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ASYMPTOTIC BEHAVIOR FOR CURVATURE FLOW WITH DRIVING FORCE WHEN CURVATURE BLOWING UP

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Abstract. In this paper, we consider the asymptotic behavior of an axisymmetric closed curve evolving by its curvature with driving force. When the curve shrinks to a point, the asymptotic behavior will be a circle. As an easy corollary, this curve will become convex eventually. The main method in this research is the comparison principle for the ratio between extrinsic and intrinsic distances.

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1 Introduction

This research aims to study the asymptotic behavior for curvature flow with driving force when the curvature blows up. Precisely, we consider the following free boundary problem

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2},\ x \in (-b(t), b(t)),\ 0 < t < T, \tag{1.1}$$

$$u(-b(t), t) = 0,\ u(b(t), t) = 0,\ 0 \le t < T, \tag{1.2}$$

$$u_x(-b(t), t) = \infty,\ u_x(b(t), t) = -\infty,\ 0 \le t < T, \tag{1.3}$$

$$u(x, 0) = u_0(x),\ -b_0 \le x \le b_0, \tag{1.4}$$

where $u_0 \in C^\infty((-b_0, b_0)) \cap C([-b_0, b_0])$ is even and satisfies $u_0(x) > 0$, $-b_0 < x < b_0$. Moreover, we assume the curve $\Gamma_0 = \{(x, y) \mid |y| = u_0(x), -b_0 \le x \le b_0\}$ is smooth. The constant A called driving force is positive.

We say (u, b) is a solution of (1.1)-(1.4), if

(1). $b(t)$ is a positive function and $b \in C([0, T)) \cap C^1((0, T))$.

(2). $u \in C(\overline{D}_T) \cap C^{2,1}(D_T)$, where $\overline{D}_T = \cup_{0 \le t < T}\big([-b(t), b(t)] \times \{t\}\big)$ and $D_T = \cup_{0<t<T}\big((-b(t), b(t)) \times \{t\}\big)(\overline{D}_T \ne \overline{D_T})$.

(3). (u, b) satisfies (1.1)-(1.4).

The constant T denotes the maximal time such that $\Gamma(t) = \{(x, y) \in \mathbb{R}^2 \mid |y| = u(x, t), -b(t) \le x \le b(t)\}$ is smooth for $0 < t < T$. And we explain the notation in (1.3) by

$$u_x(-b(t), t) = \lim_{x \to -b(t)} u(x, t),\ u_x(b(t), t) = \lim_{x \to b(t)} u(x, t).$$

If $T < \infty$, seeing Corollary 6.6 in [12], there exists t_0 such that $u(x, t)$ loses all its local minimum, $t_0 < t < T$ and $\Gamma(t)$ shrinks to the origin, as $t \to T$. More precisely, if we let

$$h(t) = \max_{-b(t) \le x \le b(t)} u(x, t),$$

there holds $h(t) \to 0$ and $b(t) \to 0$, $t \to T$.

Noting that the initial function u_0 is even, $u(x, t)$ is also even. Therefore for every $t > t_0$, $u(x, t)$ is increasing for $x \in (-b(t), 0)$ and $u(x, t)$ is decreasing for $x \in (0, b(t))$. Moreover, $h(t) = u(0, t)$, $t > t_0$.

Main results. Next, under the case $T < \infty$, we introduce the following similarity transformation(first used by [4]):

$$z = \frac{x}{\sqrt{2(T-t)}},\ \tau = -\frac{1}{2}\ln(T-t) \tag{1.5}$$

and

$$w(z, \tau) = \frac{1}{\sqrt{2}} e^\tau u(\sqrt{2} e^{-\tau} z, T - e^{-2\tau}). \tag{1.6}$$

We also define

$$r(\tau) = \frac{1}{\sqrt{2}} e^\tau h(T - e^{-2\tau}) \text{ and } q(\tau) = \frac{1}{\sqrt{2}} e^\tau b(T - e^{-2\tau}).$$

Obviously, $r(\tau) = w(0,\tau) = \max_{-q(\tau)\le z\le q(\tau)} w(z,\tau)$, $\tau > -\frac{1}{2}\ln(T-t_0)$. Then u satisfies (1.1), (1.2), (1.3), (1.4) if and only if w satisfies

$$w_\tau = \frac{w_{zz}}{1+w_z^2} - zw_z + w + \sqrt{2}Ae^{-\tau}\sqrt{1+w_z^2},\ z \in (-q(\tau), q(\tau)),\ \tau > \tau_0, \tag{1.7}$$

$$w(-q(\tau),\tau) = w(q(\tau),\tau) = 0,\ \tau > \tau_0, \tag{1.8}$$

$$w_z(-q(\tau),\tau) = \infty,\ w_z(q(\tau),\tau) = -\infty,\ \tau > \tau_0, \tag{1.9}$$

$$w_0(z) := w(z,\tau_0) = \frac{1}{\sqrt{2T}}u_0(\sqrt{2T}z),\ z \in [-b(0)/\sqrt{2T}, b(0)/\sqrt{2T}], \tag{1.10}$$

where $\tau_0 = -\frac{1}{2}\ln T$. The stationary problem for (1.7), (1.8), (1.9), (1.10) is given by

$$\frac{\varphi_{zz}}{1+\varphi_z^2} - z\varphi_z + \varphi = 0,\ z \in (-\overline{q}, \overline{q}), \tag{1.11}$$

$$\varphi(-\overline{q}) = \varphi(\overline{q}) = 0, \tag{1.12}$$

$$\varphi_z(-\overline{q}) = \infty,\ \varphi_z(\overline{q}) = -\infty, \tag{1.13}$$

for some $\overline{q}$. Obviously, $\varphi(z) = \sqrt{1-z^2}$ and $\overline{q} = 1$ are the unique solution of the above stationary problem (1.11)-(1.13).

Here we give our main result.

Theorem 1.1 (Asymptotic behavior)**.** *The solution $(w(z,\tau), q(\tau))$ of problem (1.7)-(1.10) converges to the unique solution $(\varphi(z), \overline{q})$ of (1.11)-(1.13) pointwise, as $\tau \to +\infty$, where w and φ are considered as 0 outside the interval.*

Furthermore, there exists t_1 such that $\Gamma(t)$ is strict convex for $t_1 < t < T$. Equivalently, $u_{xx}(x,t) < 0$, for $-b(t) < x < b(t)$, $t_1 < t < T$.

Remark 1.2. Indeed, we can prove the graph of $w(z,\tau)$ converges to the graph of $\varphi(z)$ under the Hausdorff distance.

Note that our result does not assume convexity for initial data as in [9]. Indeed, we assume symmetry to the the initial curve.

Background Recently, the paper [12] has considered an axisymmetric closed curve evolving by its mean curvature flow with driving force. It classifies the solution in three categories and gives the asymptotic behavior for the case expanding and bounded. Here we recall the results in [12].

Theorem 1.3 in [12] shows that the solution (u,b) of the free boundary problem (1.1), (1.2), (1.3), (1.4) must fulfill one of the following situations.

(1) (Expanding) $T = \infty$ and that both $h(t)$ and $b(t)$ tend to ∞ as $t \to \infty$. Then there exist $R_1(t)$, $R_2(t)$ such that

$$B_{R_1(t)}((0,0)) \subset D(t) \subset B_{R_2(t)}((0,0)),$$

where $D(t) = \{(x,y) \in \mathbb{R}^2 \mid |y| < u(x,t), -b(t) < x < b(t)\}$. Moreover $\lim_{t\to\infty} R_1(t)/t = \lim_{t\to\infty} R_2(t)/t = A$.

(2) (Bounded) $T=\infty$ and that both $h(t)$ and $b(t)$ are bounded from above and below by two positive constants for $t>0$. Then $\lim_{t\to\infty} d_H(\Gamma(t), \partial B_{1/A}((0,0)))=0$.

(3) (Shrinking) $T<\infty$ and that both $h(t)$ and $b(t)$ tend to 0 as $t\to T$. Then $\Gamma(t)$ shrinks to a point at $t=T$.

We recall $h(t)=\max_{-b(t)\le x\le b(t)} u(x,t)$. Here $d_H(A,B)$ denotes the Hausdorff metric defined as

$$d_H(A,B)=\max\{\sup_{x\in A}\inf_{y\in B} d(x,y), \sup_{y\in B}\inf_{x\in A} d(x,y)\},$$

where A, B are subsets in $\mathbb{R}^2$.

The results in [12] do not contain the asymptotic behavior for the shrinking condition. Therefore, this paper is a continuation of the research in [12].

A short review for mean curvature flow. For the classical mean curvature flow $V=-\kappa$, where V denotes the outer normal velocity and κ denotes the mean curvature. Concerning this problem, Huisken [6] showed that any solution that starts out as a smooth, compact and convex surface remains so until it shrinks to a "round point", its asymptotic shape is a sphere just before it disappears. He proves this result for hypersurfaces of $\mathbb{R}^{n+1}$ with $n\ge 2$, but Gage and Hamilton [2] showed that it still holds when $n=1$, the curves in the plane. Gage and Hamilton also showed that embedded curve remains embedded, i.e. the curve will not intersect itself. Grayson [5] proved the remarkable fact that such family must become convex eventually. Thus, any embedded curve in the plane will shrink to "round point" under the curvature flow.

For the problem $V=-\kappa+A$, where $A>0$, in [3], they investigate the equation (1.1) with the following free boundary condition and appropriate initial data

$$u(l_-(t),t)=0=u(l_+(t),t)=0,\ 0\le t<T, \tag{1.14}$$

$$u_x(l_-(t),t)=\tan\psi_-,\ u_x(l_+(t),t)=-\tan\psi_+,\ 0\le t<T, \tag{1.15}$$

where $0<\psi_-,\psi_+<\pi/2$. They also give the classifications of the solution into three types like the results in [12]. Moreover, for the shrinking case, they prove that the curve will become convex eventually. Since the contact angles $0<\psi_-,\psi_+<\pi/2$, it is easy to get $|u_x(x,t)|\le M$, $l_-(t)\le x\le l_+(t)$, $0<t<T$, for some $M>0$. Then there exists $C>0$ such that

$$l_+(t)-l_-(t)\ge C\int_{l_-(t)}^{l_+(t)}\sqrt{1+u_x^2(x,t)}dx.$$

But in our condition, seeing $\psi_-=\psi_+=\pi/2$, the derivative u_x is unbounded. We have to find a new method to get the conclusion

$$b(t)\ge C\int_{l_-(t)}^{l_+(t)}\sqrt{1+u_x^2(x,t)}dx.$$

The most important tool in this paper is the comparison principle for extrinsic and intrinsic distances. Let the flow $\mathbf{G}:[0,L_*(t)]\times[0,T)\to\mathbb{R}^2$ be the smooth closed curves evolving by the classical curve shortening flow

$$\frac{\partial}{\partial t}\mathbf{G}(s,t)=\frac{\partial^2}{\partial s^2}\mathbf{G}(s,t),$$

where s denotes the arc length parameter, $L_*(t)$ denotes the perimeter of $G(\cdot, t)$. For any two points on mean curvature flow $\mathbf{G}(s,t)$, denoted by $\mathbf{G}(s_1,t)$, $\mathbf{G}(s_2,t)$. Denote $d = |\mathbf{G}(s_1,t) - \mathbf{G}(s_2,t)|$. $l = |s_2 - s_1|$ is the length of the curve between $\mathbf{G}(s_1,t)$, $\mathbf{G}(s_2,t)$. More precisely, l and d are called the intrinsic and extrinsic distances, respectively. The paper [7] shows that $m(t) = \min_{(s_1,s_2)\in[0,L_*(t)]\times[0,L_*(t)]} (d/l)(s_1,s_2,t)$ is non-decreasing in time. The ratio between extrinsic and intrinsic distance is also used by [10] and [11].

In our problem, if we let $\mathbf{F}$ satisfy

$$\Gamma(t) = \{(x,y) \in \mathbb{R}^2 \mid |y| = u(x,t), -b(t) \leq x \leq b(t)\} = \{\mathbf{F}(s,t) \in \mathbb{R}^2 \mid s \in [0, L(t)]\},$$

$L(t)$ denotes the perimeter of $\Gamma(t)$. Then $\mathbf{F}$ satisfies

$$\frac{\partial}{\partial t}\mathbf{F}(s,t) = \frac{\partial^2}{\partial s^2}\mathbf{F}(s,t) - A\mathbf{N},$$

where $\mathbf{N}$ denotes the unit inner normal vector. We will see the result in [7] does not hold. In section 2, we can see the curvature flow with driving force does not intersect itself interior, but could intersects itself exterior(Section 4).

The rest of this paper is organized as follows. In Section 2, we first recall the basic facts for curvature flow with driving force. The similar results for $A = 0$ are given in [2]. Next, a special comparison principle for extrinsic and intrinsic distances will be proved in this section. In Section 3, the proof of Theorem 1.1 will be given by using Lyapunov function. In section 4, we give an example for the comparison principle for extrinsic and intrinsic distances not holding.

2 Comparison principle between extrinsic and intrinsic distances

In this section, we will give the proof of the comparison principle between extrinsic and intrinsic distances.

First, we give some basic results for general mean curvature flow with driving force. For $A = 0$, the results are proved by Gage and Hamilton in [2]. Let M be an one-dimension Riemannian manifold and $\mathbf{F} : M \times [0,T) \to \mathbb{R}^2$ be a smooth map. $\mathbf{F}$ satisfies

$$\frac{\partial}{\partial t}\mathbf{F}(p,t) = \kappa\mathbf{N} - A\mathbf{N}, \tag{2.1}$$

where the sign of κ is determined by

$$\frac{\partial^2}{\partial s^2}\mathbf{F}(s,t) = \kappa\mathbf{N},$$

where we recall $\mathbf{N}$ is the unit inner normal velocity, s is the arc length parameter.

In this section, for convenience, we take $M = \mathbb{S}^1$ with parameter p. Let $\mathbf{F} : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a closed embedded curve moving by (2.1).

Using the arclength parameter s,

$$\frac{\partial}{\partial s}=\frac{1}{v}\frac{\partial}{\partial p},$$

where $v=|\partial\mathbf{F}/\partial p|$. The sign of κ will be determined by

$$\frac{\partial^2\mathbf{F}}{\partial s^2}=\kappa\mathbf{N}.$$

Let $\mathbf{T}$ be the unit tangent vector given by

$$\mathbf{T}=\frac{\partial\mathbf{F}/\partial p}{|\partial\mathbf{F}/\partial p|}.$$

The Frenet equations show that

$$\frac{1}{v}\frac{\partial}{\partial p}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\end{pmatrix}=\begin{pmatrix}0 & \kappa\\ -\kappa & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\end{pmatrix}.$$

Define θ by $\mathbf{T}=(\cos\theta,\sin\theta)$. We can deduce that

$$\frac{\partial s}{\partial\theta}=\frac{1}{\kappa}.$$

Lemma 2.1.

$$\frac{\partial v}{\partial t}=-\kappa^2 v+A\kappa v.$$

Proof. By (2.1) and the Frenet equations,

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial|\partial\mathbf{F}/\partial p|}{\partial t}=\langle\mathbf{T},\mathbf{F}_{pt}\rangle=\langle\mathbf{T},\mathbf{F}_{tp}\rangle=\langle\mathbf{T},(\kappa\mathbf{N}-A\mathbf{N})_p\rangle\\ &= \langle\mathbf{T},(\kappa-A)\mathbf{N}_p\rangle=\langle\mathbf{T},-v\kappa(\kappa-A)\mathbf{T}\rangle=-\kappa^2 v+A\kappa v.\end{aligned}$$

□

Lemma 2.2. *Denote* $l=\int_{p_1}^{p_2} v dp=s(p_2)-s(p_1)$, $p_1,p_2\in\mathbb{S}^1$, *then*

$$\frac{\partial l}{\partial t}=A\int_{s(p_1)}^{s(p_2)}\kappa ds-\int_{s(p_1)}^{s(p_2)}\kappa^2 ds.$$

In particular, $dL(t)/dt=2\pi A-\int_0^{L(t)}\kappa^2 ds$, *where we recall* $L(t)$ *is the perimeter of the curve.*

Proof. Using $\partial v/\partial t=-\kappa^2 v+A\kappa v$ and $\partial\theta/\partial s=\kappa$, this lemma can be proved at once. □

We note that the arc length parameter s depends on t, then $\partial/\partial t$ does not commute with $\partial/\partial s$. The following lemma gives the relation between them.

Lemma 2.3.

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}=\frac{\partial}{\partial s}\frac{\partial}{\partial t}+(\kappa^2-A\kappa)\frac{\partial}{\partial s}.$$

Proof. Apply Lemma 2.1, we get

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial s} &= \frac{\partial}{\partial t}\left(\frac{1}{v}\frac{\partial}{\partial p}\right) = \frac{\partial}{\partial s}\frac{\partial}{\partial t} + \frac{\partial}{\partial t}\left(\frac{1}{v}\right)\frac{\partial}{\partial p} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} - \frac{\partial v/\partial t}{v^2}\frac{\partial}{\partial p}\\ &= \frac{\partial}{\partial s}\frac{\partial}{\partial t} + (\kappa^2 - A\kappa)\frac{\partial}{\partial s}.\end{aligned}$$

□

The derivatives of **T** and **N** are related as follows:

Lemma 2.4.

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \kappa}{\partial s}\mathbf{N} \text{ and } \frac{\partial \mathbf{N}}{\partial t} = -\frac{\partial \kappa}{\partial s}\mathbf{T}.$$

Proof. By Lemma 2.3, (2.1) and Frenet equations,

$$\begin{aligned}\frac{\partial \mathbf{T}}{\partial t} &= \frac{\partial^2 \mathbf{F}}{\partial t \partial s} = \frac{\partial^2 \mathbf{F}}{\partial s \partial t} + (\kappa^2 - A\kappa)\frac{\partial \mathbf{F}}{\partial s} = \frac{\partial}{\partial s}(\kappa\mathbf{N} - A\mathbf{N}) + (\kappa^2 - A\kappa)\mathbf{T}\\ &= \frac{\partial \kappa}{\partial s}\mathbf{N} - (\kappa^2 - A\kappa)\mathbf{T} + (\kappa^2 - A\kappa)\mathbf{T} = \frac{\partial \kappa}{\partial s}\mathbf{N}.\end{aligned}$$

On the other hand,

$$0 = \frac{\partial}{\partial t}\langle \mathbf{T}, \mathbf{N}\rangle = \langle \frac{\partial \kappa}{\partial s}\mathbf{N}, \mathbf{N}\rangle + \langle \mathbf{T}, \frac{\partial \mathbf{N}}{\partial t}\rangle.$$

Note that $\partial \mathbf{N}/\partial t$ must be perpendicular to **N**. We complete the proof. □

Lemma 2.5.

$$\frac{\partial \theta}{\partial t} = \frac{\partial \kappa}{\partial s}$$

Proof. Since $\mathbf{T} = (\cos\theta, \sin\theta)$

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \theta}{\partial t}(-\sin\theta, \cos\theta).$$

On the other hand, we use the formula in Lemma 2.4 to calculate

$$\frac{\partial \mathbf{T}}{\partial t} = -\frac{\partial \kappa}{\partial s}\mathbf{N} = \frac{\partial \kappa}{\partial s}(-\sin\theta, \cos\theta).$$

Comparing components the proof is completed. □

Lemma 2.6. *Let $S(t)$ be the area enclosed by the curve $\mathbf{F}(\cdot, t)$. Then*

$$\frac{d}{dt}S(t) = -2\pi + AL(t).$$

Proof. By Gauss-Green's Theorem,

$$S(t) = -\frac{1}{2}\int_0^{L(t)} \langle \mathbf{F}, \mathbf{N}\rangle ds.$$

Using above lemmas, we get

$$\begin{aligned}
\frac{d}{dt}S(t) &= -\frac{1}{2}\int_0^{2\pi}\langle\frac{\partial \mathbf{F}}{\partial t}, v\mathbf{N}\rangle dp - \frac{1}{2}\int_0^{2\pi}\langle\mathbf{F}, \frac{\partial v}{\partial t}\mathbf{N}\rangle dp - \frac{1}{2}\int_0^{2\pi}\langle\mathbf{F}, v\frac{\partial \mathbf{N}}{\partial t}\rangle dp \\
&= -\frac{1}{2}\int_0^{2\pi}\langle\kappa\mathbf{N} - A\mathbf{N}, v\mathbf{N}\rangle dp - \frac{1}{2}\int_0^{2\pi}\langle\mathbf{F}, (-\kappa^2 v + A\kappa v)\mathbf{N}\rangle dp \\
&+ \frac{1}{2}\int_0^{2\pi}\langle\mathbf{F}, v\frac{\partial \kappa}{\partial s}\mathbf{T}\rangle dp = -\pi + \frac{1}{2}AL(t) - \frac{1}{2}\int_0^{L(t)}\langle\mathbf{F}, A\kappa\mathbf{N}\rangle ds \\
&+ \frac{1}{2}\int_0^{L(t)}\langle\mathbf{F}, \kappa^2\mathbf{N}\rangle ds - \frac{1}{2}\int_0^{L(t)}\kappa ds - \frac{1}{2}\int_0^{L(t)}\kappa^2\langle\mathbf{F}, \mathbf{N}\rangle ds \\
&= -2\pi + \frac{1}{2}AL(t) - \frac{1}{2}\int_0^{L(t)}\langle\mathbf{F}, A\frac{\partial \mathbf{T}}{\partial s}\rangle ds = -2\pi + \frac{1}{2}AL(t) + \frac{A}{2}\int_0^{L(t)} ds \\
&= -2\pi + AL(t).
\end{aligned}$$

In the third and fifth equalities, we use the integral by parts. □

Next we are going to prove the comparison principle for extrinsic and intrinsic distances under mean curvature flow with driving force in a special case.

Theorem 2.7. *For our flow*

$$\Gamma(t) = \{(x, y) \mid |y| = u(x, t), -b(t) \le x \le b(t)\}, t_0 < t < T,$$

let $d = 2x$ *and* $l(x, t) = \int_{-x}^{x}\sqrt{1 + u_x^2}dx$, $0 \le x \le b(t)$. *Then*

$$m(t) = \min_{0\le x\le b(t)} d/\psi$$

is strictly increasing provided that $m(t) < 1$, *for* $t_0 < t < T$, *where*

$$\psi = \frac{L}{\pi}\sin\frac{l\pi}{L},$$

where we recall t_0 *is defined in Section 1 such that* $u(x, t)$ *loses all its local minimum,* $t_0 < t < T$.

Remark 2.8. (1) The quantities d and l are the extrinsic and intrinsic distances between $(-x, u(x, t))$ and $(x, u(x, t))$ and $l \le L(t)/2$. Hence $d = 2x$ and $l = 2\int_0^x\sqrt{1 + u_x^2}dx$.

(2) Noting that $\lim_{x\to 0^+} d/\psi = 1$, d/ψ can not attain its minimum which is less than 1 at $x = 0$.

Proof. Case 1: Let $0 < x_0 < b(t)$ be a minimum point of d/ψ defined through the relation

$$m(t) = (d/\psi)(x_0, t).$$

Then

$$\frac{\partial^2}{\partial x^2}\frac{d}{\psi}(x_0, t) \ge 0$$

and

$$0 = \frac{\partial}{\partial x}\frac{d}{\psi}(x_0, t) = \frac{2}{\psi} - \frac{2d\cos\alpha}{\psi^2}\sqrt{1+u_x^2},$$

where $\alpha = l(x_0, t)\pi/L$. Consequently,

$$\frac{1}{\sqrt{1+u_x^2}} = \frac{d}{\psi}\cos\alpha,$$

at $x = x_0$. Let $0 < \beta < \pi/2$ satisfy $\tan\beta = -u_x(x_0, t)$(recall $u_x(x_0, t) < 0$), then

$$\cos\beta = \frac{1}{\sqrt{1+u_x^2}}(x_0, t) = \left(\frac{d}{\psi}\cos\alpha\right)(x_0, t). \tag{2.2}$$

Since $d/\psi(x_0, t) < 1$, we observe that $0 < \alpha < \beta < \pi/2$. Moreover,

$$\begin{aligned} 0 &\leq \frac{\partial^2}{\partial x^2}\frac{d}{\psi}(x_0, t) = -\frac{4\cos\alpha}{\psi^2}\sqrt{1+u_x^2} - \frac{4\cos\alpha}{\psi^2}\sqrt{1+u_x^2} + \frac{8d}{\psi^3}\cos^2\alpha(1+u_x^2) \\ &+ \frac{4\pi d\sin\alpha}{L\psi^2}(1+u_x^2) - \frac{2d\cos\alpha}{\psi^2}\frac{u_x u_{xx}}{\sqrt{1+u_x^2}} = \frac{4\pi d\sin\alpha}{L\psi^2}(1+u_x^2) - \frac{2d\cos\alpha}{\psi^2}\frac{u_x u_{xx}}{\sqrt{1+u_x^2}} \\ &= \frac{4\pi^2 d}{L^2\psi}(1+u_x^2) - \frac{2d\cos\alpha}{\psi^2}\frac{u_x u_{xx}}{\sqrt{1+u_x^2}}, \end{aligned}$$

where we invoked (2.2) and $\psi = L/\pi\sin(l\pi/L)$. Consequently,

$$-\frac{2d\cos\alpha}{\psi^2}\frac{u_x u_{xx}}{(1+u_x^2)^{3/2}}(x_0, t) \geq -\frac{4\pi^2 d}{L^2\psi}(x_0, t). \tag{2.3}$$

$$\frac{\partial l}{\partial t}(x_0, t) = \frac{\partial}{\partial t}\left(\int_{-x}^{x}\sqrt{1+u_x^2}dx\right)(x_0, t) = \int_{-x_0}^{x_0}\frac{u_x}{\sqrt{1+u_x^2}}du_t = \frac{2u_x u_t}{\sqrt{1+u_x^2}}(x_0, t)$$

$$\begin{aligned} &- \int_{-x_0}^{x_0}\frac{u_t u_{xx}}{(1+u_x^2)^{3/2}}dx = \frac{2u_x u_{xx}}{(1+u_x^2)^{3/2}}(x_0, t) + 2Au_x(x_0, t) - \int_0^l \kappa^2 ds \\ &- 2A\arctan u_x(x_0, t) = \frac{2u_x u_{xx}}{(1+u_x^2)^{3/2}}(x_0, t) - 2A\tan\beta - \int_0^l \kappa^2 ds + 2A\beta, \end{aligned}$$

where we again invoked (2.2) and $\tan\beta = -u_x(x_0, t)$. Using the Hölder inequality, we have

$$l\int_0^l \kappa^2 ds \geq \left(\int_0^l \kappa ds\right)^2 = 4\beta^2 \tag{2.4}$$

and

$$L\int_0^L \kappa^2 ds \geq \left(\int_0^L \kappa ds\right)^2 = 4\pi^2. \tag{2.5}$$

$$\begin{aligned}
m'(t) &= \frac{d}{dt}\left(\frac{d}{\psi}\right)(x_0,t) = -\frac{d\sin\alpha}{\psi^2\pi}\left(2\pi A - \int_0^L \kappa^2 ds\right) + \frac{dl\cos\alpha}{\psi^2 L}\left(2\pi A - \int_0^L \kappa^2 ds\right)\\
&- \frac{d\cos\alpha}{\psi^2}\left(\frac{2u_x u_{xx}}{(1+u_x^2)^{3/2}}(x_0,t) - 2A\tan\beta - \int_0^l \kappa^2 ds + 2A\beta\right)\\
&= \frac{2Ad\cos\alpha}{\psi^2}\big((\tan\beta-\beta)-(\tan\alpha-\alpha)\big) - \frac{d\cos\alpha}{\psi^2}\frac{2u_x u_{xx}}{(1+u_x^2)^{3/2}}(x_0,t) + \frac{d\sin\alpha}{\psi^2\pi}\int_0^L \kappa^2 ds\\
&- \frac{dl\cos\alpha}{\psi^2 L}\int_0^L \kappa^2 ds + \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds > -\frac{4d\pi^2}{\psi L^2} + \frac{d\cos\alpha}{\psi^2\pi}(\tan\alpha-\alpha)\int_0^L \kappa^2 ds\\
&+ \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds \geq -\frac{4d\pi^2}{\psi L^2} + \frac{4\pi d\cos\alpha}{\psi^2 L}(\tan\alpha-\alpha) + \frac{4\beta^2 d\cos\alpha}{l\psi^2} = -\frac{4\pi d\alpha\cos\alpha}{\psi^2 L}\\
&+ \frac{4\beta^2 d\cos\alpha}{l\psi^2} = \frac{4\beta^2 d\cos\alpha}{l\psi^2} - \frac{4\alpha^2 d\cos\alpha}{l\psi^2} > 0,
\end{aligned}$$

where we use (2.2), (2.3), (2.4), (2.5) and $\tan\alpha - \alpha$ is increasing, $0 < \alpha < \pi/2$.

Case 2: For $x_0 = b(t)$ such that

$$m(t) = (d/\psi)(x_0,t).$$

Since $u(x,t)$ is increasing for $-b(t) < x < 0$ and decreasing for $0 < x < b(t)$, $t_0 < t < T$, we let $x = v(y,t)$ be the inverse of $y = u(x,t)$ in the first quadrant. Consider

$$\mathcal{L}(y,t) = \begin{cases} 2\int_y^{h(t)} \sqrt{1+v_y^2(y,t)}dy, \ y > 0,\\ L(t) - 2\int_y^{h(t)} \sqrt{1+v_y^2(y,t)}dy, \ y \leq 0,\end{cases}$$

recalling $h(t) = u(0,t) = \max_{-b(t)<x<b(t)} u(x,t)$.

It is easy to see $l(x,t) = \mathcal{L}(u(x,t),t)$, for $0 \leq x \leq b(t)$, specially, $l(b(t),t) = \mathcal{L}(0,t)$. Since $y = 0$ is an interior point and $\psi = \frac{L}{\pi}\sin\frac{\mathcal{L}\pi}{L}$ is smooth, we can prove this case similarly as in case 1. The proof is now complete. □

Similarly, we can obtain

Theorem 2.9. *For our flow*

$$\Gamma(t) = \{(x,y) \mid |y| = u(x,t), -b(t) \leq x \leq b(t)\}, \ t_0 < t < T,$$

where t_0 is the same as in Theorem 2.7. Let

$$d = 2y, \ \text{and } l = 2\int_0^y \sqrt{1+v_y^2(y,t)}dy, \ 0 \leq y \leq h(t),$$

where $v(y,t)$ is the inverse of $u(x,t)$ in the first quadrant as in the proof of Theorem 2.7. Then

$$m(t) = \min_{0\leq y\leq h(t)} d/\psi$$

is strictly increasing provided that $m(t) < 1$, $t_0 < t < T$.

Using Theorems 2.7 and 2.9, we obtain

Corollary 2.10. *There exists a constant $C > 0$ such that*

$$d \geq Cl,\ t_0 < t < T,$$

where d and l are the extrinsic and intrinsic distances in Theorem 2.7 or 2.9. In particular,

$$h(t) \geq CL(t) \text{ and } b(t) \geq CL(t), t_0 < t < T.$$

Remark 2.11. To explain the geometric meaning in the proof of Theorem 2.7, we will give the calculation in geometric method for closed curve moving by (2.1).

Let $\mathbf{F} : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a closed embedded curve moving by (2.1). In this remark, we let

$$d(p_1, p_2, t) = |\mathbf{F}(p_1, t) - \mathbf{F}(p_2, t)|,\ l(p_1, p_2, t) = |s(p_1) - s(p_2)|,$$

where s denotes the arc length parameter at time t. ψ is also defined as in Theorem 2.7 by

$$\psi = \frac{L}{\pi} \sin \frac{l\pi}{L}.$$

We define

$$m(t) = \min_{(p_1,p_2)\in\mathbb{S}^1\times\mathbb{S}^1} d/\psi(p_1, p_2, t).$$

Assume that d/ψ attains its minimum at $(p_1, p_2) \in \mathbb{S}^1 \times \mathbb{S}^1$, i.e.,

$$m(t) = (d/\psi)(p_1, p_2, t) < 1.$$

Here we abuse the notation (p_1, p_2) to shorten the notations in the following argument.

Let s be the arc length parameter at time t and without loss of generality $0 \leq s(p_1) < s(p_2) < L/2$ such that $l(p_1, p_2, t) = s(p_2) - s(p_1)$. Next we represent l, d by arclength parameter

$$l = s_2 - s_1 \text{ and } d = |\mathbf{F}(s_1, t) - \mathbf{F}(s_2, t)|.$$

Then

$$\frac{\partial}{\partial s_i}(d/\psi)(p_1, p_2, t) = 0,\ i = 1, 2 \text{ and } \big(\frac{\partial^2}{\partial s_i \partial s_j}(d/\psi)\big)_{2\times 2}(p_1, p_2, t) \geq 0.$$

Let

$$e_i := \frac{\partial \mathbf{F}}{\partial s_i}(p_1, p_2, t) \text{ and } \omega := \frac{\mathbf{F}(p_2, t) - \mathbf{F}(p_1, t)}{d(p_1, p_2, t)}.$$

Then there holds

$$0 = \frac{\partial}{\partial s_1}(d/\psi)(p_1, p_2, t) = -\frac{\langle \omega, e_1 \rangle}{\psi} + \frac{d}{\psi^2} \cos \alpha,$$

where $\alpha = l(p_1, p_2, t)\pi/L = (s(p_2) - s(p_1))\pi/L \in (0, \pi/2)$. Consequently,

$$\langle \omega, e_i \rangle = \frac{d}{\psi} \cos \alpha,\ i = 1, 2 \tag{2.6}$$

at (p_1, p_2, t). We can choose $0 < \beta < \pi/2$ such that

$$\cos\beta = \langle \omega, e_i \rangle = d/\psi \cos\alpha < \cos\alpha. \tag{2.7}$$

Then $\beta > \alpha$.

Since matrix $\big(\frac{\partial^2}{\partial s_i \partial s_j}(d/\psi)\big)_{2\times 2}(p_1, p_2, t)$ is non-negative, then for every vector $\xi \in \mathbb{R}^2$ there holds

$$\xi\Big(\frac{\partial^2}{\partial s_i \partial s_j}(d/\psi)\Big)_{2\times 2}(p_1, p_2, t)\xi^t \geq 0, \tag{2.8}$$

where ξ^t denotes the transposition of ξ.

In view of relations of (2.6), there are two possible cases:
Case 1: $e_1 = e_2$. We choose $\xi = (1, 1)$ in (2.8).

$$0 \leq (1,1)\Big(\frac{\partial^2}{\partial s_i \partial s_j}(d/\psi)\Big)_{2\times 2}(p_1, p_2, t)(1,1)^t = \frac{1}{\psi}\langle \omega, (\kappa\mathbf{N})(p_2, t) - (\kappa\mathbf{N})(p_1, t)\rangle. \tag{2.9}$$

Case 2: $e_1 \neq e_2$. We choose $\xi = (1, -1)$ in (2.8).

$$\begin{aligned} 0 &\leq (1,-1)\Big(\frac{\partial^2}{\partial s_i \partial s_j}(d/\psi)\Big)_{2\times 2}(p_1, p_2, t)(1,-1)^t \\ &= \frac{1}{\psi}\langle \omega, (\kappa\mathbf{N})(p_2, t) - (\kappa\mathbf{N})(p_1, t)\rangle + \frac{4\pi^2 d}{L^2\psi}. \end{aligned}$$

Then

$$-\frac{4\pi^2 d}{L^2\psi} \leq \frac{1}{\psi}\langle \omega, (\kappa\mathbf{N})(p_2, t) - (\kappa\mathbf{N})(p_1, t)\rangle. \tag{2.10}$$

Since there is no t derivative in above calculation, more precise calculation is necessary which is found in [7], Theorem 2.3. Here we safely omit it.

Therefore, by (2.1) and Lemma 2.2

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{d}{\psi}\right) &= -\frac{d}{\psi^2}\frac{\partial \psi}{\partial t} + \frac{1}{\psi}\frac{\partial d}{\partial t} = -\frac{d}{\psi^2}\left(\frac{1}{\pi}\frac{dL}{dt}\sin\alpha + \frac{\partial l}{\partial t}\cos\alpha - \frac{l}{L}\frac{dL}{dt}\cos\alpha\right) \\ &+ \frac{1}{d\psi}\langle \omega, \frac{\partial}{\partial t}\mathbf{F}(p_2, t) - \frac{\partial}{\partial t}\mathbf{F}(p_1, t)\rangle = -\frac{d}{\psi^2}\left(\frac{1}{\pi}(2\pi A - \int_0^L \kappa^2 ds)\sin\alpha\right. \\ &+ \left.(A\int_0^l \kappa ds - \int_0^l \kappa^2 ds)\cos\alpha - \frac{l}{L}(2\pi A - \int_0^L \kappa^2 ds)\cos\alpha\right) \\ &+ \frac{1}{d\psi}\langle \omega, (\kappa - A)\mathbf{N}(p_2, t) - (\kappa - A)\mathbf{N}(p_1, t)\rangle = -\frac{2Ad}{\psi^2}\sin\alpha \\ &- \frac{dA}{\psi^2}\cos\alpha\int_0^l \kappa ds + \frac{2\pi dlA}{\psi^2 L}\cos\alpha - \frac{A}{\psi}\langle \omega, \mathbf{N}(p_2, t) - \mathbf{N}(p_1, t)\rangle \\ &+ \frac{1}{\psi}\langle \omega, (\kappa\mathbf{N})(p_2, t) - (\kappa\mathbf{N})(p_1, t)\rangle + \frac{d\sin\alpha}{\pi\psi^2}\int_0^L \kappa^2 ds + \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds \\ &- \frac{dl}{\psi^2 L}\cos\alpha\int_0^L \kappa^2 ds. \end{aligned}$$

In the following step, we assume that

$$-\frac{A}{\psi}\langle \omega, \mathbf{N}(p_2,t)-\mathbf{N}(p_1,t)\rangle > 0. \tag{2.11}$$

Seeing Figure 1, there holds

$$-\frac{A}{\psi}\langle \omega, \mathbf{N}(p_2,t)-\mathbf{N}(p_1,t)\rangle = \frac{2A}{\psi}\sin\beta. \tag{2.12}$$

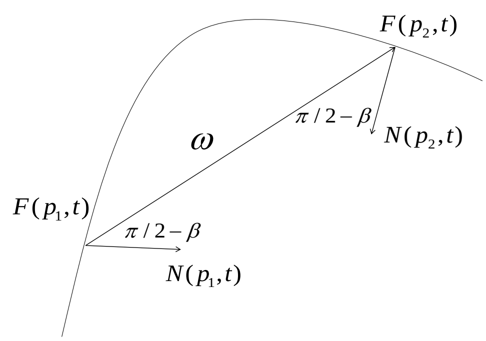

Figure 1: Assumption (2.11)

Case 1: $e_1 = e_2$. By calculation,

$$\frac{dA}{\psi^2}\cos\alpha\int_0^l \kappa ds = 0.$$

Then

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{d}{\psi}\right) &\geq -\frac{2Ad}{\psi^2}\sin\alpha + \frac{2\pi dlA}{\psi^2 L}\cos\alpha + \frac{2A}{\psi}\sin\beta + \frac{d\sin\alpha}{\pi\psi^2}\int_0^L \kappa^2 ds + \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds \\ &- \frac{dl}{\psi^2 L}\cos\alpha\int_0^L \kappa^2 ds \geq \frac{2A}{\psi}\left(\sin\beta - \frac{d}{\psi}\sin\alpha\right) + \frac{d}{\pi\psi^2}(\sin\alpha - \alpha\cos\alpha)\int_0^L \kappa^2 ds \\ &> 0,\end{aligned}$$

where we use (2.7), (2.9), $d/\psi < 1$ and $\sin\alpha - \alpha\cos\alpha > 0$, for $0 < \alpha < \pi/2$.
Case 2: $e_1 \neq e_2$.

Using Hölder inequality,

$$l\int_0^l \kappa^2 ds \geq \left(\int_0^l \kappa ds\right)^2 = 4\beta^2$$

and

$$L\int_0^L \kappa^2 ds \geq \left(\int_0^L \kappa ds\right)^2 = 4\pi^2.$$

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{d}{\psi}\right) &\geq -\frac{2Ad}{\psi^2}\sin\alpha - \frac{2\beta dA}{\psi^2}\cos\alpha + \frac{2\pi dlA}{\psi^2 L}\cos\alpha + \frac{2A}{\psi}\sin\beta + \frac{d\sin\alpha}{\pi\psi^2}\int_0^L \kappa^2 ds \\
&+ \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds - \frac{dl}{\psi^2 L}\cos\alpha\int_0^L \kappa^2 ds - \frac{4\pi^2 d}{L^2\psi} \\
&\geq \frac{2A}{\psi}\left(\sin\beta - \beta\cos\beta - (\frac{d}{\psi})(\sin\alpha - \alpha\cos\alpha)\right) + \frac{d}{\pi\psi^2}(\sin\alpha - \alpha\cos\alpha)\int_0^L \kappa^2 ds \\
&+ \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds - \frac{4\pi^2 d}{L^2\psi} \geq \frac{4\pi^2 d}{\pi L\psi^2}(\sin\alpha - \alpha\cos\alpha) + \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds - \frac{4\pi^2 d}{L^2\psi} \\
&= -\frac{4d\alpha^2\cos\alpha}{l\psi^2} + \frac{d\cos\alpha}{\psi^2}\int_0^l \kappa^2 ds \geq -\frac{4d\alpha^2\cos\alpha}{l\psi^2} + \frac{4d\beta^2\cos\alpha}{l\psi^2} > 0,
\end{aligned}$$

where we use (2.7), (2.10), (2.12), $d/\psi < 1$, $\beta > \alpha$ and $\sin\alpha - \alpha\cos\alpha$ is increasing for $0 < \alpha < \pi/2$.

A sufficient condition for the assumption (2.11) is that the line connecting $\mathbf{F}(p_2, t)$ and $\mathbf{F}(p_1, t)$ lies in the domain surrounded by the curve. In Theorem 2.7, the conclusion that d/ψ is increasing provided that $d/\psi < 1$ is true in the direction $(2x_0, 0)$ instead of all directions, since the line connecting $(-x_0, u(x_0, t))$ and $(x_0, u(x_0, t))$ just enough lies in the domain surrounded by the curve $\Gamma(t)$. This is the key point under the condition $A > 0$. We cannot guarantee that d/ψ is non-decreasing in every direction even if d/ψ is very small. We construct such an example in Section 4.

3 Proof of Theorem 1.1

Lemma 3.1. *For the shrinking case in Theorem C, there exist $C_1, C_2 > 0$ such that*

$$C_1 \leq \frac{b(t)}{\sqrt{T-t}} \leq C_2 \text{ and } C_1 \leq \frac{h(t)}{\sqrt{T-t}} \leq C_2, t_0 < t < T.$$

Proof. Since $u(x, t)$ has only one maximum at $x = 0$, it is easy to see that $0 \leq L(t) \leq 4h(t) + 4b(t) \to 0$, $0 \leq S(t) \leq 4b(t)h(t) \to 0$, $t \to T$. Using Lemma 2.6 and $S(t) \to 0$, $L(t) \to 0$ as $t \to T$, there holds

$$S(t) = 2\pi(T-t) - A\int_t^T L(s)ds = 2\pi(T-t) + o(T-t).$$

By isoperimeter inequality $L(t)^2 \geq 4\pi S(t)$,

$$\liminf_{t\to T}\frac{L(t)^2}{T-t} \geq \lim_{t\to T}\frac{4\pi S(t)}{T-t} = 8\pi^2.$$

Using Corollary 2.10, there exists $C > 0$ such that

$$h(t) \geq CL(t) \text{ and } b(t) \geq CL(t).$$

Then there exists $C_1 > 0$ such that

$$\liminf_{t\to T}\frac{b(t)}{\sqrt{T-t}} \geq C_1 \text{ and } \liminf_{t\to T}\frac{h(t)}{\sqrt{T-t}} \geq C_1.$$

Using similarity transformation (1.5) and (1.6), there exists $\widetilde{C_1} > 0$ such that

$$r(\tau) \geq \widetilde{C_1} \text{ and } q(\tau) \geq \widetilde{C_1}.$$

We next prove upper bounds for $r(\tau)$, $q(\tau)$ by contradiction argument. Assume that if there exists a sequence $\tau_k \to \infty$ such that $r(\tau_k) \to \infty$. $\widetilde{S}(\tau)$ denotes the area enclosed by $w(z,\tau)$ and axis z. By calculation,

$$\widetilde{S}(\tau) = 2\int_0^{q(\tau)} w(z,\tau)dz = \frac{\int_0^{b(t)} u(x,t)dx}{T-t} = \frac{S(t)}{4(T-t)} \leq C,$$

for some C. Since $w(z,\tau_k)$ is even in z and $w(z,\tau_k)$ is monotone decreasing for $z > 0$,

$$\widetilde{C_1} w(-\frac{\widetilde{C_1}}{2}, \tau_k) \leq \widetilde{S}(\tau_k) \leq C, \ \forall k.$$

Consequently, $w(-\widetilde{C_1}/2, \tau_k)$ is bounded for all k. Consider the extrinsic and intrinsic distances between $(-\widetilde{C_1}/2, w(-\widetilde{C_1}/2, \tau_k))$ and $(\widetilde{C_1}/2, w(\widetilde{C_1}/2, \tau_k))$ after transformation, denoted by $\widetilde{d}(\tau_k)$ and $\widetilde{l}(\tau_k)$, respectively. Then there hold $\widetilde{d}(\tau_k) = \widetilde{C_1}$ and $r(\tau_k) - w(-\widetilde{C_1}/2, \tau_k) < \widetilde{l}(\tau_k)$. By the argument above, since $w(-\widetilde{C_1}/2, \tau_k)$ is bounded, $\widetilde{l}(\tau_k) \to \infty$, as $k \to \infty$. Then $\widetilde{d}(\tau_k)/\widetilde{l}(\tau_k) \to 0$, as $k \to \infty$.

Consider the extrinsic and intrinsic distance between

$$(-\sqrt{2(T-t_k)}\widetilde{C}_1/2, u(-\sqrt{2(T-t_k)}\widetilde{C}_1/2, t_k)) \text{ and } (\sqrt{2(T-t_k)}\widetilde{C}_1/2, u(\sqrt{2(T-t_k)}\widetilde{C}_1/2, t_k)),$$

denoted by $d(t_k)$ and $l(t_k) < L(t_k)/2$, respectively. By calculation,

$$d(t_k) = \sqrt{2(T-t_k)}\widetilde{d}(\tau_k) \text{ and } l(t_k) = \sqrt{2(T-t_k)}\widetilde{l}(\tau_k).$$

Then $d(t_k)/l(t_k) = \widetilde{d}(\tau_k)/\widetilde{l}(\tau_k) \to 0$, as $k \to \infty$, which contradicts to Corollary 2.10. Therefore, $r(\tau)$ is bounded. Similarly it also holds for $q(\tau)$. Consequently,

$$C_1 \leq \frac{b(t)}{\sqrt{T-t}} \leq C_2 \text{ and } C_1 \leq \frac{h(t)}{\sqrt{T-t}} \leq C_2.$$

□

For the lemma above, it is obvious that there exist $D_1, D_2 > 0$ such that $D_1 < r(\tau) < D_2$ and $D_1 < q(\tau) < D_2$.

Since $w(z,\tau)$ is increasing for $-q(\tau) < z < 0$ and decreasing for $0 < z < q(\tau)$, $\tau > -\frac{1}{2}\ln(T-t_0)$, we can represent $w = w(z,\tau)$ under polar coordinate,

$$\begin{cases} z = \rho(\theta,\tau)\cos\theta, \\ w(z,\tau) = \rho(\theta,\tau)\sin\theta, \end{cases}$$

$0 \leq \theta \leq \pi$, $\tau > -\frac{1}{2}\ln(T-t_0)$. Consequently, $\rho(\theta,\tau)$ satisfies

$$\rho_\tau = \frac{\rho_{\theta\theta}}{\rho^2+\rho_\theta^2} - \frac{2\rho_\theta^2+\rho^2}{\rho(\rho_\theta^2+\rho^2)} + \rho + \frac{\sqrt{2}}{\rho}Ae^{-\tau}\sqrt{\rho_\theta^2+\rho^2}, \ 0 < \theta < \pi, \ \tau > -\frac{1}{2}\ln(T-t_0), \quad (3.1)$$

$$\rho_\theta(0,\tau) = \rho_\theta(\pi,\tau) = 0, \ \tau > -\frac{1}{2}\ln(T-t_0). \quad (3.2)$$

Lemma 3.2. *For any given $\epsilon > 0$, there exist positive constant C_k and B_k such that*

$$|\frac{\partial^k}{\partial\theta^k}\rho(\theta,\tau)| < C_k,\ |\frac{\partial^k}{\partial\tau^k}\rho(\theta,\tau)| < B_k,\ k = 1,2,\cdots,\ 0 \le \theta \le \pi,\ \tau \ge -\frac{1}{2}\ln(T - t_0) + \epsilon.$$

Proof. Firstly, we prove that there exist constants ρ_1, $\rho_2 > 0$ such that $\rho_1 \le \rho \le \rho_2$.

Since $r(\tau) < D_2$, $q(\tau) < D_2$ and $w(z,\tau)$ has only one maximum point at $x = 0$, it is easy to get $\rho < \sqrt{2}D_2 := \rho_2$.

Consider the intrinsic and extrinsic distances, $\widetilde{l}(\tau)$ and $\widetilde{d}(\tau)$, respectively, between $(W(D_1/2,\tau), D_1/2)$ and $(-W(D_1/2,\tau), D_1/2)$, where $z = W(r,\tau)$ is the inverse of $r = w(z,\tau)$, for $z \ge 0$. By Corollary 2.10, $\widetilde{d}(\tau) \ge C\widetilde{l}(\tau)$. Note that $\widetilde{d}(\tau) = 2W(D_1/2,\tau)$ and $\widetilde{l}(\tau) \ge r(\tau) - D_1/2 \ge D_1/2$. Then $W(D_1/2,\tau) \ge CD_1/4$. Since $z = W(r,\tau)$ is decreasing with respective to r, $W(r,\tau) \ge W(D_1/2,\tau) \ge CD_1/4$, $0 \le r \le D_1/2$. It is easy to see $\rho > \min\{D_1/2, CD_1/4\} := \rho_1$.

Next, we are going to prove our main result. We extend ρ by even and periodic in θ. Using the interior estimates in [8], we can get

$$|\frac{\partial^k}{\partial\theta^k}\rho(\theta,\tau)| < C_k,\ |\frac{\partial^k}{\partial\tau^k}\rho(\theta,\tau)| < B_k,\ 0 \le \theta \le \pi,\ \tau \ge -\frac{1}{2}\ln(T - t_0) + \epsilon.$$

□

Proof of Theorem 1.1. Firstly, We introduce the following Lyapunov functional borrowed from [6](The Lyapunov functional also is used by [3]):

$$E[w(\cdot,\tau)] = \int_{-q(\tau)}^{q(\tau)} \exp\left\{-\frac{z^2 + w^2(z,\tau)}{2}\right\}\sqrt{1 + w_z^2(z,\tau)}dz.$$

We can compute that

$$\frac{d}{d\tau}E[w(\cdot,\tau)] = -\int_{-q(\tau)}^{q(\tau)} w_\tau^2(z,\tau)\exp\left\{-\frac{z^2 + w^2(z,\tau)}{2}\right\}(1 + w_z^2(z,\tau))^{-1/2}dz + J,$$

where

$$J = \sqrt{2}Ae^{-\tau}\int_{-q(\tau)}^{q(\tau)} \exp\left\{-\frac{z^2 + w^2(z,\tau)}{2}\right\} w_\tau(z,\tau)dz.$$

We consider the following integral

$$\left|\int_{-q(\tau)}^{q(\tau)} \exp\left\{-\frac{z^2 + w^2(z,\tau)}{2}\right\} w_\tau(z,\tau)dz\right| \le \int_{-q(\tau)}^{q(\tau)} \left|\frac{w_{zz}}{1 + w_z^2} - zw_z + w + \sqrt{2}Ae^{-\tau}\sqrt{1 + w_z^2}\right| dz$$

$$\le \left\{\int_{-q(\tau)}^{q(\tau)} \left|\frac{w_{zz}}{(1 + w_z^2)^{3/2}}\right| + |z|\frac{|w_z|}{\sqrt{1 + w_z^2}} + \frac{w}{\sqrt{1 + w_z^2}} + \sqrt{2}A\right\}\sqrt{1 + w_z^2}dz.$$

We note that $|q(\tau)|$, $|w(z,\tau)|$ are bounded. By Lemma 3.2, the curvature $|w_{zz}/(1 + w_z^2)^{3/2}| = |(-\rho\rho_{\theta\theta} + 2\rho_\theta^2 + \rho^2)/(\rho_\theta^2 + \rho^2)^{3/2}|$ is bounded, $0 \le \theta \le \pi$, $\tau > -\frac{1}{2}\ln(T - t_0) + \epsilon$.

Then

$$|J| \le C_1\sqrt{2}Ae^{-\tau}\int_{-q(\tau)}^{q(\tau)}\sqrt{1+w_z^2}dz \le C_1\sqrt{2}Ae^{-\tau}(2r(\tau)+2q(\tau)) \le Ce^{-\tau},$$

for $\tau > -\frac{1}{2}\ln(T-t_0)+\epsilon$. Consequently,

$$\int_{-\frac{1}{2}\ln(T-t_0)+\epsilon}^{\infty}|J|d\tau < \infty.$$

We note that

$$E(w(\cdot,\tau)) \le 2r(\tau)+2q(\tau) \le C,\ \tau > -\frac{1}{2}\ln(T-t_0)+\epsilon.$$

Therefore

$$\int_{-\frac{1}{2}\ln(T-t_0)+\epsilon}^{\infty}\int_{-q(\tau)}^{q(\tau)} w_\tau^2(z,\tau)\exp\left\{-\frac{z^2+w^2(z,\tau)}{2}\right\}(1+w_z(z,\tau))^{-1/2}dzd\tau < \infty.$$

Finally, it suffices to show that, for any sequence $\tau_n \to +\infty$, the sequence $(w(z,\tau_n), q(\tau_n))$ has a subsequence that converges to $(\varphi,\overline{q})$, as $n\to\infty$, where $(\varphi,\overline{q})$ is the solution of (1.11)-(1.13)(more precisely, the graph of $r=w(z,\tau_n)$ converges to the graph of $r=\varphi(z)$ under the Hausdorff distance).

We set

$$w_n(z,\tau)=w(z,\tau+\tau_n),\quad q_n(\tau)=q(\tau+\tau_n),\ \rho_n(\theta,\tau)=\rho(\theta,\tau+\tau_n),\ \tau\in[a,a+1],$$

where $a > -\frac{1}{2}\ln(T-t_0)+\epsilon$. By Lemma 3.2, $\frac{\partial^k}{\partial\theta^k}\rho_n(\theta,\tau)$ and $\frac{\partial^j}{\partial\tau^j}\rho_n(\theta,\tau)$ are uniformly bounded for n, $\theta\in[0,\pi]$, $\tau\in[a,a+1]$, $k=1,2,3,\ j=1,2$. Then there exists $\rho^*(\theta,\tau)$ such that ρ_n converges to ρ^* in $C^{2,1}([0,\pi]\times[a,a+1])$ as $n\to\infty$. Consequently, $w_n(z,\tau)$ converges to $w^*(z,\tau)$ as $n\to\infty$, where $w^*(z,\tau)=\rho^*(\theta,\tau)\sin\theta$. Obviously, $w^*(z,\tau)$ satisfies

$$w_\tau = \frac{w_{zz}}{1+w_z^2} - zw_z + w,\ z\in(-q^*(\tau),q^*(\tau)),\ \tau\in[a,a+1], \tag{3.3}$$

$$w(-q^*(\tau),\tau) = w(q^*(\tau),\tau) = 0,\ \tau\in[a,a+1], \tag{3.4}$$

$$w_z(-q^*(\tau),\tau)=\infty,\ w_z(q^*(\tau),\tau)=-\infty,\ \tau\in[a,a+1], \tag{3.5}$$

where $q^*(\tau)$ denotes the limit of $q_n(\tau)$ defined as above.

We next prove $w_\tau^*(z,\tau)=0$. By the argument of Lyapunov function above,

$$\int_a^{a+1}\int_{-q(\tau+\tau_n)}^{q(\tau+\tau_n)} w_\tau^2(z,\tau+\tau_n)\exp\left\{-\frac{z^2+w^2(z,\tau+\tau_n)}{2}\right\}(1+w_z^2(z,\tau+\tau_n))^{-1/2}dzd\tau$$

$$\le \int_{\tau_n+a}^{\infty}\int_{-q(\tau)}^{q(\tau)} w_\tau^2(z,\tau)\exp\left\{-\frac{z^2+w^2(z,\tau)}{2}\right\}(1+w_z^2(z,\tau))^{-1/2}dzd\tau.$$

Using ρ_n converges to ρ^* in $C^{2,1}([0.\pi] \times [a, a+1])$ and letting $n \to \infty$,

$$\int_a^{a+1} \int_{-q^*(\tau)}^{q^*(\tau)} (w^*_\tau)^2(z,\tau) \exp\left\{-\frac{z^2 + (w^*)^2(z,\tau)}{2}\right\} (1 + (w^*_z)^2(z,\tau))^{-1/2} dz d\tau = 0,$$

which implies $w^*_\tau \equiv 0$ for $-q^*(\tau) < z < q^*(\tau)$. So $(w^*, q(\tau))$ is a stationary solution of (3.3)-(3.5). Since the problem (1.11)-(1.13) is unique, $q^*(\tau) = \overline{q}$, where $\overline{q}$ is a constant. Therefore, we prove that $(w(z, \tau_n), q(\tau_n))$ converges to $(\varphi, \overline{q})$ up to a sequence. Therefore, we have $(w(z,\tau), q(\tau)) \to (\varphi, \overline{q})$, as $\tau \to \infty$. Indeed, $(\varphi, \overline{q}) = (\sqrt{1 - z^2}, 1)$. The proof of Theorem 1.1 is complete.

Since $\Gamma(t)$ can be represented by $\mathbf{F}(p,t) : \mathbb{S}^1 \times [0,T)$. Seeing the proof of Theorem 1.1,

$$\kappa(p,\tau) = \frac{-w_{zz}}{(1 + w_z^2)^{3/2}} \to 1, \text{ uniformly on } \mathbb{S}^1 \cap \{y \geq 0\},$$

as $\tau \to \infty$. Then for τ large enough $w_{zz} < 0$ for $-q(\tau) < z < q(\tau)$. Consequently, seeing the relation between w and u, there exists t_1 such that $u_{xx} < 0$, for $-b(t) < x < b(t)$, $t_1 < t < T$. □

4 An example for $\min d/\psi = 0$

In this section we give an example that the comparison principle for extrinsic and intrinsic distances does not hold for $A > 0$. First, we give some curves.

$$\gamma_1 = \{(x,y) \mid (x - \frac{2}{A})^2 + y^2 = R^2, -L \leq y \leq R\}.$$

where $L > 1/A$ and $L < R < 2/A$.

$$\gamma_2 = \{(x,y) \mid |x - \frac{2}{A}| = \frac{1}{2}\sqrt{R^2 - L^2}, -2L - \delta < y < -L - \delta\},$$

where $0 < \delta < \min\{L/4, 2/A - \frac{1}{2}\sqrt{(2/A)^2 - L^2}\}$.

$$\gamma_3 = \{(x,y) \mid |y + 2L + 3\delta| = \delta, 0 \leq x < \frac{2}{A} - \frac{1}{2}\sqrt{R^2 - L^2} - \delta\}.$$

We connect γ_1, γ_2, γ_3 smoothly by short curves, called Γ_1. Extend Γ_1 by even, denoted by Γ_0. Let $\Gamma(t)$ be the maximal smooth solution of $V = -\kappa + A$ with initial curve Γ_0 and we show that the curve $\Gamma(t)$ will intersect itself in a finite time. By the construction of Γ_0, there exist $1/A < R_1 < R$ such that

$$B_{R_1}(2/A, 0) \subset U, \; B_{R_1}(-2/A, 0) \subset U,$$

where U is the domain surrounded by Γ_0. Let $R_1(t)$ be the solution of

$$R_1'(t) = A - \frac{1}{R_1(t)},$$

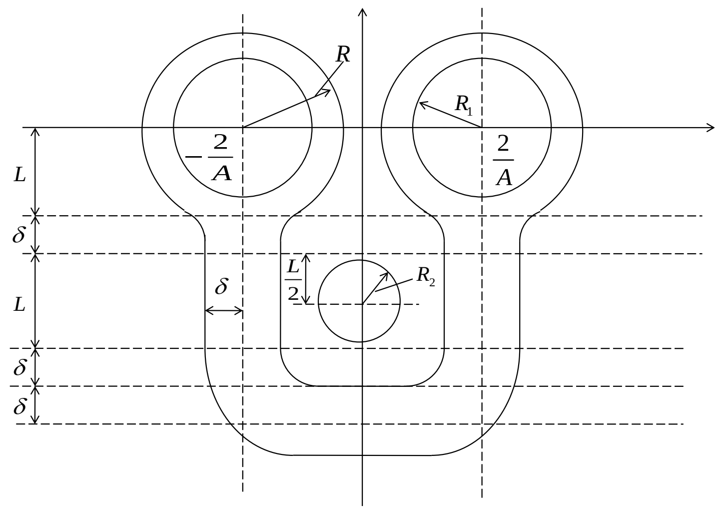

Figure 2: Initial curve Γ_0

with $R(0) = R_1$. Then $\partial B_{R_1(t)}$ evolves by $V = -\kappa + A$ with ∂B_{R_1}. By comparison principle,

$$B_{R_1(t)}(2/A, 0) \subset U(t),\ B_{R_1(t)}(-2/A, 0) \subset U(t),$$

where $U(t)$ is the domain surrounded by $\Gamma(t)$. Let $R_2(t)$ be the solution of

$$R_2'(t) = -A - \frac{1}{R_2(t)},$$

with $R_2(0) = R_2 := \min\{2/A - \sqrt{(2/A)^2 - L^2} - \delta, L/2\}$. Then $\partial B_{R_2(t)}$ evolves by $V = -\kappa - A$ with ∂B_{R_2}. Here we note the direction of the driving force must be reversed. Since $U \subset \mathbb{R}^2 \setminus B_{R_2}(0, -3L/2 - \delta)$, by comparison principle, $U(t) \subset \mathbb{R}^2 \setminus B_{R_2(t)}(0, -3L/2 - \delta)$, $0 \le t < t_2$, where t_2 is the maximal existence time of $R_2(t)$. Note that t_2 is independent on R and R_1. We can choose R and R_1 very closed to $2/A$ and seeing $R_1(t) \to \infty$ as $t \to \infty$, then there exists t_0, $t_0 < t_2$ such that

$$B_{R_1(t_0)}(2/A, 0) \cap B_{R_1(t_0)}(-2/A, 0) \neq \emptyset.$$

Combining $U(t) \subset \mathbb{R}^2 \setminus B_{R_2(t)}(0, -3L/2 - \delta)$, $0 \le t < t_2$, this implies there exists t_1, $t_1 < t_0 < t_2$ such that $\Gamma(t_1)$ intersects itself at origin. It means that $m(t_1) = \min d/\psi = 0$.

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References

[1] L. C. Evans and J. Spruck, Motion of level sets by mean curvature III, J. Geom. Anal., **2**(1992), 121-150.

[2] M. Gage and R. S. Hamilton, The heat equation shrinking convex plane curves, J. Diff. Geom., **23**(1986), 69-96.

[3] J. S. Guo, H. Matano, M. Shimojo and C.H. Wu, On a free boundary problem for the curvature flow with driving force, Archive for Rational Mechanics and Analysis March 2016, Volume 219, Issue 3, pp 1207-1272.

[4] Y. Giga, R.V. Kohn, Asymptotically self-similar blowup of semilinear heat equations, Comm. Pure Appl. Math., 38 (1985), 297-319.

[5] M. Grayson, The heat equation shrinks emmbedded plane curves to round points, J. Diff. Geom., **26**(1987), 285-314.

[6] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, J. Diff. Geom. **31**(1990), 285-299.

[7] G. Huisken, A distance comparison principle for evolving curves, Asian J. Math., **2**(1998), 127C133.

[8] O. A. Ladyzhenskaya, V. Solonnikov and N. Ural'ceva, Linear and quasilinear equations of parabolic type. Translations of Mathematical Monographs, 23 A. M. S.(1968).

[9] Z. Liu, Mean curvature flow with a constant forcing, Nonlinear Differ. Equ. Appl. ,**20**(2013), 621-649.

[10] C. Mantegazza, M. Novaga, A. Pluda, Motion by curvature of networks with two triple junctions, arXiv:1606.08011v5, 2016, 1-38.

[11] O. Schnurer, A. Azouani, M. Georgi, J. Hell, N. Jangle, A. Koller, T. Marxen, S. Ritthaler, M. Saez, F. Schulze and B. Smith, Evolution of convex lens-shaped networks under curve shortening flow. Trans. Amer. Math. Soc. **363**(2010), 2265-2294.

[12] L. J. Zhang, On curvature flow with driving force under Neumann condition in the plane, http://arxiv.org/abs/1703.10707, 1-50.