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GLOBAL WELL-POSEDNESS FOR A CAHN–HILLIARD EQUATION ON BOUNDED DOMAINS WITH PERMEABLE AND NON-PERMEABLE WALLS IN MAXIMAL REGULARITY SPACES

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Abstract. We consider the strong solution of the Cahn–Hilliard equation on bounded domains with permeable and non-permeable walls in maximal L_p regularity spaces. From the maximal L_p regularity result of the linear equation with the dynamic boundary condition, the fixed point theorem and a priori estimate, we prove that the solution exists uniquely and globally in time.

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1 Introduction

Let $0 < T < \infty$ be a some fixed time, $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain whose boundary $\Gamma := \partial\Omega$ is smooth. Denote $J := (0, T)$, $Q := J \times \Omega$ and $\Sigma := J \times \Gamma$. We consider the following Cahn–Hilliard equation

$$\partial_t u - \Delta\mu = 0 \qquad \text{in } Q, \tag{1}$$

$$\mu = -\Delta u + F'(u) \qquad \text{in } Q. \tag{2}$$

Here u is the order parameter, μ and F are the chemical and physical potentials, respectively. In this paper we improve one of following two boundary conditions:

$$\Delta\mu + b\partial_\nu\mu + c\mu = 0 \qquad \text{on } \Sigma, \tag{3}$$

$$-\alpha\Delta_\Gamma u + \partial_\nu u + G'(u) = \mu/b \qquad \text{on } \Sigma, \tag{4}$$

or

$$\Delta\mu + b\partial_\nu\mu - c\Delta_\Gamma\mu = 0 \qquad \text{on } \Sigma, \tag{5}$$

$$-\alpha\Delta_\Gamma u + \partial_\nu u + G'(u) = \mu/b \qquad \text{on } \Sigma. \tag{6}$$

The first one appears the case that the domain has porous (permeable) walls and the second one corresponds to non-permeable walls.

In the boundary conditions, α, b, c are positive constants, Δ_Γ is the Laplace–Beltrami operator on Γ, ν is the unit outward normal vector to Γ and G is the nonlinear term which comes from the surface energy. A typical example of F and G are $F(u) = (1/4)(u^2 - 1)^2$ and $G(u) = (g_s/2)u^2 - h_s u$ with $g_s > 0, h_s \neq 0$. We also treat the case $c = 0$ in subsection 2.4.

Our aim of this paper is to prove existence and uniqueness of this Cahn–Hilliard equation with these boundary conditions in maximal L_p spaces for $1 < p < \infty$. So far, the study of the Cahn–Hilliard equation has been considered in L_2 frameworks. The L_p approach has been done by the papers [18, 19] but only for the classical dynamic boundary condition. In the last decades, other type of boundary conditions has been considered and discussed in L_2 frameworks. See the next paragraph for the previous works. However, as far as we know, the study of L_p frameworks has not been treated under our boundary conditions yet. In this paper we fill this gap by a simple approach using the linear theory of abstract parabolic equations constructed in the paper [5]. The authors considered the equations called *relaxation type*, which contains our linearized Cahn–Hilliard equation with the boundary conditions we consider. So we obtain the maximal L_p regularity result on the linearized equations. For the nonlinear Cahn–Hilliard equation (1)–(2) on permeable walls (3)–(4) and on non-permeable walls (5)–(6), we prove local existence and uniqueness of solutions by fixed point argument. The key is to show the contraction property of non-linear term by restricting a small time interval and taking exponent p large, see Proposition 2.2 and Proposition 3.2. To extend global solutions, we use energy estimates from integration by parts. Combining with a priori estimates, we claim that the unique local solution does not blow up at any time, which means the solution is a global solution.

The Cahn–Hilliard equation is known as describing the spinodal decomposition of binary mixtures, which we can see in the cooling processes of alloys, glasses or polymer mixtures (see [1, 13, 16, 17]). For the study of the Cahn–Hilliard equation, various boundary conditions has been considered. At first, we would like to mention the following usual boundary conditions:

$$\partial_\nu \mu = 0 \qquad \text{on } \Sigma, \tag{7}$$

$$\partial_\nu u = 0 \qquad \text{on } \Sigma. \tag{8}$$

The condition (7) derives that the total mass $\int_\Omega u dx$ does not change for all time $t > 0$. The other condition (8) is called the variational boundary condition since it derives that the following bulk free energy

$$E_\Omega(u) := \int_\Omega \left(\frac{1}{2} |\nabla u|^2 + F(u) \right) dx \tag{9}$$

does not increase with (7). For the Cahn–Hilliard equation (1)–(2) with (7)–(8), the global well-posedness result and large time behavior were constructed. See [6, 21, 22].

However, in [13] it was proposed by physicists that one should add the following surface free energy

$$E_\Gamma(u) := \int_\Gamma \left(\frac{\alpha}{2} |\nabla_\Gamma u|^2 + G(u) \right) dS \tag{10}$$

to the bulk free energy $E_\Omega(u)$, where ∇_Γ is the surface gradient. Together with the no-flux boundary condition (7), the total energy $E(u) = E_\Omega(u) + E_\Gamma(u)$ makes non-increasing when the dynamic boundary condition

$$\alpha \Delta_\Gamma u - \partial_\nu u + G'(u) = \frac{1}{\Gamma_s} u_t \qquad \text{on } \Sigma, \tag{11}$$

is posed, with some $\Gamma_s > 0$. For this problem, see e.g., [3, 18, 20, 23]. We would like to mention the paper [18]. The authors of [18] obtained results on the maximal L_p regularity of the solution and asymptotic behavior of the solution of this problem. Moreover it has shown the existence of a global attractor. These results was extended to the non-isothermal setting by a similar maximal regularity result in [19].

The Wentzell boundary condition (3) we would like to study was proposed in the paper [8]. Thanks to the boundary condition (4), the total energy $E(u)$ is non-increasing:

$$\frac{d}{dt} E(u(t)) = -\int_\Omega |\nabla \mu|^2 dx - \frac{c}{b} \int_\Gamma \mu^2 dS \le 0 \quad (t > 0). \tag{12}$$

Since $\frac{d}{dt}(\int_\Omega u dx + \int_\Gamma u \frac{dS}{b}) = -c \int_\Gamma \mu \frac{dS}{b}$, the case $c = 0$ corresponds to the case of the conservation of the total mass in the bulk and on the boundary. In the paper [8] the existence and uniqueness of a global solution were proved via the Caginalp type equation, which is the similar method in [20]. Later in [9], these results were extended under more general assumptions. In the papers [23]($c > 0$) and [10]($c = 0$), it was shown that each solution of this model converges to a steady state as time goes to infinity and their convergence rate by using Łojasiewicz–Simon inequality.

In contrast to permeable walls, recently, the Cahn–Hilliard equation (1)–(2) with (5)–(6) in the non-permeable walls was considered, e.g., [2, 11, 12]. The first boundary condition (5) represents the Cahn–Hilliard equation on the boundary Γ. The second boundary condition (6) called the variational boundary condition (4) leads non-increasing for $E(u)$. In this system, $\int_\Omega u dx + \int_\Gamma u \frac{dS}{b}$ is constant. The existence and uniqueness of weak solutions and their asymptotic behavior were shown in [12]. The well-posedness results for this equation with singular potentials in [4] and numerical results in [7] were also studied. More recently, another boundary condition was proposed in [14] via an energetic variational approach that combines the least action principle and Onsager's principle of maximum energy dissipation.

In this paper we prove the global existence and uniqueness of the Cahn–Hilliard equation on permeable and non-permeable walls in maximal L_p regularity spaces. This article is organized as follows. In Section 2, we study the equation on permeable walls. In subsection 2.1, we give the linear theory. We use the general theory of maximal regularity of relaxation type proved by Denk–Prüss–Zacher [5]. We collect their result in Appendix A and apply it for the Cahn–Hilliard equation on permeable walls in Appendix B. In subsection 2.2, we give local well-popsedness of this equation by using usual fixed point argument. The estimate we use is essentially based on the paper in [19]. In subsection 2.3, we extend this local solution to the global solution by a energy estimate and a priori estimate. In subsection 2.4, we focus on the case $c = 0$ in the boundary condition (3). Since the estimates used in subsection 2.3 are different from the case $c > 0$, we calculate the case $c = 0$ again. We are able to get existence and uniqueness result as well. In Section 3, we study the equation on non-permeable walls. The strategy for non-permeable walls is almost same as Section 2, so we show a few estimates and give some comment, then we state our results.

Before we study the Cahn–Hilliard equation, we would like to mention about the equation on the boundary. In this paper we distinguish u, μ in the domain and u_Γ, μ_Γ on the boundary, but $u|_\Gamma = u_\Gamma, \mu|_\Gamma = \mu_\Gamma$, where "$|_\Gamma$" is the trace operator on the boundary Γ. Moreover for the boundary condition (3) and (5), we replace $(\Delta\mu)|_\Gamma$ with $\partial_t u_\Gamma$ since $\partial_t u = \Delta\mu$ in the domain Ω. So the equations we analyze are as follows

$$\begin{cases} \partial_t u = \Delta\mu, \quad \mu = -\Delta u + F'(u) & \text{in} \quad Q, \\ \partial_t u_\Gamma + b\partial_\nu \mu + c\mu_\Gamma = 0, \quad -\alpha\Delta_\Gamma u_\Gamma + \partial_\nu u + G'(u_\Gamma) = \frac{\mu_\Gamma}{b} & \text{on} \quad \Sigma, \\ u|_\Gamma = u_\Gamma, \quad \mu|_\Gamma = \mu_\Gamma & \text{on} \quad \Sigma, \\ u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

and

$$\begin{cases} \partial_t u = \Delta\mu, \quad \mu = -\Delta u + F'(u) & \text{in} \quad Q, \\ \partial_t u_\Gamma + b\partial_\nu \mu - c\Delta_\Gamma\mu_\Gamma = 0, \quad -\alpha\Delta_\Gamma u_\Gamma + \partial_\nu u + G'(u_\Gamma) = \frac{\mu_\Gamma}{b} & \text{on} \quad \Sigma, \\ u|_\Gamma = u_\Gamma, \quad \mu|_\Gamma = \mu_\Gamma & \text{on} \quad \Sigma, \\ u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

Here note that the unknown functions are u and u_Γ. We do not use the functions μ and μ_Γ except for energy estimates.

Throughout this paper, we use fractional Sobolev space $W_p^s(J, X)$ for a Banach space X, $s \in \mathbb{R}_{\geq 0} \setminus \mathbb{N}$ and $1 < p < \infty$, which is characterize as follows. Let $[s] \in \mathbb{N} \cup \{0\}$ and $\{s\} \in (0,1)$ be $s = [s] + \{s\}$. Then by using real interpolation method, it is

$$W_p^s(J, X) := (W_p^{[s]}(J, X), W_p^{[s]+1}(J, X))_{\{s\},p}.$$

Similarly, Besov space is defined as follows.

$$B_{p,p}^s(\Omega) := (W_p^{[s]}(\Omega), W_p^{[s]+1}(\Omega))_{\{s\},p}.$$

To treat nonlinear term, let $C^{m-}(\mathbb{R})(m \in \mathbb{N})$ be the space of all functions $f \in C^{m-1}(\mathbb{R})$ such that $\partial^\alpha f$ is Lipschitz continuous for each $|\alpha| = m$.

2 A Cahn–Hilliard equation on permeable walls

2.1 The linear theory

In this section we study the following linearized equation of the form

$$(*)\begin{cases} \partial_t v + \Delta^2 v = f & \text{in} \quad Q, \\ \partial_t v_\Gamma - b\partial_\nu \Delta v + bc\partial_\nu v - \alpha bc \Delta_\Gamma v_\Gamma = g & \text{on} \quad \Sigma, \\ v|_\Gamma = v_\Gamma, \quad -(\Delta v)|_\Gamma - b\partial_\nu v + \alpha b \Delta_\Gamma v_\Gamma = h & \text{on} \quad \Sigma, \\ v(0) = v_0 \quad \text{in } \Omega, \quad v_\Gamma(0) = v_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

Here the functions $f, g, h, v_0, v_{0\Gamma}$ are given and v, v_Γ are unknown. Since this linearized equation is included in the general framework studied by [5], we collect and write down these results in Appendix A, and apply it in Appendix B. Then we get the following linear theory.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^4 and $1 < p < \infty$ be $p \neq 5/4, 5/2, 5$. Let $\kappa_0 = 1/4 - 1/(4p), \kappa_1 = 1/2 - 1/(4p)$. Then the linearized Cahn–Hilliard equation $(*)$ admits a unique solution*

$$\begin{aligned}(v, v_\Gamma) \in Z \times Z_\Gamma := &\left(W_p^1(J, L_p(\Omega)) \cap L_p(J, W_p^4(\Omega))\right) \\ &\times \left(W_p^{1+\kappa_0}(J, L_p(\Gamma)) \cap W_p^1(J, W_p^{4\kappa_0}(\Gamma)) \cap L_p(J, W_p^{3+4\kappa_0}(\Gamma))\right)\end{aligned}$$

if and only if

$$\begin{aligned}(f, g, h) \in\ & X \times Y_0 \times Y_1 \\ :=\ & L_p(J, L_p(\Omega)) \times (W_p^{\kappa_0}(J, L_p(\Gamma)) \cap L_p(J, W^{4\kappa_0}(\Gamma))) \\ & \qquad \times (W_p^{\kappa_1}(J, L_p(\Gamma)) \cap L_p(J, W^{4\kappa_1}(\Gamma))), \\ (v_0, v_{0\Gamma}) \in\ & \pi Z \times \pi Z_\Gamma := B_{p,p}^{4-4/p}(\Omega) \times B_{p,p}^{4-4/p}(\Gamma),\end{aligned}$$

and the compatibility conditions

$$\begin{aligned}
&v_0|_\Gamma = v_{0\Gamma} && \text{on } \Gamma \quad \text{if} \quad p > 5/4,\\
&-(\Delta v_0)|_\Gamma - b\partial_\nu v_{0\Gamma} + \alpha b \Delta_\Gamma v_{0\Gamma} = h|_{t=0} && \text{on } \Gamma \quad \text{if} \quad p > 5/2,\\
&g|_{t=0} + b\partial_\nu \Delta v_0 - bc\partial_\nu v_0 + \alpha bc\Delta_\Gamma v_{0\Gamma} \in B^{1-5/p}_{p,p}(\Gamma) && \text{if} \quad p > 5.
\end{aligned}$$

are satisfied.

Remark 2.2. If we use time weighted L_p maximal regularity result, then we are able to relax the compatibility conditions while the regularity class of the solution for $t > 0$ is same, see [15].

2.2 Local well-posedness

In this section we prove the local well-posedness for the Cahn–Hilliard equation on permeable walls

$$(\mathrm{CH})_{\mathrm{per.}}\begin{cases}
\partial_t u + \Delta^2 u = \Delta F'(u) + f & \text{in } Q,\\
\partial_t u_\Gamma - b\partial_\nu \Delta u + bc\partial_\nu u - \alpha bc\Delta_\Gamma u_\Gamma = -b\partial_\nu F'(u) - bcG'(u_\Gamma) + g & \text{on } \Sigma,\\
u|_\Gamma = u_\Gamma, \quad -(\Delta u)|_\Gamma - b\partial_\nu u + \alpha b\Delta_\Gamma u_\Gamma = -F'(u)|_\Gamma + bG'(u_\Gamma) & \text{on } \Sigma,\\
u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma.
\end{cases}$$

Here $F \in C^{4-}(\mathbb{R}), G \in C^{2-}(\mathbb{R})$. The original equation we explained in the introduction is the case $f = g = 0$, but we are able to add non-homogeneous terms f, g. We will prove existence and uniqueness of this solution. So first we need to consider the compatibility conditions for the boundary. Let $(g, u_0, u_{0\Gamma}) \in Y_0 \times \pi Z \times \pi Z_\Gamma$ satisfy the following compatibility conditions

$$u_0|_\Gamma = u_{0\Gamma} \qquad \text{on } \Gamma \quad \text{if} \quad p > 5/4, \tag{13}$$

$$-(\Delta u_0)|_\Gamma - b\partial_\nu u_{0\Gamma} + \alpha b\Delta_\Gamma u_{0\Gamma} = -F'(u_0)|_\Gamma + bG'(u_{0\Gamma}) \quad \text{on } \Gamma \quad \text{if} \quad p > 5/2, \tag{14}$$

$$\begin{aligned} g|_{t=0} &+ b\partial_\nu \Delta u_0 - bc\partial_\nu u_0 + \alpha bc\Delta_\Gamma u_{0\Gamma}\\ &- b\partial_\nu F'(u_0) - bcG'(u_{0\Gamma}) \in B^{1-5/p}_{p,p}(\Gamma) \qquad \text{if} \quad p > 5. \end{aligned} \tag{15}$$

We use the notation $J_a := (0, a) \subset J$, $X(a)$, $Y_i(a)(i = 0, 1)$ and $Z(a)$, $Z_\Gamma(a)$ to indicate the time interval under consideration.

We can state now the following main result of this section.

Theorem 2.3. *Let $1 < p < \infty$ be $p > (n + 4)/4$ and $p \neq 5/2, 5$, and let $(f, g, u_0, u_{0\Gamma}) \in X(T) \times Y_0(T) \times \pi Z \times \pi Z_\Gamma$ satisfy the compatibility conditions* (13)–(15) *and $F \in C^{4-}(\mathbb{R}), G \in C^{2-}(\mathbb{R})$. Then there is an $a \in (0, T]$ and a unique solution $(u, u_\Gamma) \in Z(a) \times Z_\Gamma(a)$ of* $(\mathrm{CH})_{\mathrm{per.}}$*. Furthermore the solution depends continuously on the data, and if the data (f, g) are independent of t, the map $(u_0, u_{0\Gamma}) \mapsto (u(t), u_\Gamma(t))$ defines a local semiflow in the natural phase manifold $\mathcal{M}$ defined by $\pi Z \times \pi Z_\Gamma$ and the compatibility conditions* (13)–(15)*.*

Proof. The proof is based on the contraction mapping theorem. At first we take the function $(u^*, u^*_\Gamma) \in Z(T) \times Z_\Gamma(T)$ that is the solution of the linearized equation

$$\begin{cases} \partial_t u^* + \Delta^2 u^* = f & \text{in} \quad Q, \\ \partial_t u^*_\Gamma - b\partial_\nu \Delta u^* + bc\partial_\nu u^* - \alpha bc\Delta_\Gamma u^*_\Gamma = g - \tilde{g} & \text{on} \quad \Sigma, \\ u^*|_\Gamma = u^*_\Gamma, \quad -(\Delta u^*)|_\Gamma - b\partial_\nu u^* + \alpha b\Delta_\Gamma u^*_\Gamma = -\tilde{h} & \text{on} \quad \Sigma, \\ u^*(0) = u_0 \quad \text{in } \Omega, \quad u^*_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

Here

$$\tilde{g} = \begin{cases} 0 & \text{if } p < 5, \\ e^{-t\Delta_\Gamma^2}(b\partial_\nu F'(u_0) + bcG'(u_{0\Gamma})) & \text{if } 5 < p, \end{cases}$$

$$\tilde{h} = \begin{cases} 0 & \text{if } p < 5/2, \\ e^{-t\Delta_\Gamma^2}(F'(u_0)|_\Gamma - bG'(u_{0\Gamma})) & \text{if } 5/2 < p, \end{cases}$$

are the modified terms, so that we are able to use linear theory. Note that $-\Delta_\Gamma^2$ is the generator of an analytic (C_0)-semigroup in $B^{1-5/p}_{p,p}(\Gamma)$ and $B^{2-5/p}_{p,p}(\Gamma)$.

For given $a \in (0, T]$ to be fixed later, we define

$$\mathbb{E} := \{(u, u_\Gamma) \in Z(a) \times Z_\Gamma(a) \mid u|_\Gamma = u_\Gamma\}, \quad {}_0\mathbb{E} := \{(u, u_\Gamma) \in \mathbb{E} \mid (u, u_\Gamma)|_{t=0} = (0, 0)\}$$

with canonical norm $\| \cdot \|_\mathbb{E}$ and

$$\mathbb{F} := X(a) \times Y_0(a) \times Y_1(a), \quad {}_0\mathbb{F} := \{(f, g, h) \in \mathbb{F} \mid h|_{t=0} = 0\}$$

with norm $\| \cdot \|_\mathbb{F}$. Define the linear operator $\mathbb{L} : \mathbb{E} \to \mathbb{F}$ by means of

$$\mathbb{L}(v, v_\Gamma) := \begin{bmatrix} \partial_t v + \Delta^2 v \\ \partial_t v_\Gamma - b\partial_\nu \Delta v + bc\partial_\nu v - \alpha bc\Delta_\Gamma v_\Gamma \\ -(\Delta v)|_\Gamma - b\partial_\nu v + \alpha b\Delta_\Gamma v_\Gamma \end{bmatrix}.$$

By theorem 2.1, $\mathbb{L} : {}_0\mathbb{E} \to {}_0\mathbb{F}$ is linear, bounded and bijective, hence an isomorphism. Next we define the nonlinear mapping $N : \mathbb{E} \times {}_0\mathbb{E} \to {}_0\mathbb{F}$ by

$$N((u^*, u^*_\Gamma), (v, v_\Gamma)) := \begin{bmatrix} \Delta F'(u^* + v) \\ -b\partial_\nu F'(u^* + v) - bcG'(u^*_\Gamma + v_\Gamma) + \tilde{g} \\ -F'(u^* + v)|_\Gamma + bG'(u^*_\Gamma + v_\Gamma) + \tilde{h} \end{bmatrix}.$$

We will show the key proposition, which needs to use contraction mapping theorem and to show the range of N is ${}_0\mathbb{F}$. Let $\mathbb{B}_R((0, 0)) \subset {}_0\mathbb{E}$ be a closed ball with center $(0, 0)$, radius $R > 0$, and set $\mathbb{B}_R((u^*, u^*_\Gamma)) := (u^*, u^*_\Gamma) + \mathbb{B}_R((0, 0))$.

Proposition 2.4. *Let $p > (n + 4)/4$, $F \in C^{4-}(\mathbb{R}), G \in C^{2-}(\mathbb{R})$, $J_a \subset J$ and $R > 0$. Then there exist functions $\lambda_j = \lambda_j(a)$ with $\lambda_j(a) \to 0$ as $a \to 0$, $j = 1, \cdots, 5$ such that* *for all* $(u, u_\Gamma), (v, v_\Gamma) \in \mathbb{B}_R((u^*, u^*_\Gamma))$ *the following statements hold:*

$$\begin{aligned}
\|\Delta F'(u) - \Delta F'(v)\|_X &\le \lambda_1(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_{\mathbb{E}}, \\
\|\partial_\nu F'(u) - \partial_\nu F'(v)\|_{Y_0} &\le \lambda_2(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_{\mathbb{E}}, \\
\|G'(u_\Gamma) - G'(v_\Gamma)\|_{Y_0} &\le \lambda_3(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_{\mathbb{E}}, \\
\|F'(u)|_\Gamma - F'(v)|_\Gamma\|_{Y_1} &\le \lambda_4(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_{\mathbb{E}}, \\
\|G'(u_\Gamma) - G'(v_\Gamma)\|_{Y_1} &\le \lambda_5(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_{\mathbb{E}}.
\end{aligned}$$

The first and second inequalities are the same in [18, Proposition 3.2] and the others are easily followed.

We see that $u = u^* + v$, $u_\Gamma = u^*_\Gamma + v_\Gamma$ is a unique solution of $(\mathrm{CH})_{\mathrm{per.}}$ if and only if

$$\mathbb{L}(v, v_\Gamma) = N((u^*, u^*_\Gamma), (v, v_\Gamma)) \ \text{ i.e. } (v, v_\Gamma) = \mathbb{L}^{-1}N((u^*, u^*_\Gamma), (v, v_\Gamma)) \tag{16}$$

since

$$\begin{aligned}
\mathbb{L}(u^* + v, u^*_\Gamma + v_\Gamma) &= \mathbb{L}(u^*, u^*_\Gamma) + \mathbb{L}(v, v_\Gamma) \\
&= \begin{bmatrix} f \\ g - \tilde{g} \\ -\tilde{h} \end{bmatrix} + \begin{bmatrix} \Delta F'(u^* + v) \\ -b\partial_\nu F'(u^* + v) - bcG'(u^*_\Gamma + v_\Gamma) + \tilde{g} \\ -F'(u^* + v)|_\Gamma + bG'(u^*_\Gamma + v_\Gamma) + \tilde{h} \end{bmatrix} \\
&= \begin{bmatrix} \Delta F'(u^* + v) + f \\ -b\partial_\nu F'(u^* + v) - bcG'(u^*_\Gamma + v_\Gamma) + g \\ -F'(u^* + v)|_\Gamma + bG'(u^*_\Gamma + v_\Gamma) \end{bmatrix}, \\
(u^* + v, u^*_\Gamma + v_\Gamma)(0) &= (u^*, u^*_\Gamma)(0) + (v, v_\Gamma)(0) = (u_0, u_{0\Gamma}).
\end{aligned}$$

Define the operator $S : \mathbb{B}_R((0,0)) \to {}_0\mathbb{E}$ by means of $S(v, v_\Gamma) := \mathbb{L}^{-1}N((u^*, u^*_\Gamma), (v, v_\Gamma))$. We show that the operator S is a contraction map on $\mathbb{B}_R((0,0))$ with small time interval J_a.

First we prove that $S\mathbb{B}_R((0,0)) \subset \mathbb{B}_R((0,0))$ by the following calculation. Let $(w, w_\Gamma) \in \mathbb{B}_R((0,0))$.

$$\begin{aligned}
&\|S(w, w_\Gamma)\|_{\mathbb{E}} \le \|\mathbb{L}^{-1}\|_{\mathcal{L}(\mathbb{F},\mathbb{E})}\|N((u^*, u^*_\Gamma), (w, w_\Gamma))\|_{\mathbb{F}} \\
&\le C(\|N((u^*, u^*_\Gamma), (w, w_\Gamma)) - N((u^*, u^*_\Gamma), (0,0))\|_{\mathbb{F}} + \|N((u^*, u^*_\Gamma), (0,0))\|_{\mathbb{F}}) \\
&\le C(\|\Delta F'(u^* + w) - \Delta F'(u^*)\|_X + \|\partial_\nu F'(u^* + w) - \partial_\nu F'(u^*)\|_{Y_0} \\
&\quad + \|G'(u^*_\Gamma + w_\Gamma) - G'(u^*_\Gamma)\|_{Y_0} + \|F'(u^* + w)|_\Gamma - F'(u^*)|_\Gamma\|_{Y_1} + \|G'(u^*_\Gamma + w_\Gamma) - G'(u^*_\Gamma)\|_{Y_1} \\
&\quad + \|\Delta F'(u^*)\|_X + \|\partial_\nu F'(u^*)\|_{Y_0} + \|G'(u^*_\Gamma)\|_{Y_0} + \|\tilde{g}\|_{Y_0} \\
&\quad + \|F'(u^*)|_\Gamma\|_{Y_1} + \|G'(u^*_\Gamma)\|_{Y_1} + \|\tilde{h}\|_{Y_1}) \\
&\le C(\lambda(a)\|(w, w_\Gamma)\|_{\mathbb{E}} + \|\Delta F'(u^*)\|_X + \|\partial_\nu F'(u^*)\|_{Y_0} + \|G'(u^*_\Gamma)\|_{Y_0} + \|\tilde{g}\|_{Y_0} \\
&\quad + \|F'(u^*)|_\Gamma\|_{Y_1} + \|G'(u^*_\Gamma)\|_{Y_1} + \|\tilde{h}\|_{Y_1})
\end{aligned}$$

for some function $\lambda(a)$, which goes to 0 as $a \to 0$, since $(u^*, u^*_\Gamma), (u^* + w, u^*_\Gamma + w_\Gamma) \in \mathbb{B}_R((u^*, u^*_\Gamma))$ and Proposition 2.2. The remaining terms $\|\Delta F'(u^*)\|_{X(a)}$, $\|\partial_\nu F'(u^*)\|_{Y_0(a)}$,

$\|G'(u_\Gamma^*)\|_{Y_0(a)}$, $\|\tilde{g}\|_{Y_0(a)}$, $\|F'(u^*)|_\Gamma\|_{Y_1(a)}$, $\|G'(u_\Gamma^*)\|_{Y_1(a)}$, $\|\tilde{h}\|_{Y_1(a)}$ also goes to 0 as $a \to 0$. So we have $\|S(w, w_\Gamma)\|_{\mathbb{E}} \le R$, i.e. $S\mathbb{B}_R((0,0)) \subset \mathbb{B}_R((0,0))$ when a is sufficiently small.

We next show the following contraction property. Let $(w_1, w_{1\Gamma}), (w_2, w_{2\Gamma}) \in \mathbb{B}_R((0,0))$.

$$\begin{aligned}
&\|S(w_1, w_{1\Gamma}) - S(w_2, w_{2\Gamma})\|_{\mathbb{E}} \\
\le&\|\mathbb{L}^{-1}\|_{\mathcal{L}(\mathbb{F},\mathbb{E})}\|N((u^*, u_\Gamma^*), (w_1, w_{1\Gamma})) - N((u^*, u_\Gamma^*), (w_2, w_{2\Gamma}))\|_{\mathbb{F}} \\
\le&C(\|\Delta F'(u^* + w_1) - \Delta F'(u^* + w_2)\|_X + \|\partial_\nu F'(u^* + w_1) - \partial_\nu F'(u^* + w_2)\|_{Y_0} \\
&\quad + \|G'(u_\Gamma^* + w_{1\Gamma}) - G'(u_\Gamma^* + w_{2\Gamma})\|_{Y_0} + \|F'(u_\Gamma^* + w_{1\Gamma})|_\Gamma - F'(u_\Gamma^* + w_{2\Gamma})|_\Gamma\|_{Y_1} \\
&\quad + \|G'(u_\Gamma^* + w_{1\Gamma}) - G'(u_\Gamma^* + w_{2\Gamma})\|_{Y_1}) \\
\le&\frac{1}{2}\|(w_1, w_{1\Gamma}) - (w_2, w_{2\Gamma})\|_{\mathbb{E}},
\end{aligned}$$

provided a is sufficiently small by Proposition 2.2.

Therefore from the fixed point theorem, we get a unique solution $(v, v_\Gamma) \in \mathbb{B}_R((0,0))$ such that $(v, v_\Gamma) = \mathbb{L}^{-1}N((u^*, u_\Gamma^*), (v, v_\Gamma))$. The function (u^*, u_Γ^*) depends continuously on the data f, g and (v, v_Γ) depends continuously on (u^*, u_Γ^*). This implies that the unique solution $u = u^* + v$ and $u_\Gamma = u_\Gamma^* + v_\Gamma$ of $(\mathrm{CH})_{\mathrm{per.}}$ depends continuously on the data as well. If the data f, g are independent of the time, then translation is invariant. So the solution map $(u_0, u_{0\Gamma}) \mapsto (u(t), u_\Gamma(t))$ defines a local semiflow in the natural phase manifold $\pi Z \times \pi Z_\Gamma$ and the compatibility conditions (13)–(15). □

Remark 2.5. This proof also show that the existence of maximal time interval $J_{\max} = (0, a_{\max})$, which is characterized by

$$\begin{cases} \lim\limits_{t \to a_{\max}} u(t) & \text{does not exist in } \pi Z \\ \lim\limits_{t \to a_{\max}} u_\Gamma(t) & \text{does not exist in } \pi Z_\Gamma \end{cases} \quad \text{and/or} \quad \|(u, u_\Gamma)\|_{\mathbb{E}(a_{\max})} = \infty,$$

if $a_{\max} < T$.

2.3 Global well-posedness

In this section we consider the global solution for the equation with non-homogeneous terms f, g;

$$\begin{cases} \partial_t u = \Delta\mu + f, \quad \mu = -\Delta u + F'(u) & \text{in} \quad Q, \\ \partial_t u_\Gamma + b\partial_\nu \mu + c\mu_\Gamma = g, \quad -\alpha\Delta_\Gamma u_\Gamma + \partial_\nu u + G'(u_\Gamma) = \frac{\mu_\Gamma}{b} & \text{on} \quad \Sigma, \\ u|_\Gamma = u_\Gamma, \quad \mu|_\Gamma = \mu_\Gamma & \text{on} \quad \Sigma, \\ u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

As we explained in introduction, the unknown functions are only u and u_Γ though we use μ and μ_Γ. By the subsection 2.2 there is a unique solution on some maximal time interval $J_{\max} = (0, a_{\max})$. We fix some arbitrary J_a for $0 < a \le a_{\max} (\le T)$ and show the boundedness near the point $t = a$ from a priori estimate derived from energy estimate.

Multiplying the equation by u and μ, integration by parts and the boundary conditions lead

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}|u|_2^2+\frac{1}{2}|\nabla u|_2^2+\int_\Omega F(u)\right)+|\nabla\mu|_2^2\\
&=-\int_\Omega \nabla\mu\cdot\nabla u+\int_\Gamma u_\Gamma\partial_\nu\mu+\int_\Gamma \partial_t u_\Gamma\partial_\nu u+\int_\Gamma \mu_\Gamma\partial_\nu\mu+\int_\Omega fu+\int_\Omega f\mu,\\
\Rightarrow &\frac{d}{dt}\left(\frac{1}{2}|u|_2^2+\frac{1}{2}|\nabla u|_2^2+\int_\Omega F(u)+\frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2\right)+|\nabla\mu|_2^2+\frac{c}{b}|\mu_\Gamma|_{2,\Gamma}^2\\
&=-\int_\Omega \nabla\mu\cdot\nabla u-\frac{c}{b}\int_\Gamma u_\Gamma\mu_\Gamma+\int_\Gamma \partial_t u_\Gamma(\partial_\nu u-\frac{\mu_\Gamma}{b})+\int_\Omega fu+\int_\Omega f\mu+\frac{1}{b}\int_\Gamma g\mu_\Gamma\\
&=-\int_\Omega \nabla\mu\cdot\nabla u-\frac{c}{b}\int_\Gamma u_\Gamma\mu_\Gamma+\int_\Gamma \partial_t u_\Gamma(\alpha\Delta_\Gamma u_\Gamma+G'(u_\Gamma))+\int_\Omega fu+\int_\Omega f\mu+\frac{1}{b}\int_\Gamma g\mu_\Gamma\\
\Rightarrow &\frac{d}{dt}\left(\frac{1}{2}|u|_2^2+\frac{1}{2}|\nabla u|_2^2+\int_\Omega F(u)+\frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2+\frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2+\int_\Gamma G(u_\Gamma)\right)+|\nabla\mu|_2^2+\frac{c}{b}|\mu_\Gamma|_{2,\Gamma}^2\\
&=-\int_\Omega \nabla\mu\cdot\nabla u-\frac{c}{b}\int_\Gamma u_\Gamma\mu_\Gamma+\int_\Omega fu+\int_\Omega f\mu+\frac{1}{b}\int_\Gamma g\mu_\Gamma.
\end{aligned}
$$

For simplicity, we set

$$E(u,u_\Gamma):=\frac{1}{2}|u|_2^2+\frac{1}{2}|\nabla u|_2^2+\int_\Omega F(u)+\frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2+\frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2+\int_\Gamma G(u_\Gamma).$$

By Poincaré's inequality $|\mu|_2\le C(|\nabla\mu|_2+|\mu_\Gamma|_{2,\Gamma})$ and Young's inequality with ε, we have

$$\frac{d}{dt}E(u,u_\Gamma)+C_1(|\nabla\mu|_2^2+|\mu_\Gamma|_{2,\Gamma}^2)\le C_2\left(\frac{1}{2}|u|_2^2+\frac{1}{2}|\nabla u|_2^2+\frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2\right)+C_3(|f|_2^2+|g|_{2,\Gamma}^2)$$

for some $C_i>0$ $(i=1,2,3)$.

To get energy estimate, we assume that F and G satisfy the following condition:

$$\begin{cases}F(s)\ge -c_1, & c_1>0,\ s\in\mathbb{R},\\ G(s)\ge -\frac{1}{2b}s^2-c_2, & c_2>0,\ s\in\mathbb{R}.\end{cases}\tag{17}$$

Note that the typical example in the introduction satisfies this assumption. Under this condition, the function $E(u,u_\Gamma)$ is bounded from below. We get the inequality

$$\frac{d}{dt}E(u,u_\Gamma)+C_1(|\nabla\mu|_2^2+|\mu_\Gamma|_{2,\Gamma}^2)\le C_2E(u,u_\Gamma)+C_3(|f|_2^2+|g|_{2,\Gamma}^2+1).$$

We apply Gronwall's lemma, then we get energy estimate

$$E(u,u_\Gamma)\le C\left(E(u_0,u_{0\Gamma})+\int_0^{a_{\max}}(|f|_2^2+|g|_{2,\Gamma}^2+1)\right)$$

and

$$(u,u_\Gamma)\in L_\infty(J_{a_{\max}},W_2^1(\Omega)\times W_2^1(\Gamma))\tag{18}$$

when $(f, g, u_0, u_{0\Gamma}) \in X(T) \times Y_0(T) \times \pi Z \times \pi Z_\Gamma$ as $p \geq 2$ and $p > (n+4)/4$. Here the constant C depends only on $T > 0$ and is independent of $a_{\max}$.

We use the following lemma, which is obtained in the paper [18, Lemma 4.1]. To do so, we have to assume that the dimension $n = 2, 3$ and some growth condition on F and G:

$$\begin{cases} |F'''(s)| \leq C(1 + |s|^\beta), & s \in \mathbb{R}, \\ |G'(s)| \leq C(1 + |s|^{\beta+2}), & s \in \mathbb{R}, \end{cases} \quad \text{with} \quad \begin{cases} \beta < 3 \text{ in the case } n = 3, \\ \beta > 0 \text{ in the case } n = 2. \end{cases} \tag{19}$$

Lemma 2.6. *Suppose $2 \leq p < \infty$, $n = 2, 3$, the function F and G satisfy (17) and (19) and let $(u, u_\Gamma) \in \mathbb{E}(a)$ be the solution of $(\mathrm{CH})_{\mathrm{per.}}$. Then there exist constants $m, C > 0$ and $\delta \in (0, 1)$, independent of $a > 0$, such that*

$$\begin{aligned} &\|\Delta F'(u)\|_{X(a)} + \|\partial_\nu F'(u)\|_{Y_0(a)} + \|G'(u_\Gamma)\|_{Y_0(a)} + \|F'(u)|_\Gamma\|_{Y_1(a)} + \|G'(u_\Gamma)\|_{Y_1(a)} \\ \leq &C(1 + \|u\|^\delta_{Z(a)} \|u\|^m_{L_\infty(J_a, W^1_2(\Omega))}). \end{aligned}$$

Proof. The estimates of the first term $\|\Delta F'(u)\|_{X(a)}$ and the second term $\|\partial_\nu F'(u)\|_{Y_0(a)}$ is just in [18, Lemma 4.1]. Since the trace operator is bounded from $W^{1/2}_p(J_a, L_p(\Omega)) \cap L_p(J_a, W^2_p(\Omega))$ to Y_1, $Y_1 \subset Y_0$ and $u|_\Gamma = u_\Gamma$, the other three terms are also estimated. See [19, Appendix (b)]. □

Combining maximal L_p regularity estimate,

$$\begin{aligned} &\|(u, u_\Gamma)\|_{\mathbb{E}(a)} \\ \leq &C(\|\Delta F'(u)\|_{X(a)} + \|\partial_\nu F'(u)\|_{Y_0(a)} + \|G'(u_\Gamma)\|_{Y_0(a)} + \|F'(u)|_\Gamma\|_{Y_1(a)} \\ &\qquad + \|G'(u_\Gamma)\|_{Y_1(a)} + \|f\|_{X(T)} + \|g\|_{Y_0(T)} + \|(u_0, u_{0\Gamma})\|_{\pi Z \times \pi Z_\Gamma}) \\ \leq &\widetilde{C}(1 + \|u\|^\delta_{Z(a)}), \end{aligned} \tag{20}$$

where the constant $\widetilde{C}$ is independent of a. Hence $\|u\|_{Z(a)}$ is bounded and it derives the boundedness of $\|u_\Gamma\|_{Z_\Gamma(a)}$. Therefore the solution $(u, u_\Gamma) \in \mathbb{E}(a)$ is global solution, i.e. $a_{\max} = T$. We obtained the following first main theorem of this paper.

Theorem 2.7. *Suppose $2 \leq p < \infty$, $p \neq 5/2, 5$, $n = 2, 3$ and that the function F and G satisfy (17) and (19). Then for any $(f, g, u_0, u_{0\Gamma}) \in X(T) \times Y_0(T) \times \pi Z \times \pi Z_\Gamma$ satisfying the compatibility conditions (13)–(15), there exists a unique global solution $(u, u_\Gamma) \in Z(T) \times Z_\Gamma(T)$ of $(\mathrm{CH})_{\mathrm{per.}}$. The solution depends continuously on the given data and if the data are independent of t, the map $(u_0, u_{0\Gamma}) \mapsto (u(t), u_\Gamma(t))$ defines a global semiflow on the natural phase manifold $\pi Z \times \pi Z_\Gamma$ and the compatibility conditions (13)–(15).*

2.4 The degenerate case: $c = 0$

In this section we focus on the case $c = 0$ in the boundary condition (3). Almost all results for now can be applied to this case. The linear theory and local well-posedness result is completely the same as the case $c > 0$. The point different from the case $c > 0$ is the energy estimate. Multiplying the equation by u and μ, integration by parts and Young's inequality lead

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}|u|_2^2+\frac{1}{2}|\nabla u|_2^2+\int_\Omega F(u)+\frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2+\frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2+\int_\Gamma G(u_\Gamma)\right)+C_1|\nabla\mu|_2^2\\ \leq& C_2\left(\frac{1}{2}|u|_2^2+\frac{1}{2}|\nabla u|_2^2+\frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2\right)+C_3|f|_2^2+\int_\Omega f\mu+\frac{1}{b}\int_\Gamma g\mu_\Gamma\end{aligned}$$

for some $C_i > 0$ $(i = 1, 2, 3)$. Here we assume $\int_\Omega f dx + \int_\Gamma g \frac{dS}{b} = 0$. Then we see

$$\begin{aligned}\frac{d}{dt}\int_\Omega u dx &= \int_\Omega (\Delta\mu + f)dx\\ &= \int_\Gamma \partial_\nu \mu dS + \int_\Omega f dx\\ \Rightarrow \frac{d}{dt}\left(\int_\Omega u dx + \int_\Gamma u_\Gamma \frac{dS}{b}\right) &= \int_\Omega f dx + \int_\Gamma g\frac{dS}{b} = 0\end{aligned}$$

and

$$\begin{aligned}\int_\Omega f\mu + \frac{1}{b}\int_\Gamma g\mu_\Gamma &= \int_\Omega f(\mu - \overline{\mu}) + \frac{1}{b}\int_\Gamma g(\mu_\Gamma - \overline{\mu})\\ &\leq \frac{C_1}{2}|\nabla\mu|_2^2 + C_4(|f|_2^2 + |g|_2^2),\end{aligned}$$

where $\overline{\mu} = \frac{1}{|\Omega|}\int_\Omega \mu dx$ and some $C_4 > 0$ by Poincaré's inequality. This implies that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}|u|_2^2+\frac{1}{2}|\nabla u|_2^2+\int_\Omega F(u)+\frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2+\frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2+\int_\Gamma G(u_\Gamma)\right)+\tilde{C}_1|\nabla\mu|_2^2\\ \leq& \tilde{C}_2\left(\frac{1}{2}|u|_2^2+\frac{1}{2}|\nabla u|_2^2+\frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2\right)+\tilde{C}_3(|f|_2^2+|g|_2^2)\end{aligned}$$

for some $\tilde{C}_i > 0$ $(i = 1, 2, 3)$. This inequality deduces a priori estimate (18) under the assumption (17). Thus we have the global well-posedness result for the case $c = 0$.

Theorem 2.8. *Suppose* $2 \leq p < \infty$, $p \neq 5/2, 5$, $n = 2, 3$ *and that the function* F *and* G *satisfy (17) and (19). Then for any* $(f, g, u_0, u_{0\Gamma}) \in X(T) \times Y_0(T) \times \pi Z \times \pi Z_\Gamma$ *satisfying the compatibility conditions (13)–(15) with* $c = 0$ *and* $\int_\Omega f dx + \int_\Gamma g \frac{dS}{b} = 0$*, there exists a unique global solution* $(u, u_\Gamma) \in Z(T) \times Z_\Gamma(T)$ *of* $(\mathrm{CH})_{\mathrm{per.}}$ *with* $c = 0$.

3 A Cahn–Hilliard equation on non-permeable walls

3.1 The linear theory

In this section we study the linear theory of the Cahn–Hilliard equation on non-permeable walls. The linear equation is as follows:

$$(**)\begin{cases} \partial_t v + \Delta^2 v = f & \text{in} \quad Q, \\ \partial_t v_\Gamma - b\partial_\nu \Delta v - bc\Delta_\Gamma \partial_\nu v + \alpha bc \Delta_\Gamma^2 v_\Gamma = g & \text{on} \quad \Sigma, \\ v|_\Gamma = v_\Gamma, \quad -(\Delta v)|_\Gamma - b\partial_\nu v + \alpha b \Delta_\Gamma v_\Gamma = h & \text{on} \quad \Sigma, \\ v(0) = v_0 \quad \text{in } \Omega, \quad v_\Gamma(0) = v_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

We again use the general theory in [5] and the assumption of the theorem is checked in Appendix C. However we have to assume a condition on the coefficients α, b, c to get (LS) condition. The assumption is the following:

Assumption (A) The coefficients $\alpha, b, c > 0$ satisfy $\alpha bc < 2(\alpha b + c)$.
Let $\overline{Z}_\Gamma := W_p^{1+\kappa_0}(J, L_p(\Gamma)) \cap L_p(J, W_p^{4+4\kappa_0}(\Gamma))$ and $\pi\overline{Z}_\Gamma := B_{p,p}^{5-5/p}(\Gamma)$

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^5 and $1 < p < \infty$ be $p \neq 5/4, 5/2, 5$. Suppose that the constants $\alpha, b, c > 0$ satisfy the Assumption (A). Then the linearized Cahn–Hilliard equation $(**)$ admits a unique solution $(v, v_\Gamma) \in Z \times \overline{Z}_\Gamma$ if and only if $(f, g, h) \in X \times Y_0 \times Y_1$ and $(v_0, v_{0\Gamma}) \in \pi Z \times \pi\overline{Z}_\Gamma$, and the compatibility conditions*

$$\begin{aligned} & v_0|_\Gamma = v_{0\Gamma} && \text{on } \Gamma && \text{if} \quad p > 5/4, \\ & -(\Delta v_0)|_\Gamma - b\partial_\nu v_{0\Gamma} + \alpha b \Delta_\Gamma v_{0\Gamma} = h|_{t=0} && \text{on } \Gamma && \text{if} \quad p > 5/2, \\ & g|_{t=0} + b\partial_\nu \Delta v_0 - bc\Delta_\Gamma \partial_\nu v_0 - \alpha bc \Delta_\Gamma^2 v_{0\Gamma} \in B_{p,p}^{1-5/p}(\Gamma) && && \text{if} \quad p > 5, \end{aligned}$$

are satisfied.

3.2 The nonlinear theory

In this subsection we state the nonlinear theory. We state the different point from the case of permeable walls. We need the estimate of the nonlinear term $\Delta_\Gamma G'(u_\Gamma)$ corresponding to Proposition 2.2 and Lemma 2.3. From now, we restrict the case that $G(u_\Gamma) = (g_s/2)u_\Gamma^2 - h_s u_\Gamma$ with $g_s > 0, h_s \neq 0$. Thus we study the Cahn–Hilliard equation on non-permeable walls.

$$(\mathrm{CH})_{\text{non-per.}} \begin{cases} \partial_t u = \Delta\mu + f, \quad \mu = -\Delta u + F'(u) & \text{in} \quad Q, \\ \partial_t u_\Gamma + b\partial_\nu \mu - c\Delta_\Gamma \mu_\Gamma = g, \quad -\alpha\Delta_\Gamma u_\Gamma + \partial_\nu u + g_s u_\Gamma^2 - h_s = \frac{\mu_\Gamma}{b} & \text{on} \quad \Sigma, \\ u|_\Gamma = u_\Gamma, \quad \mu|_\Gamma = \mu_\Gamma & \text{on} \quad \Sigma, \\ u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

We see the following proposition.

Proposition 3.2. *Let $p > (n+4)/4$, $J_a \subset J$ and $R > 0$. Then there exist functions $\lambda_6 = \lambda_6(a)$ with $\lambda_6(a) \to 0$ as $a \to 0$ such that for all $(u, u_\Gamma), (v, v_\Gamma) \in \mathbb{B}_R((u^*, u_\Gamma^*))$ the following statements hold:*

$$\|\Delta_\Gamma u_\Gamma - \Delta_\Gamma v_\Gamma\|_{Y_0} \leq \lambda_6(a)\|(u, u_\Gamma) - (v, v_\Gamma)\|_{\mathbb{E}}.$$

This proposition is enough to show the local well-posedness result. To extend the global solution, we show the energy estimate. Multiplying the equation by u and μ, integration by parts and the boundary conditions lead

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \int_\Omega F(u) + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2 + \frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2 + \int_\Gamma G(u_\Gamma)\right)\\ &\quad + |\nabla \mu|_2^2 + \frac{c}{b}|\nabla_\Gamma \mu_\Gamma|_{2,\Gamma}^2\\ &= -\int_\Omega \nabla\mu \cdot \nabla u - \frac{c}{b}\int_\Gamma \nabla_\Gamma u_\Gamma \cdot \nabla_\Gamma \mu_\Gamma + \int_\Omega fu + \int_\Omega f\mu + \frac{1}{b}\int_\Gamma g u_\Gamma + \frac{1}{b}\int_\Gamma g\mu_\Gamma.\end{aligned}$$

Here as the case $c = 0$, we assume that $\int_\Omega f\,dx + \int_\Gamma g\frac{dS}{b} = 0$. Then we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \int_\Omega F(u) + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2 + \frac{\alpha}{2}|\nabla_\Gamma u_\Gamma|^2 + \int_\Gamma G(u_\Gamma)\right)\\ &\quad + C_1(|\nabla \mu|_2^2 + |\nabla_\Gamma \mu_\Gamma|_{2,\Gamma}^2)\\ &\leq C_2\left(\frac{1}{2}|u|_2^2 + \frac{1}{2}|\nabla u|_2^2 + \frac{1}{2b}|u_\Gamma|_{2,\Gamma}^2\right) + C_3(|f|_2^2 + |g|_2^2)\end{aligned}$$

for some $C_i > 0$ $(i = 1, 2, 3)$.

Under the assumption (17) on F, we see $(u, u_\Gamma) \in L_\infty(J_{a_{\max}}, W_2^1(\Omega) \times W_2^1(\Gamma))$. We prepare the following lemma.

Lemma 3.3. *Suppose $2 \leq p < \infty$, $n = 2, 3$, let $(u, u_\Gamma) \in \mathbb{E}(a)$ be the solution of* $(\mathrm{CH})_{\text{non-per.}}$. *Then there exist constants $C > 0$ and $\delta \in (0, 1)$, independent of $a > 0$, such that*

$$\|\Delta_\Gamma u_\Gamma\|_{Y_0(a)} \leq C(1 + \|u\|_{Z(a)}^\delta \|u\|_{L_\infty(J_a, W_2^1(\Omega))}^{1-\delta}).$$

Proof. By the trace theory and the mixed derivative theorem, it is enough to see the existence of $0 < \delta < 1$

$$\|u\|_{W_p^{3/4}(J_a, L_p(\Omega))} \leq C\|u\|_{W_p^{7/8}(J_a, W_p^{1/2}(\Omega))}^\delta \|u\|_{L_\infty(J_a, W_2^1(\Omega))}^{1-\delta}.$$

By Gagliardo–Nirenberg's inequality, we check the existence of δ satisfying

$$\begin{cases}\frac{3}{4} - \frac{1}{p} \leq \delta(\frac{7}{8} - \frac{1}{p})\\ -\frac{n}{p} \leq \delta(\frac{1}{2} - \frac{n}{p}) + (1-\delta)(1 - \frac{n}{2}).\end{cases}$$

Since the second inequality is $\frac{n}{2} - \frac{n}{p} - 1 \leq \delta(\frac{n}{2} - \frac{n}{p} - \frac{1}{2})$, we choose δ is sufficiently close to 1, then the inequalities are satisfied. $\square$

Combining the estimates in (2.3), we are able to prove the global well-posedness result.

Theorem 3.4. *Suppose $2 \le p < \infty$, $p \ne 5/2, 5$, $n = 2, 3$ and that the function F satisfy (17) and (19). Suppose that the constants $\alpha, b, c > 0$ satisfy the Assumption (A). Then for any $(f, g, u_0, u_{0\Gamma}) \in X(T) \times Y_0(T) \times \pi Z \times \pi \overline{Z}_\Gamma$ satisfying the compatibility conditions*

$$\begin{aligned} &u_0|_\Gamma = u_{0\Gamma} && \text{on } \Gamma \quad \text{if} \quad p > 5/4, \\ &-(\Delta u_0)|_\Gamma - b\partial_\nu u_{0\Gamma} + \alpha b \Delta_\Gamma u_{0\Gamma} = -F'(u_0)|_\Gamma + b g_s u_{0\Gamma} - b h_s && \text{on } \Gamma \quad \text{if} \quad p > 5/2, \\ &g|_{t=0} + b\partial_\nu \Delta u_0 + bc\partial_\nu u_0 - \alpha bc \Delta_\Gamma^2 u_{0\Gamma} \\ &\qquad - b\partial_\nu F'(u_0) + bc g_s \Delta_\Gamma u_{0\Gamma} \in B^{1-5/p}_{p,p}(\Gamma) && \quad \text{if} \quad p > 5, \end{aligned}$$

and $\int_\Omega f dx + \int_\Gamma g \frac{dS}{b} = 0$, there exists a unique global solution $(u, u_\Gamma) \in Z(T) \times \overline{Z}_\Gamma(T)$ of $(\mathrm{CH})_{\text{non-per.}}$.

Appendix A

We collect the linear theory of the dynamic boundary condition proved in the papers [5]. We represent the simplified their result to fit our equations. They studied the parabolic initial boundary value problems of the general form (so called *relaxation type*)

$$\begin{cases} \partial_t u + \mathcal{A}(t, x, D)u = f(t, x) & \text{in } Q, \\ \partial_t \rho + \mathcal{B}_0(t, x, D)u + \mathcal{C}_0(t, x, D_\Gamma)\rho = g_0(t, x) & \text{on } \Sigma, \\ \mathcal{B}_j(t, x, D)u + \mathcal{C}_j(t, x, D_\Gamma)\rho = g_j(t, x) \quad (j = 1, \cdots, m) & \text{on } \Sigma, \\ u(0, x) = u_0(x) & \text{in } \Omega, \\ \rho(0, x) = \rho_0(x) & \text{on } \Gamma, \end{cases}$$

where

$$\begin{aligned} \mathcal{A}(t, x, D) &= \sum_{|\alpha| \le 2m} a_\alpha(t, x) D^\alpha, \\ \mathcal{B}_j(t, x, D) &= \sum_{|\beta| \le m_j} b_{j\beta}(t, x) D^\beta, \\ \mathcal{C}_j(t, x, D_\Gamma) &= \sum_{|\gamma| \le k_j} c_{j\gamma}(t, x) D_\Gamma^\gamma, \end{aligned}$$

are differential operators of order $2m$, $0 \le m_j < 2m$, $0 \le k_j$ $(j = 0, 1, \cdots, m)$, respectively, with $m \in \mathbb{N}$ and m_j, $k_j \in \mathbb{N}_0$. The symbols D, respectively D_Γ mean $-i\nabla$, respectively $-i\nabla_\Gamma$, where ∇ denotes the gradient in Ω and ∇_Γ the surface gradient on Γ. Assume that all boundary operators $\mathcal{B}_j$ and at least one $\mathcal{C}_j$ are nontrivial, and set $k_j = -\infty$ in case $\mathcal{C}_j = 0$. The initial values u_0, ρ_0 as well as the right-hand sides f and g_j are given functions.

Let $\kappa_j := 1 - m_j/(2m) - 1/(2mp)$, $l_j := k_j - m_j + m_0$ and $l := \max_{j=0,1,\cdots,m} l_j$. We state their results limited to the case $l \le 2m$, the coefficients a_α, $b_{j\beta}$ and $c_{j\gamma}$ are smooth, Ω is a bounded domain and u and ρ are $\mathbb{C}$-valued functions, which adopt our case.

The essential assumptions are *the normally ellipticity condition* (E) and *the Lopatinskii–Shapiro condition* (LS), which are necessary for the maximal L_p regularity, hence are unavoidable. For the case $\ell < 2m$, which is just applied to the linearized Cahn–Hilliard

equation on permeable walls, we need another necessary condition called *the asymptotic Lopatinskii–Shapiro condition* (LS^-_∞). Let the subscript # be the principal part of the corresponding differential operator. The assumptions are as follows.
(E) For all $t \in J$, $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$, $|\xi| = 1$, we have

$$\sigma\left(\mathcal{A}_\#(t,x,\xi)\right) \subset \mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}.$$

(LS) For each fixed $t \in J$ and $x \in \Gamma$, and for all $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in \overline{\mathbb{C}}_+$ with $|\xi'| + |\lambda| \neq 0$, the ordinary differential equation in $\mathbb{R}_+ = [0,\infty)$ given by

$$\begin{cases} (\lambda + \mathcal{A}_\#(t,x,\xi',D_y))\, v(y) = 0 \quad (y > 0), \\ \mathcal{B}_{0\#}(t,x,\xi',D_y)v(0) + (\lambda + \mathcal{C}_{0\#}(t,x,\xi'))\,\sigma = 0, \\ \mathcal{B}_{j\#}(t,x,\xi',D_y)v(0) + \mathcal{C}_{j\#}(t,x,\xi')\sigma = 0 \quad (j = 1,\cdots,m) \end{cases}$$

has only the trivial solution $(v,\sigma) = (0,0)$.
($\mathbf{LS}^-_\infty$) Let $\ell < 2m$. For all fixed $t \in J$ and $x \in \Gamma$, and for all $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in \overline{\mathbb{C}}_+$ with $|\xi'| + |\lambda| \neq 0$, the ordinary differential equation in $\mathbb{R}_+ = [0,\infty)$ given by

$$\begin{cases} (\lambda + \mathcal{A}_\#(t,x,\xi',D_y))\, v(y) = 0 \quad (y > 0), \\ \mathcal{B}_{j\#}(t,x,\xi',D_y)v(0) = 0 \quad (j = 1,\cdots,m) \end{cases}$$

and for $|\xi'| = 1$ and $\lambda \in \overline{\mathbb{C}}_+$,

$$\begin{cases} \mathcal{A}_\#(t,x,\xi',D_y)v(y) = 0 \quad (y > 0), \\ \mathcal{B}_{0\#}(t,x,\xi',D_y)v(0) + (\lambda + \mathcal{C}_{0\#}(t,x,\xi'))\,\sigma = 0, \\ \mathcal{B}_{j\#}(t,x,\xi',D_y)v(0) + \mathcal{C}_{j\#}(t,x,\xi')\sigma = 0 \quad (j = 1,\cdots,m) \end{cases}$$

admit the unique trivial solution $(v,\sigma) = (0,0)$.

The existence and uniqueness results of this boundary condition are as follows.

Theorem [Denk–Prüss–Zacher] *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^{2m+l-m_0}. Assume* (E), (LS) *and for $\ell < 2m$ the condition* (LS^-_∞) *and the coefficients a_α, $b_{j\beta}$, $c_{j\gamma}$ are smooth. Let $1 < p < \infty$ be such that $\kappa_j \neq 1/p$, $j = 0,1,\cdots,m$. Then the linear equation admits a unique solution*

$$\begin{aligned}(u,\rho) \in Z \times Z_\rho := &\left(W^1_p(J,L_p(\Omega)) \cap L_p(J,W^{2m}_p(\Omega))\right) \\ &\times \left(W^{1+\kappa_0}_p(J,L_p(\Gamma)) \cap W^1_p(J,W^{2m\kappa_0}_p(\Gamma)) \cap L_p(J,W^{\ell+2m\kappa_0}_p(\Gamma))\right)\end{aligned}$$

if and only if

$$\begin{aligned}(f,g_0,g_1,\cdots,g_m) &\in X \times Y_0 \times Y_1 \times \cdots \times Y_m \\ &:= L_p(J,L_p(\Omega)) \times \otimes^m_{j=0}\left(W^{\kappa_j}_p(J,L_p(\Gamma)) \cap L_p(J,W^{2m\kappa_j}_p(\Gamma))\right) \\ (u_0,\rho_0) &\in \pi Z_u \times \pi Z_\rho := B^{2m(1-1/p)}_{p,p}(\Omega) \times B^{2m\kappa_0+\ell(1-1/p)}_{p,p}(\Gamma),\end{aligned}$$

and the compatibility conditions

$$\mathcal{B}_j(0,x)u_0(x) + \mathcal{C}_j(0,x)\rho_0(x) = g_j(0,x), \quad x \in \Gamma, \text{ if } \kappa_j > 1/p, \quad j = 1,2,\cdots,m,$$
$$g_0(0,\cdot) - \mathcal{B}_0(0,\cdot)u_0 - \mathcal{C}_0(0,\cdot)\rho_0 \in \pi_1 Z_\rho := B^{2m(\kappa_0-1/p)}_{p,p}(\Gamma), \text{ if } \kappa_0 > 1/p,$$

are satisfied.

In [5], they treated the case $l > 2m$, non-smooth coefficients case and u, ρ are $\mathcal{HT}$ Banach valued case. However it is sufficient to consider above statement. By the Newton polygon method, they characterized

$$Z_\rho = W^{1+\kappa_0}_p(J, L_p(\Gamma)) \cap L_p(J, W^{\ell+2m\kappa_0}_p(\Gamma))$$

when $\ell = 2m$, which is applied to the linearized Cahn–Hilliard equation on non-permeable walls.

Appendix B

We apply this general linear theory for the linearized Cahn–Hilliard equation on permeable walls:

$$(*)_{\text{per.}} \begin{cases} \partial_t v + \Delta^2 v = f & \text{in} \quad Q, \\ \partial_t v_\Gamma - b\partial_\nu \Delta v + bc\partial_\nu v - \alpha bc \Delta_\Gamma v_\Gamma = g & \text{on} \quad \Sigma, \\ v|_\Gamma = v_\Gamma, \quad -(\Delta v)|_\Gamma - b\partial_\nu v + \alpha b \Delta_\Gamma v_\Gamma = h & \text{on} \quad \Sigma, \\ v(0) = v_0 \quad \text{in } \Omega, \quad v_\Gamma(0) = v_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

This problem fits into the setting $\mathcal{A} = \Delta^2$, $\mathcal{B}_0 = -b\partial_\nu\Delta$, $\mathcal{C}_0 = -\alpha bc\Delta_\Gamma$, $\mathcal{B}_1 = -(\Delta\cdot)|_\Gamma$, $\mathcal{C}_1 = \alpha b\Delta_\Gamma$, $\mathcal{B}_2 = 1$, $\mathcal{C}_2 = -1$, $g_2 = 0$ and $m = 2$, $m_0 = 3$, $k_0 = 2$, $m_1 = 2$, $k_1 = 2$, $m_2 = 0$, $k_2 = 0$, $\ell_0 = 2$, $\ell_1 = 3$, $\ell_2 = 3$. Then $\ell = \ell_1 = \ell_2 = 3 < 2m$, $\kappa_0 = 1/4 - 1/(4p)$, $\kappa_1 = 1/2 - 1/(4p)$ and $\kappa_2 = 1 - 1/(4p)$.

We check the conditions (E) and (LS). Since $\sigma(\mathcal{A}_\#(t,x,\xi)) = \sigma(|\xi|^4) = \{1\} \subset \mathbb{C}_+$ for $\xi \in \mathbb{R}^n$, $|\xi| = 1$, the condition (E) is satisfied.

To see (LS) condition, we need to solve the ordinary differential equation

$$\begin{cases} \left((\lambda + (-|\xi'|^2 + \partial_y^2)^2\right) v(y) = 0 \quad (y > 0), & (21) \\ -b(-\partial_y)(-|\xi'|^2 + \partial_y^2)v(0) + \left((\lambda - \alpha bc(-|\xi'|^2)\right)\sigma = 0, & (22) \\ v(0) - \sigma = 0, & (23) \\ -(-|\xi'|^2 + \partial_y^2)v(0) + \alpha b(-|\xi'|^2)\sigma = 0. & (24) \end{cases}$$

For the case $\lambda = 0$, from (21), $v(y) = (c_1 + c_2 y)e^{-|\xi'|y}$ for some $c_1, c_2 \in \mathbb{C}$. By the boundary conditions (22)–(24),

$$\begin{cases} -b|\xi'|^2(c_2 - |\xi'|c_1) + b(3|\xi'|^2 c_2 - |\xi'|^3 c_1) + \alpha bc|\xi'|^2 c_1 = 0, \\ |\xi'|^2 c_1 - (-2|\xi'|c_2 + |\xi'|^2 c_1) - \alpha b|\xi'|^2 c_1 = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \alpha c c_1 + 2c_2 = 0, \\ \alpha b|\xi'|c_1 - 2c_2 = 0. \end{cases}$$

The determinant of the coefficient matrix is $-2(\alpha c+\alpha b|\xi'|)\neq 0$. So we have $(c_1,c_2)=(0,0)$, which implies the unique trivial solution $(v,\sigma)=(0,0)$.

For the case $\lambda\neq 0$, $v(y)=c_1e^{z_1y}+c_2e^{z_2y}$ with

$$z_k:=-\sqrt{|\xi'|^2+(-1)^{k-1}\sqrt{-\lambda}}\quad (k=1,2).$$

Here and hereafter we shall use the argument of the square root of complex numbers belongs $(-\pi/2,\pi/2]$, so that the real part of the square root of complex numbers is non-negative. By the boundary conditions (22)–(24),

$$\begin{cases}-b|\xi'|^2(c_1z_1+c_2z_2)+b(c_1z_1^3+c_2z_2^3)+(\lambda+\alpha bc|\xi'|^2)(c_1+c_2)=0,\\ |\xi'|^2(c_1+c_2)-(c_1z_1^2+c_2z_2^2)-\alpha b|\xi'|^2(c_1+c_2)=0.\end{cases}$$
$$\Rightarrow\begin{cases}bcz_1(z_1^2-|\xi'|^2)+bcz_2(z_2^2-|\xi'|^2)+(\lambda+\alpha bc|\xi'|^2)(c_1+c_2)=0,\\ -c_1(z_1^2-|\xi'|^2)-c_2(z_2^2-|\xi'|^2)-\alpha b|\xi'|^2(c_1+c_2)=0.\end{cases}$$

Since $z_k^2-|\xi'|^2=(-1)^{k-1}\sqrt{-\lambda}$ $(k=1,2)$, we see

$$\begin{cases}(\lambda+\alpha bc|\xi'|^2+b\sqrt{-\lambda}z_1)c_1+(\lambda+\alpha bc|\xi'|^2-b\sqrt{-\lambda}z_2)c_2=0\\ (\alpha b|\xi'|^2+\sqrt{-\lambda})c_1+(\alpha b|\xi'|^2-\sqrt{-\lambda})c_2=0.\end{cases}$$

We calculate the determinant of the coefficient matrix:

$$\begin{aligned}&\begin{vmatrix}\lambda+\alpha bc|\xi'|^2+b\sqrt{-\lambda}z_1 & \lambda+\alpha bc|\xi'|^2-b\sqrt{-\lambda}z_2\\ \alpha b|\xi'|^2+\sqrt{-\lambda} & \alpha b|\xi'|^2-\sqrt{-\lambda}\end{vmatrix}\\
=&\begin{vmatrix}\lambda+\alpha bc|\xi'|^2+b\sqrt{-\lambda}z_1 & -b\sqrt{-\lambda}(z_1+z_2)\\ \alpha b|\xi'|^2+\sqrt{-\lambda} & -2\sqrt{-\lambda}\end{vmatrix}\\
=&-\sqrt{-\lambda}\left\{2(\lambda+\alpha bc|\xi'|^2)+2b\sqrt{-\lambda}z_1-b(z_1+z_2)(\alpha b|\xi'|^2+\sqrt{-\lambda})\right\}\\
=&-\sqrt{-\lambda}\left\{2(\lambda+\alpha bc|\xi'|^2)-\alpha b^2|\xi'|^2(z_1+z_2)+b\sqrt{-\lambda}(z_1-z_2)\right\}\\
=:&-\sqrt{-\lambda}((\mathrm{I})+(\mathrm{II})+(\mathrm{III})).\end{aligned}$$

We claim that the real part of the last term (III) is non-negative. Then the determinant never become zero since the real part of the first term (I) and the second term (II) is positive. We focus on the sign of the term (III). From the equality $\sqrt{-\lambda}(z_1-z_2)=-2\lambda(z_1+z_2)^{-1}$,

$$\mathrm{sign}\,(\mathrm{Re}\,(\mathrm{III}))=\mathrm{sign}\,(\mathrm{Re}\,(-\lambda)\mathrm{Re}\,(\mathrm{z}_1+\mathrm{z}_2)+\mathrm{Im}\,(-\lambda)\mathrm{Im}\,(\mathrm{z}_1+\mathrm{z}_2)).$$

Here $\mathrm{Re}\,(-\lambda),\mathrm{Re}\,(z_1+z_2)\leq 0$ and $\mathrm{sign}\,\mathrm{Im}\,(-\lambda)=\mathrm{sign}\,\mathrm{Im}\,(z_1+z_2)$ since

$$z_1+z_2=-\sqrt{2|\xi'|^2+2\sqrt{|\xi'|^4+\lambda}}.$$

This implies $\mathrm{sign}\,(\mathrm{Re}\,(\mathrm{III}))$ is non-negative. This means that $(v,\sigma)=(0,0)$, which concludes that the (LS) condition is satisfied. The other condition (LS_∞^-) is easily checked, so we skip the calculation.

Appendix C

We apply this general linear theory for the linearized Cahn–Hilliard equation on non-permeable walls:

$$(*)_{\text{non-per.}}\begin{cases}\partial_t v + \Delta^2 v = f & \text{in } \quad Q,\\ \partial_t v_\Gamma - b\partial_\nu \Delta v - bc\Delta_\Gamma \partial_\nu v + \alpha bc \Delta_\Gamma^2 v_\Gamma = g & \text{on } \quad \Sigma,\\ v|_\Gamma = v_\Gamma, \quad -(\Delta v)|_\Gamma - b\partial_\nu v + \alpha b \Delta_\Gamma v_\Gamma = h & \text{on } \quad \Sigma,\\ v(0) = v_0 \quad \text{in } \Omega, \quad v_\Gamma(0) = v_{0\Gamma} \quad \text{on } \Gamma. \end{cases}$$

This problem fits into the setting $\mathcal{A} = \Delta^2$, $\mathcal{B}_0 = -b\partial_\nu\Delta - bc\Delta_\Gamma\partial_\nu$, $\mathcal{C}_0 = \alpha bc\Delta_\Gamma^2$, $\mathcal{B}_1 = -(\Delta\cdot)|_\Gamma$, $\mathcal{C}_1 = \alpha b\Delta_\Gamma$, $\mathcal{B}_2 = 1$, $\mathcal{C}_2 = -1$, $g_2 = 0$ and $m = 2$, $m_0 = 3$, $k_0 = 4$, $m_1 = 2$, $k_1 = 2$, $m_2 = 0$, $k_2 = 0$, $\ell_0 = 4$, $\ell_1 = 3$, $\ell_2 = 3$. And then $\ell = \ell_0 = 4 = 2m$, $\kappa_0 = 1/4 - 1/(4p)$, $\kappa_1 = 1/2 - 1/(4p)$ and $\kappa_2 = 1 - 1/(4p)$. The condition (E) is satisfied as before.

To see (LS) condition, we need to solve the ordinary differential equation

$$\begin{cases}\big((\lambda + (-|\xi'|^2 + \partial_y^2)^2\big)\, v(y) = 0 \quad (y > 0), & (25)\\ -b(-\partial_y)(-|\xi'|^2 + \partial_y^2)v(0) - bc(-|\xi'|^2)(-\partial_y)v(0) + \big((\lambda + \alpha bc(-|\xi'|^2)^2\big)\,\sigma = 0, & (26)\\ v(0) - \sigma = 0, & (27)\\ -(-|\xi'|^2 + \partial_y^2)v(0) + \alpha b(-|\xi'|^2)\sigma = 0. & (28)\end{cases}$$

For the case $\lambda = 0$, $v(y) = (c_1 + c_2 y)e^{-|\xi'|y}$ for some $c_1, c_2 \in \mathbb{C}$. By the boundary conditions (26)–(28),

$$\begin{cases}-b|\xi'|^2(c_2 - |\xi'|c_1) + b(3|\xi'|^2 c_2 - |\xi'|^3 c_1) - bc|\xi'|^2(c_2 - |\xi'|c_1) + \alpha bc|\xi'|^4 c_1 = 0,\\ |\xi'|^2 c_1 - (-2|\xi'|c_2 + |\xi'|^2 c_1) - \alpha b|\xi'|^2 c_1 = 0.\end{cases}$$

$$\Rightarrow \begin{cases} c|\xi'|(\alpha|\xi'| + 1)c_1 + (2 - c)c_2 = 0,\\ \alpha b|\xi'|c_1 - 2c_2 = 0.\end{cases}$$

The determinant of the coefficient matrix is $-|\xi'|(2\alpha c|\xi'| + 2\alpha b + 2c - \alpha bc)$. So we assume $\alpha bc < 2(\alpha b + c)$ (Assumption A), then we have $(c_1, c_2) = (0, 0)$, which implies the unique trivial solution $(v, \sigma) = (0, 0)$.

For the case $\lambda \neq 0$, $v(y) = c_1 e^{z_1 y} + c_2 e^{z_2 y}$ with the same z_k as before. By the boundary conditions (26)–(28),

$$\begin{cases}-b|\xi'|^2(c_1 z_1 + c_2 z_2) + b(c_1 z_1^3 + c_2 z_2^3)\\ \qquad\qquad -bc|\xi'|^2(c_1 z_1 + c_2 z_2) + (\lambda + \alpha bc(-|\xi'|^2)^2)(c_1 + c_2) = 0,\\ |\xi'|^2(c_1 + c_2) - (c_1 z_1^2 + c_2 z_2^2) - \alpha b|\xi'|^2(c_1 + c_2) = 0.\end{cases}$$

$$\Rightarrow \begin{cases}(\lambda + \alpha bc|\xi'|^4 + b\sqrt{-\lambda}z_1 - bc|\xi'|^2 z_1)c_1 + (\lambda + \alpha bc|\xi'|^2 - b\sqrt{-\lambda}z_2 - bc|\xi'|^2)c_2 = 0,\\ (\alpha b|\xi'|^2 + \sqrt{-\lambda})c_1 + (\alpha b|\xi'|^2 - \sqrt{-\lambda})c_2 = 0.\end{cases}$$

We calculate the determinant of the coefficient matrix:

$$\begin{aligned}
&\begin{vmatrix} \lambda+\alpha bc|\xi'|^4+b\sqrt{-\lambda}z_1-bc|\xi'|^2 z_1 & \lambda+\alpha bc|\xi'|^2-b\sqrt{-\lambda}z_2-bc|\xi'|^2 z_2 \\ \alpha b|\xi'|^2+\sqrt{-\lambda} & \alpha b|\xi'|^2-\sqrt{-\lambda} \end{vmatrix}\\
=&\begin{vmatrix} \lambda+\alpha bc|\xi'|^4+b\sqrt{-\lambda}z_1-bc|\xi'|^2 z_1 & -b\sqrt{-\lambda}(z_1+z_2)+bc|\xi'|^2(z_1-z_2) \\ \alpha b|\xi'|^2+\sqrt{-\lambda} & -2\sqrt{-\lambda} \end{vmatrix}\\
=&-\sqrt{-\lambda}\Big\{2(\lambda+\alpha bc|\xi'|^4)+b\sqrt{-\lambda}(z_1-z_2)\\
&\qquad -bc|\xi'|^2(z_1+z_2)-\alpha b^2|\xi'|^2(z_1+z_2)+\alpha b^2 c|\xi'|^4\frac{2}{z_1+z_2}\Big\},
\end{aligned}$$

where we used $z_1-z_2=2\sqrt{-\lambda}(z_1+z_2)^{-1}$. We see the real part of $2(\lambda+\alpha bc|\xi'|^4)+b\sqrt{-\lambda}(z_1-z_2)$ is positive. We claim that the real part of the others is non-negative by using the Assumption (A). From the Assumption (A),

$$\begin{aligned}
&\mathrm{Re}\left(-bc|\xi'|^2(z_1+z_2)-\alpha b^2|\xi'|^2(z_1+z_2)+\alpha b^2 c|\xi'|^4\frac{2}{z_1+z_2}\right)\\
\geq&\mathrm{Re}\left(-bc|\xi'|^2(z_1+z_2)-\alpha b^2|\xi'|^2(z_1+z_2)+2(\alpha b+c)b|\xi'|^4\frac{2}{z_1+z_2}\right)\\
=&(bc|\xi'|^2+\alpha b^2)\mathrm{Re}\left(\frac{4|\xi'|^2}{z_1+z_2}-(z_1+z_2)\right).
\end{aligned}$$

Note that

$$\frac{4|\xi'|^2}{z_1+z_2}-(z_1+z_2)=2(z_1+z_2)^{-1}(|\xi'|^2-z_1z_2),$$

$z_1z_2=\sqrt{\lambda+|\xi'|^4}$, $\mathrm{Re}\,(|\xi'|^2-z_1z_2)\leq 0$ and $\mathrm{Im}\,(z_1+z_2)\mathrm{Im}\,(|\xi'|^2-z_1z_2)\geq 0$. So we have

$$\begin{aligned}
&\mathrm{Sign}\,\mathrm{Re}\,\big((z_1+z_2)^{-1}(|\xi'|^2-z_1z_2)\big)\\
=&\mathrm{Sign}\big(\mathrm{Re}\,(z_1+z_2)\mathrm{Re}\,(|\xi'|^2-z_1z_2)+\mathrm{Im}\,(z_1+z_2)\mathrm{Im}\,(|\xi'|^2-z_1z_2)\big)\\
\geq&0.
\end{aligned}$$

This implies that the determinant of the coefficients never 0 and $(v,\sigma)=(0,0)$. So it was shown (LS) condition.

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