Advances in Mathematical Sciences and Applications
Vol. 27, No. 1 (2018), pp. 1–28

GAKKOTOSHO
TOKYO JAPAN

OSCILLATION TESTS FOR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

G. E. Chatzarakis *
Department of Electrical and Electronic Engineering Educators
School of Pedagogical and Technological Education (ASPETE)
14121, N. Heraklio, Athens, Greece
(E-mail: geaxatz@otenet.gr, gea.xatz@aspete.gr)

I. K. Purnaras †
Department of Mathematics
University of Ioannina
451 10 Ioannina, Greece
(E-mail: ipurnara@uoi.gr)

and

I. P. Stavroulakis ‡
Department of Mathematics
University of Ioannina
451 10 Ioannina, Greece
(E-mail: ipstav@uoi.gr)

Abstract. Consider the first-order linear retarded [advanced] differential equation of the form

$$x'(t) + p(t)x(\tau(t)) = 0 \; [x'(t) - q(t)x(\sigma(t)) = 0] \,, \quad t \geq t_0,$$

where $p(t) \geq 0$, $q(t) \geq 0$ and $\tau(t)$, $\sigma(t)$ are functions of positive real numbers such that $\tau(t) < t$ for $t \geq t_0$, $\lim_{t\to\infty} \tau(t) = \infty$ and $\sigma(t) > t$ for $t \geq t_0$. Sufficient conditions, involving lim sup and lim inf guaranteeing the oscillation of all solutions of each equation, are established. Examples illustrating the significance of the results are also given.

Communicated by Y. Giga; Received September 11 , 2017.
AMS Subject Classification: 34K11, 34K06.
Keywords: differential equation, non-monotone argument, oscillatory solution, nonoscillatory solution.

1 Introduction

Consider the first-order linear differential equation with variable retarded argument

$$x'(t) + p(t)x\left(\tau(t)\right) = 0, \quad t \geq t_0, \tag{E}$$

where $p(t) \geq 0$ and $\tau(t)$ are continuous functions and τ satisfies

$$0 < \tau(t) < t \;\text{ for }\; t \geq t_0, \quad \text{and} \quad \lim_{t\to\infty} \tau(t) = \infty. \tag{1.1}$$

By a solution of (E) we mean a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (E) is satisfied for $t \geq T_0$. A solution of (E) is called *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*. When all solutions of (E) oscillate we say that (E) is *oscillatory*.

When $\tau(t) \equiv t$, equation (E) reduces to an ordinary differential equation (o.d.e.), and it is well known that a first-order o.d.e. with constant coefficients does not possess oscillatory solutions. On the contrary, presence of even a very small delay in the argument of (E) may create oscillatory solutions. After the pioneering work of Myshkis [27], the study of the oscillatory character of (E) has attracted considerable interest and the problem of establishing sufficient conditions for the oscillation of all solutions of equation (E) has been the subject of many investigations. Besides its mathematical interest, considerable attention to this problem is given by the fact that the mathematical modelling of several real-world problems leads to differential equations that depend on the past history rather than only on the current state. The reader is referred to [1]−[6], [9], [10], [13]−[23], [25]−[27], [29]-[32], and the references cited therein. For the general oscillation theory of differential equations with deviating arguments we refer to the monographs [8], [11], and [24].

While most of the papers cited above concern the case where the arguments are nondecreasing, only a small number of papers are dealing with the general case where the arguments are not necessarily monotone. See, for example, [1]−[6], [18], [26], [29] and the references cited therein. The interest of considering equation (E) with non-monotone arguments is justified not only by its the pure mathematical interest, but also because such equations describe in a more realistic way a wide class of natural phenomena as natural disturbances (e.g. noise in communication systems) affecting parameters of the equation cause non-monotone deviations in the argument of the solutions. In the present paper we establish a number of oscillation criteria for all solutions of (E) when the argument is not necessarily monotone. Our results essentially improve several known criteria existing in the literature.

A parallel problem to that of establishing oscillating criteria for the solutions of the equation (E) is the one concerning the solutions to the equation of advanced type

$$x'(t) - q(t)x\left(\sigma(t)\right) = 0, \quad t \geq t_0, \tag{E'}$$

where $q(t) \geq 0$ and $\sigma(t)$ are continuous functions defined on $[t_0, \infty)$ and σ satisfies

$$\sigma(t) > t \quad \text{for} \;\; t \geq t_0. \tag{1.2}$$

Perhaps not widely known, the idea of advanced arguments seems to originate as early as 1903 in a paper by Schwarschild presenting a model where charged particles influence each other at a distance via both retarded and advanced arguments (see, [7], [12], [28]). For instance, the appearance of advanced arguments in an equation bends on the consideration that if two or more classical charged particles are moving in space, each particle's motion is influenced by the electromagnetic field of the other. If one assumes that the basic laws of Physics are symmetric with respect to time reversal, then the existence of time delays caused by interactions implies that there should also be advanced terms in the equations.

Dual criteria for the oscillation of all solutions of (E') may be established by following parallel argumentation to that employed for obtaining the results concerning the retarded equation (E). Thus, we focus on the ones concerning the equation (E) presenting them in detail, and consider them as our main results, while we simply cite the corresponding theorems for (E') omitting their proofs.

2 Retarded differential equations: History and motivation

The first systematic study for the oscillation of all solutions to equation (E) was made by Myshkis in 1950 [27] when he proved that every solution of (E) oscillates if

$$\limsup_{t\to\infty}[t-\tau(t)]<\infty \qquad \text{and} \qquad \liminf_{t\to\infty}[t-\tau(t)]\liminf_{t\to\infty}p(t)>\frac{1}{e}. \tag{2.1}$$

In 1972, Ladas, Lakshmikantham and Papadakis [22] proved that, if

$$\tau \text{ is nondecreasing} \qquad \text{and} \qquad \limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>1, \tag{2.2}$$

then all solutions of (E) oscillate. The next essential step was taken by Ladas [21] in 1979, and, by Koplatadze and Chanturija [17] in 1982 who improved (2.1) to

$$\alpha:=\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>\frac{1}{e}. \tag{2.3}$$

Conserning the constant $\frac{1}{e}$ in (2.3), it is to be pointed out that if the inequality

$$\int_{\tau(t)}^{t}p(s)ds\leq\frac{1}{e}$$

holds eventually, then, according to a result in [17], (E) has a nonoscillatory solution.

Obviously when the limit

$$\lim_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds$$

does not exist a gap appears between the conditions (2.2) and (2.3). How to fill this gap is an interesting problem which has attracted the attention of several authors. For example, in 1999, Jaroš and Stavroulakis [15] proved that, if τ is nondecreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.4}$$

where λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$, then all solutions of (E) oscillate.

Now we come to the case that the argument $\tau(t)$ is not necessarily monotone. Set

$$h(t) := \sup_{s\le t} \tau(s), \quad t \ge t_0. \tag{2.5}$$

Clearly, h is nondecreasing and $\tau(t) \le h(t) < t$ for all $t \ge t_0$, while $\tau \equiv h$ when τ is nondecreasing. Essential progress was made by Koplatadze and Kvinikadze [18] in 1994 who proved that, if

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{h(s)}^{h(t)} p(u)\psi_j(u)du\right) ds > 1 - D(\alpha), \tag{2.6}$$

where

$$D(\alpha) := \begin{cases} 0, & \text{if } \alpha > 1/e \\ \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, & \text{if } \alpha \in [0, 1/e] \end{cases}, \tag{2.7}$$

and

$$\psi_1(t) = 0, \qquad \psi_j(t) = \exp\left(\int_{\tau(t)}^{t} p(u)\psi_{j-1}(u)du\right), \quad j \ge 2, \tag{2.8}$$

then all solutions of (E) oscillate.

In 2011 Braverman and Karpuz [3], proved that if

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right) ds > 1, \tag{2.9}$$

then all solutions of (E) oscillate, while Stavroulakis [29] in 2014 improved (2.9) to

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right) ds > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{2.10}$$

Recently, Morshedy and Attia [26] proved that, if $D(\alpha)$ is defined by (2.7) and

$$\limsup_{t\to\infty} \left[\int_{g(t)}^{t} p_n(s)ds + D(\alpha)\exp\left(\int_{g(t)}^{t} \sum_{j=0}^{n-1} p_j(s)ds\right)\right] > 1, \tag{2.11}$$

with $p_0(t) = p(t)$ and

$$p_n(t) = p_{n-1}(t)\int_{g(t)}^{t} p_{n-1}(s)\exp\left(\int_{g(s)}^{t} p_{n-1}(u)du\right) ds, \;\; n \ge 1, \tag{2.12}$$

then all solutions of (E) oscillate. Here, $g(t)$ is a nondecreasing continuous function such that $\tau(t) \leq g(t) \leq t$, $t \geq t_1$ for some $t_1 \geq t_0$. Clearly, $g(t)$ is more general than $h(t)$ defined by (2.5).

Chatzarakis [4], [5] proved that if for some $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p_j(u)du \right) ds > 1 \tag{2.13}$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p_j(u)du \right) ds > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.14}$$

where

$$p_j(t) = p(t) \left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p_{j-1}(u)du \right) ds \right], \tag{2.15}$$

with $p_0(t) = p(t)$, then all solutions of (E) oscillate.

3 Main results

A trivial observation concerning the solutions of equation (E) is that if x is a solution of (E) then $-x$ is also a solution. Hence, assuming that a nonoscillatory solution of (E) does exist we may confine our study only to the case where x is eventually positive. Then there exists $t_1 \geq t_0$ such that $x(t)$, $x(\tau(t)) > 0$, for all $t \geq t_1$. Thus, from (E) we have

$$x'(t) = -p(t)x(\tau(t)) \leq 0, \quad t \geq t_1,$$

meaning that x is an eventually nonincreasing function of positive numbers. So, in view of the fact that $\lim_{t\to\infty} \tau(t) = \infty$ we see that there is no loss of generality in assuming that if x is a nonoscillatory solution of (E) then x *is nonincreasing and positive for all* $t \geq t_1$ with $t > \tau(t) \geq t_1$ and this is what we are going to use throughout the paper without repeating argumentation.

Theorem 1. *Assume that h is defined by* (2.5)*, and for some $j \in \mathbb{N}$ it holds*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} P_j(u)du \right) ds > 1, \tag{3.1}$$

where

$$P_j(t) = p(t) \left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} P_{j-1}(\xi)d\xi \right) du \right) ds \right], \tag{3.2}$$

with $P_0(t) = p(t)$. Then all solutions of (E)oscillate.

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution x of (E) and let $t_1 \geq t_0$ such that $x(t),\ x(\tau(t)) > 0$, for all $t \geq t_1$. Taking into accout the fact that $\tau(t) < t$, (E) implies

$$x'(t) + p(t)x(t) \leq 0, \quad t_1 \leq \tau(t) < t. \tag{3.3}$$

Dividing the last inequality by $x(t) > 0$ and integrating inequality (3.3) on $[s,t]$ we obtain

$$\int_s^t \frac{x'(u)}{x(u)}du + \int_s^t p(\xi)\, d\xi \leq 0$$

or

$$x(s) \geq x(t) \exp\left(\int_s^t p(\xi)d\xi\right), \qquad t_1 \leq \tau(s) < t. \tag{3.4}$$

Now we divide (E) by $x(t) > 0$ and integrate on $[s,t]$, so

$$-\int_s^t \frac{x'(u)}{x(u)}du = \int_s^t p(u)\frac{x(\tau(u))}{x(u)}du$$

or

$$\ln\frac{x(s)}{x(t)} = \int_s^t p(u)\frac{x(\tau(u))}{x(u)}du, \quad t_1 \leq \tau(t) < t. \tag{3.5}$$

Since $\tau(u) < u$, setting $u = t$, $s = \tau(u)$ in (3.4) we take

$$x(\tau(u)) \geq x(u) \exp\left(\int_{\tau(u)}^u p(\xi)d\xi\right), \quad t_1 \leq \tau(u) < u. \tag{3.6}$$

Combining (3.5) and (3.6) we obtain, for sufficiently large t

$$\ln\frac{x(s)}{x(t)} \geq \int_s^t p(u)\exp\left(\int_{\tau(u)}^u p(\xi)d\xi\right)du$$

or

$$x(s) \geq x(t)\exp\left(\int_s^t p(u)\exp\left(\int_{\tau(u)}^u p(\xi)d\xi\right)du\right). \tag{3.7}$$

Integrating (E) from $\tau(t)$ to t, we have

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^t p(s)x(\tau(s))\, ds = 0, \tag{3.8}$$

while inequality (3.7) by replacing s by $\tau(s)$ gives

$$x(t)\exp\left(\int_{\tau(s)}^t p(u)\exp\left(\int_{\tau(u)}^u p(\xi)d\xi\right)du\right) \leq x(\tau(s)).$$

Since $\tau(s) < t$, from (3.8) and the last inequality we find, for sufficiently larte t

$$x(t) - x(\tau(t)) + x(t)\int_{\tau(t)}^t p(s)\exp\left(\int_{\tau(s)}^t p(u)\exp\left(\int_{\tau(u)}^u p(\xi)d\xi\right)du\right)ds \leq 0.$$

Multiplying the last inequality by $p(t)$, (cf. [26]), we find

$$p(t)x(t) - p(t)x(\tau(t)) + p(t)x(t)\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} p(\xi)d\xi\right)du\right)ds \le 0,$$

which, in view of (E), becomes

$$x'(t) + p(t)x(t) + p(t)x(t)\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} p(\xi)d\xi\right)du\right)ds \le 0.$$

Hence,

$$x'(t) + p(t)\left[1 + \int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} p(\xi)d\xi\right)du\right)ds\right]x(t) \le 0,$$

or

$$x'(t) + P_1(t)x(t) \le 0, \tag{3.9}$$

where

$$P_1(t) = p(t)\left[1 + \int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} p(\xi)d\xi\right)du\right)ds\right], \quad t_2 \le t.$$

Clearly (3.9) resembles (3.3) with p replaced by P_1, so an integration of (3.9) on $[s,t]$ leads to

$$x(s) \ge x(t)\exp\left(\int_{s}^{t} P_1(\xi)d\xi\right), \qquad 0 \le s \le t. \tag{3.10}$$

Taking the steps starting from (3.3) to (3.6) we may see that x satisfies the inequality

$$x(\tau(u)) \ge x(u)\exp\left(\int_{\tau(u)}^{u} P_1(\xi)d\xi\right). \tag{3.11}$$

Combining now (3.5) and (3.11), we obtain

$$\ln\frac{x(s)}{x(t)} \ge \int_{s}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_1(\xi)d\xi\right)du$$

or

$$x(s) \ge x(t)\exp\left(\int_{s}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_1(\xi)d\xi\right)du\right),$$

from which we take

$$x(\tau(s)) \ge x(t)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_1(\xi)d\xi\right)du\right). \tag{3.12}$$

Since $\tau(s) \le h(s) \le h(t) < t$, (3.8) and (3.12) give

$$x(t) - x(\tau(t)) + x(t)\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_1(\xi)d\xi\right)du\right)ds \le 0.$$

Multiplying the last inequality by $p(t)$, as before, we find

$$x'(t)+p(t)\left[1+\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_1(\xi)d\xi\right)du\right)ds\right]x(t)\leq 0.$$

Therefore, for sufficiently large t

$$x'(t)+P_2(t)x(t)\leq 0, \tag{3.13}$$

where

$$P_2(t)=p(t)\left[1+\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_1(\xi)d\xi\right)du\right)ds\right],\ \ t_3\leq t.$$

It is apparent, now, that the steps leading from (3.3) to (3.9), then to inequality (3.13), can be repeated, and the inductive procedure leads to the conclusion that for a sufficiently large t_{j+1}, the positive solution x satisfies the inequality

$$x'(t)+P_j(t)x(t)\leq 0,\quad t\geq t_{j+1},\quad (j\in\mathbb{N})\,, \tag{3.14}$$

where

$$P_j(t)=p(t)\left[1+\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_{j-1}(\xi)d\xi\right)du\right)ds\right],\ t\geq t_{j+1}.$$

In order to take our final step, we recall that

$$h\left(t\right):=\sup_{s\leq t}\tau\left(t\right)$$

and note that h is a nondecreasing function. Moreover, since $\tau\left(s\right)\leq h\left(s\right)\leq h\left(t\right)$ from (3.10)we have

$$x\left(h\left(t\right)\right)\exp\left(\int_{\tau(s)}^{h(t)} P_j\left(u\right)du\right)\leq x\left(\tau\left(s\right)\right).$$

Integrating (E) from $h(t)$ to t and repeating the same procedure as in (3.9), in view of the last inequality we see that for sufficiently large t it holds

$$x(t)-x(h(t))+x(h(t))\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} P_j(u)du\right)ds\leq 0. \tag{3.15}$$

Since $x(t)>0$, the last inequality implies

$$x(h(t))\left[\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} P_j(u)du\right)ds-1\right]<0,$$

which contradicts (3.1).

The proof of the theorem is complete. □

We now cite two lemmas which will be used in the proof of our next results. We note that the first one (see, [32]) provides a lower estimate for the ratio $x(t)/x(h(t))$ in terms of the smaller root of the equation $\xi^2 - (1-\alpha)\xi + \alpha^2/2 = 0$, where α is given by (2.3).

Lemma 1. *[32] Assume that x is an eventually positive solution of (E) and*

$$0 < \alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \le \frac{1}{e}. \tag{3.16}$$

Then

$$\liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \ge \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{3.17}$$

Lemma 2. *([8], Lemma 2.1.1) In addition to hypothesis (1.1), assume that h is defined by (2.5). Then*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \liminf_{t\to\infty} \int_{h(t)}^{t} p(s)ds. \tag{3.18}$$

Based on the above lemmas, we establish the following two theorems.

Theorem 2. *Assume that (3.16) holds and for some $j \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} P_j(u)du\right) ds > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{3.19}$$

where P_j is defined by (3.2). Then all solutions of (E) oscillate.

Proof. Let x be an eventually positive solution of (E). Then, as in the proof of Theorem 1, we obtain (3.15), i.e, for sufficiently large t we have

$$x(t) - x(h(t)) + x(h(t)) \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} P_j(u)du\right) ds \le 0.$$

That is,

$$\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} P_j(u)du\right) ds \le 1 - \frac{x(t)}{x(h(t))},$$

which gives

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} P_j(u)du\right) ds \le 1 - \liminf_{t\to\infty} \frac{x(t)}{x(h(t))}.$$

In view of (3.17), the last inequality contradicts (3.19).
The proof of the theorem is complete. □

It is clear that the left-hand sides of both conditions (3.1) and (3.19) are identical, also the right hand side of condition (3.19) reduces to (3.1) in case that $\alpha = 0$. So it seems that Theorem 2 is the same as Theorem 1 when $\alpha = 0$. However, one may notice that condition (3.16) is required in Theorem 2 but not in Theorem 1

Theorem 3. *Assume that* (3.16) *holds and for some* $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} P_j(u)du \right) ds > \frac{2}{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}, \tag{3.20}$$

where P_j *is defined by* (3.2). *Then all solutions of* (E) *oscillate.*

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution x of (E) and that x is eventually positive. Then, as in the proof of Theorem 1, and similarly to (3.10), from (3.14) we get that for sufficiently large values of s, t we have

$$x(s) \geq x(t) \exp\left(\int_{s}^{t} P_j(\xi)d\xi \right), \tag{3.21}$$

from which for $\tau(s)$ in place of s we take

$$x(\tau(s)) \geq x(t) \exp\left(\int_{\tau(s)}^{t} P_j(\xi)d\xi \right). \tag{3.22}$$

Integrating (E) from $h(t)$ to t, we have

$$x(t) - x(h(t)) + \int_{h(t)}^{t} p(s)x(\tau(s))ds = 0,$$

which, in view of (3.22), gives

$$x(t) - x(h(t)) + \int_{h(t)}^{t} p(s)x(t) \exp\left(\int_{\tau(s)}^{t} P_j(u)du \right) ds \leq 0.$$

Since $x(t) > 0$, the last inequality leads to

$$x(h(t)) \left[\frac{x(t)}{x(h(t))} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} P_j(u)du \right) ds - 1 \right] < 0.$$

That is, for all sufficiently large t it holds

$$\int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} P_j(u)du \right) ds < \frac{x(h(t))}{x(t)},$$

and therefore

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} P_j(u)du \right) ds \leq \limsup_{t\to\infty} \frac{x(h(t))}{x(t)}.$$

In view of (3.17), the last inequality contradicts (3.20).
The proof of the theorem is complete. □

The next lemma (see [20]) provides a lower estimate for the ratio $x(h(t))/x(t)$ in terms of the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$.

Lemma 3. *Assume that* (3.16) *holds and let* x *be a positive solution of* (E)*. Then*

$$\liminf_{t\to\infty} \frac{x(h(t))}{x(t)} \geq \lambda_0, \tag{3.23}$$

where λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\alpha\lambda}$*.*

Based on inequality (3.23), we establish the following theorem.

Theorem 4. *Assume that* (3.16) *holds and for some* $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds > \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{3.24}$$

where P_j *is defined by* (3.2) *and* λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\alpha\lambda}$*. Then all solutions of* (E) *oscillate.*

Proof. Let x be an eventually positive solution and obtain (3.22) as in Theorem 1.
Observe that (3.23) implies that for each $\epsilon > 0$ there exists a t_ϵ such that

$$\lambda_0 - \epsilon < \frac{x(h(t))}{x(t)} \quad \text{for all } t \geq t_\epsilon\,. \tag{3.25}$$

Noting that by nondecreasing nature of the function $\frac{x(h(t))}{x(s)}$ in s, it holds

$$1 = \frac{x(h(t))}{x(h(t))} \leq \frac{x(h(t))}{x(s)} \leq \frac{x(h(t))}{x(t)}, \quad t_\varepsilon \leq h(t) \leq s \leq t,$$

in particular for $\epsilon \in (0, \lambda_0 - 1)$, by continuity we see that there exists a $t^* \in (h(t), t]$ such that

$$1 < \lambda_0 - \epsilon = \frac{x(h(t))}{x(t^*)}. \tag{3.26}$$

Employing (3.21) with $\tau(s)$ and $h(s)$ in place of s and t respectively (we always have $\tau(s) \leq h(s)$), we see that there exists a $t_1 \geq t_\varepsilon \geq t_0$ such that

$$x(\tau(s)) \geq x(h(s))\exp\left(\int_{\tau(s)}^{h(s)} P_j(\xi)d\xi\right), \quad t_1 \leq \tau(s) \leq h(s) \leq t. \tag{3.27}$$

Integrating (E) from t^* to t we have

$$x(t) - x(t^*) + \int_{t^*}^{t} p(s)x(\tau(s))ds = 0,$$

so, by using (3.27) along with $h(s) \leq h(t)$ in combination with the nonincreasingness of x, we have

$$x(t) - x(t^*) + x(h(t))\int_{t^*}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds \leq 0,$$

or

$$\int_{t^*}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds \le \frac{x(t^*)}{x(h(t))} - \frac{x(t)}{x(h(t))}.$$

In view of (3.26) and Lemma 1, for the ϵ considered, there exists $t'_\epsilon \ge t_\varepsilon$ such that

$$\int_{t^*}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds < \frac{1}{\lambda_0 - \epsilon} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} + \epsilon, \tag{3.28}$$

for $t \ge t'_\epsilon$.

Dividing (E) by $x(t)$ and integrating from $h(t)$ to t^* we find

$$\int_{h(t)}^{t^*} p\left(s\right) \frac{x(\tau\left(s\right))}{x(s)} ds = -\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds,$$

and using (3.27), we find

$$\int_{h(t)}^{t^*} p(s) \frac{x(h(s))}{x(s)} \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds \le -\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds. \tag{3.29}$$

By (3.25), for $s \ge h(t) \ge t'_\epsilon$, we have $\frac{x(h(s))}{x(s)} > \lambda_0 - \epsilon$, so from (3.29) we get

$$(\lambda_0 - \epsilon) \int_{h(t)}^{t^*} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds < -\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds\,.$$

Hence, for all sufficiently large t we have

$$\int_{h(t)}^{t^*} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds$$
$$< -\frac{1}{\lambda_0 - \epsilon} \int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds = \frac{1}{\lambda_0 - \epsilon} \ln \frac{x(h(t))}{x(t^*)} = \frac{\ln\left(\lambda_0 - \epsilon\right)}{\lambda_0 - \epsilon},$$

i.e.,

$$\int_{h(t)}^{t^*} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds < \frac{\ln\left(\lambda_0 - \epsilon\right)}{\lambda_0 - \epsilon}. \tag{3.30}$$

Adding (3.28) and (3.30), and then taking the limit as $t \to \infty$, we have

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds$$
$$\le \frac{1 + \ln(\lambda_0 - \epsilon)}{\lambda_0 - \epsilon} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} + \epsilon.$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (3.24).

The proof of the theorem is complete. □

Theorem 5. *Assume that for some $j \in \mathbb{N}$*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du \right) ds > \frac{1}{e}, \tag{3.31}$$

where P_j is defined by (3.2). *Then all solutions of* (E) *oscillate.*

Proof. For the sake of contradiction, let x be a nonincreasing eventually positive solution and $t_1 > t_0$ be such that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \geq t_1$.

Firstly we note that we may obtain (3.22) in the way discribed in Theorem 3, i.e.,

$$x(\tau(s)) \geq x(t) \exp\left(\int_{\tau(s)}^{t} P_j(\xi)d\xi \right).$$

Dividing (E) by $x(t)$ and integrating from $h(t)$ to t we have

$$\ln\left(\frac{x(h(t))}{x(t)}\right) = \int_{h(t)}^{t} p(s) \frac{x(\tau(s))}{x(s)} ds \quad \text{for all } t \geq t_2 \geq t_1,$$

from which in view of $\tau(s) \leq h(s)$ and by (3.22), we obtain

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq \int_{h(t)}^{t} p(s) \frac{x(h(s))}{x(s)} \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du \right) ds,$$

where P_j is defined by (3.2).

Taking into account that x is nonincreasing and $h(s) < s$, the last inequality leads to

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du \right) ds. \tag{3.32}$$

From (3.31), it follows that there exists a constant $c > 0$ such that for a sufficiently large t_3 it holds

$$\int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du \right) ds \geq c > \frac{1}{e}, \quad t \geq t_3. \tag{3.33}$$

Combining inequalities (3.32) and (3.33), we obtain

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq c, \quad t \geq t_3.$$

Thus

$$\frac{x(h(t))}{x(t)} \geq e^c \geq ec > 1,$$

which implies for some $t \geq t_4 \geq t_3$

$$x(h(t)) \geq (ec)x(t)$$

Repeating the above procedure, it follows by induction that for any positive integer k ,

$$\frac{x(h(t))}{x(t)} \geq (ec)^k, \quad \text{for sufficiently large } t,$$

Since $ec > 1$, there is $k \in \mathbb{N}$ satisfying $k > \frac{2[\ln 2 - \ln c]}{1 + \ln c}$ such that for t sufficiently large

$$\frac{x(h(t))}{x(t)} \geq (ec)^k > \frac{4}{c^2}. \tag{3.34}$$

Further (cf. [17], [2], [11]), for sufficiently large t, there exists a $t_m \in (h(t), t)$ such that

$$\begin{aligned} &\int_{h(t)}^{t_m} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds \geq \tfrac{c}{2}, \\ &\int_{t_m}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds \geq \tfrac{c}{2}. \end{aligned} \tag{3.35}$$

Integrating (E) from $h(t)$ to t_m, using (3.22) and the fact that $x(t) > 0$, we obtain

$$x(h(t)) > x(h(t_m)) \int_{h(t)}^{t_m} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds,$$

which, in view of the first inequality in (3.35), implies

$$x(h(t)) > \frac{c}{2} x(h(t_m)). \tag{3.36}$$

Similarly, integrating (E) from t_m to t, using (3.22) and the fact that $x(t) > 0$, we have

$$x(t_m) > x(h(t)) \int_{t_m}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_j(u)du\right) ds,$$

which, in view of the second inequality in (3.35), implies

$$x(t_m) > \frac{c}{2} x(h(t)). \tag{3.37}$$

Combining inequalities (3.36) and (3.37), we obtain

$$x(h(t_m)) < \frac{2}{c} x(h(t)) < \frac{4}{c^2} x(t_m),$$

which contradicts (3.34).

The proof of the theorem is complete. □

Before closing this section we note that one can easily see that conditions (3.1), (3.19), (3.24), and (3.31) substantially improve conditions (2.2) (also, (2.9), (2.13), (2.10), (2.4) and (2.3)). That can immediately be observed, if we compare the corresponding parts on the left-hand side of these conditions.

4 Examples and comments

The examples below illustrate that our conditions essentially improve known results in the literature yet indicate a type of independence among some of them. Not to pursue complexity any further, we choose to present examples with constant coefficients and variable non-monotone delays. These examples not only illustrate the significancy of our results but also indicate high level of improvement in the oscillation criteria. The calculations were made by the use of MATLAB software.

Example 1. *Consider the retarded differential equation*

$$x'(t) + \frac{1}{8}x(\tau(t)) = 0, \qquad t \geq 0, \tag{4.1}$$

with (see Fig. 1, (a))

$$\tau(t) = \begin{cases} t-1, & \text{if } t \in [8k, 8k+2] \\ -4t+40k+9, & \text{if } t \in [8k+2, 8k+3] \\ 5t-32k-18, & \text{if } t \in [8k+3, 8k+4] \\ -4t+40k+18, & \text{if } t \in [8k+4, 8k+5] \\ 5t-32k-27, & \text{if } t \in [8k+5, 8k+6] \\ -2t+24k+15, & \text{if } t \in [8k+6, 8k+7] \\ 6t-40k-41, & \text{if } t \in [8k+7, 8k+8] \end{cases}, \qquad k \in \mathbb{N}_0,$$

where $\mathbb{N}_0$ *is the set of non-negative integers.*

By (1.6)*, we see (Fig. 1, (b)) that*

$$h(t) = \begin{cases} t-1, & \text{if } t \in [8k, 8k+2] \\ 8k+1, & \text{if } t \in [8k+2, 8k+19/5] \\ 5t-32k-18, & \text{if } t \in [8k+19/5, 8k+4] \\ 8k+2, & \text{if } t \in [8k+4, 8k+29/5] \\ 5t-32k-27, & \text{if } t \in [8k+29/5, 8k+6] \\ 8k+3, & \text{if } t \in [8k+6, 8k+44/6] \\ 6t-40k-41, & \text{if } t \in [8k+44/6, 8k+8] \end{cases}, \qquad k \in \mathbb{N}_0.$$

Let the function $F_j : \mathbb{R}_0 \to \mathbb{R}_+$ $(j \in \mathbb{N})$ *be defined by*

$$F_j(t) = \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} P_j(u)du \right) ds, \tag{4.2}$$

with P_j *given by* (3.2)*. Noting that* F_j *attains its maximum at* $t = 8k + 44/6$, $k \in \mathbb{N}_0$*, for every* $j \in \mathbb{N}$*, and using an algorithm on MATLAB software, we obtain*

$$\limsup_{t\to\infty} F_1(t) = \limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} P_1(u)du \right) ds \simeq 1.0097 > 1.$$

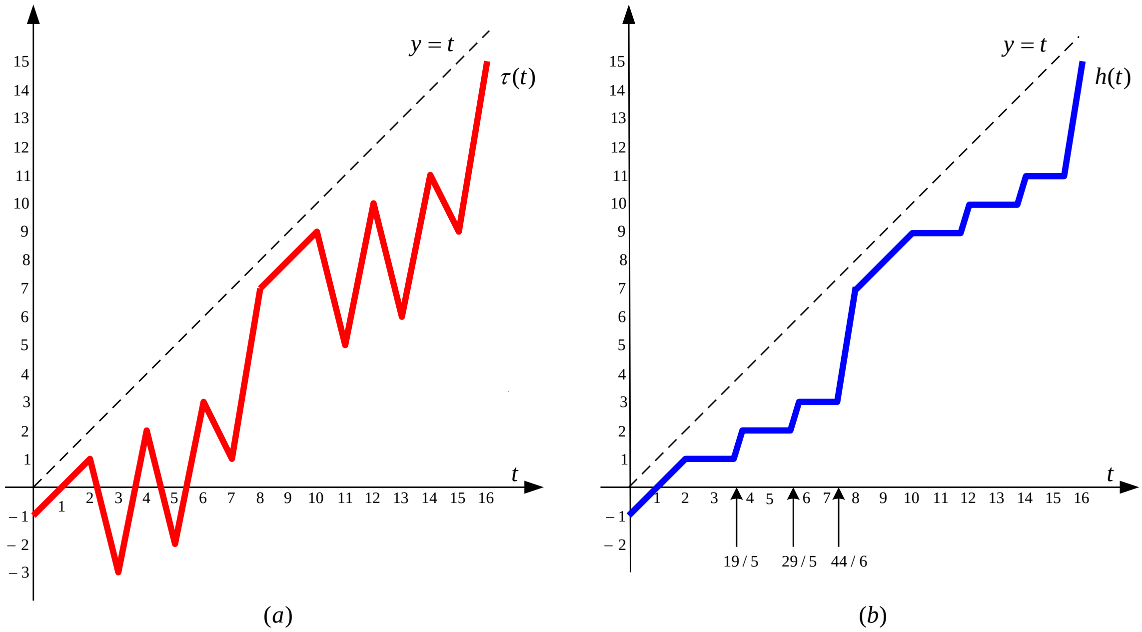

Figure 1: The graphs of $\tau(t)$ and $h(t)$

That is, condition (3.1) *of Theorem 1 is satisfied for* $j = 1$, *and therefore all solutions of* (4.1) *oscillate.*

Observe, however, that

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds = \limsup_{k\to\infty} \int_{8k+3}^{8k+44/6} \frac{1}{8} ds = 0.5417 < 1,$$

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \liminf_{k\to\infty} \int_{8k+1}^{8k+2} \frac{1}{8} ds = 0.125 < \frac{1}{e},$$

$$0.5417 < \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.9815,$$

where $\lambda_0 = 1.15537$ *is the smaller solution of* $e^{0.125\lambda} = \lambda$.

Noting that the function Φ_j *defined by*

$$\Phi_j(t) = \int_{h(t)}^{t} p(s) \exp\left(\int_{h(s)}^{h(t)} p(u)\psi_j(u)du \right) ds, \quad (j \geq 2), \tag{4.3}$$

(with ψ_j *defined by* (2.8)*) attains its maximum at* $t = 8k + 44/6$, $k \in \mathbb{N}_0$ *for every* $j \geq 2$. *Specifically, we find*

$$\limsup_{t\to\infty} \Phi_2(t) \simeq 0.6450 < 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.99098.$$

Also

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right)ds \simeq 0.74354 < 1$$

and

$$0.74354 < 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.99098.$$

As each one of the functions G_j $(j \in \mathbb{N})$ *defined by*

$$G_j(t) = \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_j(u)du\right)ds, \qquad (j \in \mathbb{N}) \tag{4.4}$$

attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$, *for every* $j \in \mathbb{N}$ *we find*

$$\limsup_{t\to\infty} G_1(t) = \limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_1(u)du\right)ds \simeq 0.8626 < 1$$

and

$$0.8626 < 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.99098.$$

That is, none of the conditions (2.2), (2.3) (2.4), (2.6) (*for* $j = 2$), (2.9), (2.10) *and* (2.13) (*for* $j = 1$), *is satisfied. In addition, observe that conditions* (2.6) *and* (2.13) *do not lead to oscillation for first iteration. On the contrary, condition* (3.1) *is satisfied from the first iteration. This means that our condition is better and much faster than* (2.6) *and* (2.13).

In addition,

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} P_1(u)du\right)ds \simeq 4.8243 < \frac{2}{1-a-\sqrt{1-2a-a^2}} \simeq 110.85,$$

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(s)} P_1(u)du\right)ds \simeq 0.7983$$

$$< \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-a-\sqrt{1-2a-a^2}}{2} \simeq 0.9815,$$

$$\liminf_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(s)} P_1(u)du\right)ds = \alpha = 0.125 < \frac{1}{e},$$

that is, none of the conditions (3.20) (*for* $j = 1$), (3.24) (*for* $j = 1$) *and* (3.31) (*for* $j = 1$), *is satisfied.*

The next example concerns the result in Theorem 2. It will be apparent that it may imply oscillation when other known criteria cited in the paper (including Theorem 1) fail.

Example 2. *Consider the retarded differential equation*

$$x'(t)+\frac{25}{27e}x(\tau(t))=0, \qquad t\geq 0, \tag{4.5}$$

with (see Fig. 2, blue line)

$$\tau(t)=t-1.5+\sin(2t), \qquad t\geq 0.$$

By (2.5)*, we see (Fig. 2, red line) that*

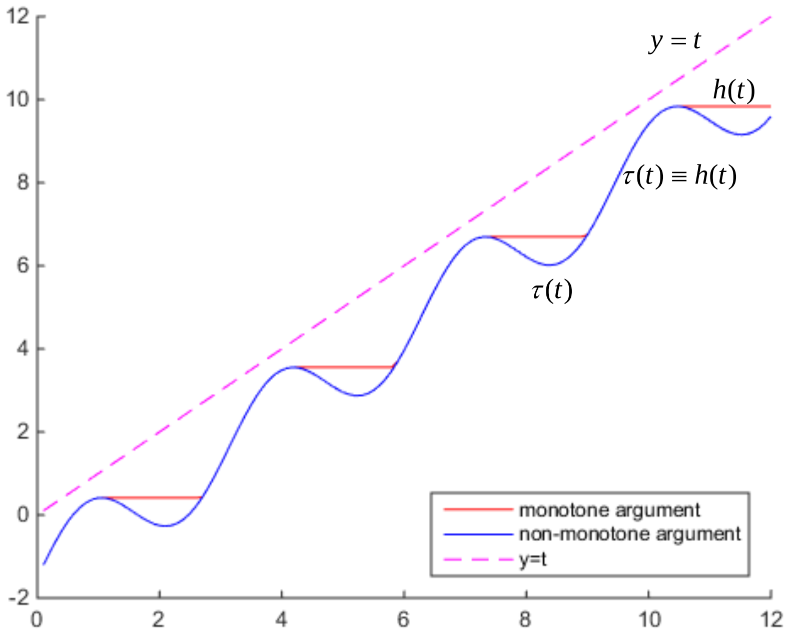

Figure 2: The graphs of $\tau(t)$ and $h(t)$

$$h(t)=\begin{cases} t-1.5+\sin(2t), & \text{if } t\in[0,\pi/3]\cup\bigcup_{k=0}^{\infty}[2.6938+k\pi,(k+1)\pi+\pi/3] \\ \frac{2\pi-9+3\sqrt{3}}{6}+k\pi & \text{if } t\in\bigcup_{k=0}^{\infty}[k\pi+\pi/3,2.6938+k\pi] \end{cases}.$$

It is easy to see that

$$\alpha=\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds=\liminf_{k\to\infty}\int_{\pi/4+k\pi-0.5}^{\pi/4+k\pi}\frac{25}{27e}ds\simeq 0.170314556<\frac{1}{e}.$$

Observe that the function F_j defined by (4.2) *in Example 1, attains its maximum at $t=2.6938+k\pi$, $k\in\mathbb{N}_0$, for every $j\in\mathbb{N}$. By using an algorithm on MATLAB software, we obtain*

$$\limsup_{t\to\infty}F_1(t)\simeq 0.9836>1-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}\simeq 0.9629.$$

That is, condition (3.19) *of Theorem 2 is satisfied for* $j = 1$*, and therefore all solutions of* (4.5) *oscillate.*

However,

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds = \limsup_{k\to\infty} \int_{\frac{2\pi-9+3\sqrt{3}}{6}+k\pi}^{2.6938+k\pi} \frac{25}{27e} ds \simeq 0.7768 < 1,$$

and the value of the constant α *is found to be*

$$\alpha \simeq 0.170314556 < \frac{1}{e}.$$

Consequently, the smaller solution of the equation $e^{\alpha\lambda} = \lambda$ *is approximately* $\lambda_0 = 1.23386$*, so*

$$0.7768 < \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.9629,$$

indicating that condition (2.4)*does not hold.*

Observe that the function Φ_2 *defined by* (4.3) *in Example 1 attains its maximum at* $t = 2.6938 + k\pi$*,* $k \in \mathbb{N}_0$*. Specifically, we find*

$$\limsup_{t\to\infty} \Phi_2(t) \simeq 0.7971 < 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.9821,$$

and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p(u)du\right) ds \simeq 0.8776 < 1,$$

$$0.8776 < 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.9821.$$

Also, specifically for the function $G_1 : \mathbb{R}_0 \to \mathbb{R}_+$ *defined by* (4.4) *in Example 1, we find*

$$\limsup_{t\to\infty} G_1(t) = \limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(t)} p_1(u)du\right) ds \simeq 0.9555 < 1,$$

so we see that

$$0.9555 < 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.9821.$$

That is, none of conditions (3.1) (*for* $j = 1$), (2.2), (2.3) (2.4), (2.6) (*for* $j = 2$), (2.9), (2.10), (2.13) (*for* $j = 1$) *and* (2.14) (*for* $j = 1$)*, is satisfied. Consequently, all criteria given in Section 2 fail to apply, neither does Theorem 1. In addition, observe that conditions* (3.1), (2.6), (2.13) *and* (2.14) *do not lead to oscillation for first iteration. On the contrary, condition* (3.19) *is satisfied from the first iteration. This means that our condition is better and much faster than* (3.1), (2.6), (2.13) *and* (2.14).

In addition,

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} P_1(u)du\right) ds \simeq 3.87 < \frac{2}{1-a-\sqrt{1-2a-a^2}} \simeq 55.974,$$

$$\liminf_{t\to\infty}\int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{h(s)} P_1(u)du\right)ds = \alpha \simeq 0.170314556 < \frac{1}{e}.$$

That is, none of conditions (3.20) *(for $j = 1$) and* (3.31) *(for $j = 1$) is satisfied, so Theorems 3 and 5 do not apply from the first iteration.*

Our last example in this section deals with the result of Theorem 3.

Example 3. *Consider the retarded differential equation*

$$x'(t) + \frac{97}{625}x(\tau(t)) = 0, \qquad t \geq 0, \tag{4.6}$$

where $\tau(t)$ is defined as in Example 1.

It is easy to see that

$$\alpha = \liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds = \liminf_{k\to\infty}\int_{7k+1}^{7k+2} p(s)ds = 0.1552 < \frac{1}{e}.$$

As before, we may see that the function $\widehat{F}_j$ ($j \in \mathbb{N}$) defined by

$$\widehat{F}_j(t) = \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} P_j(u)du\right)ds, \quad (j \in \mathbb{N}),$$

attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$, for every $j \in \mathbb{N}$. An algorithm on MATLAB software gives

$$\limsup_{t\to\infty}\widehat{F}_1(t) \simeq 69.8327 > \frac{2}{1-\alpha-\sqrt{1-2\alpha-\alpha^2}} \simeq 68.9412,$$

that is, condition (3.20) *of Theorem 3 is satisfied for $j = 1$, and therefore all solutions of (4.6) oscillate.*

However, we find

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)ds = \limsup_{k\to\infty}\int_{8k+3}^{8k+44/6}\frac{97}{625}ds \simeq 0.6725 < 1,$$

and

$$\alpha = 0.1552 < \frac{1}{e},$$

so

$$0.6725 < \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.97,$$

where $\lambda_0 = 1.2058$ is the smaller solution of $e^{0.1552\lambda} = \lambda$.

Recalling that the function Φ_j defined as in Example 1, attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$, for every $j \geq 2$. Specifically, we find

$$\limsup_{t\to\infty}\Phi_2(t) \simeq 0.84 < 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.9855,$$

that is, none of conditions (2.2), (2.3), (2.4) *and* (2.6) (*for* $j = 1$) *is satisfied.*

In addition,

$$\liminf_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} P_1(u)du \right) ds = \alpha = 0.1552 < \frac{1}{e},$$

that is, condition (3.31) (*for* $j = 1$) *is not satisfied.*

In conclusion, Theorem 3 yields oscillation while none of the criteria in Section 2 applies, neihter does Theorem 5.

Finally, let us consider the differential inequalities

$$x'(t) + p(t)x(\tau(t)) \le 0, \quad t \ge t_0 \tag{4.7}$$

and

$$x'(t) + p(t)x(\tau(t)) \ge 0, \quad t \ge t_0 \tag{4.8}$$

with p and τ as in the differential equation (E). Similarly to the corresponding definitions for equation (E) we may briefly say that by the term *solution* x of either of these inequalities we mean a continuously differentiable function which satisfies (4.7) or (4.8) for all $t \ge t_0$. It is not difficult to see that slight modifications in the proofs of Theorems 1-5 may lead to the following oscillation results.

Theorem 6. *Assume that the conditions of Theorem 1 (or 2, or 3, or 4, or 5) hold. Then, the retarded differential inequality (4.7) has no eventually positive solutions, and, the retarded differential inequality (4.8) has no eventually negative solutions.*

5 Advanced differential equations

We now consider the advanced differential equation

$$x'(t) - q(t)x(\sigma(t)) = 0, \quad t \ge t_0, \tag{E'}$$

where $q(t) \ge 0$ and $\sigma(t)$ are continuous functions with (1.2) holding.

An early oscillation result for the equation (E') is Theorem 2.4.3 in [25], stating that if

$$\sigma \text{ is nondecreasing} \qquad \text{and} \qquad \limsup_{t\to\infty} \int_{t}^{\sigma(t)} q(s)ds > 1, \tag{5.1}$$

then all solutions of (E') oscillate.

In correspondance with the results by Ladas in [21], and, by Koplatadze and Chanturija [17] for the retarded equation (E), Fukagai and Kusano [10] proved that if

$$\beta := \liminf_{t\to\infty} \int_{t}^{\sigma(t)} q(s)ds > \frac{1}{e}, \tag{5.2}$$

then all solutions of (E') oscillate, while if

$$\int_{t}^{\sigma(t)} q(s)ds \le \frac{1}{e} \quad \text{for all sufficiently large } t,$$

then Eq. (E') has a nonoscillatory solution.

When the argument $\sigma(t)$ is not necessarily monotone, we set

$$\rho(t) = \inf_{s \geq t} \sigma(s), \quad t \geq t_0, \tag{5.3}$$

and we immediately see that the function $\rho(t)$ is nondecreasing and $\sigma(t) \geq \rho(t) > t$ for all $t \geq t_0$.

In 2015, Chatzarakis and Ocalan [6], proved that if

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q(u)du \right) ds > 1, \tag{5.4}$$

or

$$\liminf_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q(u)du \right) ds > \frac{1}{e}, \tag{5.5}$$

then all solutions of (E') oscillate.

Recently, Chatzarakis [4], [5] proved that if for some $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q_j(u)du \right) ds > 1, \tag{5.6}$$

or

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q_j(u)du \right) ds > 1 - \frac{1-\beta-\sqrt{1-2\beta-\beta^2}}{2}, \tag{5.7}$$

where

$$q_j(t) = q(t) \left[1 + \int_t^{\sigma(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q_{j-1}(u)du \right) ds \right], \tag{5.8}$$

with $q_0(t) = q(t)$, then all solutions of (E') oscillate.

Oscillation conditions analogous to those obtained for the retarded equation (E) may be derived for the (dual) advanced differential equation (E') by following similar arguments with the ones employed for obtaining Theorems 1-5. The corresponding Theorems are stated below while their proofs are omitted, as they are quite similar to those for Theorems 1-5.

Theorem 7. *Assume that $\rho(t)$ is defined by* (5.3)*, and for some $j \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} Q_j(u)du \right) ds > 1, \tag{5.9}$$

where

$$Q_j(t) = q(t) \left[1 + \int_t^{\sigma(t)} q(s) \exp\left(\int_t^{\sigma(s)} q(u) \exp\left(\int_u^{\sigma(u)} Q_{j-1}(\xi)d\xi \right) du \right) ds \right], \tag{5.10}$$

with $Q_0(t) = q(t)$. Then all solutions of (E') oscillate.

Theorem 8. *Assume that*

$$0 < \beta := \liminf_{t\to\infty} \int_t^{\sigma(t)} q(s)ds \le \frac{1}{e} \tag{5.11}$$

and for some $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(t)}^{\sigma(s)} Q_j(u)du\right) ds > 1 - \frac{1-\beta-\sqrt{1-2\beta-\beta^2}}{2}, \tag{5.12}$$

where Q_j *is defined by* (5.10) *and* $\rho(t)$ *by* (5.3)*. Then all solutions of* (E') *oscillate.*

Theorem 9. *Assume that* (5.11) *holds and for some* $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s)\exp\left(\int_t^{\sigma(s)} Q_j(u)du\right) ds > \frac{2}{1-\beta-\sqrt{1-2\beta-\beta^2}}, \tag{5.13}$$

where Q_j *is defined by* (5.10) *and* $\rho(t)$ *is defined by* (5.3)*. Then all solutions of* (E') *oscillate.*

Theorem 10. *Assume that* (5.11) *holds and for some* $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(s)}^{\sigma(s)} Q_j(u)du\right) ds > \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-\beta-\sqrt{1-2\beta-\beta^2}}{2}, \tag{5.14}$$

where Q_j *is defined by* (5.10)*,* $\rho(t)$ *is defined by* (5.3)*, and* λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\beta\lambda}$*. Then all solutions of* (E') *oscillate.*

Theorem 11. *Assume that for some* $j \in \mathbb{N}$

$$\liminf_{t\to\infty} \int_t^{\rho(t)} q(s)\exp\left(\int_{\rho(s)}^{\sigma(s)} Q_j(u)du\right) ds > \frac{1}{e}, \tag{5.15}$$

where Q_j *is defined by* (5.10) *and* $\rho(t)$ *is defined by* (5.3)*. Then all solutions of* (E') *oscillate.*

It is evident that comments analogous with those presented for the retarded equation (E) can also be made for the advanced equation (E'). Instead, we choose to present an example illustrating the result of Theorem 11, also compare (5.15) with some of the conditions found in other oscillation criteria.

Example 4. *Consider the advanced differential equation*

$$x'(t) - \frac{321}{1000}x(\sigma(t)) = 0, \quad t \ge 0, \tag{5.16}$$

where (see Fig.3, a)

$$\sigma(t) = \begin{cases} 3t - 2k + 1, & \text{if } t \in [k, k+1/2] \\ -t + 2k + 3, & \text{if } t \in [k+1/2, k+1] \end{cases}.$$

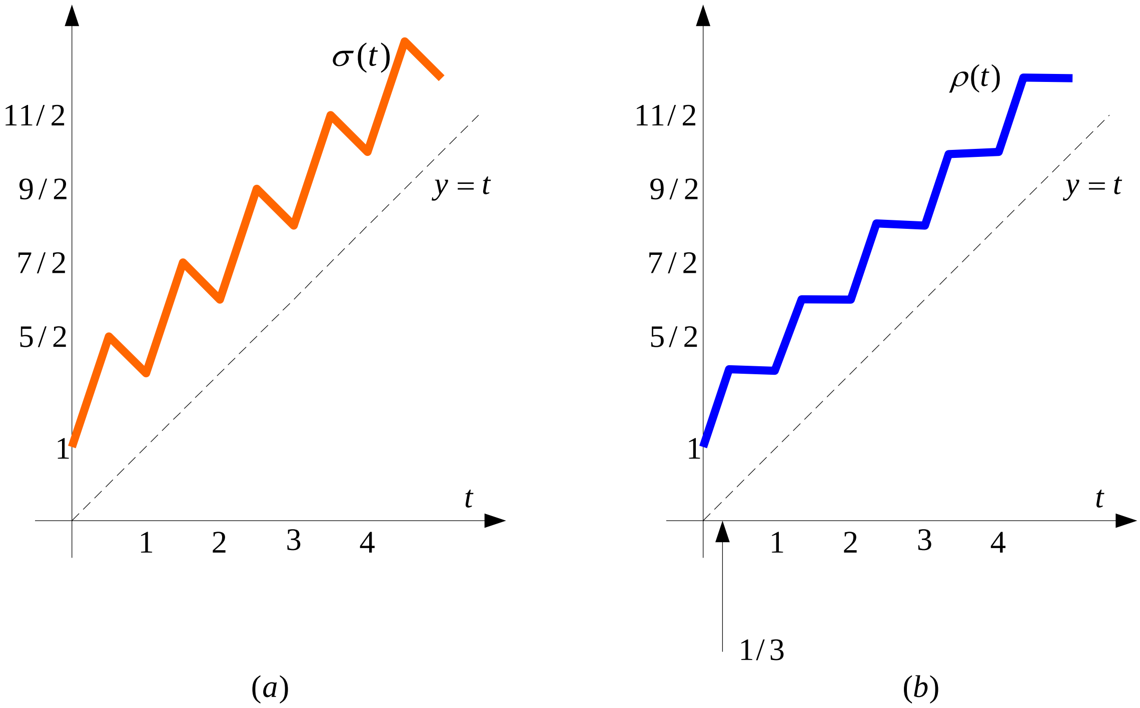

Figure 3: The graphs of $\sigma(t)$ and $\rho(t)$

By (5.3), we see (Fig.3, b) that

$$\rho(t) = \begin{cases} 3t - 2k + 1, & \text{if } t \in [k, k + 1/3] \\ k + 2, & \text{if } t \in [k + 1/3, k + 1] \end{cases} .$$

The function $\widetilde{F}_j$ $(j \in \mathbb{N})$ *defined by*

$$\widetilde{F}_j(t) = \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(s)}^{\sigma(s)} Q_j(u)du\right) ds, \quad t \geq t_0$$

attais its minimum at $t = k+1$, $k \in \mathbb{N}_0$, *for every* $j \in \mathbb{N}$. *Using an algorithm on MATLAB software, one may find that*

$$\liminf_{t\to\infty} \widetilde{F}_1(t) = \liminf_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(s)}^{\sigma(s)} Q_1(u)du\right) ds \simeq 0.3685 > \frac{1}{e}$$

that is, condition (5.15) *of Theorem 11 is satisfied for* $j = 1$, *and therefore all solutions of* (5.16) *oscillate.*

However,

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s)ds = \limsup_{k\to\infty} \int_{k+1/3}^{k+2} \frac{321}{1000} ds = 0.535 < 1,$$

$$\beta = \liminf_{t\to\infty} \int_t^{\sigma(t)} q(s)ds = \liminf_{k\to\infty} \int_{k+1}^{k+2} \frac{321}{1000} ds = 0.321 < \frac{1}{e},$$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q(u)du \right) ds \simeq 0.6783 < 1.$$

Also, the function $\widetilde{G}_j$ $(j \in \mathbb{N})$ *defined by*

$$\widetilde{G}_j(t) = \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q_j(u)du \right) ds, \quad t \geq t_0,$$

attains its maximum at $t = k + 1/3$, $k \in \mathbb{N}_0$, *for every* $j \in \mathbb{N}$. *Specifically,*

$$\limsup_{t\to\infty}(t) = \limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q_1(u)du \right) ds \simeq 0.75 < 1,$$

and

$$0.75 < 1 - \frac{1 - \beta - \sqrt{1 - 2\beta - \beta^2}}{2} \simeq 0.9129.$$

That is, none of conditions (5.1), (5.2), (5.4), (5.6) (*for* $j = 1$) *and* (5.7) (*for* $j = 1$) *is satisfied.*

In addition, we have

$$\begin{aligned} \limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_t^{\sigma(s)} Q_1(u)du \right) ds &\simeq 4.3068 \\ &< \frac{2}{1 - \beta - \sqrt{1 - 2\beta - \beta^2}} \simeq 11.4899 \end{aligned}$$

and

$$\begin{aligned} \limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \exp\left(\int_{\rho(s)}^{\sigma(s)} Q_1(u)du \right) ds &\simeq 0.6299 \\ &< \frac{1 + \ln \lambda_0}{\lambda_0} - \frac{1 - \beta - \sqrt{1 - 2\beta - \beta^2}}{2} \simeq 0.8026, \end{aligned}$$

where $\lambda_0 = 1.75857$ *is the smaller solution of* $e^{0.321\lambda} = \lambda$ *and* Q_j *is defined in* (5.10).

That is, none of conditions (5.13) (*for* $j = 1$) *and* (5.14) (*for* $j = 1$) *is satisfied.*

In conclusion, Theorem 11 applies yielding that all solutions of (5.16) oscillate while none of the criteria involving (5.1), (5.2), (5.4), (5.6) (*for* $j = 1$) *and* (5.7) (*for* $j = 1$) *are applicable, yet neither Theorem 9 nor Theorem 10 can be applied for* $j = 1$. *In addition, observe that conditions* (5.6) *and* (5.7) *do not lead to oscillation for first iteration. On the contrary, condition* (5.15) *is satisfied from the first iteration. This means that our condition is better and much faster than* (5.6) *and* (5.7).

Parallel to the differential inequalities (4.7) and (4.8) and we may consider the advanced differential inequalities

$$x'(t) - q(t)x(\sigma(t)) \geq 0, \quad t \geq t_0, \tag{5.17}$$

and

$$x'(t) - q(t)x(\sigma(t)) \leq 0, \quad t \geq t_0. \tag{5.18}$$

It is not difficult to see that by slight modifications in the proofs of Theorems 7-11 lead to the following oscillation results.

Theorem 12. *Assume that all the conditions of Theorem 7, (or Theorem 8, or Theorem 9, or Theorem 10, or Theorem 11) hold. Then the advanced differential inequality* (5.17) *has no eventually positive solutions, and, the advanced differential inequality* (5.18) *has no eventually negative solutions.*

Acknowledgement 1. *The authors would like to thank the referee for the constructive remarks which greatly improved the paper.*

References

[1] H. Akca, G. E. Chatzarakis and I. P. Stavroulakis, An oscillation criterion for delay differential equations with several non-monotone arguments, *Appl. Math. Lett.,* **59** (2016), 101−108.

[2] E. Braverman, G. E. Chatzarakis and I. P. Stavroulakis, Iterative oscillation tests for differential equations with several non-monotone arguments, *Adv. Difference Equ.*, 2016, DOI 10.1186/s13662-016-0817-3, pages 18.

[3] E. Braverman, B. Karpuz, On oscillation of differential and difference equations with non-monotone delays, *Appl. Math. Comput.*, **218** (2011) 3880–3887.

[4] G.E. Chatzarakis, Differential equations with non-monotone arguments: Iterative Oscillation results, *J. Math. Comput. Sci.,* **6** (2016), No. 5, 953−964.

[5] G.E. Chatzarakis, On oscillation of differential equations with non-monotone deviating arguments, Mediterr. J. Math., (2017) 14:82. doi: 10.1007/s00009-017-0883-0.2017.

[6] G. E. Chatzarakis and Ö. Öcalan, Oscillations of differential equations with several non-monotone advanced arguments, *Dynamical Systems: An International Journal*, DOI: 10.1080/14689367.2015.1036007, (2015), 14 pages.

[7] R. D. Driver, Can the future influence the present?, Phys. Rev., D19 (1979) 1098-1007.

[8] L. H. Erbe, Qingkai Kong and B.G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York, 1995.

[9] L. H. Erbe and B. G. Zhang, Oscillation of first order linear differential equations with deviating arguments, *Differential Integral Equations*, **1** (1988), 305−314.

[10] N. Fukagai and T. Kusano, Oscillation theory of first order functional-differential equations with deviating arguments, *Ann. Mat. Pura Appl.* **136** (1984), 95−117.

[11] I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford (1991).

[12] J. T. Hoag and R. D. Driver, A delayed-advanced model for the electrodynamics two-body problem, Nonlinear Anal. 15 (1995), 165-184.

[13] B. R. Hunt and J. A. Yorke, When all solutions of $x'(t) = -\sum q_i(t)x(t - T_i(t))$ oscillate, *J. Differential Equations* **53** (1984), 139−145.

[14] C. Jian, On the oscillation of linear differential equations with deviating arguments, *Math. in Practice and Theory,* **1** (1991), 32−40.

[15] J. Jaroš and I. P. Stavroulakis, Oscillation tests for delay equations, *Rocky Mountain J. Math.,* **29** (1999), 197−207.

[16] M. Kon, Y. G. Sficas and I. P. Stavroulakis, Oscillation criteria for delay equations, *Proc. Amer. Math. Soc.,* **128** (1994), 675−685.

[17] R. G. Koplatadze and T. A. Chanturija, Oscillating and monotone solutions of first-order differential equations with deviating argument, (Russian), *Differentsial′nyeUravneniya* **18** (1982), 1463−1465, 1472.

[18] R.G. Koplatadze and G. Kvinikadze, On the oscillation of solutions of first order delay differential inequalities and equations, *Georgian Math. J.,* **3** (1994), 675−685.

[19] T. Kusano, On even-order functional-differential equations with advanced and retarded arguments, *J. Differential Equations* **45** (1982), 75−84.

[20] M. K. Kwong, Oscillation of first-order delay equations, *J. Math. Anal. Appl.* **156** (1991), 274−286.

[21] G. Ladas, Sharp conditions for oscillations caused by delays, *Appl. Anal.* **9** (1979), 93−98.

[22] G. Ladas, V. Lakshmikantham and L. S. Papadakis, Oscillations of higher-order retarded differential equations generated by the retarded arguments, *Delay and Functional Differential Equations and their Applications*, Academic Press, New York, 1972, 219-231.

[23] G.S. Ladde, Oscillations caused by retarded perturbations of first order linear ordinary differential equations, *Atti Acad. Naz. Lincei Rendiconti* **63** (1978), 351−359.

[24] G.S. Ladde, V. Lakshmikantham, B.G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 110, Marcel Dekker, Inc., New York, 1987.

[25] X. Li and D. Zhu, Oscillation and nonoscillation of advanced differential equations with variable coefficients, *J. Math. Anal. Appl.* **269** (2002), 462−488.

[26] H. A. El-Morshedy and E. R. Attia, New oscillation criterion for delay differential equations with non-monotone arguments, *Appl. Math. Lett.,* **54** (2016), 54−59.

[27] A. D. Myshkis, Linear homogeneous differential equations of first order with deviating arguments, *Uspekhi Mat. Nauk*, **5** (1950), 160-162 (Russian).

[28] L. Page, Advanced potentials and their applications to atomic models, Phys. Rev. 24 (1924), 296-305.

[29] I. P. Stavroulakis, Oscillation criteria for delay and difference equations with non-monotone arguments, *Appl. Math. Comput.,* **226** (2014), 661−672.

[30] J. S. Yu, Z. C. Wang, B. G. Zhang and X. Z. Qian, Oscillations of differential equations with deviating arguments, *Panamer. Math. J.* **2** (1992), no. 2, 59−78.

[31] B. G. Zhang, Oscillation of solutions of the first-order advanced type differential equations, *Science exploration* **2** (1982), 79−82.

[32] D. Zhou, On some problems on oscillation of functional differential equations of first order, *J. Shandong University* 25 (1990), 434–442.