Advances in Mathematical Sciences and Applications
Vol. 27, No. 1 (2018), pp. 193–224

GAKKOTOSHO
TOKYO JAPAN

ON SOME PARABOLIC SYSTEMS ARISING FROM A NUCLEAR REACTOR MODEL WITH NONLINEAR BOUNDARY CONDITIONS

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Dedicated to the memory of the late Professor Kyûya Masuda

Abstract. In this paper, we are concerned with a reaction diffusion system arising from a nuclear reactor model in bounded domains with nonlinear boundary conditions. We show the existence of a stationary solution and its ordered uniqueness. It is also shown that every positive stationary solution possesses threshold property to determine blow-up or globally existence for solutions of nonstationary problem.

†Partly supported by the Grant-in-Aid for Scientific Research, # 15K13451, the Ministry of Education, Culture, Sports, Science and Technology, Japan.
Communicated by Editors; Received May 11, 2018
2010 *Mathematics Subject Classification.* Primary: 35K57; Secondary: 35K61, 35Q79.
Keywords: initial-boundary problem, reaction diffusion system, nonlinear boundary conditions, threshold property.

1 Introduction

We consider the following initial-boundary value problem for a nonlinear reaction diffusion system:

$$\begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & x \in \Omega,\ t > 0, \\ \partial_t u_2 - \Delta u_2 = a u_1, & x \in \Omega,\ t > 0, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta |u_2|^{\gamma-2} u_2 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x) \geq 0,\ u_2(x,0) = u_{20}(x) \geq 0, & x \in \Omega, \end{cases} \tag{NR}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, ν denotes the unit outward normal vector on $\partial\Omega$ and ∂_ν is outward normal derivative, i.e., $\partial_\nu u_i = \nabla u_i \cdot \nu$ $(i = 1, 2)$. Moreover u_1, u_2 are real-valued unknown functions, a and b are given positive constants. As for the parameters appearing in the boundary condition, we assume $\alpha \in [0, \infty)$, $\beta \in (0, \infty)$ and $\gamma \in [2, \infty)$. We note that the boundary condition for u_1 becomes the homogeneous Neumann boundary condition when $\alpha = 0$, and the boundary condition for u_2 gives the Robin boundary condition when $\gamma = 2$. Finally, u_{10}, $u_{20} \in L^\infty(\Omega)$ are given nonnegative initial data.

This system describes diffusion phenomena of neutrons and heat in nuclear reactors by taking the heat conduction into consideration, introduced by Kastenberg and Chambré [11]. In this model u_1 and u_2 represent the neutron density and the temperature in nuclear reactors respectively. There are many studies on this model under various boundary conditions, for example, [3], [4], [7], [8], [10], [20] and [22]. Many of them are concerned with the existence of positive steady-state solutions and the long-time behavior of solutions.

The original problem for (NR):

$$\begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & x \in \Omega,\ t > 0, \\ \partial_t u_2 = a u_1 - c u_2, & x \in \Omega,\ t > 0, \\ u_1 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x) \geq 0,\ u_2(x,0) = u_{20}(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

for some $c > 0$ is studied by [20]. In (1.1), the negative feedback $-cu_2$ from the heat into itself is considered instead of the diffusion term. In Rothe's book [20], the boundedness and the convergence to equilibrium for (1.1) are examined in detail.

In [7], our system is studied with $\alpha = 0$ and $\gamma = 2$, i.e., with the homogeneous Neumann boundary condition and Robin boundary condition:

$$\begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & x \in \Omega,\ t > 0, \\ \partial_t u_2 - \Delta u_2 = a u_1, & x \in \Omega,\ t > 0, \\ \partial_\nu u_1 = \partial_\nu u_2 + \beta u_2 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x),\ u_2(x,0) = u_{20}(x), & x \in \overline{\Omega}. \end{cases} \tag{1.2}$$

They showed the existence and the ordered uniqueness of positive stationary solution for $N \in [2, 5]$. They also investigated some threshold property to determine blow-up or globally existence. Moreover, in [22] the case where $\beta = 0$, that is, the homogeneous Neumann boundary condition for u_2 is studied. The author of [22] discussed the stability region and the instability region of (1.2) and give an upper bound and a lower bound on the blowing-up time for a solution which blows up in finite time.

The following system with the homogeneous Dirichlet boundary conditions:

$$\begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2^p - b u_1, & x \in \Omega,\ t > 0, \\ \partial_t u_2 - \Delta u_2 = a u_1, & x \in \Omega,\ t > 0, \\ u_1 = u_2 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x),\ u_2(x,0) = u_{20}(x), & x \in \Omega, \end{cases} \tag{1.3}$$

is studied by [8] and [10]. In [8], they showed the existence of positive stationary solutions for the case where $p = 1$ and $N = 2$, 3 or Ω is bounded convex domain with $N \in [2, 5]$. Furthermore, they obtained the threshold property of stationary solution announced in [7] when Ω is ball. In [10], the existence and ordered uniqueness of positive stationary solutions are considered for general $p > 0$ and some threshold result is obtained. Moreover the blow-up rate estimate is given for positive blowing-up solutions when Ω is ball and $p \geq 1$.

In this paper, we are concerned with the nonlinear boundary condition. From physical point of view it could be more natural to consider the nonlinear boundary condition rather than the homogeneous Dirichlet boundary condition or Neumann boundary condition. Indeed, if there is no control of the heat flux on the boundary, it is well known that the power type nonlinearity for u_2 is justified by Stefan-Boltzmann's law, which says that the heat energy radiation from the surface of the body is proportional to the fourth power of temperature when $N = 3$.

The outline of this paper is as follows. In Section 2, we consider the stationary problem associated with (NR) and show the existence of positive solutions by applying an abstract fixed point theorem based on Krasnosel'skii [12]. In order to apply this fixed point theorem, we need to estimate L^∞-norm of solutions. To do this, since we are concerned with nonlinear boundary conditions, we can not rely on the standard linear theory. To cope with this difficulty, we introduce a new approach, which enables us to obtain strong summability of solutions on the boundary. Next, we prove the ordered uniqueness for the positive stationary solutions of (NR). We here use the property of first eigenfunction for the eigenvalue problem associated with the Robin boundary condition.

In Section 3, we study the nonstationary problem. In the first subsection, we show the existence of local solutions in time for (NR) by abstract theory of maximal monotone operators associated with subdifferential operators together with L^∞-energy method [18]. In the second subsection, we discuss the large time behavior of solutions to (NR) and prove that every positive stationary solution plays a role of threshold to separate global solutions and finite time blowing-up solutions. In this procedure, we essentially rely on the comparison theorem. Furthermore in order to show the finite time blow-up of solutions of (NR), the crucial point is to construct an appropriate subsolution.

2 Stationary problem

In this section, we are going to show the existence of the positive stationary solutions for (NR) and prove the ordered uniqueness of them. The stationary problem for (NR) is given by

$$\begin{cases} -\Delta u_1 = u_1 u_2 - b u_1, & x \in \Omega, \\ -\Delta u_2 = a u_1, & x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta |u_2|^{\gamma-2} u_2 = 0, & x \in \partial\Omega. \end{cases} \tag{S-NR}$$

It should be noticed that since (S-NR) has no variational structure, it is not possible to apply the variational method to (S-NR). In order to show the existence of positive stationary solutions to (NR), we rely on the abstract fixed point theorem developed by Krasnosel'skii. The crucial step in proving the existence of positive stationary solutions is how to obtain L^∞-estimates of solutions.

We state a couple of lemmas to prove our results for (S-NR).

Lemma 2.1 (Krasnosel'skii-type fixed point theorem [12], [13]). *Suppose that E is a real Banach space with norm $\|\cdot\|$, $K \subset E$ is a positive cone, and $\Phi : K \to K$ is a compact mapping satisfying $\Phi(0) = 0$. Assume that there exists two constants $R > r > 0$ and an element $\varphi \in K \setminus \{0\}$, such that*

$$\text{(i) } u \neq \lambda \Phi(u), \ \ \forall \lambda \in (0,1), \ \ \text{if } \ u \in K \ \text{and } \|u\| = r,$$

$$\text{(ii) } u \neq \Phi(u) + \lambda\varphi, \ \ \forall \lambda \geq 0, \ \ \text{if } \ u \in K \ \text{and } \|u\| = R.$$

Then the mapping Φ possesses at least one fixed point in $K_1 := \{u \in K;\ 0 < r < \|u\| < R\}$.

Lemma 2.2 ([6]). *Let λ_1 and φ_1 be the first eigenvalue and the corresponding eigenfunction for the problem:*

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \Omega, \\ \partial_\nu\varphi + \alpha\varphi = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is smooth bounded domain in $\mathbb{R}^N$ and $\alpha > 0$. Then $\lambda_1 > 0$ and there exists a constant $C_\alpha > 0$ such that

$$\varphi_1(x) \geq C_\alpha \qquad x \in \overline{\Omega}.$$

Indeed, it is well known that $\varphi_1 > 0$ in Ω by the strong maximum principle. Suppose that there exists $x_0 \in \partial\Omega$ such that $\varphi_1(x_0) = 0$, then the boundary condition assures $\partial_\nu\varphi_1(x_0) = -\alpha\varphi_1(x_0) = 0$. On the other hand, Hopf's strong maximum principle assures that $\partial_\nu\varphi_1(x_0) < 0$. This is contradiction, i.e., $\varphi_1(x) > 0$ on $\overline{\Omega}$.

2.1 Existence of positive solutions

Theorem 2.1. *Let $1 \leq N \leq 5$ and suppose that either* (A) *or* (B) *is satisfied* :

$$\begin{cases} \text{(A)} \qquad \gamma = 2, \ \ \alpha \leq 2\beta, \\ \text{(B)} \qquad \gamma > 2. \end{cases}$$

Then (S-NR) *has at least one positive solution.*

We rely on Lemma 2.1 to prove this theorem. In order to apply Lemma 2.1, we here fix our setting:

$$E = C(\overline{\Omega}) \times C(\overline{\Omega}), \qquad u = (u_1, u_2)^{\mathrm{T}} \in E,$$
$$\|u\| = \|u_1\|_{C(\overline{\Omega})} + \|u_2\|_{C(\overline{\Omega})}, \qquad K = \{u \in E; u_1 \geq 0, u_2 \geq 0\}.$$

Set $\varphi = (\varphi_1, 0)^{\mathrm{T}} \in K \backslash \{0\}$, where λ_1 and φ_1 are the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem:

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \Omega, \\ \partial_\nu \varphi + \alpha\varphi = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

In section 2, we normalize $\varphi_1(x)$ such that $\|\varphi_1\|_{L^2} = 1$. For given $u = (u_1, u_2)^{\mathrm{T}} \in K$, let $v = (v_1, v_2)^{\mathrm{T}} = \Psi(u)$ be the unique nonnegative solution (see Brézis [2]) of

$$\begin{cases} -\Delta v_1 + bv_1 = u_1 u_2, & x \in \Omega, \\ -\Delta v_2 = au_1, & x \in \Omega, \\ \partial_\nu v_1 + \alpha v_1 = \partial_\nu v_2 + \beta|v_2|^{\gamma-2}v_2 = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

It is clear that $\Psi(0) = 0$. Moreover $\Psi : K \to K$ is compact. In order to prove the compactness of Ψ, we use the next Lemma for the following problem:

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \partial_\nu u = g, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

Lemma 2.3. ([17]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Suppose that $f \in L^{\frac{p}{2}}(\Omega)$ and $g \in L^{p-1}(\partial\Omega)$ with $p > N \geq 2$, then there exist $\delta > 0$ and a positive constant C such that every weak solution u of* (2.3) *belongs to $C^{0,\delta}(\overline{\Omega})$ and satisfies*

$$\|u\|_{C^{0,\delta}(\overline{\Omega})} \leq C \left(\|u\|_{L^2(\Omega)} + \|f\|_{L^{\frac{p}{2}}(\Omega)} + \|g\|_{L^{p-1}(\partial\Omega)} \right).$$

Since Ω is bounded and $(u_1, u_2) \in C(\overline{\Omega}) \times C(\overline{\Omega})$, it follows from elliptic estimate that $v_1 \in W^{2,p}(\Omega)$ for any p. Since $W^{2,p}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$ for $p > \frac{N}{2}$, the mapping $(u_1, u_2) \mapsto v_1$ is compact. Next we assume that $N \geq 2$ and consider the following equation:

$$\begin{cases} -\Delta v_2 = au_1 \in L^\infty(\Omega), & x \in \Omega, \\ \partial_\nu v_2 + \beta|v_2|^{\gamma-2}v_2 = 0, & x \in \partial\Omega. \end{cases}$$

Multiplying the equation by $|v_2|^{r-2}v_2$ and applying integration by parts, we get

$$(r-1)\int_\Omega |v_2|^{r-2}|\nabla v_2|^2 dx + \beta \int_{\partial\Omega} |v_2|^{r+\gamma-2} dS = a \int_\Omega u_1 |v_2|^{r-2} v_2 dx. \tag{2.4}$$

Noting that $(\|\nabla v_2\|^2_{L^2(\Omega)} + \int_{\partial\Omega} v_2^2 dS)^{1/2}$ is equivalent to the usual H^1-norm by Poincaré-Friedrichs type inequality, we obtain

$$\begin{aligned}
(l.h.s.) &= (r-1)\int_\Omega \left| |v_2|^{\frac{r-2}{2}} |\nabla v_2| \right|^2 dx + \beta \int_{\partial\Omega} |v_2|^{r+\gamma-2} dS \\
&\geq \frac{4(r-1)}{r^2} \int_\Omega \left|\nabla |v_2|^{\frac{r}{2}}\right|^2 dx + \beta \int_{\partial\Omega} |v_2|^r dS - \beta|\partial\Omega| \\
&\geq C_r \left(\int_\Omega \left|\nabla |v_2|^{\frac{r}{2}}\right|^2 dx + \int_{\partial\Omega} \left||v_2|^{\frac{r}{2}}\right|^2 dS \right) - \beta|\partial\Omega| \\
&\geq C_r \int_\Omega \left||v_2|^{\frac{r}{2}}\right|^2 dx - \beta|\partial\Omega| = C_r \|v_2\|^r_{L^r(\Omega)} - \beta|\partial\Omega|,
\end{aligned}$$

where $C_r = \min\{\frac{4(r-1)}{r^2}, \beta\} > 0$ and we used the estimate:

$$\begin{aligned}
\beta \int_{\partial\Omega} |v_2|^{r+\gamma-2} dS \geq \beta \int_{\{|v_2|\geq 1\}} |v_2|^{r+\gamma-2} dS &\geq \beta \int_{\{|v_2|\geq 1\}} |v_2|^r dS \\
&= \beta \int_{\partial\Omega} |v_2|^r dS - \beta \int_{\{|v_2|\leq 1\}} |v_2|^r dS \\
&\geq \beta \int_{\partial\Omega} |v_2|^r dS - \beta|\partial\Omega|.
\end{aligned}$$

Hence Hölder's inequality, Young's inequality and (2.4) yield

$$\|v_2\|_{L^r(\Omega)} \leq \left\{ \beta|\partial\Omega| \left(\frac{C_r}{2}\right)^{-1} + \frac{1}{r}\left(\frac{C_r}{2}\right)^{-r} \|au_1\|^r_{L^r(\Omega)} \right\}^{\frac{1}{r}} \qquad \forall r < \infty.$$

Therefore by (2.4) we have

$$\int_{\partial\Omega} |v_2|^{r+\gamma-2} dS \leq \frac{1}{\beta}\|au_1\|_{L^r(\Omega)} \left\{ \beta|\partial\Omega| \left(\frac{C_r}{2}\right)^{-1} + \frac{1}{r}\left(\frac{C_r}{2}\right)^{-r} \|au_1\|^r_{L^r(\Omega)} \right\}^{\frac{r-1}{r}} \quad \forall r < \infty.$$

Thus we see that $v_2 \in L^r(\partial\Omega)$ for all large $r < \infty$ and we can apply Lemma 2.3 to get $v_2 \in C^{0,\delta}(\overline{\Omega})$ for some $\delta > 0$. Note that $C^{0,\delta}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ is compact. As for the case where $N = 1$, (2.4) with $r = 2$ gives the a priori bound for $\|v_2\|_{H^1(\Omega)}$. Since the embedding $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact, the compactness of Ψ is easily derived. Thus we see that $\Psi : K \to K$ is compact.

In order to show the existence of positive stationary solutions for (S-NR), it suffices to prove that Ψ has a fixed point in K. Therefore, to prove Theorem 2.1 we are going to verify conditions (i) and (ii) of Lemma 2.1.

We first check condition (i).

Lemma 2.4. *Let $r = \frac{b}{2}$, then $u \neq \lambda\Psi(u)$ for any $\lambda \in (0,1)$ and $u \in K$ satisfying $\|u\| = r$. That is, condition* (i) *of* Lemma 2.1 *with $\Phi = \Psi$ holds.*

Proof. We prove the statement by contradiction. Suppose that there exist $\lambda \in (0,1)$ and $u \in K$ with $\|u\| = r$ such that $u = \lambda \Psi(u)$, that is, u_1 and u_2 satisfy

$$\begin{cases} -\Delta u_1 + b u_1 = \lambda u_1 u_2, & x \in \Omega, \\ -\Delta u_2 = \lambda a u_1, & x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta \left| \dfrac{u_2}{\lambda} \right|^{\gamma-2} u_2 = 0, & x \in \partial\Omega. \end{cases} \tag{2.5}$$

Multiplying the first equation of (2.5) by u_1 and using integration by parts, we obtain

$$\begin{aligned} \|\nabla u_1\|^2_{L^2(\Omega)} + \alpha \int_{\partial\Omega} u_1^2 dS + b\|u_1\|^2_{L^2(\Omega)} &= \lambda \int_\Omega u_1^2 u_2 dx \\ &\le \|u_2\|_{L^\infty(\Omega)} \|u_1\|^2_{L^2(\Omega)} \\ &\le \frac{b}{2} \|u_1\|^2_{L^2(\Omega)}, \end{aligned}$$

where we use the fact

$$\|u_2\|_{L^\infty(\Omega)} \le \|u\| = r = \frac{b}{2}.$$

Then

$$\|\nabla u_1\|^2_{L^2(\Omega)} + \alpha \int_{\partial\Omega} u_1^2 dS + \frac{b}{2} \|u_1\|^2_{L^2(\Omega)} \le 0.$$

Hence we have $u_1 = 0$. By the second equation of (2.5), we see that u_2 satisfies

$$\begin{cases} -\Delta u_2 = 0, & x \in \Omega, \\ \partial_\nu u_2 + \beta \left| \dfrac{u_2}{\lambda} \right|^{\gamma-2} u_2 = 0, & x \in \partial\Omega. \end{cases}$$

Multiplying this equation by u_2 and integration by parts, we obtain

$$\|\nabla u_2\|^2_{L^2(\Omega)} + \frac{\beta}{|\lambda|^{\gamma-2}} \int_{\partial\Omega} |u_2|^\gamma dS = 0, \quad \text{i.e.,} \qquad \|\nabla u_2\|_{L^2(\Omega)} = 0, \qquad u_2 \,|_{\partial\Omega} = 0.$$

By the use of Poincaré's inequality, we also get $u_2 = 0$. Thus $u_1 = u_2 = 0$. This contradicts the assumption $\|u\| = \frac{b}{2} > 0$. □

In order to verify condition (ii), we here claim the following lemma.

Lemma 2.5. *Let* $1 \le N \le 5$ *and suppose that either* (A) *or* (B) *is satisfied* :

$$\begin{cases} \text{(A)} \qquad \gamma = 2, \quad \alpha \le 2\beta, \\ \text{(B)} \qquad \gamma > 2. \end{cases}$$

Then there exists a constant $R (> r = \frac{b}{2})$ *such that for any* $\lambda > 0$ *and any solution* u *of* $u = \Psi(u) + \lambda\varphi$, *it holds that*

$$\|u\| < R.$$

Proof. We rewrite $u = \Psi(u) + \lambda\varphi$ in terms of each component:

$$\begin{cases} -\Delta u_1 + bu_1 = u_1 u_2 + \lambda(b+\lambda_1)\varphi_1, & x \in \Omega, \\ -\Delta u_2 = au_1, & x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta|u_2|^{\gamma-2}u_2 = 0, & x \in \partial\Omega. \end{cases} \tag{2.6}$$

In what follows, we denote by C a general constant which differs from place to place. First, we derive H^1-estimate for u_2. Replacing u_1 in the first equation of (2.6) by $-\frac{1}{a}\Delta u_2$, we get

$$\begin{cases} \Delta^2 u_2 - b\Delta u_2 = -u_2\Delta u_2 + \lambda a(b+\lambda_1)\varphi_1, & x \in \Omega \\ \partial_\nu u_2 + \beta|u_2|^{\gamma-2}u_2 = \partial_\nu \Delta u_2 + \alpha\Delta u_2 = 0, & x \in \partial\Omega. \end{cases} \tag{2.7}$$

Multiplying (2.7) by φ_1, using integration by parts and noting that the boundary conditions $\partial_\nu\varphi_1 + \alpha\varphi_1 = \partial_\nu u_2 + \beta|u_2|^{\gamma-2}u_2 = 0$, we have

$$\begin{aligned}
(l.h.s) &= \int_\Omega \Delta^2 u_2\varphi_1 dx - b\int_\Omega \Delta u_2\varphi_1 dx \\
&= -\int_\Omega \nabla(\Delta u_2)\cdot\nabla\varphi_1 dx + \int_{\partial\Omega}(\partial_\nu\Delta u_2)\varphi_1 dS \\
&\quad + b\int_\Omega \nabla u_2\cdot\nabla\varphi_1 dx - b\int_{\partial\Omega}(\partial_\nu u_2)\varphi_1 dS \\
&= \int_\Omega \Delta u_2\Delta\varphi_1 dx - \int_{\partial\Omega}\Delta u_2(\partial_\nu\varphi_1)dS + \int_{\partial\Omega}(\partial_\nu\Delta u_2)\varphi_1 dS \\
&\quad - b\int_\Omega u_2\Delta\varphi_1 dx + b\int_{\partial\Omega}u_2(\partial_\nu\varphi_1)dS - b\int_{\partial\Omega}(\partial_\nu u_2)\varphi_1 dS \\
&= -\lambda_1\int_\Omega \Delta u_2\varphi_1 dx + \alpha\int_{\partial\Omega}\Delta u_2\varphi_1 dS - \alpha\int_{\partial\Omega}\Delta u_2\varphi_1 dS \\
&\quad + b\lambda_1\int_\Omega u_2\varphi_1 dx - \alpha b\int_{\partial\Omega}u_2\varphi_1 dS + \beta b\int_{\partial\Omega}u_2^{\gamma-1}\varphi_1 dS \\
&= \lambda_1\int_\Omega \nabla u_2\cdot\nabla\varphi_1 dx - \lambda_1\int_{\partial\Omega}(\partial_\nu u_2)\varphi_1 dS \\
&\quad + b\lambda_1\int_\Omega u_2\varphi_1 dx - \alpha b\int_{\partial\Omega}u_2\varphi_1 dS + \beta b\int_{\partial\Omega}u_2^{\gamma-1}\varphi_1 dS \\
&= -\lambda_1\int_\Omega u_2\Delta\varphi_1 dx + \lambda_1\int_{\partial\Omega}u_2(\partial_\nu\varphi_1)dS - \lambda_1\int_{\partial\Omega}(\partial_\nu u_2)\varphi_1 dS \\
&\quad + b\lambda_1\int_\Omega u_2\varphi_1 dx - \alpha b\int_{\partial\Omega}u_2\varphi_1 dS + \beta b\int_{\partial\Omega}u_2^{\gamma-1}\varphi_1 dS \\
&= \lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx + \beta(b+\lambda_1)\int_{\partial\Omega}u_2^{\gamma-1}\varphi_1 dS - \alpha(b+\lambda_1)\int_{\partial\Omega}u_2\varphi_1 dS,
\end{aligned}$$

and

$$
\begin{aligned}
(r.h.s) &= -\int_\Omega u_2\Delta u_2\varphi_1 dx + \lambda a(b+\lambda_1)\|\varphi_1\|^2_{L^2(\Omega)} \\
&= \int_\Omega \nabla u_2\cdot\nabla(u_2\varphi_1)dx - \int_{\partial\Omega}(\partial_\nu u_2)u_2\varphi_1 dS + \lambda a(b+\lambda_1) \\
&= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \int_\Omega u_2\nabla u_2\cdot\nabla\varphi_1 dx + \beta\int_{\partial\Omega} u_2^\gamma\varphi_1 dS + \lambda a(b+\lambda_1) \\
&= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{1}{2}\int_\Omega \nabla u_2^2\cdot\nabla\varphi_1 dx + \beta\int_{\partial\Omega} u_2^\gamma\varphi_1 dS + \lambda a(b+\lambda_1) \\
&= \int_\Omega |\nabla u_2|^2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2\Delta\varphi_1 dx + \frac{1}{2}\int_{\partial\Omega} u_2^2(\partial_\nu\varphi_1)dS \\
&\quad + \beta\int_{\partial\Omega} u_2^\gamma\varphi_1 dS + \lambda a(b+\lambda_1) \\
&= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + \beta\int_{\partial\Omega} u_2^\gamma\varphi_1 dS - \frac{\alpha}{2}\int_{\partial\Omega} u_2^2\varphi_1 dS + \lambda a(b+\lambda_1).
\end{aligned}
$$

Therefore the following equality holds.

$$
\begin{aligned}
\lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx = \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + a(b+\lambda_1)\lambda \\
+ \int_{\partial\Omega}\left\{\beta u_2^\gamma - \beta\,(b+\lambda_1)\,u_2^{\gamma-1} - \frac{\alpha}{2}u_2^2 + \alpha\,(b+\lambda_1)\,u_2\right\}\varphi_1 dS.
\end{aligned} \tag{2.8}
$$

Since (A) : $\gamma = 2$, $\alpha \le 2\beta$ or (B) : $\gamma > 2$ holds, we get

$$
\inf_{u_2\ge 0}\left\{\beta u_2^\gamma - \beta\,(b+\lambda_1)\,u_2^{\gamma-1} - \frac{\alpha}{2}u_2^2 + \alpha\,(b+\lambda_1)\,u_2\right\} \ge -C > -\infty.
$$

Moreover, we see that due to the boundedness of φ_1 (cf. Lemma 2.2)

$$
\lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx \ge \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + a(b+\lambda_1)\lambda - C.
$$

By Schwarz's inequality and Young's inequality, it is easy to see that

$$
\begin{aligned}
\int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + a(b+\lambda_1)\lambda &\le \lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx + C \\
&\le \lambda_1(b+\lambda_1)\left(\int_\Omega u_2^2\varphi_1 dx\right)^{\frac{1}{2}}\|\varphi_1\|^{\frac{1}{2}}_{L^1(\Omega)} + C \\
&\le \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + C.
\end{aligned}
$$

Hence we obtain

$$
\int_\Omega |\nabla u_2|^2\varphi_1 dx \le C, \qquad \int_\Omega u_2^2\varphi_1 dx \le C, \qquad \lambda \le C, \tag{2.9}
$$

and

$$\int_\Omega u_2\varphi_1 dx \le \left(\int_\Omega u_2^2\varphi_1 dx\right)^{\frac{1}{2}} \left(\int_\Omega \varphi_1 dx\right)^{\frac{1}{2}} \le C. \tag{2.10}$$

Furthermore it follows from Lemma 2.2 and (2.9)

$$C_\alpha \left(\int_\Omega |\nabla u_2|^2 dx + \int_\Omega u_2^2 dx\right) \le \int_\Omega |\nabla u_2|^2 \varphi_1 dx + \int_\Omega u_2^2 \varphi_1 dx \le C,$$

whence follows

$$\|u_2\|_{H^1(\Omega)} \le C. \tag{2.11}$$

By (2.10) and (2.8), we also have

$$\int_{\partial\Omega} \left\{\beta u_2^\gamma - \beta\,(b+\lambda_1)\,u_2^{\gamma-1} - \frac{\alpha}{2}u_2^2 + \alpha\,(b+\lambda_1)\,u_2\right\}\varphi_1 dS \le C. \tag{2.12}$$

Hence we can obtain

$$\begin{cases} \displaystyle\int_{\partial\Omega} u_2^\gamma \varphi_1 dS \le C & (\gamma > 2 \ \text{ or } \ \gamma = 2,\ \alpha < 2\beta), \\ \displaystyle\int_{\partial\Omega} u_2 \varphi_1 dS \le C & (\gamma = 2,\ \alpha = 2\beta). \end{cases} \tag{2.13}$$

Indeed, if $\gamma > 2$, then by Hölder's inequality and Young's inequality, we get

$$\begin{aligned} \beta \int_{\partial\Omega} u_2^\gamma \varphi_1 dS + \alpha(b+\lambda_1)\int_{\partial\Omega} u_2\varphi_1 dS &\le C + \beta(b+\lambda_1)\int_{\partial\Omega} u_2^{\gamma-1}\varphi_1 dS + \frac{\alpha}{2}\int_{\partial\Omega} u_2^2\varphi_1 dS \\ &\le C + \beta(b+\lambda_1)\left(\int_{\partial\Omega} u_2^\gamma\varphi_1 dS\right)^{\frac{\gamma-1}{\gamma}} \left(\int_{\partial\Omega}\varphi_1 dS\right)^{\frac{1}{\gamma}} \\ &\quad + \frac{\alpha}{2}\left(\int_{\partial\Omega} u_2^\gamma\varphi_1 dS\right)^{\frac{2}{\gamma}} \left(\int_{\partial\Omega}\varphi_1 dS\right)^{\frac{\gamma-2}{\gamma}} \\ &\le C + \beta(b+\lambda_1)\|\varphi_1\|_{L^\infty(\Omega)}^{\frac{1}{\gamma}} |\partial\Omega|^{\frac{1}{\gamma}} \left(\int_{\partial\Omega} u_2^\gamma\varphi_1 dS\right)^{\frac{\gamma-1}{\gamma}} \\ &\quad + \frac{\alpha}{2}\|\varphi_1\|_{L^\infty(\Omega)}^{\frac{\gamma-2}{\gamma}} |\partial\Omega|^{\frac{\gamma-2}{\gamma}} \left(\int_{\partial\Omega} u_2^\gamma\varphi_1 dS\right)^{\frac{2}{\gamma}} \\ &\le C + \frac{\beta}{2}\int_{\partial\Omega} u_2^\gamma\varphi_1 dS, \end{aligned}$$

where we denote by $|\partial\Omega|$ a volume of $\partial\Omega$ and use the following property (see [9]):

$$\|\varphi_1\|_{L^\infty(\partial\Omega)} \le \|\varphi_1\|_{L^\infty(\Omega)}.$$

On the other hand, if $\gamma = 2$ and $\alpha < 2\beta$, then it follows from Schwarz's inequality and Young's inequality

$$
\begin{aligned}
&\left(\beta - \frac{\alpha}{2}\right) \int_{\partial\Omega} u_2^2 \varphi_1 dS + \alpha(b+\lambda_1) \int_{\partial\Omega} u_2 \varphi_1 dS \\
&\leq C + \beta(b+\lambda_1) \int_{\partial\Omega} u_2 \varphi_1 dS \\
&\leq C + \beta(b+\lambda_1) \left(\int_{\partial\Omega} u_2^2 \varphi_1 dS\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} \varphi_1 dS\right)^{\frac{1}{2}} \\
&\leq C + \beta(b+\lambda_1) \|\varphi_1\|_{L^\infty(\Omega)} |\partial\Omega|^{\frac{1}{2}} \left(\int_{\partial\Omega} u_2^2 \varphi_1 dS\right)^{\frac{1}{2}} \\
&\leq C + \frac{1}{2}\left(\beta - \frac{\alpha}{2}\right) \int_{\partial\Omega} u_2^2 \varphi_1 dS.
\end{aligned}
$$

For the case of $\gamma = 2$ and $\alpha = 2\beta$, from (2.12) it is clear that

$$
\beta \int_{\partial\Omega} u_2 \varphi_1 dS \leq C.
$$

Thus we obtain (2.13).

Now, we derive H^1-estimate for u_1. Multiplying the first equation of (2.6) by φ_1 and using integration by parts, we get

$$
(\lambda_1 + b) \int_\Omega u_1 \varphi_1 dx = \int_\Omega u_1 u_2 \varphi_1 dx + \lambda(\lambda_1 + b) \tag{2.14}
$$

Similarly, multiplying the second equation of (2.6) by φ_1, we get

$$
\lambda_1 \int_\Omega u_2 \varphi_1 dx + \beta \int_{\partial\Omega} u_2^{\gamma-1} \varphi_1 dS - \alpha \int_{\partial\Omega} u_2 \varphi_1 dS = a \int_\Omega u_1 \varphi_1 dx. \tag{2.15}
$$

Then by (2.14), (2.15), (2.11) and (2.13), we obtain

$$
\int_\Omega u_1 \varphi_1 dx \leq C, \qquad \int_\Omega u_1 u_2 \varphi_1 dx \leq C. \tag{2.16}
$$

Hence, by Lemma 2.2, we get a priori bounds for $\int_\Omega u_1 dx$ and $\int_\Omega u_1 u_2 dx$. Now we are going to establish a priori bound of u_1 in $H^1(\Omega)$ for the case of $N \in [3,5]$. Multiplying the first equation of (2.6) by u_1 and using integration by parts, we obtain

$$
\begin{aligned}
\|\nabla u_1\|_{L^2(\Omega)}^2 + \alpha \int_{\partial\Omega} u_1^2 ds + b\|u_1\|_{L^2(\Omega)}^2 &= \int_\Omega u_1^2 u_2 dx + \lambda(b+\lambda_1) \int_\Omega u_1 \varphi_1 dx \\
&\leq \int_\Omega (u_1 u_2)^\theta \left(u_1^{\frac{2-\theta}{1-\theta}} u_2\right)^{1-\theta} dx + C \\
&\leq \left(\int_\Omega u_1 u_2 dx\right)^\theta \left(\int_\Omega u_1^{\frac{2-\theta}{1-\theta}} u_2 dx\right)^{1-\theta} + C, \quad (2.17)
\end{aligned}
$$

where we apply Hölder's inequality with exponent $(\frac{1}{\theta}, \frac{1}{1-\theta})$ for the first term on the right hand side. Here we take $\theta = \frac{6-N}{4} \in (0,1)$, then by applying Hölder's inequality with exponent $(\frac{2N}{N+2}, \frac{2N}{N-2})$,

$$\left(\int_\Omega u_1^{\frac{2-\theta}{1-\theta}} u_2 dx\right)^{1-\theta} = \left(\int_\Omega u_1^{\frac{N+2}{N-2}} u_2 dx\right)^{\frac{N-2}{4}} \le \|u_1\|_{L^{2^*}(\Omega)}^{\frac{N+2}{4}} \|u_2\|_{L^{2^*}(\Omega)}^{\frac{N-2}{4}}.$$

where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. Using Sobolev's embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and (2.11), we obtain

$$\|u_1\|_{L^{2^*}(\Omega)}^{\frac{N+2}{4}} \|u_2\|_{L^{2^*}(\Omega)}^{\frac{N-2}{4}} \le C\|u_1\|_{H^1(\Omega)}^{\frac{N+2}{4}}.$$

Since $(\|\nabla u_1\|^2_{L^2(\Omega)} + \alpha \int_{\partial\Omega} u_1^2 ds + b\|u_1\|^2_{L^2(\Omega)})^{1/2}$ is equivalent to the usual H^1-norm of u_1 due to trace inequality and Poincaré-Friedrichs type inequality, as a consequence we have

$$\|u_1\|^2_{H^1(\Omega)} \le C\|u_1\|_{H^1(\Omega)}^{\frac{N+2}{4}} + C.$$

Since $N \in [3,5]$, we have $\frac{N+2}{4} < 2$. Hence it follows from Young's inequality

$$\|u_1\|^2_{H^1(\Omega)} \le C\|u_1\|_{H^1(\Omega)}^{\frac{N+2}{4}} + C \le \frac{1}{2}\|u_1\|^2_{H^1(\Omega)} + C.$$

Thus we derive

$$\|u_1\|_{H^1(\Omega)} \le C. \tag{2.18}$$

Next, we derive L^∞-estimates for u_1 as for the case $N \in [3,5]$. From Sobolev's embedding $H^1(\Omega) \hookrightarrow L^{\frac{10}{3}}(\Omega)$, we can see that u_1, $u_2 \in L^{\frac{10}{3}}(\Omega)$ and $u_1u_2 \in L^{\frac{5}{3}}(\Omega)$. We get $u_1 \in W^{2,\frac{5}{3}}(\Omega)$ by the elliptic estimate for the first equation of (2.6). Moreover, $u_1 \in L^5(\Omega)$ by Sobolev's embedding $W^{2,\frac{5}{3}}(\Omega) \hookrightarrow L^5(\Omega)$. Then by Hölder's inequality,

$$\int_\Omega u_1^2 u_2^2 dx \le \left(\int_\Omega u_1^{2\cdot\frac{5}{2}} dx\right)^{\frac{2}{5}} \left(\int_\Omega u_2^{2\cdot\frac{5}{3}}\right)^{\frac{3}{5}},$$

we can see that $u_1u_2 \in L^2(\Omega)$. By the same reason as before, we know that $u_1 \in W^{2,2}(\Omega) \hookrightarrow L^{10}(\Omega)$. By Hölder's inequality, we have $u_1u_2 \in L^{\frac{5}{2}}(\Omega)$. Hence applying elliptic estimate and Sobolev's embedding again, we get $u_1 \in W^{2,\frac{5}{2}}(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1,\infty)$. Therefore $u_1u_2 \in L^{\frac{10q}{3q+10}}(\Omega)$ and $u_1 \in W^{2,\frac{10q}{3q+10}}(\Omega)$. Choosing $q > 10$, we have

$$\|u_1\|_{L^\infty(\Omega)} \le C_1,$$

where we use the Sobolev's embedding $W^{2,\frac{10q}{3q+10}}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $q > 10$.

Thus we obtain L^∞-estimate of u_1 for the case of $N \in [3,5]$. About the regularity for u_2, it suffices to consider the following problem for given $u_1 \in L^\infty(\Omega)$:

$$\begin{cases} -\Delta u_2 = au_1 \in L^\infty(\Omega), & x \in \Omega, \\ \partial_\nu u_2 + \beta|u_2|^{\gamma-2}u_2 = 0, & x \in \partial\Omega. \end{cases}$$

Therefore we can derive L^∞-estimate for u_2, i.e.,

$$\|u_2\|_{L^\infty(\Omega)} \le C_2$$

by the same arguments as for the compactness of Ψ applying Lemma 2.3. Choosing $R > C_1 + C_2$, we can see that the conclusion of this lemma holds.

As for the case $N = 1, 2$, it suffices to obtain L^∞-estimate for each component. First, let $N = 2$. Choosing $\theta = \frac{1}{2}$ in (2.17), we see that it follows from Sobolev's embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ (for all $p \in [1, \infty)$)

$$\begin{aligned}
\|\nabla u_1\|^2_{L^2(\Omega)} + \alpha \int_{\partial\Omega} u_1^2 ds + b\|u_1\|^2_{L^2(\Omega)} &= \int_\Omega u_1^2 u_2 dx + \lambda(b+\lambda_1)\int_\Omega u_1\varphi_1 dx \\
&\le \int_\Omega (u_1u_2)^{\frac{1}{2}} \left(u_1^3 u_2\right)^{\frac{1}{2}} dx + C \\
&\le \left(\int_\Omega u_1u_2 dx\right)^{\frac{1}{2}} \left(\int_\Omega u_1^3 u_2 dx\right)^{\frac{1}{2}} + C \\
&\le C\left(\int_\Omega u_1^3 u_2 dx\right)^{\frac{1}{2}} + C \\
&\le C\|u_1\|^{\frac{3}{2}}_{L^6(\Omega)}\|u_2\|^{\frac{1}{2}}_{L^2(\Omega)} + C \\
&\le C\|u_1\|^{\frac{3}{2}}_{H^1(\Omega)} + C.
\end{aligned}$$

Here we note that we have already had H^1-estimate for u_2 without restrictions on the space dimension. Thus we also get H^1-estimate for u_1. In the similar way as for the previous case $N \in [3, 5]$, we can derive L^∞-estimates for u_1 and u_2.

Let $N = 1$ and $\Omega = (a_0, b_0)$ with $a_0 < b_0$. Since $u_1 \in C(\overline{\Omega})$, there exists $x_0 \in \overline{\Omega}$ such that

$$u_1(x_0) = \min_{x\in\overline{\Omega}} u_1(x).$$

Furthermore, since it holds that $\|u_1\|_{L^1(\Omega)} \le C$ for any space dimension, we have

$$\min_{x\in\overline{\Omega}} u_1(x) \le \frac{1}{|\Omega|}\int_\Omega u_1 dx \le C.$$

Here by the fundamental theorem of calculus,

$$u_1(x) = u_1(x_0) + \int_{x_0}^x u_1'(\xi) d\xi.$$

Therefore we get the following inequality:

$$\|u_1\|_{L^\infty(\Omega)} \le \int_{a_0}^{b_0} |u_1'(\xi)|d\xi + |u_1(x_0)| \le \|u_1'\|_{L^1(\Omega)} + C. \tag{2.19}$$

From (2.19), Schwarz's inequality and Young's inequality, we see that

$$\begin{aligned}\|u_1'\|_{L^2}^2 + \alpha \int_{\partial\Omega} u_1^2 ds + b\|u_1\|_{L^2}^2 &= \int_\Omega u_1^2 u_2 dx + \lambda(b+\lambda_1)\int_\Omega u_1\varphi_1 dx \\ &\le \|u_1\|_{L^\infty}\int_\Omega u_1u_2 dx + C \\ &\le C\left(\|u_1'\|_{L^1} + C\right) + C \\ &\le C\|u_1'\|_{L^2} + C \le \frac{1}{2}\|u_1'\|_{L^2}^2 + C.\end{aligned}$$

Hence we obtain a priori bound for $\|u_1\|_{H^1(\Omega)}$. Since Sobolev's embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ holds for $N = 1$, we obtain the desired estimates. □

Proof of Theorem 2.1. By applying Lemma 2.4, Lemma 2.5 and Lemma 2.1, we can verify that Theorem 2.1 holds. □

2.2 Ordered Uniqueness

Next, we discuss the ordered uniqueness of the positive solutions for (S-NR). We now prepare the following inequality.

Lemma 2.6. ([5]) *For any $\gamma \in [2,\infty)$, there exists $C_\gamma > 0$ such that*

$$(x-y)\cdot\left(|x|^{\gamma-2}x - |y|^{\gamma-2}y\right) \ge C_\gamma |x-y|^\gamma$$

for all x, $y \in \mathbb{R}^N$.

Theorem 2.2. *Let (u_1, u_2) and (v_1, v_2) be two positive solutions of* (S-NR) *satisfying $u_1 \le v_1$ or $u_2 \le v_2$. Then $u_1 \equiv v_1$ and $u_2 \equiv v_2$.*

Proof. Suppose that $u_1 \not\equiv v_1$ or $u_2 \not\equiv v_2$. Without loss of generality, we only have to consider the case where $u_2 \not\equiv v_2$ and $u_2 \le v_2$. In fact, if $u_1 \le v_1$, by the second equation of (S-NR) we have

$$-\Delta(u_2 - v_2) = a(u_1 - v_1) \le 0. \tag{2.20}$$

Multiplying (2.20) by $[u_2 - v_2]^+ := \max\{u_2 - v_2, 0\}$ and using integration by parts, we obtain

$$\|\nabla[u_2 - v_2]^+\|_{L^2(\Omega)}^2 + \beta\int_{\partial\Omega}[u_2 - v_2]^+\left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) dS \le 0. \tag{2.21}$$

Note that by Lemma 2.6

$$\begin{aligned}\int_{\partial\Omega}[u_2 - v_2]^+\left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) dS &= \int_{\{u_2\ge v_2\}}(u_2 - v_2)\left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) dS \\ &\ge \int_{\{u_2\ge v_2\}} C_\gamma(u_2 - v_2)^\gamma dS \\ &= C_\gamma\int_{\partial\Omega}\left([u_2 - v_2]^+\right)^\gamma dS.\end{aligned}$$

By this inequality and (2.21), we get

$$\|\nabla[u_2 - v_2]^+\|^2_{L^2(\Omega)} + C_\gamma \int_{\partial\Omega} \left([u_2 - v_2]^+\right)^\gamma dS \le 0.$$

Therefore we have

$$\nabla[u_2 - v_2]^+ = 0,$$

$$[u_2 - v_2]^+\big|_{\partial\Omega} = 0.$$

Hence we deduce $[u_2 - v_2]^+ \equiv 0$, i.e., $u_2 \le v_2$.

Next we consider the following eigenvalue problems:

$$\begin{cases} -\Delta w + (b - u_2(x))\, w = \mu' w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.22}$$

and

$$\begin{cases} -\Delta w + (b - v_2(x))\, w = \eta' w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.23}$$

If necessary, we take some nonnegative constant $L \ge 0$ and add both sides of equations of (2.22) and (2.23) by L, and we can assume $U(x) := b - u_2(x) + L \ge 1$ and $V(x) := b - v_2(x) + L \ge 1$. Thus we consider the following problems in stead of (2.22) and (2.23):

$$\begin{cases} -\Delta w + U(x) w = \mu w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.24}$$

and

$$\begin{cases} -\Delta w + V(x) w = \eta w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.25}$$

By applying the compactness argument for the associate Rayleigh's quotients of (2.24) and (2.25) , we know that the smallest positive eigenvalues of (2.24) and (2.25) are attained and we denote them by μ_0 and η_0. Moreover, thanks to $u_2 \not\equiv v_2$ and $u_2 \le v_2$, we see that $\eta_0 < \mu_0$. On the other hand, since (u_1, u_2) and (v_1, v_2) are positive stationary solutions for (S-NR), $u_1 > 0$ and $v_1 > 0$ satisfy

$$\begin{cases} -\Delta u_1 + (b - u_2(x) + L)\, u_1 = L u_1 & \text{in } \Omega, \\ \partial_\nu u_1 + \alpha u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta v_1 + (b - v_2(x) + L)\, v_1 = L v_1 & \text{in } \Omega, \\ \partial_\nu v_1 + \alpha v_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

By the fact that the eigenvalue corresponding to the positive eigenfunction is the smallest one, we deduce $\mu_0 = L = \eta_0$. This contradicts $\eta_0 < \mu_0$. Thus the proof is completed. $\square$

3 Nonstationary Problem

In this section, we investigate the large time behavior of solutions to (NR) and prove that the positive stationary solution plays a role of threshold to classify initial data into two groups; namely corresponding solutions of (NR) blow up in finite time or exist globally.

3.1 Local Well-posedness

First we state the local well-posedness of problem (NR).

Theorem 3.1. *Assume* $(u_{10}, u_{20}) \in L^\infty(\Omega) \times L^\infty(\Omega)$. *Then there exists* $T > 0$ *such that* (NR) *possesses a unique solution* $(u_1, u_2) \in (L^\infty(0, T; L^\infty(\Omega)) \cap C([0, T); L^2(\Omega)))^2$ *satisfying*

$$\sqrt{t}\partial_t u_1, \sqrt{t}\partial_t u_2, \sqrt{t}\Delta u_1, \sqrt{t}\Delta u_2 \in L^2(0, T; L^2(\Omega)). \tag{3.1}$$

Furthermore, if the initial data is nonnegative, then the local solution (u_1, u_2) *for* (NR) *is nonnegative.*

In order to prove this theorem, we rely on L^∞-energy method developed in [18]. To this end, we prepare some crucial lemmas.

Lemma 3.1. ([18]) *Let* Ω *be any domain in* $\mathbb{R}^N$ *and assume that exists a number* $r_0 \geq 1$ *and a constant* C *independent of* $r \in [r_0, \infty)$ *such that*

$$\|u\|_{L^r(\Omega)} \leq C \qquad \forall r \in [r_0, \infty),$$

then u *belongs to* $L^\infty(\Omega)$ *and the following property holds.*

$$\lim_{r\to\infty} \|u\|_{L^r(\Omega)} = \|u\|_{L^\infty(\Omega)}. \tag{3.2}$$

Conversely, assume that $u \in L^{r_0}(\Omega) \cap L^\infty(\Omega)$ *for some* $r_0 \in [1, \infty)$, *then* u *satisfies* (3.2).

Lemma 3.2. ([18]) *Let* $y(t)$ *be a bounded measurable non-negative function on* $[0, T]$ *and suppose that there exists* $y_0 \geq 0$ *and a monotone non-decreasing function* $m(\cdot) : [0, +\infty) \to [0, +\infty)$ *such that*

$$y(t) \leq y_0 + \int_0^t m(y(s))ds \qquad a.e.\ t \in (0, T).$$

Then there exists a number $T_0 = T_0(y_0, m(\cdot)) \in (0, T]$ *such that*

$$y(t) \leq y_0 + 1 \qquad a.e.\ t \in [0, T_0].$$

Proof of Theorem 3.1. (Existence and regularity) We consider the following approximate problem:

$$\begin{cases} \partial_t u_1 - \Delta u_1 = [u_1]_M [u_2]_M - bu_1, & x \in \Omega,\ t > 0, \\ \partial_t u_2 - \Delta u_2 = au_1, & x \in \Omega,\ t > 0, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta |u_2|^{\gamma-2} u_2 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x, 0) = u_{10}(x),\ u_2(x, 0) = u_{20}(x), & x \in \Omega, \end{cases} \tag{3.3}$$

where $M > 0$ is a given constant and the cut-off function $[u]_M$ is defined by

$$[u]_M = \begin{cases} M, & u \geq M, \\ u, & |u| \leq M, \\ -M, & u \leq -M. \end{cases}$$

Since $u \mapsto [u]_M$ is Lipschitz continuous from $L^2(\Omega)$ into itself, it is well known that (3.3) has a unique global solution (u_1, u_2) satisfying (3.1) by applying the abstract theory on maximal monotone operators developed by H. Brézis [2].

By multiplying the first equation of (3.3) by $|u_1|^{r-2}u_1$ and using integration by parts,

$$\frac{1}{r}\frac{d}{dt}\|u_1(t)\|_{L^r}^r + (r-1)\int_\Omega |\nabla u_1|^2 u_1^{r-2} dx + \alpha \int_{\partial\Omega} |u_1|^r dS \leq \int_\Omega |u_1|^r |u_2| dx - b\int_\Omega |u_1|^r dx.$$

Hence

$$\frac{1}{r}\frac{d}{dt}\|u_1(t)\|_{L^r}^r \leq \|u_2(t)\|_{L^\infty}\|u_1(t)\|_{L^r}^r.$$

Divide both sides by $\|u_1\|_{L^r}^{r-1}$ and integrate with respect to t on $[0, t]$, then we get

$$\|u_1(t)\|_{L^r} \leq \|u_{10}\|_{L^r} + \int_0^t \|u_1(\tau)\|_{L^r}\|u_2(\tau)\|_{L^\infty} d\tau.$$

Letting r tend to ∞ (Lemma 3.1), we derive

$$\|u_1(t)\|_{L^\infty} \leq \|u_{10}\|_{L^\infty} + \int_0^t \|u_1(\tau)\|_{L^\infty}\|u_2(\tau)\|_{L^\infty} d\tau.$$

Similarly, we can get the following L^∞ estimate for u_2 ;

$$\|u_2(t)\|_{L^\infty} \leq \|u_{20}\|_{L^\infty} + \int_0^t a\|u_1(\tau)\|_{L^\infty} d\tau.$$

Therefore setting $y(t) = \|u_1(t)\|_{L^\infty(\Omega)} + \|u_2(t)\|_{L^\infty(\Omega)}$, we get

$$y(t) \leq y(0) + \int_0^t \left(y^2(\tau) + ay(\tau)\right) d\tau.$$

Thus applying Lemma 3.2, we find that there exists a number $T > 0$ depending only on $\|u_{10}\|_{L^\infty(\Omega)}$ and $\|u_{20}\|_{L^\infty(\Omega)}$ such that

$$y(t) \leq y(0) + 1 \qquad \text{a.e. } t \in [0, T].$$

In other words, we get

$$\|u_1(t)\|_{L^\infty(\Omega)} + \|u_2(t)\|_{L^\infty(\Omega)} \leq \|u_{10}\|_{L^\infty(\Omega)} + \|u_{20}\|_{L^\infty(\Omega)} + 1 \qquad \text{a.e. } t \in [0, T].$$

Hence choosing $M > \|u_{10}\|_{L^\infty(\Omega)} + \|u_{20}\|_{L^\infty(\Omega)} + 1$, we can see that (u_1, u_2) gives a solution for (NR) on $[0, T]$ by the definition of the cut-off function $[u]_M$. Note that even though $\|u_1(t)\|_{L^r}^{r-1}$ attains zero, we can justify this argument by Proposition 1 in [16]. To get the regularity estimate of the solution for (NR) is standard, so we omit the details.

(Uniqueness) Let (u_1, u_2) and (v_1, v_2) be two solutions to (NR) with initial data (u_{10}, u_{20}) and (v_{10}, v_{20}) respectively. We set $w_1 = u_1 - v_1$ and $w_2 = u_2 - v_2$. From (NR), we have

$$\partial_t w_1 - \Delta w_1 = w_1 u_2 + v_1 w_2 - b w_1, \tag{3.4}$$

$$\partial_t w_2 - \Delta w_2 = a w_1, \tag{3.5}$$

$$\partial_\nu w_1 + \alpha w_1 = \partial_\nu w_2 + \beta \left(|u_2|^{\gamma-2} u_2 - |v_2|^{\gamma-2} v_2\right) = 0, \qquad \text{on } \partial\Omega.$$

We multiply (3.4) and (3.5) by w_1 and w_2 respectively, integrate over Ω and use integration by parts. Then we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_1(t)\|^2_{L^2(\Omega)} + \|\nabla w_1\|^2_{L^2(\Omega)} + \alpha \int_{\partial\Omega} w_1^2 dS \\
&\leq \int_\Omega w_1^2 u_2 dx + \int_\Omega v_1 w_1 w_2 dx \\
&\leq \|u_2\|_{L^\infty(0,T;L^\infty(\Omega))} \int_\Omega w_1^2 dx + \|v_1\|_{L^\infty(0,T;L^\infty(\Omega))} \int_\Omega w_1 w_2 dx \\
&\leq C \left(\|w_1(t)\|^2_{L^2(\Omega)} + \|w_2(t)\|^2_{L^2(\Omega)}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_2(t)\|^2_{L^2(\Omega)} + \|\nabla w_2\|^2_{L^2(\Omega)} + \beta \int_{\partial\Omega} \left(|u_2|^{\gamma-2} u_2 - |v_2|^{\gamma-2} v_2\right)(u_2 - v_2)\, dS \\
&\leq a \int_\Omega w_1 w_2 dx \\
&\leq \frac{a}{2} \left(\|w_1(t)\|^2_{L^2(\Omega)} + \|w_2(t)\|^2_{L^2(\Omega)}\right).
\end{aligned}$$

Noting that

$$\int_{\partial\Omega} \left(|u_2|^{\gamma-2} u_2 - |v_2|^{\gamma-2} v_2\right)(u_2 - v_2)\, dS \geq \int_{\partial\Omega} C_\gamma |w_2|^\gamma dS \geq 0$$

by Lemma 2.6, we can get the following differential inequality:

$$\frac{d}{dt}\left(\|w_1(t)\|^2_{L^2(\Omega)} + \|w_2(t)\|^2_{L^2(\Omega)}\right) \leq C \left(\|w_1(t)\|^2_{L^2(\Omega)} + \|w_2(t)\|^2_{L^2(\Omega)}\right),$$

whence, from Gronwall's inequality,

$$\left(\|w_1(t)\|^2_{L^2(\Omega)} + \|w_2(t)\|^2_{L^2(\Omega)}\right) \leq \left(\|u_{10} - v_{10}\|^2_{L^2(\Omega)} + \|u_{20} - v_{20}\|^2_{L^2(\Omega)}\right) e^{Ct} \qquad t \in [0, T).$$

This yields the uniqueness of the solution for (NR).

(Nonnegativity) Multiplying the first equation of (NR) by $u_1^- := \max\{-u_1, 0\}$, we get

$$\int_\Omega \partial_t u_1 u_1^- dx - \int_\Omega \Delta u_1 u_1^- dx \geq -\int_\Omega |u_1^-|^2 |u_2| dx - b \int_\Omega u_1 u_1^- dx.$$

Here, we can see that

$$\int_\Omega \partial_t u_1 u_1^- dx = \int_{\{u_1 \le 0\}} \partial_t u_1 (-u_1) dx = -\frac{1}{2}\frac{d}{dt}\int_{\{u_1 \le 0\}} (-u_1)^2 dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega \left(u_1^-\right)^2 dx,$$

and

$$\begin{aligned} -\int_\Omega \Delta u_1 u_1^- dx &= \int_\Omega \nabla u_1 \cdot \nabla u_1^- dx + \alpha \int_{\partial\Omega} u_1 u_1^- dS \\ &= -\int_\Omega |\nabla u_1^-|^2 dx - \alpha \int_{\{u_1 \le 0\}} u_1^2 dS = -\int_\Omega |\nabla u_1^-|^2 dx - \alpha \int_{\partial\Omega} \left(u_1^-\right)^2 dS. \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_1^-(t)\|_{L^2(\Omega)}^2 + \|\nabla u_1^-\|_{L^2(\Omega)}^2 + \alpha \int_{\partial\Omega} \left(u_1^-\right)^2 dS &= \int_\Omega |u_1^-|^2 |u_2| dx - b\|u_1^-(t)\|_{L^2(\Omega)}^2 \\ &\le \|u_2\|_{L^\infty(0,T;L^\infty(\Omega))} \|u_1^-(t)\|_{L^2(\Omega)}^2. \end{aligned}$$

Applying Gronwall's inequality, we obtain

$$\|u_1^-(t)\|_{L^2(\Omega)}^2 \le \|u_1^-(0)\|_{L^2(\Omega)}^2 e^{2\|u_2\|_{L^\infty(0,T;L^\infty(\Omega))} t} \qquad t \in [0,T),$$

where T is maximal existence time for (NR). Since $u_{10} \ge 0$, i.e., $\|u_1^-(0)\|_{L^2(\Omega)} = 0$, it holds that

$$u_1^-(t) = 0 \qquad a.e. \text{ in } \Omega \quad \forall t \in [0,T).$$

Hence $u_1 \ge 0$. Similarly, multiplying the second equation of (NR) by $-u_2^-$, we get

$$\frac{1}{2}\frac{d}{dt}\|u_2^-(t)\|_{L^2(\Omega)}^2 + \|\nabla u_2^-\|_{L^2(\Omega)}^2 + \beta \int_{\partial\Omega} |u_2|^{\gamma-2} |u_2^-|^2 dS = -a \int_\Omega u_1 u_2^- dx \le 0.$$

Therefore $\|u_2^-(t)\|_{L^2(\Omega)}^2 \le \|u_2^-(0)\|_{L^2(\Omega)}^2 = 0$, i.e., $u_2 \ge 0$. □

3.2 Threshold Property

Finally, we study the threshold property and prove that every positive stationary solution for (NR) gives a threshold for the blow up of solutions in the following sense.

Theorem 3.2. *Let* $(\overline{u}_1, \overline{u}_2)$ *be a positive stationary solution of* (NR)*, then the followings hold.*

(1) *Let* $0 \le u_{10}(x) \le \overline{u}_1(x)$*,* $0 \le u_{20}(x) \le \overline{u}_2(x)$*, then the solution* (u_1, u_2) *of* (NR) *exists globally. In addition, if* $0 \le u_{10}(x) \le l_1\overline{u}_1(x)$*,* $0 \le u_{20}(x) \le l_2\overline{u}_2(x)$ *for some* $0 < l_1 < l_2 \le 1$*, then*

$$\lim_{t \to +\infty} (u_1(x,t), u_2(x,t)) = (0,0) \quad \text{pointwisely on } \overline{\Omega}.$$

(2) *Assume further* $\gamma = 2$*,* $\alpha \le 2\beta$ *and let* $u_{10}(x) \ge l_1\overline{u}_1(x)$*,* $u_{20}(x) \ge l_2\overline{u}_2(x)$ *for some* $l_1 > l_2 > 1$*, then the solution* (u_1, u_2) *of* (NR) *blows up in finite time.*

Remark 3.3. The second assertion of Theorem 3.2 is also announced in [7] for the case where $\alpha = 0$ and $\gamma = 2$. However it seems that their proof contains some serious gaps.

We first prepare the following comparison theorem.

Lemma 3.3 (Comparison theorem). *If* (u_{10}, u_{20}), (v_{10}, v_{20}) *are two initial data for* (NR) *satisfying*

$$0 \le u_{10} \le v_{10}, \quad 0 \le u_{20} \le v_{20} \qquad on\ \overline{\Omega},$$

then the corresponding solutions (u_1, u_2), (v_1, v_2) *remain in the initial data order in time interval where the solutions exist, i.e.,* $u_1(x,t) \le v_1(x,t)$ *and* $u_2(x,t) \le v_2(x,t)$ *a.e.* $x \in \Omega$ *as long as* (u_1, u_2) *and* (v_1, v_2) *exist.*

Proof. Let $w_1 = u_1 - v_1$, $w_2 = u_2 - v_2$. By (NR) we have

$$\begin{cases} \partial_t w_1 - \Delta w_1 = w_1 u_2 + v_1 w_2 - b w_1, & x \in \Omega,\ t \in (0, T_m), \\ \partial_t w_2 - \Delta w_2 = a w_1, & x \in \Omega,\ t \in (0, T_m), \\ \partial_\nu w_1 + \alpha w_1 = \partial_\nu w_2 + \beta\left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) = 0, & x \in \partial\Omega,\ t \in (0, T_m), \\ w_1(x,0) \le 0, \quad w_2(x,0) \le 0, & x \in \overline{\Omega}, \end{cases} \tag{3.6}$$

where $T_m > 0$ is the maximum existence time for (u_1, u_2) and (v_1, v_2). We set

$$w^+ = w \vee 0, \quad w^- = (-w) \vee 0,$$

where $a \vee b = \max\{a, b\}$. It is easy to see that w^+, $w^- \ge 0$ and

$$w = w^+ - w^-, \quad |w| = w^+ + w^-.$$

Multiplying the first equation of (3.6) by w_1^+, we get

$$\int_\Omega \partial_t w_1 w_1^+ dx - \int_\Omega \Delta w_1 w_1^+ dx = \int_\Omega w_1 u_2 w_1^+ dx + \int_\Omega v_1 w_2 w_1^+ dx - b \int_\Omega w_1 w_1^+ dx.$$

Here, we see that

$$\int_\Omega \partial_t w_1 w_1^+ dx = \int_{\{w_1 \ge 0\}} \partial_t w_1 w_1 dx = \frac{1}{2}\frac{d}{dt}\int_{\{w_1 \ge 0\}} w_1^2 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \left(w_1^+\right)^2 dx.$$

Similarly,

$$\begin{aligned} -\int_\Omega \Delta w_1 w_1^+ dx &= \int_\Omega \nabla w_1 \cdot \nabla w_1^+ dx + \alpha \int_{\partial\Omega} w_1 w_1^+ dS \\ &= \int_{\{w_1 \ge 0\}} |\nabla w_1|^2 dx + \alpha \int_{\{w_1 \ge 0\}} w_1^2 dS = \int_\Omega |\nabla w_1^+|^2 dx + \alpha \int_{\partial\Omega} \left(w_1^+\right)^2 dS. \end{aligned}$$

Hence noting that $v_1 \geq 0$, we obtain for any $T \in (0, T_m)$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left(w_1^+\right)^2 dx + \int_\Omega |\nabla w_1^+|^2 dx + \alpha \int_{\partial\Omega} \left(w_1^+\right)^2 dS \\
&= \int_\Omega w_1 u_2 w_1^+ dx + \int_\Omega v_1 w_2 w_1^+ dx - b\int_\Omega w_1 w_1^+ dx \\
&= \int_\Omega \left(w_1^+ - w_1^-\right) u_2 w_1^+ dx + \int_\Omega v_1 \left(w_2^+ - w_2^-\right) w_1^+ dx - b\int_\Omega \left(w_1^+\right)^2 dx \\
&\leq \|u_2\|_{L^\infty(0,T;L^\infty(\Omega))} \int_\Omega \left(w_1^+\right)^2 dx + \|v_1\|_{L^\infty(0,T;L^\infty(\Omega))} \int_\Omega w_1^+ w_2^+ dx \\
&\leq C\left(\|w_1^+(t)\|^2_{L^2(\Omega)} + \|w_2^+(t)\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

Hence we get

$$
\frac{1}{2}\frac{d}{dt}\|w_1^+(t)\|^2_{L^2(\Omega)} \leq C\left(\|w_1^+(t)\|^2_{L^2(\Omega)} + \|w_2^+(t)\|^2_{L^2(\Omega)}\right). \tag{3.7}
$$

Next we do the same calculation for the second equation of (3.6) and get

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega \left(w_2^+\right)^2 dx + \int_\Omega |\nabla w_2^+|^2 dx - \int_{\partial\Omega} (\partial_\nu w_2) w_2^+ dS \leq \frac{a}{2}\left(\|w_1^+(t)\|^2_{L^2(\Omega)} + \|w_2^+(t)\|^2_{L^2(\Omega)}\right),
$$

and

$$
\begin{aligned}
-\int_{\partial\Omega} (\partial_\nu w_2) w_2^+ dS &= \beta \int_{\partial\Omega} \left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) w_2^+ dS \\
&= \beta \int_{\{u_2 \geq v_2\}} \left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right)(u_2 - v_2)\, dS \geq 0.
\end{aligned}
$$

Therefore

$$
\frac{1}{2}\frac{d}{dt}\|w_2^+(t)\|^2_{L^2(\Omega)} \leq \frac{a}{2}\left(\|w_1^+(t)\|^2_{L^2(\Omega)} + \|w_2^+(t)\|^2_{L^2(\Omega)}\right). \tag{3.8}
$$

Thus by (3.7), (3.8) and Gronwall's inequality, we get

$$
\|w_1^+(t)\|^2_{L^2(\Omega)} + \|w_2^+(t)\|^2_{L^2(\Omega)} \leq \left(\|w_1^+(0)\|^2_{L^2(\Omega)} + \|w_2^+(0)\|^2_{L^2(\Omega)}\right) e^{Ct} \quad \forall t \in [0, T_m).
$$

Since $w_1^+(0) = w_2^+(0) = 0$, the above inequality means $w_1^+ = w_2^+ = 0$. Hence, we have the desired result. □

Proof of Theorem 3.2. (1) If $0 \leq u_{10} \leq \overline{u}_1$ and $0 \leq u_{20} \leq \overline{u}_2$, then since $(\overline{u}_1, \overline{u}_2)$ is a global solution for (NR), $0 \leq u_1(x,t) \leq \overline{u}_1(x)$ and $0 \leq u_2(x,t) \leq \overline{u}_2(x)$ follow directly from Lemma 3.3. That is, we have

$$
\sup_{t\in[0,T)} \|u_i(\cdot,t)\|_{L^\infty(\Omega)} \leq \|\overline{u}_i\|_{L^\infty(\Omega)} \quad (i = 1, 2).
$$

Hence the solution (u_1, u_2) exists globally.

In addition, let $u_{10}(x) \le l_1\overline{u}_1(x)$, $u_{20}(x) \le l_2\overline{u}_2(x)$ for some $0 < l_1 < l_2 \le 1$. Since the comparison theorem holds, without loss of generality, we can assume that $u_{10}(x) = l_1\overline{u}_1(x)$, $u_{20}(x) = l_2\overline{u}_2(x)$ and $l_1 < l_2 \le 1$. We here note that $\delta u_1 := u_1(t+h) - u_1(t)$ and $\delta u_2 := u_2(t+h) - u_2(t)$ for $h > 0$ satisfy the following equations:

$$\begin{cases} \partial_t (\delta u_1) - \Delta (\delta u_1) = (\delta u_1)\, u_2(t+h) + u_1(t)\,(\delta u_2) - b\,(\delta u_1)\,, \\ \partial_t (\delta u_2) - \Delta (\delta u_2) = a\,(\delta u_1)\,, \\ \partial_\nu (\delta u_1) + \alpha\,(\delta u_1) = \partial_\nu (\delta u_2) + \beta\left(|u_2(t+h)|^{\gamma-2}u_2(t+h) - |u_2(t)|^{\gamma-2}u_2(t)\right) = 0, \\ \delta u_1(0) = u_1(0+h) - u_1(0), \quad \delta u_2(0) = u_2(0+h) - u_2(0). \end{cases} \tag{3.9}$$

Multiplying the first and second equation of (3.9) by $[\delta u_1]^+$ and $[\delta u_2]^+$ respectively and using integration by parts and repeating the same argument as for (3.7), we obtain the following inequality:

$$\|[\delta u_1]^+\|^2_{L^2(\Omega)} + \|[\delta u_2]^+\|^2_{L^2(\Omega)} \le \left(\|[\delta u_1(0)]^+\|^2_{L^2(\Omega)} + \|[\delta u_2(0)]^+\|^2_{L^2(\Omega)}\right) e^{Ct} \qquad \forall t \in [0,\infty).$$

We divide both sides of this inequality by h^2:

$$\left\|\left[\frac{\delta u_1}{h}\right]^+\right\|^2_{L^2(\Omega)} + \left\|\left[\frac{\delta u_2}{h}\right]^+\right\|^2_{L^2(\Omega)} \le \left(\left\|\left[\frac{\delta u_1(0)}{h}\right]^+\right\|^2_{L^2(\Omega)} + \left\|\left[\frac{\delta u_2(0)}{h}\right]^+\right\|^2_{L^2(\Omega)}\right) e^{Ct}.$$

Since we know that u_1, u_2 is differentiable on *a.e.* t by the regularity results of Theorem 4.1, by letting $h \searrow 0$, we obtain

$$\|[\partial_t u_1]^+\|^2_{L^2} + \|[\partial_t u_2]^+\|^2_{L^2} \le \left(\|[\partial_t u_1(0)]^+\|^2_{L^2} + \|[\partial_t u_2(0)]^+\|^2_{L^2}\right) e^{Ct} \quad a.e.\ t \in [0,\infty).$$

We here note that since $(l_1\overline{u}_1, l_2\overline{u}_2)$ is strict upper solution for (S-NR), it holds that

$$\begin{aligned} \partial_t u_1(0) &= \Delta u_{10} + u_{10}u_{20} - bu_{10} \\ &= l_1\Delta\overline{u}_1 + l_1l_2\overline{u}_1\overline{u}_2 - bl_1\overline{u}_1 \\ &\le l_1\left(\Delta\overline{u}_1 + \overline{u}_1\overline{u}_2 - b\overline{u}_1\right) = 0, \\ \partial_t u_2(0) &= \Delta u_{20} + au_{10} \\ &= l_2\Delta\overline{u}_2 + al_1\overline{u}_1 \\ &< l_2\left(\Delta\overline{u}_2 + a\overline{u}_1\right) = 0, \end{aligned}$$

which imply that $[\partial_t u_1(0)]^+ = [\partial_t u_2(0)]^+ = 0$. Hence we find that $\partial_t u_1 \le 0$ and $\partial_t u_2 \le 0$, i.e., $u_1(x,t)$ and $u_2(x,t)$ are monotone decreasing in t for a.e. $x \in \Omega$. Thus

$$\lim_{t\to\infty} (u_1(x,t), u_2(x,t)) =: (\tilde{u}_1(x), \tilde{u}_2(x))$$

exists and satisfies $(0,0) \le (\tilde{u}_1, \tilde{u}_2) \le (l_1\overline{u}_1, l_2\overline{u}_2) < (\overline{u}_1, \overline{u}_2)$. Now we prove that $(\tilde{u}_1, \tilde{u}_2)$ is a nonnegative stationary solution of (NR). First we note that

$$u_i(t) \to \tilde{u}_i \quad \text{strongly in } L^p(\Omega) \quad \text{as } k \to \infty \quad \forall p \in (1,\infty) \qquad (i = 1,2). \tag{3.10}$$

In fact, since $|u_i(x,t)-\tilde{u}_i(x)|^p \to 0$ *a.e.* $x \in \Omega$ as $t \to \infty$ and $|u_i(x,t)-\tilde{u}_i(x)|^p \le 2^p|\overline{u}_i(x)|^p \le 2^p\|\overline{u}_i\|^p_{L^\infty(\Omega)}$ *a.e.* $x \in \Omega$, Lebesgue's dominant convergence theorem assures (3.10). Next multiplying the first and the second equations of (NR) by $\partial_t u_1$ and $\partial_t u_2$ respectively, we get

$$\|\partial_t u_1(t)\|^2_{L^2(\Omega)} + \frac{d}{dt}\left\{\frac{1}{2}\|\nabla u_1(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u_1(t)\|^2_{L^2(\partial\Omega)} + \frac{b}{2}\|u_1(t)\|^2_{L^2(\Omega)}\right\}$$
$$= \int_\Omega u_1u_2\partial_t u_1 dx \le 0,$$

$$\|\partial_t u_2(t)\|^2_{L^2(\Omega)} + \frac{d}{dt}\left\{\frac{1}{2}\|\nabla u_2(t)\|^2_{L^2(\Omega)} + \frac{\beta}{\gamma}\|u_2(t)\|^\gamma_{L^\gamma(\partial\Omega)}\right\} = a\int_\Omega u_1\partial_t u_2 dx \le 0.$$

Then integration of these over $(0,T)$ for any $T>0$ gives

$$\int_0^\infty \|\partial_t u_1(t)\|^2_{L^2(\Omega)}dt + \int_0^\infty \|\partial_t u_2(t)\|^2_{L^2(\Omega)}dt \le C_0, \tag{3.11}$$

$$\sup_{t>0}\left\{\|u_1(t)\|^2_{H^1(\Omega)} + \|u_2(t)\|^2_{H^1(\Omega)}\right\} \le C_0, \tag{3.12}$$

where C_0 is a positive constant depending on $\|u_{10}\|_{H^1(\Omega)}$, $\|u_{20}\|_{H^1(\Omega)}$ and $\|u_{20}\|_{L^\gamma(\partial\Omega)}$. Hence since $u_i \in L^\infty(0,\infty;L^\infty(\Omega))$ $(i=1,2)$, from equation (NR), we derive

$$\int_n^{n+1}\left\{\|\partial_t u_1(t)\|^2_{L^2(\Omega)} + \|\partial_t u_2(t)\|^2_{L^2(\Omega)}\right\}dt \to 0 \qquad \text{as} \quad n\to\infty, \tag{3.13}$$

$$\sup_n \int_n^{n+1}\left\{\|\Delta u_1(t)\|^2_{L^2(\Omega)} + \|\Delta u_2(t)\|^2_{L^2(\Omega)}\right\}dt \le C_0. \tag{3.14}$$

Furthermore, since $\|u_2(t)\|_{L^\infty(\partial\Omega)} \le \|u_2(t)\|_{L^\infty(\Omega)}$ (see [9]), we obtain

$$\sup_{t>0}\|u_2(t)\|_{L^\infty(\partial\Omega)} \le \|\overline{u}_2\|_{L^\infty(\Omega)}. \tag{3.15}$$

Here we put

$$u_i^n(x,t) = u_i(x,n+t) \in \mathscr{H} := L^2(0,1;L^2(\Omega)) \quad t\in(0,1) \qquad (i=1,2). \tag{3.16}$$

Then $u_i^n(t)$ satisfy

$$\begin{cases} \partial_t u_1^n(t) - \Delta u_1^n(t) = u_1^n(t)u_2^n(t) - bu_1^n(t), & x\in\Omega,\ t\in(0,1), \\ \partial_t u_2^n(t) - \Delta u_2^n(t) = au_1^n(t), & x\in\Omega,\ t\in(0,1), \\ \partial_\nu u_1^n(t) + \alpha u_1^n(t) = \partial_\nu u_2^n(t) + \beta|u_2^n(t)|^{\gamma-2}u_2^n(t) = 0, & x\in\partial\Omega,\ t\in(0,1). \end{cases} \tag{3.17}$$

Then, by virtue of (3.10), (3.12), (3.13), (3.14) and (3.15), there exists subsequence of $\{u_i^n(t)\}$ denoted again by $\{u_i^n(t)\}$ such that

$$\partial_t u_i^n(t) \to 0 \qquad \text{strongly in } \mathscr{H} \text{ as } n \to \infty, \tag{3.18}$$

$$u_i^n(t) \to \tilde{u}_i(t) \equiv \tilde{u}_i \qquad \text{strongly in } \mathscr{H} \text{ as } n \to \infty, \tag{3.19}$$

$$u_1^n(t)u_2^n(t) \to \tilde{u}_1(t)\tilde{u}_2(t) \equiv \tilde{u}_1\tilde{u}_2 \qquad \text{strongly in } \mathscr{H} \text{ as } n \to \infty, \tag{3.20}$$

$$\Delta u_i^n(t) \rightharpoonup \Delta\tilde{u}_i(t) \equiv \Delta\tilde{u}_i \qquad \text{weakly in } \mathscr{H} \text{ as } n \to \infty, \tag{3.21}$$

$$u_i^n(t) \to \tilde{u}_i(t) \equiv \tilde{u}_i \qquad \text{strongly in } L^2(0,1;L^2(\partial\Omega)) \text{ as } n \to \infty, \tag{3.22}$$

$$|u_2^n(t)|^{\gamma-2}u_2^n(t) \rightharpoonup |\tilde{u}_2|^{\gamma-2}\tilde{u}_2 \qquad \text{weakly in } L^2(0,1;L^2(\partial\Omega)) \text{ as } n \to \infty, \tag{3.23}$$

$$\partial_\nu u_i^n(t) \rightharpoonup \partial_\nu \tilde{u}_i \qquad \text{weakly in } L^2(0,1;L^2(\partial\Omega)) \text{ as } n \to \infty. \tag{3.24}$$

Thus $\tilde{u}_1$ and $\tilde{u}_2$ satisfy

$$\begin{cases} -\Delta\tilde{u}_1 = \tilde{u}_1\tilde{u}_2 - b\tilde{u}_1, & x \in \Omega, \\ -\Delta\tilde{u}_2 = a\tilde{u}_1, & x \in \Omega, \\ \partial_\nu\tilde{u}_1 + \alpha\tilde{u}_1 = \partial_\nu\tilde{u}_2 + \beta|\tilde{u}_2|^{\gamma-2}\tilde{u}_2 = 0, & x \in \partial\Omega. \end{cases}$$

Since $(0,0) \le (\tilde{u}_1,\tilde{u}_2) \le (l_1\overline{u}_1, l_2\overline{u}_2) < (\overline{u}_1,\overline{u}_2)$, it follows that $(\tilde{u}_1(x),\tilde{u}_2(x))$ is nothing but $(0,0)$ from the ordered uniqueness of positive stationary solutions.

(2) Let $\gamma = 2$ and $\alpha \le 2\beta$. By the comparison theorem, we can assume without loss of generality that $u_{10}(x) = l_1\overline{u}_1(x)$, $u_{20}(x) = l_2\overline{u}_2(x)$ for some $l_1 > l_2 > 1$. Suppose that the solution (u_1,u_2) of (NR) exists globally, i.e.,

$$\sup_{t\in[0,T]} \|u_i(\cdot,t)\|_{L^\infty(\Omega)} < \infty, \qquad (i=1,2) \quad \forall\, T > 0. \tag{3.25}$$

Now we are going to construct a subsolution. For this purpose, we first note that there exists a sufficiently small number $\varepsilon > 0$ such that

$$\begin{cases} a(l_2-l_1)\overline{u}_1 + \varepsilon l_2\overline{u}_2 < 0 & \text{on } \overline{\Omega}, \\ \varepsilon + (1-l_2)\overline{u}_2 < 0 & \text{on } \overline{\Omega}. \end{cases} \tag{3.26}$$

Here we used the fact that $\overline{u}_1(x) > 0$, $\overline{u}_2(x) > 0$ on $\overline{\Omega}$, which is assured by Hopf's type maximum principle. Let $u_1^*(x,t) = l_1e^{\varepsilon t}\overline{u}_1(x)$ and $u_2^*(x,t) = l_2e^{\varepsilon t}\overline{u}_2(x)$. Then using (3.26), we get

$$\begin{aligned} \partial_t u_1^* - \Delta u_1^* - u_1^*u_2^* + bu_1^* &= \varepsilon l_1e^{\varepsilon t}\overline{u}_1 - l_1e^{\varepsilon t}\Delta\overline{u}_1 - l_1e^{\varepsilon t}\overline{u}_1 l_2e^{\varepsilon t}\overline{u}_2 + bl_1e^{\varepsilon t}\overline{u}_1 \\ &= \varepsilon l_1e^{\varepsilon t}\overline{u}_1 + l_1e^{\varepsilon t}\left(\overline{u}_1\overline{u}_2 - b\overline{u}_1\right) - l_1e^{\varepsilon t}\overline{u}_1 l_2e^{\varepsilon t}\overline{u}_2 + bl_1e^{\varepsilon t}\overline{u}_1 \\ &\le \varepsilon l_1e^{\varepsilon t}\overline{u}_1 + l_1e^{\varepsilon t}\overline{u}_1\overline{u}_2 - l_1l_2e^{\varepsilon t}\overline{u}_1\overline{u}_2 \\ &= \left\{\varepsilon + (1-l_2)\,\overline{u}_2\right\} l_1e^{\varepsilon t}\overline{u}_1 < 0, \\ \partial_t u_2^* - \Delta u_2^* - au_1^* &= \varepsilon l_2e^{\varepsilon t}\overline{u}_2 - l_2e^{\varepsilon t}\Delta\overline{u}_2 - al_1e^{\varepsilon t}\overline{u}_1 \\ &= \varepsilon l_2e^{\varepsilon t}\overline{u}_2 + l_2e^{\varepsilon t}a\overline{u}_1 - al_1e^{\varepsilon t}\overline{u}_1 \\ &= \left\{\varepsilon l_2\overline{u}_2 + a\,(l_2-l_1)\,\overline{u}_1\right\} e^{\varepsilon t} < 0, \end{aligned}$$

where we used the fact that $(\overline{u}_1, \overline{u}_2)$ satisfies

$$\begin{cases} -\Delta \overline{u}_1 = \overline{u}_1 \overline{u}_2 - b\overline{u}_1, \\ -\Delta \overline{u}_2 = a\overline{u}_1. \end{cases}$$

Moreover $\partial_\nu u_1^* + \alpha u_1^* = 0$, $\partial_\nu u_2^* + \beta u_2^* = 0$ on $\partial\Omega$ and $u_1^*(x,0) = l_1\overline{u}_1(x)$, $u_2^*(x,0) = l_2\overline{u}_2(x)$. Hence by the comparison principle, we have

$$l_1 e^{\varepsilon t}\overline{u}_1(x) = u_1^*(x,t) \le u_1(x,t), \quad l_2 e^{\varepsilon t}\overline{u}_2(x) = u_2^*(x,t) \le u_2(x,t). \tag{3.27}$$

Multiplication of equations of (NR) by φ_1 and integration by parts yield

$$\frac{d}{dt}\left(\int_\Omega u_1\varphi_1 dx\right) + (b+\lambda_1)\int_\Omega u_1\varphi_1 dx = \int_\Omega u_1 u_2 \varphi_1 dx, \tag{3.28}$$

$$\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1 \int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 dS = a\int_\Omega u_1\varphi_1 dx, \tag{3.29}$$

where λ_1 and φ_1 are the first eigenvalue and the corresponding eigenfunction for (2.1). We here normalize φ_1 so that $\|\varphi_1\|_{L^1(\Omega)} = 1$. Substituting (3.29) and $u_1 = \frac{1}{a}(\partial_t u_2 - \Delta u_2)$ in (3.28) and using integration by parts, we get

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 dS\right\} \\ &\quad + (b+\lambda_1)\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 dS\right\} \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega u_2^2\varphi_1 dx + \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + \left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS, \end{aligned} \tag{3.30}$$

where we used the fact that

$$\begin{aligned} -\int_\Omega (\Delta u_2)u_2\varphi_1 dx &= \int_\Omega \nabla u_2\cdot\nabla(u_2\varphi_1)dx - \int_{\partial\Omega}(\partial_\nu u_2)u_2\varphi_1 dS \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \int_\Omega u_2\nabla u_2\cdot\nabla\varphi_1 dx + \beta\int_{\partial\Omega} u_2^2\varphi_1 dS \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{1}{2}\int_\Omega \nabla u_2^2\cdot\nabla\varphi_1 dx + \beta\int_{\partial\Omega} u_2^2\varphi_1 dS \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2\Delta\varphi_1 dx - \frac{\alpha}{2}\int_{\partial\Omega} u_2^2\varphi_1 dS + \beta\int_{\partial\Omega} u_2^2\varphi_1 dS \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + \left(\beta-\frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS. \end{aligned}$$

We here assume $\beta - \alpha > 0$. From (3.27), it follows that

$$\begin{aligned}
&\frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx - (b+\lambda_1)\lambda_1\int_\Omega u_2\varphi_1 dx \\
&= \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \lambda_1\int_\Omega\left\{\frac{1}{4}u_2 - (b+\lambda_1)\right\}u_2\varphi_1 dx \\
&\geq \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \lambda_1\int_\Omega\left\{\frac{1}{4}u_2^* - (b+\lambda_1)\right\}u_2\varphi_1 dx \\
&\geq \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \lambda_1\int_\Omega\left\{\frac{1}{4}me^{\varepsilon t} - (b+\lambda_1)\right\}u_2\varphi_1 dx,
\end{aligned}$$

where $m := \min_{x\in\overline{\Omega}} l_2\overline{u}_2(x) > 0$. Hence there exists $t_1 > 0$ such that

$$\frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx - (b+\lambda_1)\lambda_1\int_\Omega u_2\varphi_1 dx \geq \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx \qquad \forall\, t \geq t_1. \tag{3.31}$$

Similarly, since

$$\begin{aligned}
&\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS - (b+\lambda_1)(\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 dS \\
=&\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS + \int_{\partial\Omega}\left\{\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)u_2 - (b+\lambda_1)(\beta-\alpha)\right\}u_2\varphi_1 dS \\
\geq&\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS + \int_{\partial\Omega}\left\{\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)me^{\varepsilon t} - (b+\lambda_1)(\beta-\alpha)\right\}u_2\varphi_1 dS,
\end{aligned}$$

there exists $t_2 > 0$ such that

$$\begin{aligned}
&\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS - (b+\lambda_1)(\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 dS \\
&\qquad\qquad \geq \frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS \qquad \forall\, t \geq t_2.
\end{aligned} \tag{3.32}$$

Therefore by (3.31), (3.32) and (3.30), we have

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right)\right\} + (b+2\lambda_1)\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + (\beta-\alpha)\frac{d}{dt}\left(\int_{\partial\Omega} u_2\varphi_1 dS\right) \\
\geq&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega u_2^2\varphi_1 dx\right) + \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS \qquad \forall\, t \geq t_3,
\end{aligned} \tag{3.33}$$

where $t_3 := t_1 \vee t_2$. Now we integrate (3.33) with respect to t over $[t_3, t]$ to get

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dS d\tau\right\} \\
&\geq \frac{1}{2}\int_\Omega u_2^2\varphi_1 dx - (b+2\lambda_1)\int_\Omega u_2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2(t_3)\varphi_1 dx \\
&\quad + \frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 dS d\tau + \int_\Omega \partial_t u_2(t_3)\varphi_1 dx,
\end{aligned} \tag{3.34}$$

where we neglected positive terms. Moreover we can see that there exists $t_4 > t_3$ such that

$$\frac{1}{2}\int_\Omega u_2^2\varphi_1 dx - (b+2\lambda_1)\int_\Omega u_2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2(t_3)\varphi_1 dx + \int_\Omega \partial_t u_2(t_3)\varphi_1 dx \geq \frac{1}{4}\int_\Omega u_2^2\varphi_1 dx \tag{3.35}$$

for $t \geq t_4$ by the same argument as before. Therefore from (3.34) and (3.35), we have

$$\frac{d}{dt}\left\{\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\} \geq \frac{1}{4}\int_\Omega u_2^2\varphi_1 dx + \frac{1}{2}\left(\beta-\frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 dSd\tau. \tag{3.36}$$

Since $\|\varphi_1\|_{L^1(\Omega)} = 1$, by Schwarz's inequality, we get

$$\frac{1}{4}\int_\Omega u_2^2\varphi_1 dx \geq \frac{1}{4}\left(\int_\Omega u_2\varphi_1 dx\right)^2,$$

and

$$\begin{aligned}
&\frac{1}{2}\left(\beta-\frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 dSd\tau \\
\geq\ & \frac{1}{2}(\beta-\frac{\alpha}{2})\frac{1}{\|\varphi_1\|_{L^\infty(\Omega)}|\partial\Omega|}\frac{1}{t-t_3}\left\{\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\}^2 \\
=\ & \frac{1}{2}\frac{\beta-\frac{\alpha}{2}}{\|\varphi_1\|_{L^\infty(\Omega)}|\partial\Omega|(\beta-\alpha)^2}\frac{1}{t-t_3}\left\{(\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\}^2.
\end{aligned}$$

By the above inequalities and (3.36), for $t \geq t_5 := t_4 \vee (t_3+1)$, we finally get

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\} \\
\geq & \frac{1}{4}\int_\Omega u_2^2\varphi_1 dx + \frac{1}{2}\left(\beta-\frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 dSd\tau \\
\geq & \frac{1}{4}\left(\int_\Omega u_2\varphi_1 dx\right)^2 + \frac{1}{2}\frac{\beta-\frac{\alpha}{2}}{\|\varphi_1\|_{L^\infty(\Omega)}|\partial\Omega|(\beta-\alpha)^2}\frac{1}{t-t_3}\left\{(\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\}^2 \\
\geq & C\frac{1}{t-t_3}\left\{\left(\int_\Omega u_2\varphi_1 dx\right)^2 + \left((\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right)^2\right\} \\
\geq & C\frac{1}{t-t_3}\left\{\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\}^2,
\end{aligned}$$

where C denotes some general positive constant independent of t. Set $y(t) := \int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t \int_{\partial\Omega} u_2\varphi_1 dS d\tau$, then the above inequality yields the following:

$$\begin{cases} \dfrac{d}{dt}y(t) \geq \dfrac{C}{t-t_3}y^2(t) & t \geq t_5, \\ y(t_5) > 0. \end{cases}$$

We can see that there exists $T^* > t_5$ such that

$$\lim_{t\to T^*} y(t) = +\infty. \tag{3.37}$$

In order to show the existence of T^* satisfying (3.37), it suffices to consider the following ordinary differential equation:

$$\begin{cases} \dfrac{d}{dt}\tilde{y}(t) = \dfrac{C}{t-t_3}\tilde{y}^2(t) & t \geq t_5, \\ \tilde{y}(t_5) > 0. \end{cases}$$

Since $\frac{d}{dt}\tilde{y}(t) > 0$ for all $t \geq t_5$ and $\tilde{y}(t_5) > 0$, it is clear that $\tilde{y}(t) > 0$ for all $t \geq t_5$. Divide both sides by $\tilde{y}^2(t)$ and integrate with respect to t on $[t_5, t]$, then we have

$$\frac{1}{\tilde{y}^2(t)}\frac{d}{dt}\tilde{y}(t) = \frac{C}{t-t_3},$$

$$\int_{\tilde{y}(t_5)}^{\tilde{y}(t)} \frac{1}{y^2}dy = C\log\frac{t-t_3}{t_5-t_3},$$

$$-\frac{1}{\tilde{y}(t)} + \frac{1}{\tilde{y}(t_5)} = C\log\frac{t-t_3}{t_5-t_3}.$$

Therefore we have

$$\tilde{y}(t) = \frac{1}{\frac{1}{\tilde{y}(t_5)} - C\log\frac{t-t_3}{t_5-t_3}}.$$

Hence there exists $\tilde{T} > t_5$ satisfying

$$\frac{1}{\tilde{y}(t_5)} - C\log\frac{\tilde{T}-t_3}{t_5-t_3} = 0$$

such that

$$\lim_{t\to\tilde{T}} \tilde{y}(t) = +\infty.$$

Thus (3.37) holds by comparison theorem for ordinary differential equations. This contradicts the assumption that (u_1, u_2) exists globally.

For the case of $\frac{\alpha}{2} \leq \beta \leq \alpha$, we can prove the same result with a slight modification. Actually, we get from (3.30)

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 dS\right\} \\ &+ (b+\lambda_1)\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1\int_\Omega u_2\varphi_1 dx\right\} \\ &\geq \frac{1}{2}\frac{d}{dt}\int_\Omega u_2^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx. \end{aligned}$$

Using (3.31) and integrating above inequality with respect to t over $[t_1, t]$, we have

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) - \int_\Omega \partial_t u_2(t_1)\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 dS - (\beta-\alpha)\int_{\partial\Omega} u_2(t_1)\varphi_1 dS \\ \geq &\frac{1}{2}\int_\Omega u_2^2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2(t_3)\varphi_1 dx - (b+2\lambda_1)\int_\Omega u_2\varphi_1 dx + (b+2\lambda_1)\int_\Omega u_2(t_1)\varphi_1 dx.\end{aligned}$$

Repeating the same arguments as for (3.31), we see that there exists $t_6 \geq t_1$ such that

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_2^2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2(t_3)\varphi_1 dx - (b+2\lambda_1)\int_\Omega u_2\varphi_1 dx \\ &+\int_\Omega \partial_t u_2(t_1)\varphi_1 dx + (\beta-\alpha)\int_{\partial\Omega} u_2(t_1)\varphi_1 dS \\ \geq\ &\frac{1}{4}\int_\Omega u_2^2\varphi_1 dx\end{aligned}$$

for all $t \geq t_6$. From these inequalities and Schwarz's inequality, it holds that

$$\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) \geq \frac{1}{4}\left(\int_\Omega u_2\varphi_1 dx\right)^2 \qquad \forall\ t \geq t_6.$$

Therefore we can get the following differential inequality:

$$\begin{cases} \dfrac{d}{dt}y(t) \geq y^2(t) & t \geq t_6, \\ y(t_6) > 0, \end{cases}$$

where $y(t) = \int_\Omega u_2\varphi_1 dx$. It is easy to see that there exists $T^{**} > t_6$ such that

$$\lim_{t\to T^{**}} y(t) = +\infty.$$

This leads to a contradiction. □

Remark 3.4. Since the blow-up result is proved by contradiction, there is no knowing if $\|u_1(t)\|_{L^\infty}$ and $\|u_2(t)\|_{L^\infty}$ blow up simultaneously. However we can show by another argument that L^∞-norms of u_1 and u_2 blow up at the same time, i.e., there exists $T > 0$ such that

$$\lim_{t\to T}\|u_1(t)\|_{L^\infty(\Omega)} = \infty \quad \text{and} \quad \lim_{t\to T}\|u_2(t)\|_{L^\infty(\Omega)} = \infty.$$

In fact, multiplying the first equation of (NR) by $|u_1|^{r-2}u_1$ and using integration by parts and similar calculation in the proof of Theorem 3.1, we obtain

$$\frac{d}{dt}\|u_1(t)\|_{L^r(\Omega)} \leq \|u_2(t)\|_{L^\infty(\Omega)}\|u_1(t)\|_{L^r(\Omega)} \qquad \forall t \in [0, T). \tag{3.38}$$

From the second equation of (NR), we also have

$$\|u_2(t)\|_{L^\infty(\Omega)} \leq \|u_{20}\|_{L^\infty(\Omega)} + a\int_0^t \|u_1(\tau)\|_{L^\infty(\Omega)} d\tau \qquad \forall t \in [0, T). \tag{3.39}$$

Suppose that

$$\lim_{t\to T} \|u_1(t)\|_{L^\infty(\Omega)} = \infty \qquad \text{and} \qquad M_2 := \sup_{0\le t\le T} \|u_2(t)\|_{L^\infty(\Omega)} < \infty,$$

then it follows from (3.38)

$$\frac{d}{dt}\|u_1(t)\|_{L^r} \le M_2\|u_1(t)\|_{L^r(\Omega)} \qquad \forall t \in [0,T).$$

By Gronwall's inequality, we get

$$\|u_1(t)\|_{L^r(\Omega)} \le \|u_{10}\|_{L^r(\Omega)}e^{M_2 t} \le \|u_{10}\|_{L^r(\Omega)}e^{M_2 T} \qquad \forall t \in [0,T).$$

Letting r tend to ∞, we obtain

$$\|u_1(t)\|_{L^\infty(\Omega)} \le \|u_{10}\|_{L^\infty(\Omega)}e^{M_2 T} \qquad \forall t \in [0,T),$$

which contradicts the fact $\lim_{t\to T} \|u_1(t)\|_{L^\infty(\Omega)} = \infty$. Next, suppose that

$$M_1 := \sup_{0\le t\le T} \|u_1(t)\|_{L^\infty(\Omega)} < \infty \qquad \text{and} \qquad \lim_{t\to T} \|u_2(t)\|_{L^\infty(\Omega)} = \infty,$$

then by (3.39) we see that

$$\|u_2(t)\|_{L^\infty(\Omega)} \le \|u_{20}\|_{L^\infty(\Omega)} + aM_1T \qquad \forall t \in [0,T).$$

Letting t tend to T, we get contradiction. Thus we see that u_1 and u_2 blow up at the same time.

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