymbol{v}) \quad \text{in } L^2(0,T;\boldsymbol{V}^*_\sigma).$$

Prior to the proof of Proposition 3.1 we prepare two lemmas.

Since $\boldsymbol{W}_\sigma$ is compactly embedded in $C(\overline{\Omega})^3$ for $i=1$ and in $C^1(\overline{\Omega})^3$ for $i=2$, there is a positive constant $\kappa_{\delta,N}$, depending on δ and N, such that

$$\kappa_{\delta,N} B_{\boldsymbol{W}_\sigma}(0) \subset K^i(\gamma_{\delta,N};t), \ \forall t \in [0,T], \tag{3.11}$$

where $B_{\boldsymbol{W}_\sigma}(0)$ is the closed unit ball around the origin in $\boldsymbol{W}_\sigma$.

The first lemma follows immediately from the definition of $\boldsymbol{G} : K^* \times \boldsymbol{V}_\sigma \to \boldsymbol{V}^*_\sigma$.

Lemma 3.1. *We have that*

$$|\langle \boldsymbol{G}(\boldsymbol{w},\boldsymbol{v}),\boldsymbol{z}\rangle_\sigma| \le C_N |\boldsymbol{v}|_{1,2}|\boldsymbol{z}|_{0,2}, \quad \forall \boldsymbol{w} \in K^*, \ \forall \boldsymbol{v}, \ \boldsymbol{z} \in \boldsymbol{V}_\sigma.$$

By condition (3.5), $\boldsymbol{v}_0 \in K^i(\gamma_{\delta,N};0)$ for all $\delta > 0$ and large N. Let us consider a functional inclusion of the form:

$$\boldsymbol{L}^i_{\boldsymbol{v}_0}(\delta,N)\boldsymbol{v} + \nu \boldsymbol{F}\boldsymbol{v} + \boldsymbol{G}(\boldsymbol{w},\boldsymbol{v}) \ni \boldsymbol{g} \quad \text{in } L^2(0,T;\boldsymbol{V}_\sigma). \tag{3.12}$$

We observe easily that the mapping $\boldsymbol{v} \mapsto \nu \boldsymbol{F}\boldsymbol{v} + \boldsymbol{G}(\boldsymbol{w},\boldsymbol{v})$ is maximal monotone, strictly monotone and coercive from $L^2(0,T;\boldsymbol{V}_\sigma)$ into $L^2(0,T;\boldsymbol{V}^*_\sigma)$. Hence, from the general theory on monotone operators in Banach spaces (cf. [25]) it follows that the range of the sum $\boldsymbol{L}^i_{\boldsymbol{v}_0}(\delta,N) + \nu\boldsymbol{F} + \boldsymbol{G}(\boldsymbol{w},\cdot)$ is the whole of $L^2(0,T;\boldsymbol{V}^*_\sigma)$, namely there is one and only one $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V}_\sigma)$ which satisfies (3.12); we denote by $\boldsymbol{S}^i\boldsymbol{w}$ the solution $\boldsymbol{v}$ and obtain from the energy estimate in Remark 2.3 that

$$\frac{1}{2}|\boldsymbol{v}(t)|^2_{0,2} + \nu \int_0^t |\boldsymbol{v}(\tau)|^2_{\boldsymbol{V}_\sigma} d\tau \le \frac{1}{2}|\boldsymbol{v}_0|^2_{0,2} + \int_0^t (\boldsymbol{g}(\tau),\boldsymbol{v}(\tau))_\sigma dt, \quad \forall t \in [0,T].$$

Hence

$$\sup_{t\in[0,T]} |\boldsymbol{v}(t)|^2_{0,2} + \nu|\boldsymbol{v}|^2_{L^2(0,T;\boldsymbol{V}_\sigma)} \le |\boldsymbol{v}_0|^2_{0,2} + \frac{C_0^2}{\nu}|\boldsymbol{g}|^2_{L^2(0,T;\boldsymbol{H}_\sigma)} =: M_0(\boldsymbol{u}_0,\boldsymbol{g}) \tag{3.13}$$

for a positive constant C_0 satisfying $|\boldsymbol{z}|_{0,2} \le C_0|\boldsymbol{z}|_{1,2}$ for all $\boldsymbol{z} \in \boldsymbol{V}_\sigma$.

On account of this result, with

$$\mathcal{K}^* := \left\{ \boldsymbol{w} \in L^2(0,T;\boldsymbol{V}_\sigma) \;\middle|\; \sup_{t\in[0,T]} |\boldsymbol{w}(t)|^2_{0,2} + \nu|\boldsymbol{w}|^2_{L^2(0,T;\boldsymbol{V}_\sigma)} \le M_0(\boldsymbol{u}_0,\boldsymbol{g}) \right\},$$

we can define an operator $\boldsymbol{\mathcal{S}}^i : \boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}) \cap \boldsymbol{\mathcal{K}}^* \to \boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}) \cap \boldsymbol{\mathcal{K}}^*$ by putting

$$\mathcal{S}^i \boldsymbol{w} = \boldsymbol{v}.$$

Lemma 3.2. *For $i = 1,\ 2$, the operator $\mathcal{S}^i$ is compact in $\boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}) \cap \boldsymbol{\mathcal{K}}^*$ with respect to the topology of $L^2(0,T;\boldsymbol{H}_\sigma)$.*

Proof. Let $\boldsymbol{w}_n$ be any sequence in $\boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}) \cap \boldsymbol{\mathcal{K}}^*$ such that $\boldsymbol{w}_n \to \boldsymbol{w}$ weakly in $L^2(0,T;\boldsymbol{H}_\sigma)$ (as $n \to \infty$). Clearly, $\boldsymbol{w} \in \boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}) \cap \boldsymbol{\mathcal{K}}^*$, since it is closed and convex in $L^2(0,T;\boldsymbol{H}_\sigma)$. Also, putting $\boldsymbol{v}_n := \mathcal{S}^i \boldsymbol{w}_n$, we observe from (3.13) that $\boldsymbol{f}_n := \boldsymbol{g} - \nu \boldsymbol{F}\boldsymbol{v}_n - \boldsymbol{G}^i(\boldsymbol{w}_n, \boldsymbol{v}_n) \in \boldsymbol{L}^i_{\boldsymbol{v}_0}(\delta, N)\boldsymbol{v}_n$ and $\{\boldsymbol{f}_n\}$ is bounded in $L^2(0,T;\boldsymbol{V}^*_\sigma)$. Noting (3.11) and applying Theorem 2.1 for the triplet

$$\boldsymbol{V}_\sigma \subset \boldsymbol{H}_\sigma \subset \boldsymbol{V}^*_\sigma,$$

we see that $\{\boldsymbol{v}_n\}$ is relatively compact in $L^2(0,T;\boldsymbol{H}_\sigma)$ and hence in $L^2(Q)$.

Now, we extract a subsequence $\{\boldsymbol{v}_{n_k}\}$ so that $\boldsymbol{v}_{n_k} \to \boldsymbol{v}$ in $L^2(0,T;\boldsymbol{H}_\sigma)$ as $k \to \infty$ for some $\boldsymbol{v}$; note that $\boldsymbol{v} \in \boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}) \cap \boldsymbol{\mathcal{K}}^*$. Also, from (3) of Theorem 2.3 it follows that for all $k,\ j$

$$\begin{aligned} &\frac{1}{2}|\boldsymbol{v}_{n_k}(t) - \boldsymbol{v}_{n_j}(t)|^2_{0,2} + \nu \int_0^t |\boldsymbol{v}_{n_k} - \boldsymbol{v}_{n_j}|^2_{1,2} d\tau \\ &\quad \le - \int_0^t \langle \boldsymbol{G}(\boldsymbol{w}_{n_k}, \boldsymbol{v}_{n_k}) - \boldsymbol{G}(\boldsymbol{w}_{n_j}, \boldsymbol{v}_{n_j}), \boldsymbol{v}_{n_k} - \boldsymbol{v}_{n_j} \rangle_\sigma d\tau =: I_{k,j}(t). \end{aligned}$$

Here we estimate $I_{k,j}(t)$ by Lemma 3.1 and (3.10) as follows:

$$\begin{aligned} I_{k,j}(t) &\le -\int_0^t \langle \boldsymbol{G}(\boldsymbol{w}_{n_k}, \boldsymbol{v}_{n_k} - \boldsymbol{v}_{n_j}), \boldsymbol{v}_{n_k} - \boldsymbol{v}_{n_j} \rangle_\sigma d\tau \\ &\qquad - \int_0^t \langle \boldsymbol{G}(\boldsymbol{w}_{n_k} - \boldsymbol{w}_{n_j}, \boldsymbol{v}_{n_j}), \boldsymbol{v}_{n_k} - \boldsymbol{v}_{n_j} \rangle_\sigma d\tau \\ &= -\int_0^t \langle \boldsymbol{G}(\boldsymbol{w}_{n_k} - \boldsymbol{w}_{n_j}, \boldsymbol{v}_{n_j}), \boldsymbol{v}_{n_k} - \boldsymbol{v}_{n_j} \rangle_\sigma d\tau \\ &\le 2C_N \int_0^T |\boldsymbol{v}_{n_j}|_{1,2} |\boldsymbol{v}_{n_k} - \boldsymbol{v}_{n_j}|_{0,2} d\tau \to 0 \text{ (as } k,\ j \to \infty). \end{aligned}$$

As a consequence, $\boldsymbol{v}_{n_k} \to \boldsymbol{v}$ in $C([0,T];\boldsymbol{H}_\sigma) \cap L^2(0,T;\boldsymbol{V}_\sigma)$ and $\boldsymbol{G}(\boldsymbol{w}_{n_k}, \boldsymbol{v}_{n_k}) \to \boldsymbol{G}(\boldsymbol{w}, \boldsymbol{v})$ weakly in $L^2(0,T;\boldsymbol{V}^*_\sigma)$ as $k \to \infty$, whence $\boldsymbol{f}_{n_k} \to \boldsymbol{f} := \boldsymbol{g} - \nu \boldsymbol{F}\boldsymbol{v} - \boldsymbol{G}(\boldsymbol{w}, \boldsymbol{v})$ weakly in $L^2(0,T;\boldsymbol{V}^*_\sigma)$ as $k \to \infty$. Hence, by the demiclosedness of maximal monotone operators, $\boldsymbol{f} \in \boldsymbol{L}^i_{\boldsymbol{v}_0}(\delta, N)\boldsymbol{v}$, namely

$$\boldsymbol{L}^i_{\boldsymbol{v}_0}(\delta, N)\boldsymbol{v} + \nu \boldsymbol{F}\boldsymbol{v} + \boldsymbol{G}(\boldsymbol{w}, \boldsymbol{v}) \ni \boldsymbol{g},$$

which shows that $\mathcal{S}^i \boldsymbol{w} = \boldsymbol{v}$. By the uniqueness of solution of this functional inclusion, it is concluded that $\mathcal{S}^i \boldsymbol{w}_n = \boldsymbol{v}_n \to \boldsymbol{v} = \mathcal{S}^i \boldsymbol{w}$ in $L^2(0,T;\boldsymbol{V}_\sigma)$ without extracting any subsequence from $\{\boldsymbol{w}_n\}$. Thus $\mathcal{S}^i$ is compact in $\boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}) \cap \boldsymbol{\mathcal{K}}^*$ with respect to the topology of $L^2(0,T;\boldsymbol{H}_\sigma)$. ♢

Proof of Proposition 3.1: By virtue of Lemma 3.2, the operator $\mathcal{S}^i$ admits at least one fixed point, $\boldsymbol{v} = \mathcal{S}^i \boldsymbol{v}$ in $\boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N})$. Besides, by the definition (3.9) of $\boldsymbol{L}^i_{\boldsymbol{v}_0}(\delta, N)$, this fixed point $\boldsymbol{v}$ satisfies (3.8), namely $\boldsymbol{v}$ is a solution of $NS^i(\gamma_{\delta,N}; \boldsymbol{g}, \boldsymbol{v}_0)$. Finally we prove the uniqueness of solution. Let $\boldsymbol{v}$ and $\bar{\boldsymbol{v}}$ be two solutions $NS^i(\gamma_{\delta,N};\boldsymbol{g},\boldsymbol{v}_0)$. Then, by (3) of Theorem 2.3, we have

$$\frac{1}{2}|\boldsymbol{v}(t)-\bar{\boldsymbol{v}}(t)|^2_{0,2}+\nu\int_0^t|\boldsymbol{v}-\bar{\boldsymbol{v}}|^2_{1,2}d\tau\le-\int_0^t\langle \boldsymbol{G}(\boldsymbol{v},\boldsymbol{v})-\boldsymbol{G}(\bar{\boldsymbol{v}},\bar{\boldsymbol{v}}),\boldsymbol{v}-\bar{\boldsymbol{v}}\rangle_\sigma d\tau=:I(t). \quad (3.14)$$

Just as in the proof of Lemma 3.2, the right hand side of (3.14) is dominated by

$$\begin{aligned} I(t) &\le -\int_0^t\langle \boldsymbol{G}(\boldsymbol{v}-\bar{\boldsymbol{v}},\boldsymbol{v}),\boldsymbol{v}-\bar{\boldsymbol{v}}\rangle_\sigma d\tau=\int_0^t\langle \boldsymbol{G}(\boldsymbol{v}-\bar{\boldsymbol{v}},\boldsymbol{v}-\bar{\boldsymbol{v}}),\boldsymbol{v}\rangle_\sigma d\tau \\ &\le 2C_N\int_0^T|\boldsymbol{v}-\bar{\boldsymbol{v}}|_{1,2}|\boldsymbol{v}-\bar{\boldsymbol{v}}|_{0,2}d\tau \\ &\le 2C_N\left\{\varepsilon\int_0^t|\boldsymbol{v}-\bar{\boldsymbol{v}}|^2_{1,2}d\tau+c_\varepsilon\int_0^t|\boldsymbol{u}-\bar{\boldsymbol{u}}|^2_{0,2}\tau\right\}, \end{aligned}$$

where ε is any small positive number and c_ε is a positive number depending only on ε. Now, taking $\varepsilon>0$ so as to satisfy $2C_N\varepsilon<\frac{\nu}{2}$, we get from (3.14) that

$$|\boldsymbol{v}(t)-\bar{\boldsymbol{v}}(t)|^2_{0,2}+\nu\int_0^t|\boldsymbol{v}-\bar{\boldsymbol{v}}|^2_{1,2}d\tau\le C'\int_0^t|\boldsymbol{v}-\bar{\boldsymbol{v}}|^2_{0,2}d\tau,\ \forall t\in[0,T].$$

where C' is a positive constant. According to the Gronwall inequality, we see that $\boldsymbol{v}=\bar{\boldsymbol{v}}$ on $[0,T]$. Thus the solution of $NS^i(\gamma_{\delta,N};\boldsymbol{g},\boldsymbol{u}_0)$ is unique and the proof of Proposition 3.1 is now complete. ♢

Remark 3.3. In the case when γ is everywhere bounded and strictly positive on $\overline{Q}$, we have $\gamma=\gamma_{\delta,N}$ for all small $\delta>0$ and large $N>0$, so that Proposition 3.1 gives that $NS^i(\gamma;\boldsymbol{g},\boldsymbol{v}_0)$ has one and only one weak solution $\boldsymbol{v}$ in $C([0,T];\boldsymbol{H}_\sigma)\cap\boldsymbol{\mathcal{K}}^i(\gamma)$.

3.3. Proof of Theorem 3.1 (Non-degenerate case)

In this subsection, we accomplish the proof of Theorem 3.1 by proving the convergence of approximate solutions constructed by Proposition 3.1.

By assumption (3.4), we see that

$$\gamma_{\delta,N}(x,t)=\gamma(x,t)\wedge N=:\gamma_N(x,t),\quad \forall\delta \text{ with } 0<\delta\le c_*,\ \forall \text{large } N>0.$$

Let $\boldsymbol{v}_N$ be the approximate solution of $NS^i(\gamma_N;\boldsymbol{g},\boldsymbol{v}_0)$, $i=1,\ 2$, for large $N>0$. Then, by (3.13) there exists sequences $\{N_n\}$ tending to ∞ (as $n\to\infty$) such that

$$\boldsymbol{v}_n:=\boldsymbol{v}_{N_n}\to\boldsymbol{v} \text{ weakly in } L^2(0,T;\boldsymbol{V}_\sigma) \text{ and weakly}^* \text{ in } L^\infty(0,T;\boldsymbol{H}_\sigma) \quad (3.15)$$

for a certain function $\boldsymbol{v}\in L^2(0,T;\boldsymbol{V}_\sigma)\cap L^\infty(0,T;\boldsymbol{H}_\sigma)$, satisfying

$$\boldsymbol{v}\in\boldsymbol{\mathcal{K}}^i(\gamma). \quad (3.16)$$

We note that $\boldsymbol{v}_n$ is the solution of

$$\boldsymbol{f}_n:=\boldsymbol{g}-\nu\boldsymbol{F}\boldsymbol{v}_n-\boldsymbol{G}(\boldsymbol{v}_n,\boldsymbol{v}_n)\in\boldsymbol{L}^i_{\boldsymbol{v}_0}(c_*,N_n)\boldsymbol{v}_n,$$

which is equivalent to the variational inequality (cf. (3.8)):

$$\boldsymbol{v}_n \in C([0,T];\boldsymbol{H}_\sigma) \cap \boldsymbol{\mathcal{K}}^i(\gamma_{N_n}), \quad \boldsymbol{v}_n(0) = \boldsymbol{v}_0,$$
$$\int_0^t \langle \boldsymbol{\xi}' - \boldsymbol{f}_n, \boldsymbol{v}_n - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}_n(t) - \boldsymbol{\xi}(t)|_{0,2}^2 \le \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|_{0,2}^2 \tag{3.17}$$
$$\forall \boldsymbol{\xi} \in \boldsymbol{\mathcal{K}}_0^i(\gamma_{N_n}), \forall t \in [0,T].$$

Here we note that for any $\boldsymbol{\xi} \in C^1([0,T];\boldsymbol{W}_\sigma) \subset C(\overline{Q})^3$

$$\begin{aligned}\int_0^T |\langle \boldsymbol{G}(\boldsymbol{v}_n,\boldsymbol{v}_n),\boldsymbol{\xi}\rangle| dt &\le C\int_0^T |\boldsymbol{v}_n|_{0,2}|\boldsymbol{v}_n|_{1,2}|\boldsymbol{\xi}|_{\boldsymbol{W}_\sigma} dt \\ &\le CM_0(\boldsymbol{g},\boldsymbol{v}_0)|\boldsymbol{\xi}|_{L^\infty(0,T;\boldsymbol{W}_\sigma)}\end{aligned}$$

for some positive constant C independent of n. This shows that $\{\boldsymbol{G}(\boldsymbol{v}_n,\boldsymbol{v}_n)\}$ is bounded in $L^1(0,T;\boldsymbol{W}_\sigma^*)$, so is $\{\boldsymbol{f}_n\}$ in $L^1(0,T;\boldsymbol{W}_\sigma^*)$. Also, by non-degenerate condition $c_* > 0$, we can choose a positive number κ so that

$$\kappa B_{\boldsymbol{W}_\sigma}(0) \subset K^i(\gamma_{N_n};t), \ \forall t \in [0,T], \quad \forall n, \ i = 1,2.$$

Hence, for a sufficient large constant $M_0' > 0$ it follows that

$$\boldsymbol{v}_n \in Z_2(\kappa, M_0', \boldsymbol{v}_0), \forall n,$$

so that $\{\mathrm{Var}_{W_\sigma^*}(\boldsymbol{v}_n)\}$ is bounded by Lemma 2.2. Moreover, by virtue of Theorem 2.1 and Lemma 2.1, $\{\boldsymbol{v}_n\}$ is relatively compact in $L^2(0,T;\boldsymbol{H}_\sigma)$ and there is a subsequence $\{\boldsymbol{v}_{n_k}\}$ such that $\boldsymbol{v}_{n_k} \to \boldsymbol{v}$ in $L^2(0,T;\boldsymbol{H}_\sigma)$ as $k \to \infty$ and $\boldsymbol{v}_{n_k}(t) \to \boldsymbol{v}(t)$ weakly in $\boldsymbol{H}_\sigma$ for every $t \in [0,T]$; we may assume that the limit function $\boldsymbol{v}$ is the same one as in (3.15). For simplicity we write this subsequence by $\{\boldsymbol{v}_n\}$, again; we have together with (3.15) and (3.16)

$$\begin{aligned}&\boldsymbol{v}_n \to \boldsymbol{v} \text{ in } L^2(0,T;\boldsymbol{H}_\sigma) \text{ and weakly in } L^2(0,T;\boldsymbol{V}_\sigma),\\ &\boldsymbol{v}_n(t) \to \boldsymbol{v}(t) \text{ weakly in } \boldsymbol{H}_\sigma, \quad \forall t \in [0,T].\end{aligned} \tag{3.18}$$

This implies that

$$\int_0^t \langle \boldsymbol{G}(\boldsymbol{v}_n,\boldsymbol{v}_n),\boldsymbol{\xi}\rangle_\sigma d\tau \to \int_0^t \langle \boldsymbol{G}(\boldsymbol{v},\boldsymbol{v}),\boldsymbol{\xi}\rangle_\sigma d\tau, \ \forall \boldsymbol{\xi} \in C^1([0,T];\boldsymbol{W}_\sigma), \ \forall t \in [0,T]. \tag{3.19}$$

Taking any function $\boldsymbol{\xi} \in \boldsymbol{\mathcal{K}}_0^i(\gamma)$, we see that $\boldsymbol{\xi}$ is a test function for (3.17), since $\boldsymbol{\xi} \in \boldsymbol{\mathcal{K}}_0^i(\gamma_{N_n})$ for all large n. Now, substitute $\boldsymbol{\xi}$ in (3.17) and pass to the limit as $n \to \infty$ to have by (3.18) and (3.19) that

$$\begin{aligned}\int_0^t \langle \boldsymbol{\xi}', \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma d\tau + \nu\int_0^t \langle \boldsymbol{F}\boldsymbol{v}, \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma + \int_0^t \langle \boldsymbol{G}(\boldsymbol{v},\boldsymbol{v}), \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma d\tau \\ + \frac{1}{2}|\boldsymbol{v}(t) - \boldsymbol{\xi}(t)|_{0,2}^2 \le \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|_{0,2}^2 + \int_0^t \langle \boldsymbol{g}, \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma d\tau\end{aligned}$$

and $\boldsymbol{v}$ is of bounded variation as a function from $[0,T]$ into $\boldsymbol{W}_\sigma^*$, whence the function $t \mapsto (\boldsymbol{v}(t),\boldsymbol{\xi}(t))_\sigma$ is of bounded variation on $[0,T]$ for each $\boldsymbol{\xi} \in \boldsymbol{\mathcal{K}}_0^i(\gamma)$ (cf. (3.3) in Remark

3.1). Clearly $\boldsymbol{v}(0) = \boldsymbol{v}_0$, because $\boldsymbol{v}_n(0) = \boldsymbol{v}_0 \to \boldsymbol{v}(0)$ weakly in $\boldsymbol{H}_\sigma$ by (3.18). Thus $\boldsymbol{v}$ is a weak solution of $NS^i(\gamma; \boldsymbol{g}, \boldsymbol{v}_0)$. ♢

Remark 3.4. The obstacle function γ was approximated by $\gamma_{\delta,N}$ and this approximation satisfies (3.7), which was used in the proof of Theorem 3.1. As is easily checked, Theorem 3.1 can be proved by means of any approximation of γ, as long as (3.7) is fulfilled, although we need some easy modifications in the proof.

Remark 3.5. When γ is so close to 0 on some region Q', for the solution $\boldsymbol{v}$ of $NS^i(\gamma; \boldsymbol{g}, \boldsymbol{v}_0)$ we see that

$$|\boldsymbol{v}| \text{ is close to 0 on } Q' \text{ in the case of } i = 1,$$

or

$$|\nabla \boldsymbol{v}| \text{ is close to 0 on } Q' \text{ in the case of } i = 2.$$

The former means that the velocity $\boldsymbol{v}$ is close to 0 on such a region Q', and the latter that $\boldsymbol{v}$ is close to a vector field independent of space variable x, namely it depend almost only on time, but $\boldsymbol{v}$ itself is not necessarily close to 0 on Q'.

4. Degenerate case of $NS^1(\gamma; \boldsymbol{g}, \boldsymbol{v}_0)$

4.1. Helmholtz decomposition

In this subsection we suppose in addition to (3.1) that γ is nonnegative and satisfies (3.6).

Given sequences $\{\delta_n\}$ with $\delta_n \downarrow 0$ and $\{N_n\}$ with $N_n \uparrow \infty$, we put

$$\gamma_n(x,t) = (\gamma(x,t) \vee \delta_n) \wedge N_n, \ \forall x \in \Omega, \ \forall t \in [0,T], \ \forall n \in \mathrm{N}.$$

Let t_0 be fixed in $[0,T]$ with $\Omega_0 := \{x \in \Omega \mid \gamma(x,t_0) > 0\} \neq \emptyset$ and in this subsection denote $\gamma(x,t_0)$ simply by $\gamma(x)$ for $x \in \Omega$.

For any function $\boldsymbol{z}$ in $L^2(\Omega_0)^3$, consider the Helmholtz-decomposition (cf. [19, 34])

$$\boldsymbol{z} = \tilde{\boldsymbol{z}} + \nabla q, \quad \tilde{\boldsymbol{z}} \in \boldsymbol{H}_\sigma(\Omega_0), \quad q \in L^2_{loc}(\Omega_0), \ \nabla q \in L^2(\Omega_0)^3;$$

we note that the correspondence $\boldsymbol{z} \to \tilde{\boldsymbol{z}}$ is the projection of $\boldsymbol{z}$ onto $\boldsymbol{H}_\sigma(\Omega_0)$ and Ω_0 is an open set in Ω which is possibly not Lipschitz.

Lemma 4.1. *Let $\boldsymbol{z}$, $\tilde{\boldsymbol{z}}$, q be as above, $M > 0$ be any number and put*

$$q^M(x) := (q(x) \wedge M) \vee (-M), \quad x \in \Omega_0.$$

Then $q^M \in H^1(\Omega_0)$ and $\boldsymbol{z}^M := \tilde{\boldsymbol{z}} + \nabla q^M \to \boldsymbol{z}$ in $L^2(\Omega_0)^3$ as $M \uparrow \infty$.

Proof. Note that $|q^M(x)| \leq M$ and $|\nabla q^M(x)| \leq |\nabla q(x)|$ for a.e. $x \in \Omega_0$. Besides, since $q^M \to q$ a.e. on Ω_0 as $M \uparrow \infty$, we have $|\nabla q^M| \uparrow |\nabla q|$ a.e. on Ω_0 as $M \uparrow \infty$. Hence we have the conclusion of the lemma. ♢

Next consider an approximation of Ω_0 by smooth (or Lipschitz) open sets. Given any positive number $\varepsilon > 0$ and any set Ω' in Ω we use the notation:

$$U_\varepsilon(\Omega') := \{x \in \Omega \mid \exists x' \in \Omega' \text{ s.t. } |x - x'| < \varepsilon\} \ \ (\varepsilon\text{−neighborhood of } \Omega').$$

Lemma 4.2. *Let $\varepsilon > 0$ be any small number. Then there is a bounded open set ω_ε, with a Lipschitz boundary, in Ω_0 such that*

$$\overline{U_\varepsilon(\omega_\varepsilon)} \subset \Omega_0,$$

and

$$\forall x \in \Omega_0 - \omega_\varepsilon,\ \exists x' \in \Omega - \Omega_0 \text{ such that } |x - x'| \le c_o\varepsilon,$$

where $c_0 > 1$ is a positive constant independent of ε, and moreover, ω_ε is increasing as $\varepsilon \downarrow 0$ and

$$\bigcup_{\varepsilon>0} \omega_\varepsilon = \Omega_0, \quad \text{hence } meas(\Omega_0 - \omega_\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

The proof of Lemma 4.2 is elementary, so it is omitted.

Lemma 4.3. *Let $\varepsilon > 0$ and ω_ε be the same as in Lemma 4.2. Then there is a cut-off function $\alpha_\varepsilon \in \mathcal{D}(\Omega)$ such that*

$$0 \le \alpha_\varepsilon \le 1 \quad on\ \Omega, \quad \mathrm{supp}(\alpha_\varepsilon) \subset \Omega_0, \tag{4.1}$$

$$\alpha_\varepsilon = 1 \quad on\ \omega_\varepsilon, \quad \alpha_\varepsilon = 0 \quad on\ \Omega - \Omega_0, \tag{4.2}$$

$$|\nabla \alpha_\varepsilon| \le \frac{C(\Omega_0)}{\varepsilon} \quad on\ \Omega, \tag{4.3}$$

where $C(\Omega_0)$ is a positive constant depending on Ω_0 (but independent of ε).

Proof. As α_ε we can choose the convolution

$$\rho_{\frac{\varepsilon}{4}} * \chi_{\overline{U_{\frac{\varepsilon}{2}}(\omega_\varepsilon)}}(x) := \int_\Omega \rho_{\frac{\varepsilon}{4}}(x-y)\chi_{\overline{U_{\frac{\varepsilon}{2}}(\omega_\varepsilon)}}(y)dy, \quad x \in \Omega,$$

where $\chi_{\overline{U_{\frac{\varepsilon}{2}}(\omega_\varepsilon)}}$ is the characteristic function of $\overline{U_{\frac{\varepsilon}{2}}(\omega_\varepsilon)}$ and $\rho_{\frac{\varepsilon}{4}}$ is the 3-D mollifier with support $|x| \le \frac{\varepsilon}{4}$. The proof is elementary. See [2; **4.19**] for the statement. $\diamondsuit$

Lemma 4.4. *Let $\{\boldsymbol{v}_n\}$ be a bounded sequence in $\boldsymbol{H}_\sigma$ such that*

$$|\boldsymbol{v}_n(x)| \le \gamma_n(x), \quad a.e.\ x \in \Omega,\ \forall n, \tag{4.4}$$

and

$$\int_\Omega (\boldsymbol{v}_n - \boldsymbol{v}_m) \cdot \boldsymbol{z} dx \to 0 \quad as\ n,\ m \to \infty, \tag{4.5}$$

for any $\boldsymbol{z} \in \boldsymbol{W}_\sigma$ with supp$(|\boldsymbol{z}|) \subset \Omega_0$. Then, there is $\boldsymbol{v} \in \boldsymbol{H}_\sigma$ such that $\boldsymbol{v}_n \to \boldsymbol{v}$ weakly in $L^2(\Omega)^3$.

Proof. Let M be any positive number and $\bar{q} \in H^1(\Omega_0)$ with $|\bar{q}(x)| \le M$ for a.e. $x \in \Omega_0$. First we shall show that

$$\lim_{n,m\to\infty} \int_{\Omega_0} (\boldsymbol{v}_n - \boldsymbol{v}_m) \cdot \nabla \bar{q} dx = 0. \tag{4.6}$$

For each small $\varepsilon > 0$ we take ω_ε with $\overline{U_\varepsilon(\omega_\varepsilon)} \subset \Omega_0$ and a cut-off function α_ε as in Lemmas 4.2 and 4.3. Since $\alpha_\varepsilon q \in H_0^1(\Omega)$, we observe with the help of estimates in Lemmas 4.2, 4.3 and (4.4) that

$$\begin{aligned}
&\left|\int_{\omega_\varepsilon} (\boldsymbol{v}_n - \boldsymbol{v}_m)\cdot\nabla\bar{q}dx\right| = \left|\int_{\omega_\varepsilon} (\boldsymbol{v}_n - \boldsymbol{v}_m)\cdot\nabla(\alpha_\varepsilon\bar{q})dx\right| \\
= \;& \left|\int_{\Omega}(\boldsymbol{v}_n - \boldsymbol{v}_m)\cdot\nabla(\alpha_\varepsilon\bar{q})dx - \int_{\Omega-\omega_\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_m)\cdot\nabla(\alpha_\varepsilon\bar{q})dx\right| \\
= \;& \left|0 - \int_{\Omega-\omega_\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_m)\cdot\nabla(\alpha_\varepsilon\bar{q})dx\right| \\
= \;& \left|\int_{\Omega_0-\omega_\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_m)\cdot\nabla(\alpha_\varepsilon\bar{q})dx\right| \\
\le \;& \int_{\Omega_0-\omega_\varepsilon}(\gamma_n+\gamma_m)|\nabla\bar{q}|dx + \int_{\Omega_0-\omega_\varepsilon}(\gamma_n+\gamma_m)M\frac{C(\Omega_0)}{\varepsilon}dx.
\end{aligned}$$

In the last inequality we note from Lemma 4.2 that for any $x \in \Omega_0 - \omega_\varepsilon$ there is $x' \in \Omega - \Omega_0$ such that $|x - x'| \le c_0\varepsilon$, so that by the Lipschitz continuity of γ we have $\gamma(x) = |\gamma(x) - \gamma(x')| \le L_\gamma c_0\varepsilon$, where $L_\gamma = L_\gamma(\kappa, t_0)$, with a small $\kappa > 0$, is the Lipschitz constant of $\gamma = \gamma(\cdot, t_0)$ in a neighborhood of $\Omega - \Omega_0$. Therefore, by (4.1)-(4.3),

$$\begin{aligned}
&\lim_{n,m\to\infty}\int_{\Omega_0-\omega_\varepsilon}(\gamma_n+\gamma_m)\cdot|\nabla\bar{q}|dx \\
= \;& \int_{\Omega_0-\omega_\varepsilon}2\gamma|\nabla\bar{q}|dx \le 2L_\gamma c_0\varepsilon\int_{\Omega_0-\omega_\varepsilon}|\nabla\bar{q}|dx \to 0 \text{ as } \varepsilon \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n,m\to\infty}\int_{\Omega_0-\omega_\varepsilon}(\gamma_n+\gamma_m)\cdot M\frac{C(\Omega_0)}{\varepsilon}dx \\
= \;& \int_{\Omega_0-\omega_\varepsilon}2\gamma M\frac{C(\Omega_0)}{\varepsilon}dx \le 2L_\gamma c_0 MC(\Omega_0)\cdot\text{meas}(\Omega_0 - \omega_\varepsilon) \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}$$

Thus we have (4.6).

Here we note that (4.5) holds for every $\boldsymbol{z} \in \boldsymbol{H}_\sigma(\Omega_0)$, since $\boldsymbol{W}_\sigma(\Omega_0)$ is dense in $\boldsymbol{H}_\sigma(\Omega_0)$. Next, let $\boldsymbol{z}$ be any function in $L^2(\Omega_0)^3$. We denote the Helmholtz decomposition of $\boldsymbol{z}$ in $L^2(\Omega_0)^3$ by $\boldsymbol{z} := \tilde{\boldsymbol{z}} + \nabla q$, $\tilde{\boldsymbol{z}} \in \boldsymbol{H}_\sigma(\Omega_0)$, $q \in L^2_{loc}(\Omega_0)$ with $\nabla q \in L^2(\Omega_0)^3$. Given large constant $M > 0$, we put

$$q^M(x) = (q(x) \wedge M) \vee (-M), \quad x \in \Omega_0.$$

By Lemma 4.1, $q^M \in H^1(\Omega_0)$ and $\nabla q^M \to \nabla q$ in $L^2(\Omega_0(t))^3$ as $M \uparrow \infty$. Then it follows that

$$\begin{aligned}
&\left|\int_{\Omega_0}(\boldsymbol{v}_n - \boldsymbol{v}_m)\cdot\boldsymbol{z}dx\right| \\
= \;& \left|\int_{\Omega_0}(\boldsymbol{v}_n - \boldsymbol{v}_m)\cdot(\tilde{\boldsymbol{z}} + \nabla q)dx\right| \\
\le \;& \left|\int_{\Omega_0}(\boldsymbol{v}_n - \boldsymbol{v}_m)\cdot\tilde{\boldsymbol{z}}dx\right| + \left|\int_{\Omega_0}(\boldsymbol{v}_n - \boldsymbol{v}_m)\cdot\nabla q^M dx\right| \\
&+2M'|\nabla q - \nabla q^M|_{L^2(\Omega_0)^3},
\end{aligned}$$

where $M' = \sup_n |\boldsymbol{v}_n|_{0,2}$.

We derive from (4.5) and (4.6) with $\bar{q} = q^M$ that

$$\limsup_{n,m\to\infty} \left| \int_{\Omega_0} (\boldsymbol{v}_n - \boldsymbol{v}_m) \cdot \boldsymbol{z} dx \right| \leq 2M' |\nabla q - \nabla q^M|_{L^2(\Omega_0)^3}. \tag{4.7}$$

Therefore, the right hand side of (4.7) converges to 0 as $M \uparrow \infty$, so that $\boldsymbol{v}_n - \boldsymbol{v}_m \to 0$ weakly in $L^2(\Omega_0)^3$ as $n, m \to \infty$. This implies that $\boldsymbol{v}_n \to \boldsymbol{v}$ in $L^2(\Omega)^3$ as well as in $\boldsymbol{H}_\sigma$, since $\boldsymbol{v}_n \to 0$ uniformly on $\Omega - \Omega_0$ by (4.4). $\diamondsuit$

4.2. Proof of Theorem 3.2 (Degenerate case)

The main idea for the proof of Theorem 3.2 is found in [20], but it will be here repeated for the completeness under the additional condition (3.6).

In the rest of this paper we use the following notation:

$$\Omega_\kappa(t) := \{x \in \Omega \mid \gamma(x,t) > \kappa\}, \quad \forall \kappa \geq 0, \ \forall t \in [0,T],$$

$$E_0 := \{t \in [0,T] \mid \Omega_0(t) \neq \emptyset\};$$

$E_0 := \bigcup_{\ell=1}^\infty E_\ell$ is relatively open in $[0,T]$, where E_ℓ is any connected component of E_0 . Also, we put

$$\hat{Q}_J(\gamma > \kappa) := \bigcup_{t\in J} \Omega_\kappa(t) \times \{t\},$$

for any subinterval J of $[0,T]$. When $J = [0,T]$, $\hat{Q}_{[0,T]}(\gamma > 0) = \hat{Q}(\gamma > 0)$.

We consider an exhaustion of the set $\hat{Q}(\gamma > 0)$ by means of 4-dimensional rectangulars (parallel to the (x,t)-coordinate axis) in $\hat{Q}(\gamma > 0)$.

For each ℓ we observe that $\hat{Q}_{E_\ell}(\gamma > 0)$ is a countable union of sets $\Omega_i^m \times J_i^m$ in the form:

$$\hat{Q}_{E_\ell}(\gamma > 0) = \bigcup_{m=1}^\infty \bigcup_{i=1}^{P_m} \Omega_i^m \times J_i^m, \tag{4.8}$$

where

(a) $E_\ell = \bigcup_{m=1}^\infty J^m$ with $J^m := \bigcup_{i=1}^{P_m} J_i^m$ such that

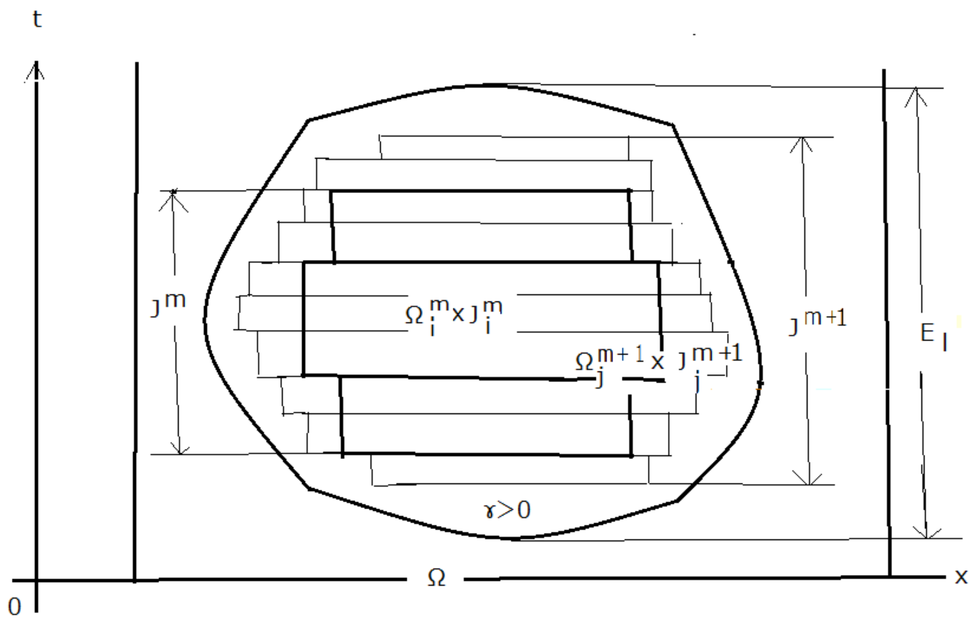
- each J_i^m is an interval of the form $[T_m, T'_m]$ or $(T_m, T'_m]$ or $[T_m, T'_m)$ or (T_m, T'_m),
- J^m is a direct sum of J_i^m and J^m is increasing in m, namely $J^m \subset J^{m+1}$,

(b) Ω_i^m is a smooth open set in Ω such that

- $\overline{\Omega_i^m} \subset \Omega_0(t_{m,i})$ for some $t_{m,i} \in J_i^m$, hence $\mathrm{dist}(\Omega_i^m, \Omega - \Omega_0(t_{m,i})) > 0$.
- Ω_i^m is increasing in m in the sense that $\Omega_i^m \subset \Omega_j^{m+1}$ if $t \in J_i^m \cap J_j^{m+1}$ for $1 \leq j \leq P_{m+1}$.

From (4.8) we see that for each $t \in E_\ell$,

$$\mathrm{meas}(\Omega_0(t) - \Omega_i^m) \to 0 \ \text{ as } m \to \infty, \ t \in J_i^m.$$



Let γ_n be the same as in the previous subsection:

$$\gamma_n := (\gamma \vee \delta_n) \wedge N_n, \quad \delta_n \downarrow 0, \ N_n \uparrow \infty.$$

By virtue of Theorem 3.1, problem $NS^1(\gamma_n; \boldsymbol{g}, \boldsymbol{v}_0)$ has a solution $\boldsymbol{v}_n$ satisfying uniform estimate (cf. (3.13)):

$$\sup_{t\in[0,T]} |\boldsymbol{v}_n(t)|_{0,2}^2 + \nu|\boldsymbol{v}_n|^2_{L^2(0,T;\boldsymbol{V}_\sigma)} \le M_0(\boldsymbol{v}_0, \boldsymbol{g}) =: M_0. \tag{4.9}$$

From (4.9) we see that $\{\boldsymbol{v}_n\}$ is bounded in $L^\infty(0,T;\boldsymbol{H}_\sigma) \cap L^2(0,T;\boldsymbol{V}_\sigma)$. Now choose a subsequence of $\{\boldsymbol{v}_n\}$, denoted by the same notation for simplicity, such that

$$\boldsymbol{v}_n \to \boldsymbol{v} \text{ weakly}^* \text{ in } L^\infty(0,T;\boldsymbol{H}_\sigma) \ \text{ and weakly in } L^2(0,T;\boldsymbol{V}_\sigma), \tag{4.10}$$

for some $\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{H}_\sigma) \cap L^2(0,T;\boldsymbol{V}_\sigma)$. We note that $\boldsymbol{v}_n \in \boldsymbol{\mathcal{K}}^1(\gamma_n)$, i.e. $|\boldsymbol{u}_n| \le \gamma_n$ a.e. on Q. Since $\gamma_n \to \gamma$ uniformly on Q (in the extended sense (3.7)), it follows that

$$|\boldsymbol{v}| \le \gamma \text{ a.e. on } Q, \quad \textit{i.e. } \boldsymbol{v} \in \boldsymbol{\mathcal{K}}^1(\gamma).$$

Besides, each $\boldsymbol{v}_n$ satisfies the following variational inequality:

$$\begin{aligned}\int_0^t (\boldsymbol{\xi}'(\tau), \boldsymbol{v}_n(\tau) - \boldsymbol{\xi}(\tau))_\sigma d\tau + \nu \int_0^t \langle \boldsymbol{F}\boldsymbol{v}_n(\tau), \boldsymbol{v}_n(\tau) - \boldsymbol{\xi}(\tau)\rangle_\sigma d\tau \\ + \int_0^t \langle \boldsymbol{G}(\boldsymbol{v}_n, \boldsymbol{v}_n), \boldsymbol{v}_n - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}_n(t) - \boldsymbol{\xi}(t)|^2_{0,2} \\ \le \int_0^t \langle \boldsymbol{g}, \boldsymbol{v}_n - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|^2_{0,2}, \quad \forall t \in [0,T], \ \forall \boldsymbol{\xi} \in \boldsymbol{\mathcal{K}}^1_0(\gamma_n).\end{aligned} \tag{4.11}$$

Putting

$$\boldsymbol{f}_n := \boldsymbol{g} - \nu \boldsymbol{F}\boldsymbol{v}_n - \boldsymbol{G}(\boldsymbol{v}_n, \boldsymbol{v}_n), \tag{4.12}$$

we observe that $\boldsymbol{f}_n \in L^2(0,T;\boldsymbol{V}_\sigma^*)$ and $\{\boldsymbol{G}(\boldsymbol{v}_n,\boldsymbol{v}_n)\}$ is bounded in $L^1(0,T;\boldsymbol{W}_\sigma^*)$, hence $\{\boldsymbol{f}_n\}$ is bounded in $L^1(0,T;\boldsymbol{W}_\sigma^*)$.

Now, for each index $\{m,i\}$ in (4.8) consider the sequence of approximate solutions $\boldsymbol{v}_n$ on the cylindrical open set $\Omega_i^m \times J_i^m$. For simplicity we write Ω' for Ω_i^m and write J $(= [T_1, T_1'])$ for J_i^m, assuming J_i^m is a closed interval $[T_{m,i}, T'_{m,i}]$; any other case can be similarly handled. With these notations, consider the spaces $\boldsymbol{V}_\sigma(\Omega')$, $\boldsymbol{H}_\sigma(\Omega')$ and $\boldsymbol{W}_\sigma(\Omega')$ built on Ω';

$$\boldsymbol{W}_\sigma(\Omega') \subset \boldsymbol{V}_\sigma(\Omega'), \quad \boldsymbol{V}_\sigma(\Omega') \subset \boldsymbol{H}_\sigma(\Omega') \subset \boldsymbol{W}_\sigma^*(\Omega')$$

with dense and compact embeddings.

By the Helmholtz decomposition of $\boldsymbol{v}_n(\cdot,t)$ in $L^2(\Omega')^3$,

$$\boldsymbol{v}_n(x,t) = \tilde{\boldsymbol{v}}_n(x,t) + \nabla q_n(x,t), \quad \tilde{\boldsymbol{v}}_n(\cdot,t) \in \boldsymbol{H}_\sigma(\Omega') \cap W^{1,2}(\Omega')^3, \ q_n(\cdot,t) \in H^1(\Omega');$$

we know (cf. [34; Chaper 1]) that $|\tilde{\boldsymbol{v}}_n|_{W^{1,2}(\Omega')^3} \le C(\Omega')|\boldsymbol{v}_n|_{W^{1,2}(\Omega')^3}$ with some positive constant $C(\Omega')$ depending only on Lipschitz open set Ω'. It is easy to see from (4.9) that

$$\sup_{t\in[0,T]} |\tilde{\boldsymbol{v}}_n(t)|^2_{\boldsymbol{H}_\sigma(\Omega')} + \nu |\tilde{\boldsymbol{v}}_n|^2_{L^2(0,T;W^{1,2}(\Omega')^3)} \le M_1(\boldsymbol{u}_0, \boldsymbol{g}) =: M_1. \tag{4.13}$$

where M_1 is a positive constant independent of n. We can regard $\boldsymbol{f}_n(t)$ given by (4.12) as a linear continuous functional, denoted by $\tilde{\boldsymbol{f}}_n$, on $\boldsymbol{W}_\sigma(\Omega')$, by putting

$$\begin{aligned}\langle \tilde{\boldsymbol{f}}_n(t), \boldsymbol{\xi}\rangle_\sigma \ := \ & \int_{\Omega'} \boldsymbol{g}(x,t)\cdot \boldsymbol{\xi}(x)dx - \nu \int_{\Omega'} \nabla \boldsymbol{v}_n(x,t)\cdot \nabla \boldsymbol{\xi}(x)dx \\ & + \int_{\Omega'} (\boldsymbol{v}_n(t)\cdot\nabla)\boldsymbol{v}_n(t)\cdot\boldsymbol{\xi}(x)dx, \quad \forall \boldsymbol{\xi} \in \boldsymbol{W}_\sigma(\Omega').\end{aligned}$$

The estimate (4.13) shows that $\{\tilde{\boldsymbol{f}}_n\}$ is bounded in $L^1(T_1, T_1'; \boldsymbol{W}_\sigma^*(\Omega'))$ and

$$\int_{T_1}^{T_1'} \langle \boldsymbol{f}_n, \boldsymbol{v}_n\rangle_\sigma dt \le \int_{T_1}^{T_1'} \langle \boldsymbol{g}, \boldsymbol{v}_n\rangle_\sigma dt$$

is bounded above; namely for a positive constant M_1' it holds that

$$|\tilde{\boldsymbol{f}}_n|_{L^1(T_1,T_1';\boldsymbol{W}_\sigma^*(\Omega'))} \le M_1', \quad \forall n.$$

Lemma 4.5. *Let $\Omega' \times [T_1, T_1'] := \Omega_i^m \times [T_{m,i}, T'_{m,i}]$ be as above, and let C_1 be a positive constant satisfying $|\boldsymbol{z}|_{C(\overline{\Omega})^3} \le C_1 |\boldsymbol{z}|_{\boldsymbol{W}_\sigma}$ for all $\boldsymbol{z} \in \boldsymbol{W}_\sigma$. Then $\tilde{\boldsymbol{v}}_n$ is of bounded variation as a function from $[T_1, T_1']$ into $\boldsymbol{W}_\sigma^*(\Omega')$ and its total variation is estimated by:*

$$\mathrm{Var}_{\boldsymbol{W}_\sigma^*(\Omega')}(\tilde{\boldsymbol{v}}_n) \le M_1' + \frac{C_1 M_0^{\frac{1}{2}}}{\kappa_{m,i}} \int_{T_1}^{T_1'} |\boldsymbol{g}|_{0,2} dt + \frac{C_1 M_0}{2\kappa_{m,i}}, \quad \forall \text{large } n, \tag{4.14}$$

with the constant $\kappa_{m,i}$ in condition (b) and M_0 the same constant as in (3.13).

Proof. We first show that there is a constant $\kappa' > 0$ such that

$$\kappa' B_{\boldsymbol{W}_\sigma(\Omega')}(0) \subset K^1(\gamma_n;t),\ \forall t \in [T_1, T_1'],\ \forall \text{large } n. \tag{4.15}$$

In fact, since $\overline{\Omega'} = \overline{\Omega_i^m} \subset \Omega_{\kappa_{m,i}}$ and $\gamma_n \to \gamma$ uniformly on Ω, we see that $\gamma > \kappa_{m,i}$ on $\overline{\Omega'}$ and hence $\gamma_n > \kappa_{m,i}$ on $\overline{\Omega}'$ for all large n. Therefore, if $\boldsymbol{z} \in B_{\boldsymbol{W}_\sigma(\Omega')}(0)$, then

$$\frac{\kappa_{m,i}}{C_1}|\boldsymbol{z}|_{C(\overline{\Omega'})^3} \le \kappa_{m,i}|\boldsymbol{z}|_{\boldsymbol{W}_\sigma(\Omega')} \le \kappa_{m,i} \le \gamma_n(\cdot,t) \text{ a.e. on } \Omega',\ \forall t \in [T_1,T_1'],\ \forall \text{ large } n.$$

Thus we have (4.15) with $\kappa' = \frac{\kappa_{m,i}}{C_1}$.

Next, let $\boldsymbol{\xi}$ be any element in $C_0^1(T_1,T_1';\boldsymbol{W}_\sigma(\Omega'))$ with $|\boldsymbol{\xi}|_{C([T_1,T_1'];\boldsymbol{W}_\sigma(\Omega'))} \le 1$. Then, $\hat{\boldsymbol{\xi}} := \pm\kappa'\boldsymbol{\xi}$ is a possible test function for (4.11), hence we get that

$$\int_{T_1}^{T_1'} \langle \hat{\boldsymbol{\xi}}', \tilde{\boldsymbol{v}}_n - \hat{\boldsymbol{\xi}}\rangle_\sigma dt \le \int_{T_1}^{T_1'} \langle \boldsymbol{f}_n, \boldsymbol{v}_n - \hat{\boldsymbol{\xi}}\rangle_\sigma dt + \frac{1}{2}|\boldsymbol{u}_n(T_1)|_{0,2}^2 - \frac{1}{2}|\boldsymbol{u}_n(T_1')|_{0,2}^2,$$

whence

$$\begin{aligned}
&\int_{T_1}^{T_1'} \langle \boldsymbol{\xi}', \tilde{\boldsymbol{v}}_n\rangle_\sigma dt \\
\le\ & -\int_{T_1}^{T_1'} \langle \tilde{\boldsymbol{f}}_n, \boldsymbol{\xi}\rangle_\sigma dt + \frac{1}{\kappa'}\int_{T_1}^{T_1'} (\boldsymbol{g}, \boldsymbol{v}_n)_\sigma dt + \frac{1}{2\kappa'}|\boldsymbol{v}_n(T_1)|_{0,2}^2 - \frac{1}{2\kappa'}|\boldsymbol{v}_n(T_1')|_{0,2}^2 \\
\le\ & \int_{T_1}^{T_1'} |\tilde{\boldsymbol{f}}_n|_{\boldsymbol{W}_\sigma^*(\Omega')} dt + \frac{M_0^{\frac{1}{2}}}{\kappa'}\int_{T_1}^{T_1'} |\boldsymbol{g}|_{0,2} dt + \frac{1}{2\kappa'}|\boldsymbol{v}_n(T_1)|_{0,2}^2 - \frac{1}{2\kappa'}|\boldsymbol{v}_n(T_1')|_{0,2}^2.
\end{aligned}$$

Consequently, we have (4.14). ♢

Corollary 4.1. *Under the same assumptions and notation as in Lemma 4.5, for each* $\{m,i\}$ *there are a subsequence* $\{\tilde{\boldsymbol{v}}_{n_{k(m,i)}}\}_{k=1}^\infty$ *of* $\{\tilde{\boldsymbol{v}}_n\}$ *(depending on* Ω_i^m*) and a function* $\tilde{\boldsymbol{v}}_{m,i}$ *in* $L^\infty(J_i^m;\boldsymbol{H}_\sigma(\Omega_i^m)) \cap L^2(J_i^m;W^{1,2}(\Omega_i^m)^3)$ *such that*

$$\tilde{\boldsymbol{v}}_{n_{k(m,i)}}(t) \to \tilde{\boldsymbol{v}}_{m,i}(t) \text{ weakly in } L^2(\Omega_i^m)^3,\ \ \forall t \in J_i^m \text{ (as } k \to \infty), \tag{4.16}$$

hence

$$\int_{\Omega_i^m} (\boldsymbol{v}_{n_{k(m,i)}}(\cdot,t) - \boldsymbol{v}_{n_{j(m,i)}}(\cdot,t))\cdot \boldsymbol{z} dx \to 0,\ \forall \boldsymbol{z} \in \boldsymbol{H}_\sigma(\Omega_i^m),\ \forall t \in J_i^m,\ \text{as } k,\ j \to \infty. \tag{4.17}$$

Proof. According to Lemma 4.5 and Lemma 2.1, there is a subsequence $\{\tilde{\boldsymbol{v}}_{n_{k(m,i)}}\}$ of $\{\tilde{\boldsymbol{v}}_n\}$ and a function $\tilde{\boldsymbol{u}}_{m,i} \in L^\infty(J_i^m;\boldsymbol{H}_\sigma(\Omega_i^m)) \cap L^2(J_i^m;W^{1,2}(\Omega_i^m)^3)$ such that

$$\tilde{\boldsymbol{v}}_{n_{k(m,i)}}(t) \to \tilde{\boldsymbol{v}}_{m,i}(t) \text{ weakly in } \boldsymbol{H}_\sigma(\Omega_i^m),\ \forall t \in J_i^m \text{ (as } k \to \infty). \tag{4.18}$$

Here, for any function $\boldsymbol{z} \in L^2(\Omega')^3$ we use the Helmholtz decomposition

$$\boldsymbol{z} = \tilde{\boldsymbol{z}} + \nabla q,\ \tilde{\boldsymbol{z}} \in \boldsymbol{H}_\sigma(\Omega_i^m),\ q \in H^1(\Omega_i^m),$$

to see by (4.18)

$$\begin{aligned}\int_{\Omega_i^m} \tilde{\boldsymbol{v}}_{n_{k(m.i)}}(\cdot,t)\cdot \boldsymbol{z}dx &= \int_{\Omega_i^m} \tilde{\boldsymbol{v}}_{n_{k(m,i)}}(\cdot,t)\cdot(\tilde{\boldsymbol{z}}+\nabla q)dx \\ &= \int_{\Omega_i^m} \tilde{\boldsymbol{v}}_{n_{k(m,i)}}(\cdot,t)\cdot \tilde{\boldsymbol{z}}dx \to \int_{\Omega_i^m} \tilde{\boldsymbol{v}}_{m,i}(\cdot,t)\cdot \tilde{\boldsymbol{z}}dx \\ &= \int_{\Omega_i^m} \tilde{\boldsymbol{v}}_{m,i}(\cdot,t)\cdot(\tilde{\boldsymbol{z}}+\nabla q)dx = \int_{\Omega_i^m} \tilde{\boldsymbol{v}}_{m,i}(\cdot,t)\cdot \boldsymbol{z}dx;\end{aligned}$$

hence (4.16) holds. The convergence (4.17) follows, since $\int_{\Omega_i^m} \boldsymbol{v}_{n_{k(m,i)}}\cdot \boldsymbol{z}dx = \int_{\Omega_i^m} \tilde{\boldsymbol{v}}_{n_{k(m,i)}}\cdot \boldsymbol{z}dx$ for $\boldsymbol{z}\in \boldsymbol{H}_\sigma(\Omega_i^m)$. ♢

Corollary 4.2. *Under the same assumptions and notation as in Corollary 4.1, there is a subsequence* $\{\boldsymbol{v}_{n_{k(m)}}\}_{k=1}^\infty$ *of* $\{\boldsymbol{v}_n\}$*, depending only on* m*, such that*

$$\int_\Omega (\boldsymbol{v}_{n_{k(m)}}(\cdot,t)-\boldsymbol{v}_{n_{j(m)}}(\cdot,t))\cdot \boldsymbol{z}dx \to 0,\quad \forall \boldsymbol{z}\in \boldsymbol{H}_\sigma(\Omega_i^m),\ \forall t\in J_i^m, \tag{4.19}$$

as $k,\ j\to\infty$.

Proof. This corollary is a direct consequence of Corollary 4.1 and the definition $\boldsymbol{v}_{n_{k(m,i)}}$; in fact, we extract a subsequence $\{\boldsymbol{v}_{n_{k(m,i+1)}}\}$ from $\{\boldsymbol{v}_{n_{k(m,i)}}\}$ so as to satisfy (4.17) with i replaced by $i+1$, repeatedly for $i=1,2,\cdots,N_m-1$. As a result, we get a subsequence $\{\boldsymbol{v}_{n_{k(m)}}\}$ for which (4.19) holds. ♢

Lemma 4.6. *Under the same assumptions and notation as in Corollary 4.2, there is a subsequence* $\{\boldsymbol{v}_{n_m}\}_{m=1}^\infty$ *with a function* $\bar{\boldsymbol{v}}: E_\ell \to \boldsymbol{H}_\sigma$ *such that*

$$\boldsymbol{v}_{n_m}(t)\to \bar{\boldsymbol{v}}(t) \text{ weakly in } L^2(\Omega)^3,\quad \forall t\in E_\ell,\ (\text{as } m\to\infty). \tag{4.20}$$

Proof. We make use of the subsequences $\{\boldsymbol{v}_{n_{k(m)}}\}$ constructed in Corollary 4.2. In the table of these subsequences:

$$\begin{array}{cccccc} \boldsymbol{v}_{n_{1(1)}} & \boldsymbol{v}_{n_{2(1)}} & \boldsymbol{v}_{n_{3(1)}} & \cdots\cdots\cdots\cdots & \boldsymbol{v}_{n_{m(1)}} & \cdots \\ \boldsymbol{v}_{n_{1(2)}} & \boldsymbol{v}_{n_{2(2)}} & \boldsymbol{v}_{n_{3(2)}} & \cdots\cdots\cdots\cdots & \boldsymbol{v}_{n_{m(2)}} & \cdots \\ \boldsymbol{v}_{n_{1(3)}} & \boldsymbol{v}_{n_{2(3)}} & \boldsymbol{v}_{n_{3(3)}} & \cdots\cdots\cdots\cdots & \boldsymbol{v}_{n_{m(3)}} & \cdots \\ & & \cdots\cdots\cdots & & & \\ & & \cdots\cdots\cdots & & & \\ \boldsymbol{v}_{n_{1(m)}} & \boldsymbol{v}_{n_{2(m)}} & \boldsymbol{u}_{n_{3(m)}} & \cdots\cdots\cdots\cdots & \boldsymbol{u}_{n_{m(m)}} & \cdots \\ & & \cdots\cdots\cdots & & & \end{array}$$

we pic up the diagonal functions $\boldsymbol{v}_{n_{m(m)}} =: \boldsymbol{v}_{n_m}$ and consider the subsequence $\{\boldsymbol{v}_{n_m}\}_{m=1}^\infty$ of $\{\boldsymbol{v}_n\}$.

We shall show below that this is a required one. Let $\boldsymbol{\xi}$ be any function $\boldsymbol{W}_\sigma$ such that $K := \text{supp}(|\boldsymbol{\xi}|) \subset \Omega_0(t),\ t \in E_\ell$. Since K is compact in $\Omega_0(t)$, there is a positive number κ such that $\gamma(x,t) > \kappa$ for all $x \in K$. Therefore by condition (a), $K \subset \Omega_i^m \subset \Omega_0(t)$ for a large m and some i with $t \in J_i^m$, which implies by Corollary 4.2 that

$$0 = \lim_{k,j\to\infty} \int_\Omega (\boldsymbol{v}_{n_{k(m)}}(\cdot,t) - \boldsymbol{v}_{n_{j(m)}}(\cdot,t)) \cdot \boldsymbol{\xi} dx = \lim_{m,m'\to\infty} \int_\Omega (\boldsymbol{v}_{n_m}(\cdot,t) - \boldsymbol{v}_{n_{m'}}(\cdot,t)) \cdot \boldsymbol{\xi} dx$$

Applying Lemma 4.4 to the sequence $\{\boldsymbol{v}_{n_m}\}$, we conclude that there is $\bar{\boldsymbol{v}}(t) \in L^2(\Omega)^3$ such that $\boldsymbol{v}_{n_m}(t) \to \bar{\boldsymbol{v}}(t)$ weakly in $L^2(\Omega)^3$ for each $t \in E_\ell$ as $m \to \infty$. Thus we have (4.20). $\diamondsuit$

Corollary 4.3. *Under the same assumptions and notation as in Lemma 4.6, we put*

$$\hat{\boldsymbol{v}}(x,t) := \begin{cases} \bar{\boldsymbol{v}}(x,t), & (x,t) \in \Omega \times E_\ell \\ 0, & \textit{otherwise}, \end{cases}$$

Then

$$\boldsymbol{v}_{n_m} \to \hat{\boldsymbol{v}} \quad \textit{in } L^2(0,T;\boldsymbol{H}_\sigma).$$

Proof. From Lemmas 4.5 and 4.6 with their corollaries adapted for every component E_ℓ we can construct a subsequence $\{\boldsymbol{v}_{n_m}\}$ of $\{\boldsymbol{v}_n\}$ with a function $\bar{\boldsymbol{v}} : E_0 \to \boldsymbol{H}_\sigma$ such that $\boldsymbol{v}_{n_m}(t) \to \bar{\boldsymbol{v}}(t)$ weakly in $L^2(\Omega)^3$ for all $t \in E_0$. Besides, since $\boldsymbol{v}_{n_m}(\cdot,t) \to 0$ uniformly on $\Omega - \Omega_0(t)$ for every $t \in [0,T] - E_0$, it follows consequently that $\boldsymbol{v}_{n_m}(t) \to \hat{\boldsymbol{v}}(t)$ weakly in $L^2(\Omega)^3$ for all $t \in [0,T]$.

Now we recall a compactness lemma [27;Lemma 5.1, Chapter 1] to get

$$|\boldsymbol{v}_{n_m}(t) - \boldsymbol{v}_{n_{m'}}(t)|_{0,2}^2 \le \varepsilon |\boldsymbol{v}_{n_m}(t) - \boldsymbol{v}_{n_{m'}}(t)|_{1,2}^2 + C_\varepsilon |\boldsymbol{v}_{n_m}(t) - \boldsymbol{v}_{n_{m'}}(t)|_{\boldsymbol{W}_\sigma^*}^2, \tag{4.21}$$

$$\forall t \in [0,T], \quad \forall m,\ m',$$

where $\varepsilon > 0$ is any positive number and C_ε is a positive constant depending only ε. Integrating the above inequality in time t and letting $m,\ m' \to \infty$ yield that $\boldsymbol{v}_{n_m} - \boldsymbol{v}_{n_{m'}} \to 0$ in $L^2(0,T;\boldsymbol{H}_\sigma)$ as $m,\ m' \to \infty$, since the last term of (4.21) tends to 0 uniformly on $[0,T]$ as $m,\ m' \to \infty$. This shows that $\boldsymbol{v}_{n_m} \to \hat{\boldsymbol{v}}$ (strongly) in $L^2(0,T;\boldsymbol{H}_\sigma)$ (hence in $L^2(Q)$) as $m \to \infty$. $\diamondsuit$

Proof of Theorem 3.2: Let $\hat{\boldsymbol{v}}$ be the same function as in Corpllary 4.3. Recalling that $\boldsymbol{v}_n \to \boldsymbol{v}$ weakly in $L^2(0,T;\boldsymbol{V}_\sigma)$ and weakly* in $L^2(0,T;\boldsymbol{H}_\sigma)$ as $n \to \infty$ (cf (4.10)), we see that $\hat{\boldsymbol{v}} = \boldsymbol{v}$ a.e. on Q and hence $\hat{\boldsymbol{v}}$ may be identified with $\boldsymbol{v}$. In the sequel, for simplicity, let us denote $\boldsymbol{v}_{n_m}$ by $\boldsymbol{v}_n$.

Let $\boldsymbol{\xi}$ be any function in $\boldsymbol{\mathcal{K}}_0^1(\gamma)$. Then $\boldsymbol{\xi}$ is a possible test function of approximate problem $NS^1(\gamma_n;\boldsymbol{g},\boldsymbol{v}_0)$ for all large n, so that

$$\int_0^t (\boldsymbol{\xi}'(\tau), \boldsymbol{v}_n(\tau) - \boldsymbol{\xi}(\tau))_\sigma d\tau + \nu \int_0^t \langle \boldsymbol{F}\boldsymbol{v}_n(\tau), \boldsymbol{v}_n(\tau) - \boldsymbol{\xi}(\tau) \rangle_\sigma d\tau$$

$$+ \int_0^t \langle \boldsymbol{G}(\boldsymbol{v}_n, \boldsymbol{v}_n), \boldsymbol{v}_n - \boldsymbol{\xi} \rangle_\sigma d\tau + \frac{1}{2} |\boldsymbol{v}_n(t) - \boldsymbol{\xi}(t)|_{0,2}^2 \tag{4.22}$$

$$\leq \int_0^t \langle \boldsymbol{g}, \boldsymbol{v}_n - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|_{0,2}^2, \quad \forall t \in [0,T],$$

Just as in the proof of Theorem 3.1 we obtain (cf. (3.18), (3.19))) that

$$\frac{1}{2}|\boldsymbol{v}(t) - \boldsymbol{\xi}(t)|_{0,2}^2 \leq \liminf_{n\to\infty} \frac{1}{2}|\boldsymbol{v}_n(t) - \boldsymbol{\xi}(t)|_{0,2}^2$$

and

$$\lim_{n\to\infty} \int_0^t \langle \boldsymbol{G}(\boldsymbol{v}_n, \boldsymbol{v}_n), \boldsymbol{\xi}\rangle d\tau = \int_0^t \langle \boldsymbol{G}(\boldsymbol{v}, \boldsymbol{v}), \boldsymbol{\xi}\rangle d\tau.$$

Hence, letting $n \to \infty$ in (4.22) gives

$$\int_0^t (\boldsymbol{\xi}'(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\xi}(\tau))_\sigma d\tau + \nu \int_0^t \langle \boldsymbol{F}\boldsymbol{v}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\xi}(\tau)\rangle_\sigma d\tau$$
$$+ \int_0^t \langle \boldsymbol{G}(\boldsymbol{v}, \boldsymbol{v}), \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}(t) - \boldsymbol{\xi}(t)|_{0,2}^2$$
$$\leq \int_0^t \langle \boldsymbol{g}, \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|_{0,2}^2, \quad \forall t \in [0,T].$$

As to the other properties of $\boldsymbol{v}$, we have that

$$\boldsymbol{v} \in \boldsymbol{\mathcal{K}}^1(\gamma),$$

$$\boldsymbol{v}_n(0) = \boldsymbol{v}_0 \to \boldsymbol{v}(0) \text{ weakly in } \boldsymbol{H}_\sigma, \text{ hence } \boldsymbol{v}(0) = \boldsymbol{v}_0.$$

Finally we show that $t \to (\boldsymbol{v}(t), \boldsymbol{\xi}(t))_\sigma$ is of bounded variation on $[0,T]$. It is enough to show it in the case of $\mathrm{supp}(|\boldsymbol{\xi}|) \subset \hat{Q}_{E_\ell}(\gamma > 0)$ for some ℓ. In this case there is m^* such that $\mathrm{supp}(|\boldsymbol{\xi}|) \subset \bigcup_{i=1}^{P_m^*} \Omega_i^{m^*} \times J_i^{m^*}$. We use the Helmholtz decomposition of $\boldsymbol{v}(t)$ in $L^2(\Omega_i^{m^*})$ which is of the form $\boldsymbol{v}(t) = \tilde{\boldsymbol{v}}(t) + \nabla q(t)$, $\tilde{\boldsymbol{v}}(t) \in \boldsymbol{H}_\sigma(\Omega_i^{m^*})$, $q(t) \in H^1(\Omega_i^{m^*})$, for each $t \in J_i^{m^*}$. By the estimate (4.14) in Lemma 4.5 we have

$$\mathrm{Var}_{\boldsymbol{W}_\sigma^*(\Omega_i^{m^*})}(\tilde{\boldsymbol{v}}) \leq M_1' + \frac{C_1 M_0^{\frac{1}{2}}}{\kappa_{m,i}} \int_{T_1}^{T_1'} |\boldsymbol{g}|_{0,2} dt + \frac{C_1 M_0}{2\kappa_{m,i}}. \tag{4.23}$$

In fact, since $\boldsymbol{v}_n \to \boldsymbol{v}$ weakly in $L^2(Q)^3$, it follows that $\tilde{\boldsymbol{v}}_n \to \tilde{\boldsymbol{v}}$ in $L^2(0,T;\boldsymbol{H}_\sigma(\Omega_i^{m^*}))$. By the lower semicontinuity of the total variation functional, (4.14) implies (4.23). Moreover, for any i we observe that

$$|(\boldsymbol{v}(t), \boldsymbol{\xi}(t))_\sigma - (\boldsymbol{v}(s), \boldsymbol{\xi}(s))_\sigma| = |(\tilde{\boldsymbol{v}}(t), \boldsymbol{\xi}(t))_\sigma - (\tilde{\boldsymbol{v}}(s), \boldsymbol{\xi}(s))_\sigma|$$
$$\leq \ |\tilde{\boldsymbol{v}}(t) - \tilde{\boldsymbol{v}}(s)|_{0,2}||\boldsymbol{\xi}(t)|_{\boldsymbol{W}_\sigma} + |\boldsymbol{v}(s)|_{0,2}|\boldsymbol{\xi}(s) - \boldsymbol{\xi}(t)|_{\boldsymbol{W}_\sigma}, \quad \forall s,\ t \in J_i^{m^*}.$$

From this we see easily that the total variation of $t \to (\boldsymbol{v}(t), \boldsymbol{\xi}(t))_\sigma$ on J^{m^*} is bounded by

$$\text{const.} \left(\sum_{i=1}^{P_{m^*}} \mathrm{Var}_{\boldsymbol{W}_\sigma^*(\Omega_i^{m^*})}(\tilde{\boldsymbol{v}}) + |\boldsymbol{\xi}'|_{L^1(0,T;\boldsymbol{W}_\sigma)} \right), \tag{4.24}$$

and (4.24) is valid for every connected component E_ℓ. Hence $t \to (\boldsymbol{v}(t), \boldsymbol{\xi}(t))_\sigma$ is of bounded variation on $[0,T]$. Thus $\boldsymbol{v}$ is a weak solution of $NS^1(\gamma; \boldsymbol{g}, \boldsymbol{v}_0)$. ♢

Remark 4.1. We suppose that

- for each $t \in [0, T]$ the closure of any connected component of $\Omega - \overline{\Omega_0(t)}$ meets the boundary Γ.

In this case, the same type of lemma as Lemma 4.4 can be proved under gradient constraint $|\nabla \boldsymbol{v}| \le \gamma$. Therefore with some modification in the proof of Theorem 3.2, an existence result can be shown for $NS^2(\gamma; \boldsymbol{g}, \boldsymbol{v}_0)$. However, this additional geometric assumption is too restrictive to apply this result to Stefan/Navier-Stokes problem discussed in the next section.

5. Stefan/Navier-Stokes problems

In this section, let us consider Stefan/Navier-Stokes problem. As was mentioned in the introduction, it is a system of the enthalpy formulation of solid-liquid phase change in fluid flows with freezing and melting effect.

We begin with the precise formulations in non-degenerate and degenerate cases of obstacle function, postulating that the region Ω is divided into three unknown time-dependent regions,

$$\Omega = \Omega_s(t) \cup \Omega_\ell(t) \cup \Omega_m(t),$$

which are respectively called the solid, liquid and mixture (mussy) regions, and velocity constraint depends on the order parameter of phase, namely the velocity field is independent of space variable x on $\Omega_s(t)$, governed by Navier-Stokes equation in $\Omega_\ell(t)$ and constrained by an order parameter dependent obstacle function in $\Omega_m(t)$. In our model, one of main ideas is that the dynamics of velocity field is described as a quasi-variational inequality of Navier-Stokes type.

5.1. Variational formulation of Stefan/Navier-Stokes problem

First of all we give the weak variational formulations of Stefan/Navier-Stokes problems with constraints on the velocity and its gradient.

Let $\beta = \beta(r)$ be a non-decreasing and Lipschitz continuous function with Lipschitz constant L_β from $\mathbf{R}$ into $\mathbf{R}$ such that

$$\begin{cases} \liminf\limits_{|r|\to\infty} \dfrac{\beta(r)}{|r|} > 0, \quad \beta(r) = 0 \text{ for } r \in [0,1], \\ \beta(r) \text{ is strictly increasing for } r < 0 \ \text{ and for } r > 1. \end{cases} \tag{5.1}$$

Let $\hat{\beta} := \hat{\beta}(r)$ be the primitive of β given by $\hat{\beta}(r) := \int_0^r \beta(s)ds$ for all $r \in \mathbf{R}$. Clearly

$$\hat{\beta}(0) = 0, \quad \beta(r) = \hat{\beta}'(r) \left(= \frac{d\hat{\beta}(r)}{dr} \right), \ \forall r \in \mathbf{R}.$$

Next, let $\gamma = \gamma(r)$ be a non-negative and non-decreasing continuous function from $\mathbf{R}$ into $[0, \infty]$ and consider the following two cases:

(Non-degenerate case)

$$\begin{cases} \gamma(r) \geq c_* > 0 \text{ for } r \in \mathbf{R}, \text{ where } c_* \text{ is a positive constant,} \\ \gamma \text{ is strictly increasing on } [0,1), \\ \gamma(r) \uparrow \infty \text{ as } r \uparrow 1, \quad \gamma(r) = \infty \text{ for } r \geq 1. \end{cases} \tag{5.2}$$

(Degenerate case)

$$\begin{cases} \gamma(r) = 0 \text{ for } r \leq 0, \quad \gamma \text{ is strictly increasing on } [0,1) \\ \gamma(r) \uparrow \infty \text{ as } r \uparrow 1, \quad \gamma(r) = \infty \text{ for } r \geq 1, \\ \gamma(r) \text{ is locally Lipschitz continuous in a neighborhood of } r = 0. \end{cases} \tag{5.3}$$

For the enthalpy formulation of the Stefan problem we introduce some function spaces. Let $V := H^1(\Omega)$ with norm

$$|z|_V := \left\{ \int_\Omega |\nabla z|^2 dx + n_0 \int_\Gamma |z|^2 d\Gamma \right\}^{\frac{1}{2}}, \quad \forall z \in V,$$

where n_0 is a fixed positive number and $d\Gamma$ is the usual surface measure on Γ. The dual space V^* is equipped with the dual norm of V. In this case

$$V \subset L^2(\Omega) \subset V^* \text{ with dense and compact embeddings}$$

and the duality mapping $F : V \to V^*$ is given by

$$\langle Fu, z \rangle := \int_\Omega \nabla u \cdot \nabla z dx + n_0 \int_\Gamma uz d\Gamma, \quad \forall u, \ z \in V, \tag{5.4}$$

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{V^*,V}$. We know that F is linear, continuous and uniformly monotone from V onto V^*. Now we set up an inner product $(\cdot, \cdot)_*$ in V^* by

$$(u^*, z^*)_* := \langle u^*, F^{-1} z^* \rangle, \quad \forall u^*, \ z^* \in V^*;$$

note that V^* is a Hilbert space with this inner product $(\cdot, \cdot)_*$. Moreover, we define a proper, l.s.c. and convex function $\varphi(\cdot)$ on V^* by

$$\varphi(z) := \begin{cases} \int_\Omega \hat{\beta}(z(x)) dx, & \text{for } z \in L^2(\Omega), \\ \infty, & \text{for } z \in V^* - L^2(\Omega). \end{cases}$$

We know (cf. [11, 12]) that the subdifferential $\partial_* \varphi(\cdot)$ of $\varphi(\cdot)$ in the Hilbert space V^* is a singlevalued mapping in V^* such that

$$D(\partial_* \varphi) := \{ z \in L^2(\Omega) \mid \beta(z) \in V \} \text{ and } \partial_* \varphi(z) = F \beta(z), \quad \forall z \in D(\partial_* \varphi). \tag{5.5}$$

The following evolution equation is considered in the space V^* as the enthalpy formulation of the Stefan problem:

$$\begin{aligned} & w'(t) + \partial_* \varphi(w(t)) + \operatorname{div}(w(t) \boldsymbol{v}(t)) = h(t) \text{ in } V^*, \quad t \in (0,T), \\ & w(0) = w_0, \end{aligned} \tag{5.6}$$

where $\boldsymbol{v} := \boldsymbol{v}(x,t)$ is a vector field on Q in $L^2(0,T;\boldsymbol{V}_\sigma)$ as well as $w_0 \in L^2(\Omega)$ and $h \in L^2(Q)$.

On account of the results in [11, 12], the problem (5.6) with $\boldsymbol{v} \equiv 0$ admits one and only one solution $w \in W^{1,2}(0,T;V^*) \cap C_w([0,T];L^2(\Omega))$ such that $t \mapsto \int_\Omega \hat{\beta}(w(\cdot,t))dx$ is absolutely continuous on $[0,T]$, where $C_w([0,T];L^2(\Omega))$ is the space of all weakly continuous functions from $[0,T]$ into $L^2(\Omega)$. We notice from (5.4) and (5.5) that (5.6) is written in the form

$$w'(t) + F(\beta(w(t)) + \operatorname{div}(w(t)\boldsymbol{v}(t)) = h(t) \ \text{ in } V^*, \text{ a.e. } t \in (0,T), \ w(0) = w_0, \tag{5.7}$$

or

$$w_t - \Delta\beta(w) + \boldsymbol{v}\cdot\nabla w = h \ \text{ in } Q, \ \frac{\partial\beta(w)}{\partial n} + n_0\beta(w) = 0 \text{ on } \Sigma, \ w(\cdot,0) = w_0 \text{ on } \Omega, \tag{5.8}$$

(in the distribution sense);

note here that $\boldsymbol{v}\cdot\nabla w = \operatorname{div}(w\boldsymbol{v})$ is the convective term for the enthalpy w caused by the fluid flow.

We are now consider a coupled system of (5.7) (hence (5.8)) and the variational inequality of Navier-Stokes type discussed in sections 2∼4:

$$\boldsymbol{v} \in \boldsymbol{\mathcal{K}}^i(\gamma(w^{\varepsilon_0})), \quad \boldsymbol{v}(0) = \boldsymbol{v}_0,$$

$$\begin{aligned} &\int_0^t (\boldsymbol{\xi}'(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\xi}(\tau))_\sigma d\tau + \nu\int_0^t \langle \boldsymbol{F}\boldsymbol{v}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\xi}(\tau)\rangle_\sigma d\tau \\ &\qquad + \int_0^t \langle \boldsymbol{G}(\boldsymbol{v},\boldsymbol{v}), \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}(t) - \boldsymbol{\xi}(t)|^2_{0,2} \\ &\le \int_0^t \langle \boldsymbol{g}, \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|^2_{0,2}, \quad \forall t \in [0,T], \ \forall \boldsymbol{\xi} \in \boldsymbol{\mathcal{K}}^i_0(\gamma(w^{\varepsilon_0})). \end{aligned} \tag{5.9}$$

In the above formulation, for $i = 1,2$ the constraint sets $K^i(\gamma(w^{\varepsilon_0});t)$, the classes $\boldsymbol{\mathcal{K}}^i(\gamma(w^{\varepsilon_0}))$ of test functions and $\boldsymbol{\mathcal{K}}^i_0(\gamma(w^{\varepsilon_0}))$ of smooth test functions are defined by as follows:

$$w^{\varepsilon_0}(x,t) = \int_\Omega \rho_{\varepsilon_0}(x-y)w(y,t)dy, \ x \in \Omega,$$

$$K^1(\gamma(w^{\varepsilon_0});t) := \{\boldsymbol{z} \in \boldsymbol{V}_\sigma \mid |\boldsymbol{z}| \le \gamma(w^{\varepsilon_0}(\cdot,t)) \text{ a.e. on } \Omega\}, \ t \in [0,T], \tag{5.10}$$

$$\boldsymbol{\mathcal{K}}^1(\gamma(w^{\varepsilon_0})) := \{\boldsymbol{\xi} \in L^2(0,T;\boldsymbol{V}_\sigma) \mid \boldsymbol{\xi}(t) \in K^1(\gamma(w^{\varepsilon_0};t) \text{ a.e. } t \in (0,T)\}, \tag{5.11}$$

$$\boldsymbol{\mathcal{K}}^1_0(\gamma(w^{\varepsilon_0})) := \left\{ \boldsymbol{\xi} \in C^1([0,T];\boldsymbol{W}_\sigma) \;\middle|\; \begin{array}{l} |\boldsymbol{\xi}| \le \gamma(w^{\varepsilon_0}) \text{ on } Q, \\ \operatorname{supp}(|\boldsymbol{\xi}|) \subset \hat{Q}(\gamma(w^{\varepsilon_0} > 0)) \end{array} \right\} \tag{5.12}$$

and

$$K^2(\gamma(w^{\varepsilon_0});t) := \{\boldsymbol{z} \in \boldsymbol{V}_\sigma \mid |\nabla\boldsymbol{z}| \le \gamma(w^{\varepsilon_0}(\cdot,t)) \text{ a.e. on } \Omega\}, \ t \in [0,T], \tag{5.13}$$

$$\boldsymbol{\mathcal{K}}^2(\gamma(w^{\varepsilon_0})) := \{\boldsymbol{\xi} \in L^2(0,T;\boldsymbol{V}_\sigma) \mid \boldsymbol{\xi}(t) \in K^2(\gamma(w^{\varepsilon_0};t) \text{ a.e. } t \in (0,T)\}, \tag{5.14}$$

$$\mathcal{K}_0^2(\gamma(w^{\varepsilon_0})) := \left\{ \boldsymbol{\xi} \in C^1([0,T]; \boldsymbol{W}_\sigma) \;\middle|\; \begin{array}{l} |\nabla \boldsymbol{\xi}| \le \gamma(w^{\varepsilon_0}) \text{ on } Q, \\ \mathrm{supp}\,(|\nabla \boldsymbol{\xi}|) \subset \hat{Q}(\gamma(w^{\varepsilon_0} > 0)) \end{array} \right\}. \tag{5.15}$$

The initial datum $\boldsymbol{v}_0$ and source $\boldsymbol{g}$ are respectively prescribed in $\boldsymbol{V}_\sigma$ and in $L^2(0,T;\boldsymbol{H}_\sigma)$.

Definition 5.1. For given data

$$w_0 \in L^2(\Omega),\ h \in L^2(Q),\ \ \boldsymbol{g} \in L^2(0,T;\boldsymbol{H}_\sigma),\ \boldsymbol{v}_0 \in \boldsymbol{H}_\sigma(\Omega),$$

our problem, referred to $SNS^i(\beta,\gamma;h,\boldsymbol{g},w_0,\boldsymbol{v}_0)$, $i=1,2$, is to find a pair of functions $\{w,\boldsymbol{v}\}$ from $[0,T]$ into $V^* \times \boldsymbol{H}_\sigma$ satisfying the following (i), (ii), (iii) and (iv):

(i) $w \in W^{1,2}(0,T;V^*) \cap L^\infty(Q) \cap C_w([0,T];L^2(\Omega))$, $w(0)=w_0$ and $t \mapsto \int_\Omega \hat{\beta}(x,t)dx$ is absolutely continuous on $[0,T]$,

(ii) $\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{H}_\sigma) \cap \mathcal{K}^i(\gamma(w^{\varepsilon_0}))$, $\boldsymbol{v}(0)=\boldsymbol{v}_0$ and $t \mapsto (\boldsymbol{v}(t),\boldsymbol{\xi}(t))_\sigma$ is of bounded variation on $[0,T]$ for every $\boldsymbol{\xi} \in C^1([0,T];\boldsymbol{W}_\sigma)$ with

$$\mathrm{supp}\,(|\boldsymbol{\xi}|) \subset \hat{Q}(\gamma(w^{\varepsilon_0})>0) \text{ for } i=1,\ \ \mathrm{supp}\,(|\nabla\boldsymbol{\xi}|) \subset \hat{Q}(\gamma(w^{\varepsilon_0})>0) \text{ for } i=2,$$

(iii) w satisfies the evolution equation of (5.6) (hence (5.7)).

(iv) $\boldsymbol{v}$ satisfies (5.9).

Such a pair of functions $\{w,\boldsymbol{v}\}$ is called a weak solution of $SNS^i(\beta,\gamma;h,\boldsymbol{g},w_0,\boldsymbol{v}_0)$.

The existence results are stated in the following two theorems.

Theorem 5.1. (Non-degenerate case) *Assume that the data w_0, h, $\boldsymbol{v}_0$ and $\boldsymbol{g}$ satisfy*

$$w_0 \in L^\infty(\Omega), \quad h \in L^\infty(Q), \tag{5.16}$$

and

$$\begin{array}{l} \boldsymbol{v}_0 \in \boldsymbol{V}_\sigma,\ \boldsymbol{g} \in L^2(0,T;\boldsymbol{H}_\sigma), \\ |\boldsymbol{v}_0| \in L^\infty(\Omega),\ |\boldsymbol{v}_0| \le \gamma(w_0^{\varepsilon_0}) \textit{ a.e. on } \Omega \textit{ in the case of } i=1, \\ |\nabla\boldsymbol{v}_0| \in L^\infty(\Omega),\ |\nabla\boldsymbol{v}_0| \le \gamma(w_0^{\varepsilon_0}) \textit{ a.e. on } \Omega \textit{ in the case of } i=2, \end{array} \tag{5.17}$$

where $w_0^{\varepsilon_0}(x) = \int_\Omega \rho_{\varepsilon_0}(x-y)w_0(y)dy$, $x \in \Omega$. Moreover, assume that β and γ satisfy (5.1) and (5.2), respectively. Then $SNS^i(\beta,\gamma;h,\boldsymbol{g},w_0,\boldsymbol{v}_0)$, $i=1,2$, admits at least one weak solution $\{w,\boldsymbol{v}\}$.

Remark 5.1. In the non-degenerate case of γ the classes $\mathcal{K}_0^i(\gamma(w^{\varepsilon_0}))$, $i=1,2$, of admissible test functions are given by

$$\mathcal{K}_0^1(\gamma(w^{\varepsilon_0})) := \{\boldsymbol{\xi} \in C^1([0,T];\boldsymbol{W}_\sigma) \mid |\boldsymbol{\xi}| \le \gamma(w^{\varepsilon_0}) \text{ on } Q\},$$

and

$$\mathcal{K}_0^2(\gamma(w^{\varepsilon_0})) := \{\boldsymbol{\xi} \in C^1([0,T];\boldsymbol{W}_\sigma) \mid |\nabla\boldsymbol{\xi}| \le \gamma(w^{\varepsilon_0}) \text{ on } Q\}.$$

Therefore, just as the case of Theorem 3.1, the mathematical treatment of non-degenerate case is much easier than the degenerate one.

Theorem 5.2. (Degenerate case) *Assume that (5.1) and (5.3) are satisfied. For the data w_0, h, $\boldsymbol{v}_0$ and $\boldsymbol{g}$, in addition to (5.16) and (5.17) for $i=1$, suppose that*

$$\text{supp}\,(|\boldsymbol{v}_0|) \subset \{x \in \Omega \mid \gamma(w_0^{\varepsilon_0}(x)) > 0\}. \tag{5.18}$$

Then $SNS^1(\beta, \gamma; h, \boldsymbol{g}, w_0, \boldsymbol{v}_0)$ admits at least one weak solution $\{w, \boldsymbol{v}\}$ in the sense of Definition 5.1 for $i=1$.

The solvability of $SNS^2(\beta, \gamma; h, \boldsymbol{g}, w_0, \boldsymbol{v}_0)$ under gradient constraint is an open question in the degenerate case in which some difficulties arise just as in the case of $NS^2(\gamma; \boldsymbol{g}, \boldsymbol{v}_0)$.

5.2. Approximate problems

We approximate the functions β and γ by β_δ and $\gamma_{\delta,N}$ for each small $\delta > 0$ and large $N > 0$, which are defined by

$$\beta_\delta(r) := \beta(r) + \delta r, \quad \forall r \in \mathbf{R}$$

and

$$\gamma_{\delta,N}(r) := (\gamma(r) \vee \delta) \wedge N, \quad \forall r \in \mathbf{R}.$$

The proper, l.s.c. and convex function $\varphi_\delta(\cdot)$ is defined by

$$\varphi_\delta(z) := \begin{cases} \displaystyle\int_\Omega \hat{\beta}_\delta(z(x))dx, & \text{for } z \in L^2(\Omega), \\ \infty, & \text{for } z \in V^* - L^2(\Omega), \end{cases}$$

where $\hat{\beta}_\delta(r) := \int_0^r \beta_\delta(s)ds$ for $r \in \mathbf{R}$. It is easy to see that φ_δ converges to φ in the sense of Mosco [28] as $\delta \downarrow 0$. Also, the evolution equation

$$w'(t) + \partial_* \varphi_\delta(w(t)) + \text{div}\,(w(t)\boldsymbol{v}(t)) = h(t) \;\; \text{in } V^*, \text{ a.e. } t \in (0,T), \quad w(0) = w_0, \tag{5.19}$$

is formulated similarly in the case (5.6). This is equivalently written in the form:

$$w'(t) + F\beta_\delta(w(t)) + \text{div}\,(w(t)\boldsymbol{v}(t)) = h(t) \;\; \text{in } V^*, \text{ a.e. } t \in (0,T), \quad w(0) = w_0. \tag{5.20}$$

Lemma 5.1. *Let $\{w_n\}$ and $\{\boldsymbol{v}_n\}$ be bounded sequences in $L^\infty(Q)$ and $L^2(0,T;\boldsymbol{V}_\sigma)$, respectively, and let $w \in L^\infty(Q)$ and $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V}_\sigma)$ such that*

$$w_n \to w \text{ weakly in } L^2(Q), \quad \boldsymbol{v}_n \to \boldsymbol{v} \text{ in } L^2(0,T;\boldsymbol{H}_\sigma).$$

Then

$$\text{div}\,(w_n \boldsymbol{v}_n) \to \text{div}\,(w\boldsymbol{v}) \;\; \text{weakly in } L^2(0,T;V^*). \tag{5.21}$$

Moreover, if $w_n \to w$ in $L^2(Q)$, then the convergence of (5.21) holds in $L^2(0,T;V^*)$.

Proof. First we note by the boundedness of $\{w_n\}$ in $L^\infty(Q)$ that $w_n \to w$ weakly* in $L^\infty(Q)$. For any function $z^* \in L^2(0,T;V^*)$ we have

$$
\begin{aligned}
& \left| \int_0^T (\operatorname{div}(w_n \boldsymbol{v}_n - w\boldsymbol{v}), z^*)_* dt \right| \\
= & \left| \int_Q \operatorname{div}(w_n \boldsymbol{v}_n - w\boldsymbol{v}) F^{-1} z^* dxdt \right| \\
= & \left| \int_Q ((w_n - w)\boldsymbol{v} + w_n(\boldsymbol{v}_n - \boldsymbol{v})) \cdot \nabla F^{-1} z^* dxdt \right| \\
\le & \left| \int_Q (w_n - w)\boldsymbol{v} \cdot \nabla F^{-1} z^* dxdt \right| + \int_Q |w_n||\boldsymbol{v}_n - \boldsymbol{v}||\nabla F^{-1} z^*| dxdt \\
\le & \left| \int_Q (w_n - w)\boldsymbol{v} \cdot \nabla F^{-1} z^* dxdt \right| + C|\boldsymbol{v}_n - \boldsymbol{v}|_{L^2(0,T;\boldsymbol{H}_\sigma)} |z^*|_{L^2(0,T;V^*)}, \\
\to & \ 0 \ \ as\ n \to \infty,
\end{aligned}
$$

where $C = \sup_{n\in\mathbf{N}} |w_n|_{L^\infty(Q)}$. Hence (5.21) is obtained. In particular, if $w_n \to w$ in $L^2(Q)$, then the last convergence holds in the strong topology of $L^2(0,T;V^*)$. ♢

Lemma 5.2. *Assume that (5.18) holds and let $\boldsymbol{v}$ be any function in $L^2(0,T;\boldsymbol{V}_\sigma)$. Then we have:*

(a) For each $\delta > 0$, problem (5.6) (hence (5.7)) has a unique solution w in $W^{1,2}(0,T;V^*) \cap L^2(0,T;V) \cap C([0,T];L^2(\Omega))$ such that $\sqrt{t}w' \in L^2(0,T;L^2(\Omega))$ and $\sqrt{t}|\nabla w|_{L^2(\Omega)} \in L^\infty(0,T)$.

(b) For each $\delta > 0$, the solution w of (5.6) satisfies that

$$
\sup_{t\in[0,T]} |w(t)|^2_{L^2(\Omega)} + \delta |w|^2_{L^2(0,T;V)} \le |w_0|^2_{L^2(\Omega)} + \frac{C_2^2}{\delta} |h|^2_{L^2(Q)}, \tag{5.22}
$$

$$
\sup_{t\in[0,T]} \int_\Omega \hat\beta_\delta(w(t))dx + \frac{1}{2}|\beta_\delta(w)|^2_{L^2(0,T;V)} \le \int_\Omega \hat\beta(w_0)dx + |w_0|^2_{L^2(\Omega)} + \frac{C_2^2}{2}|h|^2_{L^2(Q)}, \tag{5.23}
$$

$$
|w|_{L^\infty(Q)} \le R_0, \quad \forall \delta \in (0,1], \tag{5.24}
$$

where C_2 is a positive constant satisfying $|z|_{L^2(\Omega)} \le C_2 |z|_V$ for all $z \in V$ and R_0 is a positive constant independent of $\delta \in (0,1]$ and $\boldsymbol{v}$.

Proof. There are several approaches to the solvability of the approximate problem (5.7). Here we are going to use the L-pseudomonotone theory (cf. [8, 9]) for it.

First we consider the case of $\boldsymbol{v} \in L^\infty(Q)^3$. In this case we observe that the operator $Aw := -\Delta(\beta_\delta(w)) + \operatorname{div}(w\boldsymbol{v})$ is coercive and L-pseudomotone from $L^2(0,T;V)$ into $L^2(0,T;V^*)$ in case $L = L_{w_0}$ is the time-derivative $\frac{d}{dt}$ associated for constraint $K(t) = V$ (see section 2). Therefore, applying a result in [8, 9], the range of $L_{w_0} + A$ is the whole of $L^2(0,T;V^*)$; namely, given $h \in L^2(Q)$, there is $w \in L^2(0,T;V)$ such that $L_{w_0}w + Aw = h$ in $L^2(0,T;V^*)$. Moreover, since $\beta_\delta(\cdot)$ is bi-Lipschitz continuous on $\mathbf{R}$, it follows from the general theory in [24; Chapter 2] that the solution w of (5.20) has the required regularity properties, and as to the uniqueness of solutions, for two solutions $w_k,\ k = 1, 2$, of (5.20) associated for the initial datum $w_{k0} \in L^2(\Omega)$ and the source $h_k \in L^2(Q)$ we have for all $t \in [0, T]$

$$|(w_1(t) - w_2(t))^+|_{L^1(\Omega)} \le |(w_{10} - w_{20})^+|_{L^1(\Omega)} + \int_0^t |(h_1(\tau) - h_2(\tau))^+|_{L^1(\Omega)} d\tau,$$

where $(\cdot)^+$ denotes the positive part of $(\cdot)$. In particular, if $w_{10} = w_{20}$ and $h_1 = h_2$, then this shows $w_1 = w_2$ on Q, namely the uniqueness of the solution of (5.20). Thus we have (a).

Next we show (b). Multiplying (5.20) by w and noting that

$$\begin{aligned}\int_\Omega \mathrm{dvi}\,(w(x,t)\boldsymbol{v}(x,t))w(x,t)dx &= -\int_\Omega w(x,t)\boldsymbol{v}(x,t)\cdot\nabla w(x,t)dx \\ &= -\int_\Omega \boldsymbol{v}(x,t)\cdot\nabla\left(\frac{1}{2}w(x,t)^2\right)dx \\ &= \frac{1}{2}\int_\Omega \mathrm{div}\,\boldsymbol{v}(x,t)w(x,t)^2 dx = 0\end{aligned}$$

we have that

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2_{L^2(\Omega)} + \delta\int_\Omega \{\beta'(w(x,t))|\nabla w(x,t)|^2 + \delta|\nabla w(x,t)|^2\}dx$$

$$+n_0\int_\Gamma \{\beta(w(x,t))w(x,t) + \delta|w(x,t)|^2\}d\Gamma \le \int_\Omega |h(x,t)||w(x,t)|dx$$

for all $t \in [0, T]$. By integrating it over $[0, T]$ in time, we have an estimate of the form (5.22). The estimate (5.23) is similarly obtained. In fact, multiply (5.20) by $\beta_\delta(w)$. Then

$$\frac{d}{dt}\int_\Omega \hat{\beta}_\delta(x,t)dx + \int_\Omega |\nabla\beta_\delta(w(x,t))|^2 dx + n_0\int_\Gamma |\beta_\delta(w)|^2 d\Gamma \le \int_\Omega h\beta_\delta(w(x,t))dx.$$

Integrating this over $[0, t]$ in time, we easily get an estimate of the form (5.23).

Next, let R_1 be a positive constant such that $|w_0|_{L^\infty(\Omega)} \le R_1$ and $|h|_{L^\infty(Q)} \le R_1$. Note that the solution w of (5.20) satisfies

$$w_t - \Delta\beta_\delta(w) + \boldsymbol{v}\cdot\nabla w = h \ \text{ a.e. in } Q,\ \frac{\partial\beta_\delta(w)}{\partial n} + n_0\beta_\delta(w) = 0 \text{ a.e. on } \Sigma.$$

Multiplying this by $(w - R_1 - R_1 t)^+$, we obtain that

$$\int_\Omega (w_t - R_1)(w - R_1 - R_1 t)^+ dx + \int_\Omega \beta'_\delta(w)|\nabla(w - R_1 - R_1 t)^+|^2 dx$$

$$+\int_\Omega \frac{1}{2}\boldsymbol{v}\cdot\nabla[(w - R_1 - R_1 t)^+]^2 dx = \int_\Omega (h - R_1)(w - R_1 - R_1 t)^+ dx \le 0.$$

In the above inequality the second integral is non-negative and third integral is zero becaose of the divergencefreeness of $\boldsymbol{v}$, so that

$$\frac{d}{dt}\int_\Omega |(w(t) - R_1 - R_1 t)^+|^2 dx \le 0 \ \text{ a.e. } t \in [0, T].$$

Hence

$$\int_\Omega |(w(t) - R_1 - R_1 t)^+|^2 dx \le \int_\Omega |(w_0 - R_1)^+|^2 dx = 0, \ \forall t \in [0,T],$$

namely $w \le R_1 + R_1 T =: R_0$ a.e. on Q and (5.24) is obtained.

In the general case of $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V}_\sigma)$, we approximate $\boldsymbol{v}$ by a bounded solenoidal function $\boldsymbol{v}_\varepsilon$, $\varepsilon > 0$, so that $|\boldsymbol{v} - \boldsymbol{v}_\varepsilon|_{L^2(0,T;\boldsymbol{V}_\sigma)} \le \varepsilon$. For each $\varepsilon > 0$ we denote by w_ε the solution of (5.18) associated for convection vector $\boldsymbol{v}_\varepsilon$. Then, by the above results, estimates (5.22), (5.23) and (5.24) hold true and they are independent of ε. By using these uniform estimates and Lemma 5.1, it is not difficult to show that w_ε converges to the solution w of (5.19) as $\varepsilon \to 0$, and we have the conclusion of the lemma. ♢

According to Lemma 5.2, we see that the solution w of (5.6) (hence (5.7)) is constructed in the class

$$\mathcal{Y} := \left\{ w \in W^{1,2}(0,T;V^*) \ \middle| \ \begin{array}{l} w(0) = w_0, \quad \beta_\delta(w) \in L^2(0,T;V), \\ (5.22),\ (5.23) \text{ and } (5.24) \text{ hold for all } \delta \in (0,1] \end{array} \right\}.$$

With the functions β_δ and $\gamma_{\delta,N}$ approximate problem $SNS^i(\beta, \gamma_{\delta,N}; h, \boldsymbol{g}, w_0, \boldsymbol{v}_0)$ is formulated as follows.

We use the notation $K^i(\gamma_{\delta,N};t)$, $\boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}(w^{\varepsilon_0}))$ and $\boldsymbol{\mathcal{K}}^i_0(\gamma_{\delta,N}(w^{\varepsilon_0}))$, etc., which are defined by (5.12)-(5.17) with γ replaced by $\gamma_{\delta,N}$.

Definition 5.2. We denote by $SNS^i(\beta_\delta, \gamma_{\delta,N}; h, \boldsymbol{g}, w_0, \boldsymbol{v}_0)$ the problem to find a pair of functions $\{w, \boldsymbol{v}\}$ which satisfies:

(i) $w \in W^{1,2}(0,T;V^*) \cap L^2(0,T;V) \cap C([0,T];L^2(\Omega))$, $w(0) = w_0$ and

$$w'(t) + F(\beta_\delta(w(t))) + \operatorname{div}(w(t)\boldsymbol{v}(t)) = h(t) \ \text{ in } V^*, \ \text{a.e. } t \in (0,T),$$

(ii) $\boldsymbol{v} \in C([0,T];\boldsymbol{H}_\sigma) \cap \boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}(w^{\varepsilon_0}))$, $\boldsymbol{v}(0) = \boldsymbol{v}_0$ and

$$\int_0^t (\boldsymbol{\xi}'(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\xi}(\tau))_\sigma d\tau + \nu \int_0^t \langle \boldsymbol{F}\boldsymbol{v}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\xi}(\tau)\rangle_\sigma d\tau$$

$$+ \int_0^t \langle \boldsymbol{G}(\boldsymbol{v},\boldsymbol{v}), \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}(t) - \boldsymbol{\xi}(t)|^2_{0,2}$$

$$\le \int_0^t \langle \boldsymbol{g}, \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|^2_{0,2}, \quad \forall t \in [0,T], \ \forall \boldsymbol{\xi} \in \boldsymbol{\mathcal{K}}^i_0(\gamma_{\delta,N}(w^{\varepsilon_0})),$$

As to the approximate problem $SNS^i(\beta_\delta, \gamma_{\delta,N}; h, \boldsymbol{g}, w_0, \boldsymbol{v}_0)$ we prove:

Proposition 5.1. *Assume (5.16) and (5.17) hold for the data. Then for each $\delta > 0$ and $N > 0$ there is at least one solution $\{w_{\delta,N}, \boldsymbol{v}_{\delta,N}\}$ of $SNS^i(\beta_\delta, \gamma_{\delta,N}; h, \boldsymbol{g}, w_0, \boldsymbol{v}_0)$ in the sense of Definition 5.2 such that*

$$|w_{\delta,N}|_{L^\infty(Q)} \le R_0 \tag{5.25}$$

and

$$\sup_{t\in[0,T]} |\boldsymbol{v}_{\delta,N}(t)|_{0,2}^2 + \nu|\boldsymbol{v}_{\delta,N}|_{L^2(0,T;\boldsymbol{V}_\sigma)}^2 \le |\boldsymbol{v}_0|_{0,2}^2 + \frac{C_0^2}{\nu}|\boldsymbol{g}|_{L^2(0,T;\boldsymbol{H}_\sigma)}^2, \tag{5.26}$$

where R_0 is the same one as in (b) of Lemma 5.1 and C_0 is a positive constant satisfying $|\boldsymbol{z}|_{0,2} \le C_0|\boldsymbol{z}|_{1,2}$ for all $\boldsymbol{z} \in \boldsymbol{V}_\sigma$.

Proof. We shall prove the proposition by the Schauder fixed point argument.
(Step 1) Let $\bar{\boldsymbol{v}}$ be any function in $L^2(0,T;\boldsymbol{V}_\sigma) \cap L^\infty(Q)^3$ and denote by w the unique solution of

$$w'(t) + F\beta_\delta(w(t)) + \mathrm{div}\,(w(t)\bar{\boldsymbol{v}}(t)) = h(t) \text{ in } V^*, \text{ a.e. } t \in (0,T), \quad w(0) = w_0. \tag{5.27}$$

By Lemma 5.1, $w \in W^{1,2}(0,T;V^*) \cap L^2(0,T;V)$ and $|w|_{L^\infty(Q)} \le R_0$. Since $\gamma_{\delta,N} \ge \delta$ on $\mathbf{R}$, we have

$$\frac{\delta}{C_1} B_{\boldsymbol{W}_\sigma^*}(0) \le K^i(\gamma_{\delta,N}(w^{\varepsilon_0});t), \quad \forall t \in [0,T], \tag{5.28}$$

where C_1 is a positive constant satisfying

$$|\boldsymbol{z}|_{C(\overline{\Omega})^3} \le C_1|\boldsymbol{z}|_{\boldsymbol{W}_\sigma} \text{ in the case of } i=1; \quad |\nabla \boldsymbol{z}|_{C(\overline{\Omega})^3} \le C_1|\boldsymbol{z}|_{\boldsymbol{W}_\sigma} \text{ in the case of } i=2.$$

In particular, if γ is non-degenerate, i.e. $\gamma \ge c_*$, then

$$\frac{c_*}{C_1} B_{\boldsymbol{W}_\sigma^*}(0) \le K^i(\gamma_{\delta,N}(w^{\varepsilon_0});t), \quad \forall t \in [0,T], \ \forall \delta \in (0,c_*], \ \forall \text{large } N > 0.$$

Now, consider the variational inequality:

$$\begin{gathered}
\boldsymbol{v} \in C([0,T];\boldsymbol{H}_\sigma) \cap \boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}(w^{\varepsilon_0})), \ \boldsymbol{v}(0) = \boldsymbol{v}_0, \\
\int_0^t (\boldsymbol{\xi}'(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\xi}(\tau))_\sigma d\tau + \nu \int_0^t \langle \boldsymbol{F}\boldsymbol{v}(\tau), \boldsymbol{v}(\tau) - \boldsymbol{\xi}(\tau)\rangle_\sigma d\tau \\
+ \int_0^t \langle \boldsymbol{G}(\boldsymbol{v},\boldsymbol{v}), \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}(t) - \boldsymbol{\xi}(t)|_{0,2}^2 \\
\le \int_0^t \langle \boldsymbol{g}, \boldsymbol{v} - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|_{0,2}^2, \quad \forall t \in [0,T], \ \forall \boldsymbol{\xi} \in \boldsymbol{\mathcal{K}}_0^i(\gamma_{\delta,N}(w^{\varepsilon_0})),
\end{gathered} \tag{5.29}$$

Then, by (5.28), Proposition 3.1 gives that this variational inequality possesses one and only one solution $\boldsymbol{v}$ together with estimates:

$$\sup_{t\in[0,T]} |\boldsymbol{v}(t)|_{0,2}^2 \le |\boldsymbol{v}_0|_{0,2}^2 + \frac{C_0}{\nu}|\boldsymbol{g}|_{L^2(0,T;\boldsymbol{H}_\sigma)}^2 =: N_1, \ \nu|\boldsymbol{v}|_{L^2(0,T;\boldsymbol{V}_\sigma)}^2 \le N_1. \tag{5.30}$$

Furthermore, as is easily seen, the weak solution $\boldsymbol{v}$ of (5.29) belongs to the set $Z_2(\frac{\delta}{C_1}, N_1', \boldsymbol{v}_0)$, where N_1' is a positive constant depending only on N_1 and $\delta > 0$. Therefore, by Lemma 2.2,

$$\mathrm{Var}_{\boldsymbol{W}_\sigma^*}(\boldsymbol{v}) \le N_1' + \frac{C_0}{\delta}N_1' + \frac{C_0}{2\delta}|\boldsymbol{v}_0|_{0,2}^2 =: C^*(\delta). \tag{5.31}$$

In particular, when γ is non-degenerate, we have

$$\mathrm{Var}_{\boldsymbol{W}_\sigma^*}(\boldsymbol{v}) \le N_1' + \frac{C_0}{c_*}N_1' + \frac{C_0}{2c_*}|\boldsymbol{v}_0|_{0,2}^2 =: C^*.$$

Now, we consider the set $\mathcal{X}_\delta$ given by:

$$\mathcal{X}_\delta := \left\{ \boldsymbol{v} \in L^2(0,T;\boldsymbol{V}_\sigma) \;\middle|\; \begin{array}{l} \nu|\boldsymbol{v}|^2_{L^2(0,T;\boldsymbol{V}_\sigma)} \le N_1, \\ |\boldsymbol{v}|^2_{L^\infty(0,T;\boldsymbol{H}_\sigma)} \le N_1, \\ \mathrm{Var}_{\boldsymbol{W}^*_\sigma}(\boldsymbol{v}) \le C^*(\delta) \end{array} \right\}. \tag{5.32}$$

By Lemma 2.1, $\mathcal{X}_\delta$ is a compact convex subset of $L^2(0,T;\boldsymbol{H}_\sigma)$. Also, in the non-degenerate case of γ, we consider

$$\mathcal{X}_{c_*} := \left\{ \boldsymbol{v} \in L^2(0,T;\boldsymbol{V}_\sigma) \;\middle|\; \begin{array}{l} \nu|\boldsymbol{v}|^2_{L^2(0,T;\boldsymbol{V}_\sigma)} \le N_1, \\ |\boldsymbol{v}|^2_{L^\infty(0,T;\boldsymbol{H}_\sigma)} \le N_1, \\ \mathrm{Var}_{\boldsymbol{W}^*_\sigma}(\boldsymbol{v}) \le C^* \end{array} \right\}. \tag{5.33}$$

We denote by $\mathcal{S}_1$ the mapping which assigns to $\bar{\boldsymbol{v}} \in \mathcal{X}_\delta$ the solution w of (5.27), namely $w = \mathcal{S}_1\bar{\boldsymbol{v}}$, and by $\mathcal{S}_2$ the mapping which assigns to w the weak solution $\boldsymbol{v}$ of (5.29), $\boldsymbol{v} = \mathcal{S}_2 w$. The composition of $\mathcal{S} = \mathcal{S}_2\mathcal{S}_1$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ is a mapping which maps $\mathcal{X}_\delta$ into itself, and $\boldsymbol{v} = \mathcal{S}\bar{\boldsymbol{v}}$.

(Step 2) We now show the continuity of $\mathcal{S}$ in $\mathcal{X}_\delta$ with respect to the topology of $L^2(0,T;\boldsymbol{H}_\sigma)$. Let $\{\bar{\boldsymbol{v}}_n\}$ be a sequence in $\mathcal{X}_\delta$ such that $\bar{\boldsymbol{v}}_n \to \bar{\boldsymbol{v}}$ in $L^2(0,T;\boldsymbol{H}_\sigma)$ and put $w_n := \mathcal{S}_1\bar{\boldsymbol{v}}_n$, which is the solution of

$$w_n' + F\beta_\delta(w_n) + \mathrm{div}\,(w_n\bar{\boldsymbol{v}}_n) = h \text{ in } V^*, \text{ a.e. on } (0,T),\ w_n(0) = w_0. \tag{5.34}$$

Note (cf. [24; Chapter 2]) that $\{w_n\}$ is bounded in $W^{1,2}(0,T;V^*) \cap C([0,T];L^2(\Omega)) \cap L^2(0,T;V)$ with $|w_n|_{L^\infty(Q)} \le R_0$, whence it follows from the Aubin compactness theorem [3, 27] that $\{w_n\}$ is relatively compact in $L^2(Q)$. Now, choose any subsequence $\{w_{n_k}\}$ of $\{w_n\}$ so that $w_{n_k} \to w$ in $L^2(Q)$; for this subsequence we have that $w_{n_k} \to w$ weakly in $W^{1,2}(0,T;V^*) \cap L^2(0,T;V)$ (as $k \to \infty$) and $\nabla\beta_\delta(w_n) = \beta_\delta'(w_{n_k})\nabla w_{n_k} \to \beta_\delta'(w)\nabla w = \nabla\beta_\delta(w)$ weakly in $L^2(Q)^3$. Besides, by Lemma 5.2, $\mathrm{div}\,(w_{n_k}\bar{\boldsymbol{v}}_{n_k}) \to \mathrm{div}\,(w\bar{\boldsymbol{v}})$ in $L^2(0,T;V^*)$. Therefore, letting $k \to \infty$ in (5.34) with $n = n_k$ yields that

$$w' + F\beta_\delta(w) + \mathrm{div}\,(w\bar{\boldsymbol{v}}) = h \text{ in } V^*, \text{ a.e. on } (0,T),\ w(0) = w_0. \tag{5.35}$$

Since the solution of (5.35) is unique, it follows that the above argument holds true without extracting any subsequence from $\{w_n\}$. Namely we have shown that $w_n = \mathcal{S}_1\bar{\boldsymbol{v}}_n$ converges to $w = \mathcal{S}_1\bar{\boldsymbol{v}}$ in weakly in $W^{1,2}(0,T;V^*) \cap L^2(0,T;V)$ and in $L^2(Q)$.

Next, put $\boldsymbol{v}_n := \mathcal{S}_2 w_n$ which is the weak solution of

$$\begin{gathered} \boldsymbol{v}_n \in C([0,T];\boldsymbol{H}_\sigma) \cap \boldsymbol{\mathcal{K}}^i(\gamma_{\delta,N}(w_n^{\varepsilon_0})),\ \boldsymbol{v}_n(0) = \boldsymbol{v}_0, \\ \int_0^t (\boldsymbol{\xi}'(\tau), \boldsymbol{v}_n(\tau) - \boldsymbol{\xi}(\tau))_\sigma d\tau + \nu\int_0^t \langle \boldsymbol{F}\boldsymbol{v}_n(\tau), \boldsymbol{v}_n(\tau) - \boldsymbol{\xi}(\tau)\rangle_\sigma d\tau \\ + \int_0^t \langle \boldsymbol{G}(\boldsymbol{v}_n,\boldsymbol{v}_n), \boldsymbol{v}_n - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}_n(t) - \boldsymbol{\xi}(t)|^2_{0,2} \\ \le \int_0^t \langle \boldsymbol{g}, \boldsymbol{v}_n - \boldsymbol{\xi}\rangle_\sigma d\tau + \frac{1}{2}|\boldsymbol{v}_0 - \boldsymbol{\xi}(0)|^2_{0,2}, \quad \forall t \in [0,T],\ \forall \boldsymbol{\xi} \in \boldsymbol{\mathcal{K}}^i_0(\gamma_{\delta,N}(w_n^{\varepsilon_0})). \end{gathered} \tag{5.36}$$

Since $w_n \to w$ in weakly in $W^{1,2}(0,T;V^*) \cap L^2(0,T;V)$, strongly in $L^2(Q)$ and $\{w_n\}$ is uniformly bounded on $\overline{Q}$, it follows that $w_n^{\varepsilon_0} \to w^{\varepsilon_0}$ in $C(\overline{Q})$ as well as $\gamma_{\delta,N}(w_n^{\varepsilon_0}) \to$

$\gamma_{\delta,N}(w^{\varepsilon_0})$ in $C(\overline{Q})$. This convergence shows that $K^i(\gamma_{\delta,N}(w_n^{\varepsilon_0});t) \Longrightarrow K^i(\gamma_{\delta,N}(w^{\varepsilon_0});t)$ on $[0,T]$ (see section 2). Therefore, by virtue of the convergence result in [17; Theorem 2.2], $\boldsymbol{v}_n$ converges to the weak solution $\boldsymbol{v}$ ($\in \mathcal{X}_\delta$) of (5.36) in $C([0,T];\boldsymbol{H}_\sigma)$, satisfying (5.30) and (5.31). Moreover, by Lemma 5.2, $\mathrm{div}\,(w_n\boldsymbol{v}_n) \to \mathrm{div}\,(w\boldsymbol{v})$ in $L^2(0,T;V^*)$, so that it follows from the general convergence result on evolution equations generated by subdifferentials (cf. [24; Chapter 1]) that w is the solution of

$$w' + F\beta_\delta(w) + \mathrm{div}\,(w\boldsymbol{v}) = h \text{ in } V^*, \text{ a.e. on } (0,T),\ w(0) = w_0.$$

Hence $\mathcal{S}_2 w_n = \boldsymbol{v}_n \to \boldsymbol{v} = \mathcal{S}_2 w$ in $L^2(0,T;\boldsymbol{H}_\sigma)$, which shows that $\boldsymbol{\mathcal{S}}\bar{\boldsymbol{v}}_n \to \boldsymbol{\mathcal{S}}\bar{\boldsymbol{v}}$ in $L^2(0,T;\boldsymbol{H}_\sigma)$. Thus $\mathcal{S}$ is continuous in $\mathcal{X}_\delta$ with respect to the topology of $L^2(Q)$.

Now, applying the fixed-point theorem to the continuous mapping $\mathcal{S} : \mathcal{X}_\delta \to \mathcal{X}_\delta$ we can find a fixed point $\boldsymbol{v}_{\delta,N}$ of $\mathcal{S}$, i.e. $\boldsymbol{v}_{\delta,N} = \boldsymbol{\mathcal{S}}\boldsymbol{v}_{\delta,N}$, which gives a weak solution $\{w_{\delta,N}, \boldsymbol{v}_{\delta,N}\}$ with $w_{\delta,N} = \mathcal{S}_1\boldsymbol{v}_{\delta,N}$. ♢

5.3. Proof of Theorems 5.1 and 5.2

(Non-degenerate case)

In the non-degenerate case we note that

$$\gamma_{\delta,N}(r) = \gamma(r) \wedge N =: \gamma_N(r), \quad \forall r \in \mathbf{R},\ \forall \delta \in (0, c_*],\ \forall \text{large } N.$$

By virtue of Proposition 5.1 and Lemma 5.2, for every $0 < \delta \le c_*$ and large N the approximate problem $SNS^i(\beta_\delta, \gamma_N; h, \boldsymbol{g}, w_0, \boldsymbol{v}_0)$ admits a pair of solutions $\{w_{\delta,N}, \boldsymbol{v}_{\delta,N}\}$ in the class $\mathcal{Y} \times \mathcal{X}_{c_*}$. Hence there are a sequence $\{\delta_n, N_n\}$ with $\delta_n \downarrow 0$ and $N_n \uparrow \infty$ (as $n \to \infty$) and a pair of function $\{w, \boldsymbol{v}\}$ with $\tilde{\beta} \in L^2(0,T;V)$ such that

$$\begin{cases} w_n := w_{\delta_n,N_n} \to w \text{ weakly}^* \text{ in } L^\infty(Q), \text{ weakly in } W^{1,2}(0,T;V^*), \\ \beta_{\delta_n}(w_n) \to \tilde{\beta} \text{ weakly in } L^2(0,T;V) \end{cases} \tag{5.37}$$

and

$$\begin{cases} \boldsymbol{v}_n := \boldsymbol{v}_{\delta_n,N_n} \to \boldsymbol{v} \text{ weakly in } L^2(0,T;\boldsymbol{V}_\sigma), \\ \qquad\qquad \text{weakly}^* \text{ in } L^\infty(0,T;\boldsymbol{H}_\sigma) \text{ and in } L^2(0,T;\boldsymbol{H}_\sigma). \end{cases} \tag{5.38}$$

Moreover, by (5.37), $w_n^{\varepsilon_0}(t) \in C^\infty(\overline{\Omega})$ and $\{w_n^{\varepsilon_0}\}$ is bounded in $W^{1,2}(0,T;C(\overline{\Omega}))$. This shows that $w_n^{\varepsilon_0} \to w^{\varepsilon_0}$ in $C(\overline{Q})$, so that $\gamma_{N_n}(w_n^{\varepsilon_0}) \to \gamma(w^{\varepsilon_0})$ in the following sense:

$$\begin{cases} \forall \kappa > 0, \gamma_{N_n}(w_n^{\varepsilon_0}) \to \gamma(w^{\varepsilon_0}) \text{ uniformly on } \overline{Q}(\gamma(w^{\varepsilon_0}) \le k), \\ \forall \text{large } M > 0,\ \exists M' \ (> M),\ \exists n_M \text{ such that} \\ \qquad \gamma_{N_n}(w_n^{\varepsilon_0}) > M \text{ on } \overline{Q}(\gamma > M'),\ \forall n \ge n_M. \end{cases} \tag{5.39}$$

First we prove:

Lemma 5.3. *$\tilde{\beta} = \beta(w)$, and $\beta_{\delta_n}(w_n) \to \beta(w)$ in $L^2(Q)$ and weakly in $L^2(0,T;V)$ as $n \to \infty$.*

Proof. Taking the inner product of the difference of two equations $w_n' - w_m' + F(\beta_{\delta_n}(w_n) - \beta_{\delta_m}(w_m)) + \operatorname{div}(w_n\boldsymbol{v}_n - w_m\boldsymbol{v}_m) = 0$ and $w_n - w_m$ in V^* and integrating the resultant in time, we have

$$\frac{1}{2}|w_n(t) - w_m(t)|^2_{V^*} + \int_0^t \int_\Omega (\beta(w_n) - \beta(w_m))(w_n - w_m)dxd\tau$$
$$+ \int_\Omega (\delta_n w_n - \delta_m w_m)(w_n - w_m)dxd\tau$$
$$= \int_0^t \int_\Omega (w_n\boldsymbol{v}_n - w_m\boldsymbol{v}_m) \cdot \nabla F^{-1}(w_n - w_m)dxd\tau, \quad \forall t \in [0,T],$$

so that

$$\frac{1}{2}|w_n(t) - w_m(t)|^2_{V^*} + \frac{1}{L_\beta}\int_0^t \int_\Omega |\beta(w_n) - \beta(w_m)|^2 dxd\tau$$
$$\leq \int_0^t \int_\Omega |\delta_n w_n - \delta_m w_m||w_n - w_m|dxd\tau \tag{5.40}$$
$$+ \int_0^t \int_\Omega (w_n\boldsymbol{v}_n - w_m\boldsymbol{v}_m) \cdot \nabla F^{-1}(w_n - w_m)dxd\tau,$$

for all $t \in [0,T]$, where L_β is a (positive) Lipschitz constant of β. We note here that $w_n(t) - w_m(t) \to 0$ weakly in $L^2(\Omega)$ for every $t \in [0,T]$ and F^{-1} is linear and bounded from $L^2(\Omega)$ into $H^2(\Omega)$ and compact from $L^2(\Omega)$ into V. This implies that $F^{-1}(w_n(t) - w_m(t)) \to 0$ in V for all $t \in [0,T]$ and the last integral in (5.40) converges to 0 as $n,\ m \to \infty$. As a consequence, $\beta(w_n)$ converges in $L^2(Q)$ as well as $\beta_{\delta_n}(w_n)$ in $L^2(Q)$ as $n \to \infty$. Since $w_n \to w$ weakly in $L^2(Q)$ and $\beta(\cdot)$ is a maximal monotone mapping in $L^2(Q)$. It follows from the demiclosedness of maximal monotone mappings that $\beta(w_n) \to \beta(w)$ in $L^2(Q)$, hence $\beta_{\delta_n}(w_n) = \beta(w_n) + \delta_n w_n \to \beta(w)$ in $L^2(Q)$.

Also, by (5.22), $\{\beta_{\delta_n}(w_n)\}$ is bounded in $L^2(0,T;V)$ which implies that $\beta_{\delta_n}(w_n) \to \beta(w)$ weakly in $L^2(0,T;V)$. ♢

Proof of Theorem 5.1: We denote $SNS^i(\beta_{\delta_n}, \gamma_{\delta_n,N_n}; h, \boldsymbol{g}, w_0, \boldsymbol{v}_0)$ by $SNS^i_{\delta_n,N_n}$. In the non-degenerate case, the approximate solution $\{w_n, \boldsymbol{v}_n\}$ of $SNS^i_{\delta_n,N_n}$ is constructed in $\mathcal{Y} \times \mathcal{X}_{c_*}$. We already have seen (5.37), (5.38) with $\tilde{\beta} = \beta(w)$ and (5.39). Since the estimates in the class $\mathcal{Y} \times \mathcal{X}_{c_*}$ are independent of parameter N_n, we can show by using convergence property (5.39) of obstacle functions $\gamma_{N_n}(w_n^{\varepsilon_0})$ (as mentioned in Remark 3.4) that the convergence of $\{w_n, \boldsymbol{v}_n\}$ are obtained just as in the proof of Proposition 5.1 and $\{w, \boldsymbol{v}\} \in \mathcal{Y} \times \mathcal{X}_{c^*}$ is a weak solution of $SNS^i(\beta, \gamma; h, \boldsymbol{g}, w_0, \boldsymbol{v}_0)$. The limit $\boldsymbol{v}$ of $\{\boldsymbol{v}_n\}$ may lose the continuity in $\boldsymbol{H}_\sigma$ in time, but it still stays in the class $\mathcal{X}_{c_*}$, whence $t \mapsto (\boldsymbol{v}(t), \boldsymbol{\xi}(t))_\sigma$ is of bounded variation on $[0,T]$ for every $\boldsymbol{\xi} \in C^1([0,T]; \boldsymbol{W}_\sigma)$. ♢

Proof of Theorem 5.2: We consider only the problem $SNS^1(\beta, \gamma; h, \boldsymbol{g}, w_0, \boldsymbol{v}_0)$. In the degenerate case, by Proposition 5.1, for given parameters $\delta_n \downarrow 0$ and $N_n \uparrow \infty$ our approximate solution $\{w_n, \boldsymbol{v}_n\} := \{w_{\delta_n,N_n}, \boldsymbol{v}_{\delta_n,N_n}\}$ was constructed in $\mathcal{Y} \times \mathcal{X}_{\delta_n}$ and we may assume by the uniform estimates (5.22), (5.23), (5.25) and (5.26) that $\{w_n, \boldsymbol{v}_n\}$ satisfies

$$w_n \to w \text{ weakly}^* \text{ in } L^\infty(Q),\ \text{ weakly in } W^{1,2}(0,T;V^*),$$
$$\beta_{\delta_n}(w_n) \to \beta(w) \text{ in } L^2(Q) \text{ and weakly in } L^2(0,T;V) \tag{5.41}$$

and

$$\boldsymbol{v}_n \to \boldsymbol{v} \text{ weakly in } L^2(0,T;\boldsymbol{V}_\sigma) \text{ and weakly}^* \text{ in } L^\infty(0,T;\boldsymbol{H}_\sigma). \tag{5.42}$$

From (5.41) it follows that

$$\begin{cases} \forall \kappa > 0, \gamma_{\delta_n,N_n}(w_n^{\varepsilon_0}) \to \gamma(w^{\varepsilon_0}) \text{ uniformly on } \overline{Q}(\gamma(w^{\varepsilon_0}) \le k), \\ \forall \text{large } M > 0, \exists M' \ (> M), \ \exists n_M \text{ such that} \\ \qquad \gamma_{\delta_n,N_n}(w_n^{\varepsilon_0}) > M \text{ on } \overline{Q}(\gamma > M'), \ \forall n \ge n_M. \end{cases} \tag{5.43}$$

Now, applying Lemma 4.6 with Corollary 4.3 to $\{\boldsymbol{v}_n\}$ and the sequence of obstacle functions $\{\gamma_{\delta_n,N_n}(w_n^{\varepsilon_0})\}$ satisfying (5.43), we conclude that

$$\boldsymbol{v}_n \to \boldsymbol{v} \ \text{ in } L^2(0,T;\boldsymbol{H}_\sigma). \tag{5.44}$$

Hence, by Lemma 5.1, $\mathrm{div}\,(w_n\boldsymbol{v}_n) \to \mathrm{div}\,(w\boldsymbol{v})$ weakly in $L^2(0,T;V^*)$ and w_n converges to w weakly in $W^{1,2}(0,T;V^*)$, which is the solution of

$$w' + F\beta(w) + \mathrm{div}\,(w\boldsymbol{v}) = h \ \text{ in } V^*, \text{ a.e. on } (0,T), \ w(0) = w_0.$$

Besides, just as in the proof of Theorem 3.2, it is obtained that $\boldsymbol{v}_n$ converges to $\boldsymbol{v}$ in the sense of (5.42) and (5.44) and the limit $\boldsymbol{v}$ satisfies the variational inequality (5.9) and (ii) for $i = 1$ in Definition 5.1. Thus $\{w, \boldsymbol{v}\}$ is a weak solution of $SNS^1(\beta,\gamma;h,\boldsymbol{g},w_0,\boldsymbol{v}_0)$. $\diamondsuit$

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