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BOUNDEDNESS IN A CHEMOTAXIS MODEL WITH NONLINEAR DIFFUSION AND LOGISTIC TYPE SOURCE FOR TUMOR INVASION

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Abstract. This paper deals with nonnegative solutions of the Neumann boundary value problem for the chemotaxis system modeling tumor invasion,

$$\begin{cases} u_t = \nabla \cdot (D(u,w)\nabla u) - \nabla \cdot (u\lambda(x,t)\nabla v) + F(x,t,u,w), & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v + zw, & \text{in } \Omega \times (0,\infty), \\ w_t = -zw, & \text{in } \Omega \times (0,\infty), \\ z_t = \Delta z + G(u,w,z), & \text{in } \Omega \times (0,\infty) \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ $(N \le 9)$, where $F(\cdot,t,\cdot,\cdot) \in C^{1-}_{\text{loc}}(\overline{\Omega} \times \mathbb{R}^2)$ for all $t \in [0,\infty)$, $G \in C^{1-}_{\text{loc}}(\mathbb{R}^3)$ such that

$$\alpha u^2 + F(x,t,u,w) \le C_F,\ F(x,t,0,w) \ge 0, \quad x \in \overline{\Omega},\ t,u,w \ge 0 \text{ with } \alpha > 0,$$
$$z + G(u,w,z) \le u + C_G,\ G(0,0,0) \ge 0, \quad u,w,z \ge 0.$$

When $N \le 3$, the case of linear diffusion was studied by Fujie [**4**]. However the methods of [**4**] cannot be directly applied to the case of nonlinear diffusion or the case $N \ge 4$. In this paper it is shown that the problem possesses a unique global-in-time classical solution which is bounded in $\Omega \times (0,\infty)$.

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1. Introduction

In this paper we consider the initial-boundary value problem

$$
(1.1)\quad \begin{cases} u_t = \nabla \cdot (D(u,w)\nabla u) - \nabla \cdot (u\lambda(x,t)\nabla v) + F(x,t,u,w), & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta v + zw, & \text{in } \Omega \times (0,\infty), \\ w_t = -zw, & \text{in } \Omega \times (0,\infty), \\ z_t = \Delta z + G(u,w,z), & \text{in } \Omega \times (0,\infty), \\ D(u,w)\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & \text{in } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0,\ w(\cdot,0) = w_0,\ z(\cdot,0) = z_0, & \text{in } \Omega, \end{cases}
$$

where $\Omega \subset \mathbb{R}^N$ $(N \le 9)$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial \nu}$ represents the differentiation with respect to the outward normal vector ν of $\partial\Omega$. We assume that the initial data satisfies

$$
(1.2)\qquad (u_0, v_0, w_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \times C^{1+\theta}(\overline{\Omega})
$$

with some $\theta > 0$ and

$$
(1.3)\qquad \begin{cases} z_0 \in C^0(\overline{\Omega}) & \text{if } N \le 3, \\ z_0 \in C^0(\overline{\Omega}) \cap H^1(\Omega) & \text{if } N = 4,5, \\ z_0 \in C^0(\overline{\Omega}) \cap W^{1,2\gamma}(\Omega) & \text{if } 6 \le N \le 9 \end{cases}
$$

with some $\gamma > 1 + \frac{10}{N+1}$. Also we suppose that $D \in C^{1+\theta}(\mathbb{R}^2)$ for the same θ as above and

$$
(1.4)\qquad D(\zeta_1, \zeta_2) \ge c_0 > 0 \quad \text{for all } \zeta_1, \zeta_2 \ge 0
$$

with some $c_0 > 0$. Moreover, we assume that $\lambda \in C^1(\overline{\Omega} \times [0,\infty)) \cap L^\infty(0,\infty; L^\infty_{\mathrm{loc}}(\Omega))$, $F(\cdot,t,\cdot,\cdot) \in C^{1-}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}^2)$ for all $t \in [0,\infty)$, where $C^{1-}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}^2)$ is defined as

$$
C^{1-}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}^2) := \{ f \in C^0(\overline{\Omega} \times \mathbb{R}^2) \,|\, f \text{ is locally Lipschitz continuous on } \overline{\Omega} \times \mathbb{R}^2 \},
$$

and F satisfies the following conditions:

$$
(1.5)\qquad F(y, \xi_1, 0, \xi_3) \ge 0 \quad \text{for all } y \in \overline{\Omega}, \text{ for all } \xi_1, \xi_3 \ge 0,
$$
$$
(1.6)\qquad \alpha \xi_2^2 + F(y, \xi_1, \xi_2, \xi_3) \le C_F \quad \text{for all } y \in \overline{\Omega}, \text{ for all } \xi_1, \xi_2, \xi_3 \ge 0
$$

with some $\alpha > 0$ and $C_F > 0$. Furthermore we let $G \in C^{1-}_{\mathrm{loc}}(\mathbb{R}^3)$ such that

$$
(1.7)\qquad G(0,0,0) \ge 0,
$$
$$
(1.8)\qquad \eta_3 + G(\eta_1, \eta_2, \eta_3) \le \eta_1 + C_G \quad \text{for all } \eta_1, \eta_2, \eta_3 \ge 0,
$$

with some $C_G > 0$.

Modeling Tumor Invasion. The model (1.1) was proposed in [**3**] as a modified tumor invasion model of Chaplain–Anderson type in [**1**]. In this model, the unknown functions u, v, w and z describe the molar concentrations of tumor cells, the active extra cellular matrix, denoted by ECM*, the extra cellular matrix, denoted by ECM, and the matrix degrading enzyme, denoted by MDE, respectively, from the view point of cancer phenomena. ECM is dissolved by the biochemical reaction with MDE. However MDE does not decrease by the biochemical reaction with ECM since MDE is an enzyme. ECM* is produced by the biochemical reaction between ECM and MDE and plays a role as an attractant of tumor cells. The coefficient $D(u,w)$ of the random motility of tumor cells is given by the function of u and w in general. If the density of tumor cells is small or the density of ECM is large, then tumor cells cannot move freely on cancer phenomena. Conversely if tumor cells gather together more and more or ECM is resolved more and more by the biochemical reaction with MDE, tumor cells can move freely. A typical example of $D(u,w)$ is given by

$$D(u,w) := \begin{cases} D_1 + \dfrac{D_2 e^u}{1+e^u} + \dfrac{D_3 e^{f(w)}}{1+e^{f(w)}} & \text{if } w < w_c, \\ D_1 + \dfrac{D_2 e^u}{1+e^u} & \text{if } w \geq w_c, \end{cases}$$

where D_1, D_2 and D_3 are nonnegative constants satisfying $D_1 + D_2 + D_3 > 0$, w_c is a positive constant or $w_c = \infty$ and $f(w)$ is a decreasing function on $[0, w_c)$ satisfying $\lim_{w \nearrow w_c} f(w) = -\infty$. This $D(u,w)$ satisfies (1.4) if $D_1 + D_2 > 0$. $\lambda(x,t)$ is a sensibility coefficient of the chemotaxis of tumor cells. $F(x,t,u,w)$ is the proliferation and apoptosis rate of tumor cells. $G(u,w,z)$ implies the production and the decay of MDE.

The result of the previous study. First of all, the case that $D \equiv 1$, $\lambda \equiv 1$, $F \equiv 0$, $G(u,w,z) = -z+u$ was considered in [**2**], [**3**], [**7**]. Local existence was shown by Fujie–Ito–Yokota [**2**]. In the case $N \leq 3$, Fujie et al. [**3**] obtained global existence and asymptotic behavior. In the case $N \geq 4$, Jin–Xiang [**7**] showed them if u_0, z_0, ∇v_0 are enough small.

Secondly, Fujie [**4**] studied the case that $D \equiv 1$, $\lambda \equiv 1$, $F(x,t,u,w) = \kappa u - \mu u^2$ $(\kappa, \mu > 0)$, $G(u,w,z) = -z+u$. In this case, Fujie [**4**] established global existence and asymptotic behavior in the case $N \leq 3$. However he did not consider global existence and asymptotic behavior in the case $N \geq 4$.

Thirdly, the case that D satisfies (1.4), $\lambda \equiv 1$, $F \equiv 0$, $G(x,t,u,w) = -z + u$ was considered in [**5**]. Fujie et al. [**5**] also established both global existence and asymptotic behavior in the case $N \leq 3$; however, they did not study them in the case $N \geq 4$. Moreover, they obtained a global weak solution in the case of nonlinear degenerate with $N \leq 3$. From the above, we can not find the result with nonlinear diffusion or logistic source in the case $N \geq 4$.

Main result. We focus on the condition that $N \geq 4$ in the case of nonlinear diffusion and logistic type source. We cannot directly obtain the L^∞-boundedness of u in the case $N \geq 4$ since we cannot use the L^1-boundedness of u. Therefore we consider that the first solution component u is bounded in $L^p(\Omega)$ with some $p > 1$ by using a differential inequality as in [**12**] and show that the solution (u,v,w,z) of (1.1) is global and uniformly bounded with respect to x.

In this paper we show global existence in the case $N \leq 9$. We divide the proof into three cases. When $N \leq 3$, we have global existence by using an argument similar to the proofs of [**3**, Lemmas 3.5–3.6], [**4**, Proposition 2.3]. When $N = 4, 5$, we prove that z is bounded in $L^{\frac{2N}{N-2}}(\Omega)$ to apply a similar way as in [**4** Section 3]. On the other hand, we need to show the boundedness of $\|u(\cdot,t)\|_{L^p(\Omega)}$ for some $p>1$ when $6\le N\le 9$. From a differential inequality with $\|\nabla v(\cdot,t)\|_{L^{p_1}(\Omega)}$ and $\|\nabla z(\cdot,t)\|_{L^{p_2}(\Omega)}$ with some $p_1,p_2>1$ we can see that $\|u(\cdot,t)\|_{L^p(\Omega)}\le C$ with some $C>0$.

The main result reads as follows.

Theorem 1.1. *Let $N\le 9$ and suppose that* (1.2)–(1.8) *hold. Then there exists a uniquely determined quadruple (u,v,w,z) of nonnegative functions*

$$
\begin{aligned}
&u\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)),\\
&v\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))\cap L^\infty_{loc}([0,\infty);W^{1,\infty}(\Omega)),\\
&w\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{1,1}(\overline{\Omega}\times(0,\infty))\quad\textit{and}\\
&z\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)),
\end{aligned}
$$

which solve (1.1) *classically in $\Omega\times(0,\infty)$. Moreover the solution (u,v,w,z) of* (1.1) *is bounded in the sense that there exists $C>0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|w(\cdot,t)\|_{L^\infty(\Omega)}+\|z(\cdot,t)\|_{L^\infty(\Omega)}\le C$$

for all $t>0$.

The following process represents how to prove Theorem 1.1. First of all, we obtain local existence in (1.1) and the basic estimates in Section 2. Secondly, we show in Section 3 that if we see the L^p-boundedness of u with some $p>\frac{N}{4}$ or the L^q-boundedness of z with some $q>\frac{N}{2}$, then $\|u(\cdot,t)\|_{L^\infty(\Omega)}$, $\|z(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}$ are bounded. In Section 4 we finally prove the boundedness of $\|u(\cdot,t)\|_{L^1(\Omega)}$ in the case $N\le 3$, $\|z(\cdot,t)\|_{L^{\frac{2N}{N-2}}(\Omega)}$ in the case $N=4,5$, and $\|u(\cdot,t)\|_{L^p(\Omega)}$ with some $p>1+\frac{10}{N-4}$ in the case $6\le N\le 9$.

2. Preliminaries. Local existence and basic estimates

The following local existence and uniqueness statement can be proved by modifying the proof of [**5**].

Lemma 2.1 (*Local existence*)**.** *Let $N\ge 1$, and assume that* (1.2), *$z_0\in C^0(\overline{\Omega})$,* (1.4)–(1.8) *hold. Then there exist $T_{\max}\in(0,\infty]$ and a unique classical solution (u,v,w,z) of* (1.1) *in $\Omega\times(0,T_{\max})$ such that the following properties hold:*

$$
\begin{aligned}
&u,v,w,z\ \textit{are nonnegative},\\
&u\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})),\\
&v\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max}))\cap L^\infty_{loc}([0,T_{\max});W^{1,\infty}(\Omega)),\\
&w\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{1+\theta,1}(\overline{\Omega}\times(0,T_{\max}))\quad\textit{and}\\
&z\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})).
\end{aligned}
$$

Moreover, if $T_{\max}<\infty$, then

$$\lim_{t\nearrow T_{\max}}\Big(\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|z(\cdot,t)\|_{L^\infty(\Omega)}\Big)=\infty. \tag{2.1}$$

Lemmas 2.2–2.5 are proved by a simple observation from the equations in (1.1). In the following, we will denote by (u, v, w, z) the corresponding solution to (1.1) given by Lemma 2.1 and by $T_{\max}$ its maximal existence time.

Lemma 2.2. *The first solution component u satisfies*

$$\int_\Omega u(x,t)\,dx \le \int_\Omega u_0(x)\,dx + (\alpha^{-1} + C_F)|\Omega| \quad \text{for all } t \in (0, T_{\max}). \tag{2.2}$$

Proof. Integrating the first equation in (1.1) over Ω and noticing $D(u,w)\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, we see from (1.6) that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u &= \int_\Omega \nabla\cdot(D(u,w)\nabla u) - \int_\Omega \nabla\cdot(u\lambda(x,t)\nabla v) + \int_\Omega F(x,t,u,w)\\ &\le -\alpha\int_\Omega u^2 + C_F|\Omega| \quad \text{for all } t \in (0, T_{\max}).\end{aligned}$$

Since $\alpha u^2 \ge u - \alpha^{-1}$, this yields

$$\frac{d}{dt}\int_\Omega u \le -\int_\Omega u + (\alpha^{-1} + C_F)|\Omega| \quad \text{for all } t \in (0, T_{\max}).$$

Using Gronwall's lemma, we obtain (2.2). □

Lemma 2.3. *The second solution component v and the third solution component w satisfy*

$$\int_\Omega v(x,t)\,dx + \int_\Omega w(x,t)\,dx = \int_\Omega v_0(x)\,dx + \int_\Omega w_0(x)\,dx$$

for all $t \in (0, T_{\max})$.

Proof. We add the second equation to the third equation in (1.1) and integrate with respect to $x \in \Omega$ to obtain

$$\frac{d}{dt}\int_\Omega (v + w) = \int_\Omega \Delta v = 0 \quad \text{for all } t \in (0, T_{\max})$$

because $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, and the claim is proved. □

Moreover, we also obtain the following property in relation to the third solution component in (1.1).

Lemma 2.4. *The third solution component w fulfills*

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le \|w_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\max}).$$

Proof. The third solution component w satisfies

$$w(\cdot,t) = w_0 \exp\left(-\int_0^t z(\cdot,s)ds\right)$$

for all $t \in (0, T_{\max})$. Since $w, z \ge 0$ and the initial data satisfies (1.2), this claim can be proved. □

Lemma 2.5. *The fourth solution component z satisfies*

$$\int_\Omega z(x,t)\,dx \le \int_\Omega z_0(x)\,dx + \int_\Omega u_0(x)\,dx + (\alpha^{-1} + C_F)|\Omega| \quad \text{for all } t \in (0, T_{\max}). \tag{2.3}$$

Proof. Integrating the fourth equation over Ω, noticing $\frac{\partial z}{\partial \nu} = 0$ and using Lemma 2.2 yield

$$\begin{aligned}\frac{d}{dt}\int_\Omega z &= \int_\Omega (\Delta z - z + u)\\ &= -\int_\Omega z + \int_\Omega u\\ &\le -\int_\Omega z + \int_\Omega u_0 + (\alpha^{-1} + C_F)|\Omega|.\end{aligned}$$

Hence Gronwall's lemma leads to (2.3). □

The next inequality called the trace inequality is found in [**10**, Remark 52.9].

Lemma 2.6. *For any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that*

$$\|\varphi\|_{L^2(\partial\Omega)} \le \varepsilon\|\nabla\varphi\|_{L^2(\Omega)} + C(\varepsilon)\|\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in H^1(\Omega).$$

Finally we recall the following lemma which is often utilized to remove the convexity of domains (see [**6**], [**9**], for instance).

Lemma 2.7. *Assume that Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $\varphi \in C^2(\overline{\Omega})$ satisfies $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$. Then*

$$\frac{\partial|\nabla\varphi|^2}{\partial\nu} \le C|\nabla\varphi|^2 \text{ on } \partial\Omega,$$

where $C > 0$ is a constant.

3. Conditional boundedness

Combining Lemmas 3.1–3.3, we see the boundedness of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$, $\|z(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ if $\|u(\cdot,t)\|_{L^p(\Omega)}$ is bounded with some $p > \frac{N}{4}$. First we find the following estimate for the fourth solution component z under an appropriate boundedness assumption on u.

Lemma 3.1. *Let $p \ge 1$ and*

$$\begin{cases} q \in \left[1, \frac{Np}{N-2p}\right) & \text{if } p \le \frac{N}{2},\\ q \in [1,\infty] & \text{if } p > \frac{N}{2}.\end{cases}$$

Then for all $M > 0$ there exists $C(p,q,M) > 0$ such that if for some $T \in (0, T_{\max})$ we have

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le M \quad \text{for all } t \in (0,T),$$

then

$$\|z(\cdot,t)\|_{L^q(\Omega)} \le C(p,q,M) \quad \text{for all } t \in (0,T).$$

Proof. Using variation of constants representation of z, from the fourth equation in (1.1) we have

$$z(\cdot,t) = e^{t(\Delta-1)}z_0 + \int_0^t e^{(t-s)(\Delta-1)}(z(\cdot,s) + G(u(\cdot,s), w(\cdot,s), z(\cdot,s)))\,ds$$

for all $t \in (0,T)$. We infer from (1.8) that

$$\begin{aligned} z(\cdot,t) &\le e^{t(\Delta-1)}z_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)\,ds + \int_0^t e^{(t-s)(\Delta-1)}C_G\,ds \\ &\le e^{t(\Delta-1)}z_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)\,ds + C_G \end{aligned}$$

for all $t \in (0,T)$. Using an argument similar to the proof of [**3**, Lemma 3.1], we complete the proof of this lemma. □

Next we show that boundedness property of z entails a certain regularity for ∇v. According to an argument similar to the proof of [**3**, Lemmas 3.2], the following claim can be proved.

Lemma 3.2. *Let $q \ge 1$ and*

$$\begin{cases} r \in \left[1, \frac{Nq}{N-q}\right) & \text{if } q \le N, \\ r \in [1,\infty] & \text{if } q > N. \end{cases}$$

Then for all $M > 0$ there exists $C(q,r,M) > 0$ with the following property: for all $T \in (0, T_{\max})$, if

$$\|z(\cdot,t)\|_{L^q(\Omega)} \le M \quad \text{for all } t \in (0,T),$$

then

$$\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \le C(q,r,M) \quad \text{for all } t \in (0,T).$$

Finally we derive an estimate for u from a present appropriate boundedness property of ∇v.

Lemma 3.3. *Suppose that $r > \max\{N,2\}$. Then for all $M > 0$ there exists $C(r,M) > 0$ such that if*

$$\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \le M \quad \text{for all } t \in (0,T),$$

with some $T \in (0, T_{\max})$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C(r,M) \quad \text{for all } t \in (0,T). \tag{3.1}$$

Proof. Let $p \geq 1$. Multiplying the first equation in (1.1) by u^{p-1}, we see that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p = -(p-1)\int_\Omega D(u,w)u^{p-2}|\nabla u|^2 + (p-1)\int_\Omega \lambda(x,t)u^{p-1}\nabla u\cdot\nabla v \\ + \int_\Omega u^{p-1}F(x,t,u,w).$$

Using the conditions (1.4), (1.6), Young's inequality and Hölder's inequality entails that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{\alpha}{p}\int_\Omega u^{p+1} - \frac{C_F}{p}\int_\Omega u^{p-1} \\
&\leq -c_0(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \Lambda(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v \\
&\leq -\frac{c_0(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{\Lambda^2(p-1)}{2c_0}\int_\Omega u^p|\nabla v|^2 \\
&\leq -\frac{c_0(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{\Lambda^2(p-1)}{2c_0}\left(\int_\Omega u^{\frac{pr}{r-2}}\right)^{\frac{r-2}{r}}\left(\int_\Omega |\nabla v|^r\right)^{\frac{2}{r}} \\
&\leq -\frac{c_0(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{\Lambda^2(p-1)M^2}{2c_0}\left(\int_\Omega u^{\frac{pr}{r-2}}\right)^{\frac{r-2}{r}} \\
&=: -I_1 + I_2,
\end{aligned}$$

where

$$\Lambda := \|\lambda\|_{L^\infty(\Omega\times(0,\infty))}.$$

Using the Gagliardo–Nirenberg inequality, we see that there exists $C_{GN} > 0$ such that

$$\int_\Omega u^p \leq C_{GN}\left(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}\right)^{2\tau_1}\|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{2(1-\tau_1)}, \tag{3.2}$$

where

$$\tau_1 := \frac{p-1}{p-1+\frac{2}{N}} \in (0,1).$$

In view of (2.2), we have that

$$\begin{aligned}
\|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)} &= \left(\int_\Omega u\right)^{\frac{p}{2}} \\
&\leq \left(\int_\Omega u_0 + (\alpha^{-1} + C_F)|\Omega|\right)^{\frac{p}{2}}.
\end{aligned}$$

The estimate (3.2) leads to

$$\int_\Omega u^{p-2}|\nabla u|^2 = \|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \geq c_1\left(\int_\Omega u^p\right)^{\frac{1}{\tau_1}} - c_2 \tag{3.3}$$

with some constants $c_1, c_2 > 0$. The Gagliardo–Nirenberg inequality and Young's inequality also entail that

$$
\begin{aligned}
(3.4)\qquad I_2 &\le C_{GN}\left(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|_{L^2(\Omega)}\right)^{2\tau_2} \|u^{\frac{p}{2}}\|^{2(1-\tau_2)}_{L^{\frac{2}{p}}(\Omega)} \\
&\le \frac{c_0(p-1)}{4}\int_\Omega u^{p-2}|\nabla u|^2 + c_3\int_\Omega u^p + c_4 \\
&= \frac{1}{2}I_1 + c_3\int_\Omega u^p + c_4
\end{aligned}
$$

with

$$
\tau_2 := \frac{p-\frac{r-2}{r}}{p-\frac{N-2}{N}} \in (0,1).
$$

We see from (3.3) and (3.4) that

$$
\begin{aligned}
-I_1 + I_2 &\le \frac{1}{2}I_1 + c_3\int_\Omega u^p + c_4 \\
&\le -\frac{c_0c_1(p-1)}{4}\left(\int_\Omega u^p\right)^{\frac{1}{\tau_1}} + c_3\int_\Omega u^p + \frac{c_0c_2(p-1)}{4} + c_4.
\end{aligned}
$$

Therefore Young's inequality yields

$$
\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &\le -\frac{c_0c_1(p-1)}{4}\left(\int_\Omega u^p\right)^{\frac{1}{\tau_1}} + c_3\int_\Omega u^p \\
&\quad - \frac{\alpha}{p}\int_\Omega u^{p+1} + \frac{C_F}{p}\int_\Omega u^{p-1} + \frac{c_0c_2(p-1)}{4} + c_4 \\
&\le -C_1\left(\int_\Omega u^p\right)^{\frac{1}{\tau_1}} + C_2\int_\Omega u^p + C_3,
\end{aligned}
$$

where $C_1 = 2^{-2}c_0c_1(p-1)$, $C_2 = c_3 + p^{-2}(C_F(p-1) - \alpha(p+1))$, $C_3 = 2^{-2}c_0c_2(p-1) + (\alpha + p^{-2}C_F)|\Omega| + c_4$. It follows from this differential inequality that there exists a constant $C(p) > 0$ satisfying $\|u(\cdot,t)\|_{L^p(\Omega)} \le C(p)$ for all $t \in (0,T)$. Finally we find from standard Moser type arguments (see [**11**], for instance) that (3.1) holds. □

If we show that $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded, we have the L^∞-boundedness of $z(\cdot,t)$ by using Lemma 3.1 again. Moreover, we see the L^∞-boundedness of $\nabla v(\cdot,t)$ from this boundedness and Lemma 3.2. Therefore we need the boundedness of $\|u(\cdot,t)\|_{L^p(\Omega)}$ with some $p > \frac{N}{4}$ or $\|z(\cdot,t)\|_{L^q(\Omega)}$ with some $q > \frac{N}{2}$ to obtain the L^∞-boundedness.

4. Boundedness. Proof of Theorem 1.1

We show the boundedness of $\|z(\cdot,t)\|_{L^{\frac{2N}{N-2}}(\Omega)}$ to prove a global existence of classical solutions of (1.1) in the case $N = 4, 5$. Moreover we also show that $\|u(\cdot,t)\|_{L^p(\Omega)}$ is bounded in the case $6 \le N \le 9$.

Lemma 4.1. *Let $4 \le N \le 9$. Assume that (1.2)–(1.8) hold. Then there exists $C > 0$ such that the first and fourth solution components u, z satisfy*

$$\int_\Omega u(x,t)dx + \int_\Omega z^2(x,t)dx + \int_\Omega |\nabla z(x,t)|^2 dx \le C \quad \textit{for all } t \in (0, T_{\max})$$

and

$$\|z(\cdot,t)\|_{L^{\frac{2N}{N-2}}(\Omega)} \le C \quad \textit{for all } t \in (0, T_{\max}). \tag{4.1}$$

Proof. Multiplying the fourth equation in (1.1) by $z, -\Delta z$ respectively, and then integrating them with respect to x, we end up with

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega z^2 = \int_\Omega z z_t &= \int_\Omega z(\Delta z - z + u) \\ &= -\int_\Omega |\nabla z|^2 - \int_\Omega z^2 + \int_\Omega uz \\ &\le -\int_\Omega |\nabla z|^2 - \frac{1}{2}\int_\Omega z^2 + \frac{1}{2}\int_\Omega u^2 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla z|^2 = \int_\Omega \nabla z_t \cdot \nabla z &= \int_\Omega z_t(-\Delta z) \\ &= -\int_\Omega |\Delta z|^2 - \int_\Omega |\nabla z|^2 + \int_\Omega u\,(-\Delta z) \\ &\le -\int_\Omega |\nabla z|^2 + \frac{1}{4}\int_\Omega u^2 \end{aligned} \tag{4.3}$$

for all $t \in (0, T_{\max})$ by Young's inequality. Combining (2.2), (4.2) and (4.3), we obtain the differential inequality

$$\frac{d}{dt}\int_\Omega \left(\frac{3}{2\alpha}u + z^2 + |\nabla z|^2\right) + \int_\Omega \left(\frac{3}{2\alpha}u + z^2 + |\nabla z|^2\right) \le \int_\Omega \frac{3}{2\alpha}u + \frac{3}{2\alpha}C_F \le C$$

for all $t \in (0, T_{\max})$ with some $C > 0$. Applying Gronwall's lemma and (1.3) to this inequality, we have the following estimate:

$$\int_\Omega \left(\frac{3}{2\alpha}u + z^2 + |\nabla z|^2\right) \le \int_\Omega \left(\frac{3}{2\alpha}u_0 + z_0^2 + |\nabla z_0|^2\right) + C$$

for all $t \in (0, T_{\max})$. Hence we obtain from this estimate that $z(\cdot,t) \in H^1(\Omega)$ for all $t \in (0, T_{\max})$. At last, we establish from Sobolev's inequality that $z(\cdot,t) \in L^{\frac{2N}{N-2}}(\Omega)$ and $\|z(\cdot,t)\|_{L^{\frac{2N}{N-2}}(\Omega)} \le C$ for all $t \in (0, T_{\max})$. $\square$

The next lemma is a key tool to prove Theorem 1.1.

Lemma 4.2. *Let $6 \le N \le 9$, $p > 1 + \frac{10}{N-4}$, and assume that* (1.2)–(1.8) *hold. The first solution component u satisfies*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max}). \tag{4.4}$$

Proof. **(Step 1)–(Step 3)** constitute the proof of Lemma 4.2.

(Step 1) Estimate for $\frac{d}{dt}\int_\Omega u^p$.

First of all, Step 1 gives the following property :

$$\frac{d}{dt}\int_\Omega u^p \le -\frac{p(p-1)}{4}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{\Lambda^2 p(p-1)}{3}\int_\Omega u^p|\nabla v|^2 - \alpha\int_\Omega u^{p+1} + c$$

for all $t \in (0, T_{\max})$, where $\Lambda := \|\lambda\|_{L^\infty(\Omega\times(0,\infty))}$. Multiplying the first equation in (1.1) by u^{p-1} and integrating by parts over Ω, then using (1.6) and Young's inequality, we end up with

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= -(p-1)\int_\Omega D(u,w)u^{p-2}|\nabla u|^2 + (p-1)\int_\Omega \lambda(x,t)u^{p-1}\nabla u\cdot\nabla v \\
&\quad + \int_\Omega u^{p-1}F(x,t,u,w) \\
&\le -c_0(p-1)\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + (p-1)\int_\Omega \lambda(x,t)u^{p-1}\left(\nabla u\cdot\nabla v\right) \\
&\quad + C_F\int_\Omega u^{p-1} - \alpha\int_\Omega u^{p+1} \\
&\le -c_0(p-1)\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + (p-1)\int_\Omega \left(\frac{3}{4}|\nabla u^{\frac{p}{2}}|^2 + \frac{\Lambda^2}{3}u^p|\nabla v|^2\right) \\
&\quad - \frac{\alpha}{p}\int_\Omega u^{p+1} + c \\
&\le -\frac{p-1}{4}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{\Lambda^2(p-1)}{3}\int_\Omega u^p|\nabla v|^2 - \frac{\alpha}{p}\int_\Omega u^{p+1} + c.
\end{aligned}$$

(Step 2) Estimate for $\frac{d}{dt}\int_\Omega |\nabla v|^{2p_1}$, $\frac{d}{dt}\int_\Omega |\nabla z|^{2p_2}$.

Next we also prove that

$$\frac{d}{dt}\int_\Omega |\nabla v|^{2p_1} + \int_\Omega |\nabla v|^{2p_1} + \frac{p_1-1}{p_1}\int_\Omega |\nabla|\nabla v|^{p_1}|^2 \le c_2\int_\Omega (wz)^2|\nabla v|^{2(p_1-1)} + c_5 \tag{4.5}$$

where some $p_1 > 1$ and

$$\frac{d}{dt}\int_\Omega |\nabla z|^{2p_2} + \int_\Omega |\nabla z|^{2p_2} + \frac{p_2-1}{p_2}\int_\Omega |\nabla|\nabla z|^{p_2}|^2 \le d_2\int_\Omega u^2|\nabla z|^{2(p_2-1)} + d_5. \tag{4.6}$$

where some $p_2 > 1$. First it follows from the second equation in (1.1) that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla v|^{2p_1} &= 2p_1\int_\Omega |\nabla v|^{2(p_1-1)}(|\nabla v|^2)_t \\
&= 2p_1\int_\Omega |\nabla v|^{2(p_1-1)}(\nabla\Delta v\cdot\nabla v) + 2p_1\int_\Omega |\nabla v|^{2(p_1-1)}(\nabla v\cdot\nabla(wz)).
\end{aligned}$$

Applying $\nabla\Delta v\cdot\nabla v=\frac{1}{2}\Delta|\nabla v|^2-|D^2v|^2$ and $\frac{1}{N}|\Delta v|^2\le|D^2v|^2$ to this equation, we have

$$\frac{d}{dt}\int_\Omega|\nabla v|^{2p_1}\le 2p_1\int_\Omega|\nabla v|^{2(p_1-1)}\Delta|\nabla v|^2-\frac{2p_1}{N}\int_\Omega|\Delta v|^2|\nabla v|^{2(p_1-1)}$$
$$+2p_1\int_\Omega|\nabla v|^{2(p_1-1)}(\nabla v\cdot\nabla(wz)).$$

The addition of $\int_\Omega|\nabla v|^{2p_1}$ to this estimate leads to the following estimate:

$$\begin{aligned}&\frac{d}{dt}\int_\Omega|\nabla v|^{2p_1}+\frac{2p_1}{N}\int_\Omega|\Delta v|^2|\nabla v|^{2(p_1-1)}+\int_\Omega|\nabla v|^{2p_1}\\&\le 2p_1\int_\Omega|\nabla v|^{2(p_1-1)}\Delta|\nabla v|^2+2p_1\int_\Omega|\nabla v|^{2(p_1-1)}\left(\nabla v\cdot\nabla(wz)\right)+\int_\Omega|\nabla v|^{2p_1}\\&=:J_1+J_2+J_3.\end{aligned}\tag{4.7}$$

We estimate the terms on the right-hand side of (4.7) respectively. We obtain from Lemma 2.5 that

$$\begin{aligned}J_1&=2p_1\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}|\nabla v|^{2(p_1-1)}-\frac{4(p_1-1)}{p_1}\int_\Omega|\nabla|\nabla v|^{p_1}|^2\\&\le 2p_1c\int_{\partial\Omega}|\nabla v|^{2p_1}-\frac{4(p_1-1)}{p_1}\int_\Omega|\nabla|\nabla v|^{p_1}|^2\\&\le-\frac{3(p_1-1)}{p_1}\int_\Omega|\nabla|\nabla v|^{p_1}|^2+c_1\int_\Omega|\nabla v|^{2p_1}.\end{aligned}$$

Using integration by parts yields that

$$J_2\le\frac{2p_1}{N}\int_\Omega|\Delta v|^2|\nabla v|^{2(p_1-1)}+\frac{p_1-1}{p_1}\int|\nabla|\nabla v|^{p_1}|^2+c_2\int_\Omega(wz)^2|\nabla v|^{2(p_1-1)}.$$

Plugging Lemma 2.5 into Lemma 3.2 entails that $\|\nabla v(\cdot,t)\|_{L^1(\Omega)}$ is bounded. The boundedness $\|\nabla v(\cdot,t)\|_{L^1(\Omega)}$, the Gagliardo–Nirenberg inequality and Young's inequality imply that there exists $\kappa=\frac{p_1-\frac{1}{2}}{p_1+\frac{1}{N}-\frac{1}{2}}\in(0,1)$ such that

$$\begin{aligned}(c_1+1)J_3&\le c_3\|\nabla|\nabla v|^{p_1}\|_{L^2(\Omega)}^{2\kappa}\|\nabla v\|_{L^1(\Omega)}^{2p_1(1-\kappa)}+c_4\|\nabla v\|_{L^1(\Omega)}^{2p_1}\\&\le\frac{p_1-1}{p_1}\int_\Omega|\nabla|\nabla v|^{p_1}|^2+c_5.\end{aligned}$$

Thus we obtain (4.5). Moreover, (4.6) can be seen as well as the proof of (4.5).
(Step 3) The completion of the differential inequality.
We show the following differential inequality:

$$\frac{d}{dt}\int_\Omega\left(u^p+|\nabla v|^{2p_1}+|\nabla z|^{2p_2}\right)+\int_\Omega\left(u^p+|\nabla v|^{2p_1}+|\nabla z|^{2p_2}\right)\le C.\tag{4.8}$$

We infer from the result of Step 1 and Step 2 that

$$
\begin{aligned}
(4.9)\qquad & \frac{d}{dt}\int_\Omega \left(u^p + |\nabla v|^{2p_1} + |\nabla z|^{2p_2}\right) + \alpha \int_\Omega u^{p+1} + \int_\Omega \left(|\nabla v|^{2p_1} + |\nabla z|^{2p_2}\right) \\
& + \frac{p(p-1)}{4}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{p_1-1}{p_1}\int_\Omega |\nabla|\nabla v|^{p_1}|^2 + \frac{p_2-1}{p_2}\int_\Omega |\nabla|\nabla z|^{p_2}|^2 \\
& \le \frac{\Lambda^2 p(p-1)}{3}\int_\Omega u^p|\nabla v|^2 + c_2\int_\Omega (wz)^2\,|\nabla v|^{2(p_1-1)} + d_2\int_\Omega u^2|\nabla z|^{2(p_2-1)} + c \\
& =: K_1 + K_2 + K_3 + c.
\end{aligned}
$$

We define $\rho_1 := 2(p+1)$, $\tilde{\rho}_1 := \frac{2(p+1)(p_2-1)}{p-1}$. Using Hölder's inequality and Young's inequality, we have

$$
\begin{aligned}
(4.10)\qquad K_1 = \frac{\Lambda^2 p(p-1)}{3}\int_\Omega u^p|\nabla v|^2 &\le \Big(\frac{\alpha(p+1)}{4p}\int_\Omega u^{p+1}\Big)^{\frac{p}{p+1}}\Big(c_6\int_\Omega |\nabla v|^{2(p+1)}\Big)^{\frac{1}{p+1}} \\
&\le \frac{\alpha}{4}\int_\Omega u^{p+1} + \tilde{c_6}\int_\Omega |\nabla v|^{\rho_1}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.11)\qquad K_3 = d_2\int_\Omega u^2|\nabla z|^{2(p_2-1)} &\le \Big(\frac{\alpha(p+1)}{8}\int_\Omega u^{p+1}\Big)^{\frac{2}{p+1}}\Big(c_7\int_\Omega |\nabla z|^{\frac{2(p+1)(p_2-1)}{p-1}}\Big)^{\frac{p-1}{p+1}} \\
&\le \frac{\alpha}{4}\int_\Omega u^{p+1} + \tilde{c_7}\int_\Omega |\nabla z|^{\tilde{\rho}_1}.
\end{aligned}
$$

Similarly, applying to Hölder's inequality, Young's inequality and the Gagliardo–Nirenberg inequality, we can find some $q_1 > \frac{N}{N-2}$ satisfying $\tilde{\kappa} = \frac{\frac{N-2}{N} - \frac{1}{2q_1}}{\frac{N-2}{N} + \frac{1}{N} - \frac{N}{2(N-2)q_1}} \in (0,1)$ and then we put $\rho_2 := \frac{2(p_1-1)q_1}{q_1-1}$, $\tilde{\rho}_2 := \frac{2(N-2)q_1}{N}$ such that

$$
\begin{aligned}
(4.12)\qquad K_2 &\le \tilde{c_2}\Big(\int_\Omega |\nabla v|^{\frac{2(p_1-1)q_1}{q_1-1}}\Big)^{\frac{q_1-1}{q_1}}\Big(\int_\Omega z^{2q_1}\Big)^{\frac{1}{q_1}} \\
&\le \tilde{\tilde{c_2}}\int_\Omega |\nabla v|^{\rho_2} + \tilde{\tilde{c_2}}\int_\Omega z^{2q_1} \\
&\le \tilde{\tilde{c_2}}\int_\Omega |\nabla v|^{\rho_2} + \bar{c_2}\|\nabla z\|^{2q_1\tilde{\kappa}}_{L^{\tilde{\rho}_2}(\Omega)}\|z\|^{2p(1-\tilde{\kappa})}_{L^{\frac{2N}{N-2}}(\Omega)} + \|z\|^{2p}_{L^{\frac{2N}{N-2}}(\Omega)} \\
&\le \tilde{\tilde{c_2}}\int_\Omega |\nabla v|^{\rho_2} + \hat{c}_2\int_\Omega |\nabla z|^{\tilde{\rho}_2} + \hat{c}_2.
\end{aligned}
$$

Let $q_1 \in \left(\frac{N(p+1)}{N+q_2}, \frac{(N+2)(p+1)}{2(N-2)}\right)$, $p_1 \in \left(p+1-\frac{q_2}{N}, r+\frac{q_2(q_1-1)}{N}\right)$, $p_2 > 3 - \frac{4}{p+1}$ for all $q_2 \in \left[1, \frac{2N}{N-4}\right)$ (For example, let $6 < p \le 7$, $q_1 = p_1 = 8$, $p_2 > \frac{17}{7}$). Using the Gagliardo–

Nirenberg inequality and Young's inequality, we have for $i = 1, 2$

$$(4.13)\qquad (\tilde{c_6} + \tilde{\tilde{c_2}}) \int_\Omega |\nabla v|^{\rho_i} \le c_8 \Big(\||\nabla v|^{p_1}\|^{\kappa_i}_{L^2(\Omega)} \||\nabla v|^{p_1}\|^{1-\kappa_i}_{L^{\frac{q_2}{p_1}}(\Omega)} + \||\nabla v|^{p_1}\|^{\frac{q_2}{p_1}}_{L^{\frac{\rho_i}{p_1}}(\Omega)} \Big)^{\frac{\rho_i}{p_1}}$$

$$\le \tilde{c_8} \Big(\int_\Omega |\nabla |\nabla v|^{p_1}|^2 \Big)^{\sigma_i} + \tilde{c_8}$$

$$\le \frac{p_1 - 1}{2p_1} \int_\Omega |\nabla |\nabla v|^{p_1}|^2 + \bar{c_8},$$

where

$$\kappa_i := \frac{2p_1}{\rho_i} \cdot \frac{\frac{\rho_i}{q_2} - 1}{\frac{2p_1}{q_2} + \frac{2}{N} - 1} \in (0,1),\ \ \sigma_i := \frac{\rho_i \kappa_i}{2p_1} \in (0,1)$$

and

$$(4.14)\qquad (\tilde{c_7} + \hat{c_2}) \int_\Omega |\nabla z|^{\tilde{\rho}_i} \le c_9 \Big(\||\nabla z|^{p_2}\|^{\tilde{\kappa}_i}_{L^2(\Omega)} \||\nabla z|^{p_2}\|^{1-\tilde{\kappa}_i}_{L^{\frac{2}{p_2}}(\Omega)} + \||\nabla z|^{p_2}\|^{\frac{2}{p_2}}_{L^{\frac{\tilde{\rho}_i}{p_2}}(\Omega)} \Big)^{\frac{\tilde{\rho}_i}{p_2}}$$

$$\le \tilde{c_9} \Big(\int_\Omega |\nabla |\nabla z|^{p_2}|^2 \Big)^{\sigma_i} + \tilde{c_9}$$

$$\le \frac{p_2 - 1}{2p_2} \int_\Omega |\nabla |\nabla z|^{p_2}|^2 + \bar{c_9},$$

where

$$\tilde{\kappa}_i := \frac{2p_2}{\tilde{\rho}_i} \cdot \frac{\frac{\tilde{\rho}_i}{2} - 1}{p_2 + \frac{2}{N} - 1} \in (0,1),\ \ \tilde{\sigma}_i := \frac{\tilde{\rho}_i \tilde{\kappa}_i}{2p_2} \in (0,1).$$

At last, Young's inequality leads to

$$(4.15)\qquad \int_\Omega u^p \le \frac{\alpha}{2} \int_\Omega u^{p+1} + c_{10}.$$

Taking into account (4.9)–(4.15), we show (4.8). Hence we see from (4.8), Gronwall's lemma and (1.3) that

$$\int_\Omega \big(u^p + |\nabla v|^{2p_1} + |\nabla z|^{2p_2} \big) \le \int_\Omega \big(u_0^p + |\nabla v_0|^{2p_1} + |\nabla z_0|^{2p_2} \big) + C$$

for all $t \in (0, T_{\max})$. Therefore we obtain the boundedness of $\|u(\cdot,t)\|_{L^p(\Omega)}$ for all $t \in (0, T_{\max})$. □

If the next proposition is shown, we can prove Theorem 1.1.

Proposition 4.3. *Let $N \le 9$ and assume that (1.2)–(1.8) hold. Then the solution (u, v, w, z) of (1.1) is global in time; that is, $T_{\max} = \infty$. Moreover, there exist $\alpha \in (0,1)$, $C > 0$ such that*

$$(4.16)\qquad \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \le C$$

for all $t > 0$, as well as

$$(4.17)\qquad \|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[t,t+1])} + \|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[t,t+1])} + \|z\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[t,t+1])} \le C$$

for all $t \ge 1$.

Proof. We divide the proof into the three cases $N \le 3$, $N = 4, 5$ and $6 \le N \le 9$ to show this proposition.

(Case $N \le 3$**)**

We have the L^1-boundedness of u from Lemma 2.2. Therefore we can complete the proof of this proposition in the case $N \le 3$ by using an argument similar to the proof of [**3**, Lemmas 3.5–3.6].

(Case $N = 4, 5$**)**

Since $N = 4, 5$, we obtain

$$\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \le C_1(N, M) \quad \text{for all } t \in (0, T),$$

where some $M > 0$ satisfies

$$\|z(\cdot, t)\|_{L^{\frac{2N}{N-2}}(\Omega)} \le M \quad \text{for all } t \in (0, T)$$

by using Lemmas 3.2 and 4.1. Furthermore, Lemma 3.3 and (4.12), $T \in (0, T_{\max})$ yield that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} < C_2(N, M) \quad \text{for all } t \in (0, T_{\max}).$$

Therefore we derive from Lemma 3.1 and (4.13) that

$$\|z(\cdot, t)\|_{L^\infty(\Omega)} \le C_3(N, M) \quad \text{for all } t \in (0, T_{\max}).$$

Combination of Lemma 2.4 and (2.1) in Lemma 2.1 shows that $T_{\max} = \infty$ and, by independence of the obtained estimate with respect to $t \in (0, \infty)$, establishes (4.16). Therefore straightforward bootstrap arguments involving standard interior parabolic regularity theory (see [**8**] for instance) readily yield (4.17).

(Case $6 \le N \le 9$**)**

Since $6 \le N \le 9$, we have (4.4) from Lemma 4.2. So we see that

$$\begin{aligned}
&\|u(\cdot, t)\|_{L^\infty(\Omega)} < C_4(N, C) && \text{for all } t \in (0, T_{\max}),\\
&\|z(\cdot, t)\|_{L^\infty(\Omega)} \le C_5(N, C) && \text{for all } t \in (0, T_{\max}),\\
&\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \le C_6(N, C) && \text{for all } t \in (0, T_{\max})
\end{aligned}$$

by using Lemmas 3.1–3.3 and (4.4). In the same way as in the case $N = 4, 5$, we obtain (4.16) and (4.17) from Lemmas 2.1 and 2.4. □

Proof of Theorem 1.1. Proposition 4.3 completes the proof of Theorem 1.1. □

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